Complexity Classification in Qualitative Temporal Constraint Reasoning

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June 13, 2002

Abstract

We study the computational complexity of the *qualitative algebra* which is a temporal formalism that combines the point algebra, the point-interval algebra and Allen's interval algebra. We identify all tractable fragments and show that every other fragment is NP-complete. The use of combinatorial techniques has enabled us to prove this result without computer-assisted case analyses.

Keywords: Temporal reasoning, computational complexity.

1 Introduction

Reasoning about temporal knowledge is a common task in many branches of computer science and elsewhere, *cf.* Golumbic and Shamir [7] for a list of examples from a wide range of applications. Knowledge of temporal constraints is typically expressed in terms of collections of relations between time points and/or time intervals. Reasoning tasks include determining the satisfiability of such collections and deducing new relations from those that are known.

Several frameworks for formalizing this type of problem have been suggested (see [19] for a survey); for instance, the *point algebra* [20] (for expressing relations between time points), the *point-interval algebra* [21] (for expressing relations between time points and intervals) and the famous *Allen's interval algebra* [1] for expressing relations between time intervals. Basic temporal formalisms can only be used for reasoning about objects of a single type—for instance, the point algebra [22] is only useful for time points and Allen's interval algebra [1] is only useful for time intervals. Such restricted languages have been studied intensively from a complexity-theoretic point of view. For instance, all tractable subclasses of Allen's interval algebra, the point-interval algebra and a number of point algebras for different time models have been identified [4, 8, 10, 12, 22]

Obviously, this kind of basic formalisms may not be sufficient for modelling real-world problems so several formalisms for multisorted temporal reasoning have been proposed [3, 9, 11, 16, 18]. It is not very surprising that the basic temporal formalisms are easier to analyse (from a complexity-theoretic standpoint) than the multisorted formalisms; in fact, virtually nothing is known about tractability in more complex formalisms. The goal of this article is to study the computational complexity of a multi-sorted formalism, namely Meiri's [16] *Qualitative Algebra*. It is a temporal formalism able to represent both time points and time intervals and it is possible to relate points with points, points with intervals and intervals with intervals using an expressive set of qualitative relations. More precisely, the algebra is an amalgamation of the point algebra, the point-interval algebra and Allen's algebra. Thus, this research follows the recent trend in artificial intelligence of combining different formalisms, cf. [2, 23].

We identify all tractable fragments of the satisfiability problem and show that all other fragments are NP-complete. By using combinatorial techniques, we can prove this result without using computer-assisted enumeration methods. The key element in our approach is reducibility via expressibility – i.e. given a set of relations, we derive new relations by different methods. By analyzing the structure of relations, we show that every non-tractable fragment of the Qualitative Algebra can express some NP-complete fragment of the point-interval algebra or of Allen's algebra. Consequently, this article shows that combinatorial methods are not only useful when classifying constraint problems (as in [12]), but also for combining complexity results for different formalisms.

The article is organised as follows: in Section 2 we give the basic definitions and present the maximal tractable subclasses. In Section 3 we formally state the classification result and prove it; Subsection 3.1 contains some tractability results and Section 3.2. contains the classification proof together with descriptions of a few proof techniques. Some concluding remarks are collected in Section 4. This article is based on an incomplete classification of the Qualitative Algebra presented by Krokhin & Jonsson in a conference paper [14].

2 Preliminaries

In the Qualitative Algebra (QA) [16], a qualitative constraint between two objects O_i and O_j (each may be a point or an interval), is a disjunction of the form

$$(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$$

where each one of the $r_i's$ is a *basic qualitative relation* that may exist between two objects. There are three types of basic relations.

1. *Point-point* (PP) relations that can hold between a pair of points.
2. *Point-interval* (PI) and *interval-point* (IP) relations that can hold between a point and an interval and vice-versa.
3. *Interval-interval* (II) relations that can hold between a pair of intervals.

The PP-relations correspond to the *point algebra* [22], PI-relations to the *point-interval algebra* [21] and II-relations to *Allen's interval algebra* [1]. The basic relations are shown in Table 1. Note that we use different fonts to distinguish between PI- and II-relations. The endpoint relation $I^- < I^+$ that is required for all intervals has been omitted. For the sake of brevity, we will write expressions of the form $(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$ as $O_i(r_1 \ldots r_k)O_j$. Let $\emptyset$ denote the empty relation. Let $\mathcal{PP}$, $\mathcal{PI}$ and $\mathcal{II}$ denote the sets of all PP-relations, PI-relations and II-relations, respectively, and let $\mathcal{QA} = \mathcal{PP} \cup \mathcal{PI} \cup \mathcal{II}$.

The problem of *satisfiability* (QA-SAT) of a set of point and interval variables with relations between them is that of deciding whether there exists an assignment of points and intervals on the real line for the variables, such that all of the relations are satisfied. This is defined as follows.

Definition 1 *Let $X \subseteq \mathcal{QA}$. An instance Π of* QA-SAT(X) *consists of a set V_p of point variables, a set V_I of interval variables and a set of constraints of the form xry where $x, y \in V_p \cup V_I$ and $r \in X$. We require that $V_p \cap V_I = \emptyset$.*

The question is whether Π is satisfiable *or not, i.e. whether there exists a function M, called a* model*, satisfying the following:*

1. *for each $v \in V_p$, $M(v) \in \mathcal{R}$;*

2. *for each $v \in V_i$, $M(v) = (I^-, I^+) \in \mathcal{R} \times \mathcal{R}$ and $I^- < I^+$.*

3. *for each constraint $xry \in C$, $M(x) r M(y)$ holds.*

We note that QA-SAT is in NP; let Π be an arbitrarily chosen instance with point variables V_p and interval variables V_I. The relations are qualitative so we do not need to consider models that assign real values to the variables, it is enough to merely consider models that assign values from the finite set $\{1, \ldots, m\}$ where $m = |V_p| + 2|V_I|$, and such a model can be guessed non-deterministically in polynomial time.

Let $X \subseteq \mathcal{QA}$ and assume that $\Pi = (V_p, V_I, C)$ is an instance of QA-SAT. We define $\mathrm{Var}(\Pi)$ as the set of variables in Π and $X_{\mathcal{PP}}$, $X_{\mathcal{PI}}$, $X_{\mathcal{II}}$ as $X \cap \mathcal{PP}$, $X \cap \mathcal{PI}$, $X \cap \mathcal{II}$, respectively. We extend the notation to sets of constraints and problem instances, i.e. $\Pi_{\mathcal{II}}$ denotes the subinstance only containing II-constraints:

$$(\emptyset, V_I, \{IrJ \in C \mid I, J \in V_I\}).$$

If there exists a polynomial-time algorithm solving all instances of QA-SAT(X) then we say that X is tractable. On the other hand, if QA-SAT(X) is NP-complete then we say that X is NP-complete. Since $\mathcal{QA}$ is finite, the problem of describing tractability in $\mathcal{QA}$ can be reduced to the problem of describing the *maximal* tractable subclasses in $\mathcal{QA}$, i.e., subclasses that cannot be extended without losing tractability.

The complexity of QA-SAT(X) has been completely determined earlier when X is a subset of $\mathcal{PP}$, $\mathcal{PI}$ or $\mathcal{II}$.

Theorem 2 (Vilain *et al.* [22]) *$\mathcal{PP}$ is tractable.*

Theorem 3 (Jonsson *et al.* [10]) *Let X be a subclass of $\mathcal{PI}$. Then X is tractable if it is contained in one of the 5 subclasses $\mathcal{V}_{\mathcal{H}}, \mathcal{V}_{\mathcal{S}}, \mathcal{V}_{\mathcal{E}}, \mathcal{V}_{\mathsf{s}}$ and $\mathcal{V}_{\mathsf{f}}$ (see Table 2). Otherwise, X is NP-complete.*

In order to simplify the presentation of tractable subclasses of II-relations, we use the symbol $\pm$, which should be interpreted as follows. A condition involving $\pm$ means the conjunction of two conditions: one corresponding to $+$ and one corresponding to $-$. For example, condition $(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r$ means that both $(\mathsf{o}) \subseteq r \Leftrightarrow (\mathsf{d}) \subseteq r$ and $(\mathsf{o}^{-1}) \subseteq r \Leftrightarrow (\mathsf{d}^{-1}) \subseteq r$ hold.

Theorem 4 (Krokhin *et al.* [12]) *Let X be a subclass of $\mathcal{II}$. Then X is tractable if it is contained in one of the 18 subclasses listed in Table 3. Otherwise, X is NP-complete.*

Let $\mathcal{II}_{\text{tr}}$ denote the set of the 18 maximal tractable subclasses of II-relations. In some previous papers, the subclasses in Tables 2 and 3 were defined in other ways. However, in all cases except for $\mathcal{H}$, it is very straightforward to verify that our definitions are equivalent to the original ones. The subclass $\mathcal{H}$ was originally defined as the 'ORD-Horn algebra' [17], but has also been characterized as the set of 'pre-convex' relations (see, e.g., [15]). Using the latter description it is not hard to show that our definition of $\mathcal{H}$ is equivalent.

3 Main Result

Our main result is the identification of all tractable subclasses X of $\mathcal{QA}$. Let $\mathcal{W} \subseteq \mathcal{II}$ and $\mathcal{V} \subseteq \mathcal{PI}$. Let $\mathcal{WV} = \mathcal{W} \cup \mathcal{V} \cup \mathcal{PP}$ and $\mathcal{WV}' = \mathcal{W} \cup \mathcal{V} \cup \{=, \leq, \geq\}$.

Theorem 5 *Let $X \subseteq \mathcal{QA}$. Then* QA-SAT(X) *is tractable if and only if X is a included in one of the subclasses defined below. Otherwise,* QA-SAT(X) *is NP-complete.*

- $\mathcal{WV}_{\mathsf{b}}$ *and* $\mathcal{WV}_{\mathsf{a}}$ *if* $\mathcal{W} \in \mathcal{II}_{\text{tr}}$
- $\mathcal{WV}_{\mathsf{d}}$ *if* $\mathcal{W} \in \mathcal{II}_{\text{tr}} - \{\mathcal{H}, \mathcal{S}_p, \mathcal{E}_p\}$
- $\mathcal{HV}_{\mathcal{H}}$, $\mathcal{S}_p\mathcal{V}_{\mathcal{S}}$, $\mathcal{E}_p\mathcal{V}_{\mathcal{E}}$
- $\mathcal{WV}_{\mathcal{SH}}$ *if* $\mathcal{W} \in \{\mathcal{S}_d, \mathcal{S}_o, \mathcal{S}^*\}$
- $\mathcal{WV}_{\mathcal{EH}}$ *if* $\mathcal{W} \in \{\mathcal{E}_d, \mathcal{E}_o, \mathcal{E}^*\}$
- $\mathcal{WV}'_s$ *if* $\mathcal{W} \in \{\mathcal{E}^*, \mathcal{A}_\equiv, \mathcal{A}_1, \ldots, \mathcal{A}_4\}$
- $\mathcal{WV}'_f$ *if* $\mathcal{W} \in \{\mathcal{S}^*, \mathcal{A}_\equiv, \mathcal{B}_1, \ldots, \mathcal{B}_4\}$

The rest of this section is structured as follows. In Subsection 3.1, we prove the tractability of a number of subclasses and we give the proof of Theorem 5 in Subsection 3.2.

Basic relation		Example	Endpoints
p before q	$<$	`p` `  q`	$p < q$
p equals q	$=$	`  p` `  q`	$p = q$
p after q	$>$	`    p` `  q`	$p > q$

Basic relation		Example	Endpoints
p before I	`b`	`p` `  III`	$p < I^-$
p starts I	`s`	`  p` `  III`	$p = I^-$
p during I	`d`	`   p` `  III`	$I^- < p < I^+$
p finishes I	`f`	`    p` `  III`	$p = I^+$
p after I	`a`	`      p` `  III`	$p > I^+$

Basic relation		Example	Endpoints
I precedes J	`p`	`III`	$I^+ < J^-$
J preceded by I	$\mathtt{p}^{-1}$	`      JJJ`	
I meets J	`m`	`IIII`	$I^+ = J^-$
J met by I	$\mathtt{m}^{-1}$	`     JJJJ`	
I overlaps J	`o`	`IIII`	$I^- < J^- < I^+$,
J overl. by I	$\mathtt{o}^{-1}$	`  JJJJ`	$I^+ < J^+$
I during J	`d`	`  III`	$I^- > J^-$,
J includes I	$\mathtt{d}^{-1}$	`JJJJJJJ`	$I^+ < J^+$
I starts J	`s`	`III`	$I^- = J^-$,
J started by I	$\mathtt{s}^{-1}$	`JJJJJJJ`	$I^+ < J^+$
I finishes J	`f`	`    III`	$I^+ = J^+$,
J finished by I	$\mathtt{f}^{-1}$	`JJJJJJJ`	$I^- > J^-$
I equals J	$\equiv$	`IIII` `JJJJ`	$I^- = J^-$, $I^+ = J^+$

Table 1: Basic PP-, PI- and II-relations.

$\mathcal{V}_{\mathcal{H}} = \{r \mid r \cap (\texttt{bs}) \neq \emptyset \;\&\; r \cap (\texttt{fa}) \neq \emptyset \Rightarrow (\texttt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{SH}} = \{r \mid \; r \cap (\texttt{fa}) \neq \emptyset \Rightarrow (\texttt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{EH}} = \{r \mid r \cap (\texttt{bs}) \neq \emptyset \Rightarrow (\texttt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{S}} = \{r \mid r \cap (\texttt{df}) \neq \emptyset \Rightarrow (\texttt{a}) \subseteq r\}$
$\mathcal{V}_{\mathcal{E}} = \{r \mid r \cap (\texttt{sd}) \neq \emptyset \Rightarrow (\texttt{b}) \subseteq r\}$
$\mathcal{V}_{\mathsf{r}} = \{r \mid r \neq \emptyset \Rightarrow (\mathsf{r}) \subseteq r\}$ where $\mathsf{r} \in \{\texttt{b}, \texttt{s}, \texttt{d}, \texttt{f}, \texttt{a}\}$

Table 2: Subsets of PI-relations.

3.1 Tractability results

We shall now show that all subclasses in Theorem 5 are tractable. In fact, Lemma 6 prove a slightly stronger result which will be useful in the proof of the main theorem.

Lemma 6 *$\mathcal{W}\mathcal{V}_{\mathsf{b}}$ and $\mathcal{W}\mathcal{V}_{\mathsf{a}}$ are tractable if and only if $\mathcal{W} \subseteq \mathcal{S}$ for some $\mathcal{S} \in \mathcal{II}_{\text{tr}}$. Otherwise, they are NP-complete.*

Proof. If $\mathcal{W}$ is not a subset of a member of $\mathcal{II}_{\text{tr}}$, then both $\mathcal{W}\mathcal{V}_{\mathsf{b}}$ and $\mathcal{W}\mathcal{V}_{\mathsf{a}}$ are NP-complete by Theorem 4. Thus, we assume $\mathcal{W}$ is tractable and give a proof for the case $X = \mathcal{W}\mathcal{V}_{\mathsf{b}}$; the other case is analogous. Let Π be an arbitrary instance of $\textsc{Qa-Sat}(X)$ and assume without loss of generality that no constraint is trivially unsatisfiable, i.e. of the form $x\emptyset y$. We claim that Π is satisfiable iff $\Pi_{\mathcal{PP}}$ and $\Pi_{\mathcal{II}}$ are satisfiable—obviously, this can be checked in polynomial time by the choice of $\mathcal{W}$.

If $\Pi_{\mathcal{PP}}$ or $\Pi_{\mathcal{II}}$ are not satisfiable, then Π is not satisfiable. Otherwise, there exists two models $M_{\mathcal{PP}}$ and $M_{\mathcal{II}}$ of $\Pi_{\mathcal{PP}}$ and $\Pi_{\mathcal{II}}$, respectively. We can, without loss of generality, assume that $M_{\mathcal{PP}}$ has the following additional property: $M_{\mathcal{PP}}(p) < M_{\mathcal{II}}(I^-)$ for all $p \in \text{Var}(\Pi_{\mathcal{PP}})$ and $I \in \text{Var}(\Pi_{\mathcal{II}})$. We construct a model M of Π as follows:

$$M(x) = \begin{cases} M_{\mathcal{PP}}(x) & \text{if } x \in \text{Var}(\Pi_{\mathcal{PP}}) \\ M_{\mathcal{II}}(x) & \text{if } x \in \text{Var}(\Pi_{\mathcal{II}}) \end{cases}$$

It follows that M is a model of Π since every constraint in $\Pi_{\mathcal{PI}}$ contains the relation $\texttt{b}$. $\square$

Lemma 7 *$\mathcal{W}\mathcal{V}_{\mathsf{d}}$ is tractable if $\mathcal{W} \in \mathcal{II}_{\text{tr}} - \{\mathcal{H}, \mathcal{S}_{\mathsf{p}}, \mathcal{E}_{\mathsf{p}}\}$.*

Proof. Assume Π is a satisfiable instance of $\textsc{Qa-Sat}(X)$ where $X \in \mathcal{II}_{\text{tr}} - \{\mathcal{H}, \mathcal{S}_{\mathsf{p}}, \mathcal{E}_{\mathsf{p}}\}$ By analyzing the correctness proofs of the algorithms for these

$$\mathcal{S}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$
$$\mathcal{S}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{S}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_1 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_2 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_3 = \{r \mid r \cap (\mathsf{pmodf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_4 = \{r \mid r \cap (\mathsf{pmodf}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$

$$\mathcal{E}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$
$$\mathcal{E}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r\}$$
$$\mathcal{E}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_1 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_2 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_3 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_4 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$

$$\mathcal{E}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ff}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{S}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ss}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{H} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset \ \&\ r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

$$\mathcal{A}_{\equiv} = \{r \mid r \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$$

Table 3: The tractable subalgebras of Allen's algebra.

subclasses [5, 6], one can notice that Π always has a model M in which the intersection of all intervals is itself a non-empty interval, say J.

Thus, we can use a similar trick as in the proof of Lemma 6: instead of moving the points to a position before or after the intervals, we scale the points and move them to a position within the interval J. $\square$

For proving tractability of the remaining subclasses, we define the function $S : \mathcal{QA} \to \mathcal{II}$ such that

$$
\begin{array}{ll}
S(<) = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) & S(=) = (\equiv \mathsf{ss}^{-1}) \\
S(>) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) & S(\mathtt{b}) = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \\
S(\mathtt{s}) = (\equiv \mathsf{ss}^{-1}) & S(\mathtt{d}) = (\mathsf{o}^{-1}\mathsf{df}) \\
S(\mathtt{f}) = (\mathsf{m}^{-1}) & S(\mathtt{a}) = (\mathsf{p}^{-1})
\end{array}
$$

and $S(r) = r$ if r is a basic II-relation. We extend S such that $S(r) = S(r_1) \cup \ldots \cup S(r_n)$ if $r = (r_1, \ldots, r_n)$, and given a set $X \subseteq \mathcal{QA}$, we define $S(X) = \{S(r) \mid r \in X\}$.

The idea is to transform instances of QA-SAT(X) into instances of QA-SAT$(X \cap \mathcal{II})$—this will avoid the need for constructing completely new algorithms.

Lemma 8 *Let* $\Pi = (V_p, V_I, C)$ *be an instance of* QA-SAT(X). *Let* $V_I' = V_I$ *and* $V_p' = \{I_p' \mid p \in V_p\}$ *(where we assume that* $V_I' \cap V_p' = \emptyset$*). Define an instance*

$$\Pi' = (\emptyset, V_I' \cup \{I_p' \mid p \in V_p\}, C')$$

of QA-SAT$(\mathcal{II})$ *where* $C' = \{I_p' S(r) I_q' \mid prq \in C_{\mathcal{PP}}\} \cup \{I_p' S(r) I' \mid prI \in C_{\mathcal{PI}}\} \cup \{I' S(r) J' \mid IrJ \in C_{\mathcal{II}}\}$.

Then, Π *is satisfiable iff* Π' *is satisfiable.*

Proof. *only-if:* Let M be a model of Π. Construct an interpretation M' of Π' as follows:

1. for each interval $I' \in V_I'$, let $M'(I') = M(I)$; and
2. for each interval $I_p' \in V_p'$, let $M'(I_p') = [M(p), M(p) + 1]$.

It is straightforward to verify that M' is a model of Π'. As an example, assume that $p(\mathtt{bs})I \in C$, $M(p) = 1$ and $M(I) = [2, 4]$. Then, $I_p'(\equiv$

$\mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})I' \in C'$, $M'(I'_p) = [1,2]$ and $M'(I') = [2,4]$; consequently, the relation between I'_p and I' is satisfied.

if: Let M' be a model of Π'. Construct an interpretation M of Π as follows:

1. for each point $p \in V_p$, let $M(p) = M'(I_p^-)$; and
2. for each interval $I \in V_I$, let $M(I) = M'(I')$.

Once again, it is straightforward to verify that M is a model of Π. We take the same example as before: Assume $I'_p(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})I' \in C'$, $M'(I'_p) = [1,2]$ and $M'(I') = [2,4]$. Then, we know that $p(\mathsf{bs})I \in C$, $M(p) = 1$ and $M(I) = [2,4]$. $\square$

As is evident in the proof, function S identifies the points with the left endpoint of intervals while the relations between the right endpoints are arbitrary; thus, we can symmetrically define a function E that identifies points with the right endpoint of intervals.

$$\begin{array}{ll} E(<) = (\mathsf{pmods}) & E(=) = (\equiv \mathsf{ff}^{-1}) \\ E(>) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1}) & E(\mathsf{b}) = (\mathsf{p}) \\ E(\mathsf{s}) = (\mathsf{m}) & E(\mathsf{d}) = (\mathsf{ods}) \\ E(\mathsf{f}) = (\equiv \mathsf{ff}^{-1}) & E(\mathsf{a}) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1}) \end{array}$$

Lemma 9 *Let X be one of the subclasses in Theorem 5 that is not covered by Lemmata 6 or 7. Then, X is tractable.*

Proof. Assume X' is a tractable subset of $\mathcal{II}$. If $S(X) \subseteq X'$ or $E(X) \subseteq X'$, then X is tractable by Lemma 8. It can be verified that either $S(X)$ or $E(X)$ is a subset of $X \cap \mathcal{II}$ and the lemma follows since $X \cap \mathcal{II}$ is tractable. $\square$

3.2 Proof of Theorem 5

The proof of Theorem 5 consists of three parts where we successively restrict the allowed PP-relations. The two first parts (where we first assume $(<) \in \mathcal{S}_{\mathcal{PP}}$ and then $(<) \notin \mathcal{S}_{\mathcal{PP}}$ but $(\neq) \in \mathcal{S}_{\mathcal{PP}}$) have a similar structure. The final part (where we assume $\mathcal{S}_{\mathcal{PP}} \subseteq \{=, \leq, \geq\}$) is slightly different.

One of our main tools for proving the result is the notion of *derivations*. Suppose $X \subseteq \mathcal{QA}$ and Π is an instance of QA-SAT(X). Let the two variables

x, y appear in Π. Furthermore, let $r \in \mathcal{QA}$ be the relation defined as follows: a basic relation r' is included in r if and only if the instance obtained from Π by adding the constraint $xr'y$ is satisfiable. In this case, we say that r is *derived* from X.

It should be noted that if the instance $\Pi_1 = \Pi \cup \{xr'y\}$ is satisfiable, then, for any two points or intervals i_1, j_1 such that $i_1 r' j_1$, there is a model M of Π such that $M(x) = i_1$ and $M(y) = j_1$. This can be established as follows: since Π_1 is satisfiable, it has a model M'. Denote $M'(x)$ by i_2 and $M'(y)$ by j_2; then $i_2 r' j_2$. There exists a continuous monotone injective transformation ϕ of the real line such that ϕ takes i_2 to i_1 and j_2 to j_1. Obviously, ϕ maps intervals to intervals, and it does not change the relative order between points and intervals. Therefore, by combining ϕ and M' we obtain the required model M.

It can easily be checked that adding a derived relation r to X does not change the complexity of QA-SAT(X) because, in any instance, any constraint involving r can be replaced by the set of constraints in Π (introducing fresh variables when needed), and this can be done in polynomial time.

Given a relation $t \in \mathcal{QA}$ and a set $\mathcal{S} \subseteq \mathcal{QA}$ such that $\mathcal{S}$ is closed under derivations, we define the relation $r_t^{\mathcal{S}} = \bigcap\{r \in \mathcal{S} \mid t \subseteq r\}$ and note that $r_t^{\mathcal{S}} \in \mathcal{S}$ since it is derived from the relations in $\mathcal{S}$. We drop the superscript whenever $\mathcal{S}$ is understood from the context.

We will sometimes use a principle of *duality* for simplifying proofs. We make use of a function reverse which is defined on the basic relations of $\mathcal{QA}$ by the following table:

r	$<$	$=$	$>$										
reverse(r)	$>$	$=$	$<$										
r	b	s	d	f	a								
reverse(r)	a	f	d	s	b								
r	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
reverse(r)	$\equiv$	p^{-1}	p	m^{-1}	m	o^{-1}	o	d	d^{-1}	f	f^{-1}	s	s^{-1}

and is defined for all other elements in $\mathcal{QA}$ by setting $\text{reverse}(R) = \bigcup_{r \in R} \text{reverse}(r)$.

Let Π be any instance of QA-SAT, and let Π' be obtained from Π by replacing every relation r with reverse(r). It is easy to check that Π has a model M if and only if Π' has a model M' given by

$$M'(x) = \begin{cases} -M(x) & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{PP}}) \\ [-M(x)^+, -M(x)^-] & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{II}}) \end{cases}$$

In other words, M' is obtained from M by redirecting the real line and leaving all points and intervals (as geometric objects) in their places. This observation leads to the following lemma.

Lemma 10 *Let $X = \{r_1, \ldots, r_n\} \subseteq \mathcal{QA}$ and $X' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{QA}$ be such that, for all $1 \leq k \leq n$, $r'_k = \mathsf{reverse}(r_k)$. Then X is tractable (NP-complete) if and only if X' is tractable (NP-complete).*

As an example of the use of Lemma 10, note that a proof of NP-completness for, say, $\{(<), (\mathtt{bf}), (\mathsf{ods}^{-1})\}$, immediately yields a proof of NP-completeness for $\{(>), (\mathtt{sa}), (\mathsf{o}^{-1}\mathsf{df}^{-1})\}$.

3.2.1 Case 1: Strict inequality

Henceforth, we assume that $(<) \in \mathcal{S}_{\mathcal{PP}}$. The classification proof of this special case has four step. In each step, it is proved that if a subclass $\mathcal{S}$ satisfies a certain condition, then either $\mathcal{S}$ is NP-complete, contained in one of the tractable subclasses or $\mathcal{S}$ satisfies the conditions of some earlier step. Throughout the proof, we assume that $\mathcal{S}$ is closed under derivations and $(<) \in \mathcal{S}$. We say that a relation is *non-trivial* if it is not equal to the empty relation.

Step 1. We begin by proving that $\mathcal{S}$ is NP-complete unless $\mathcal{S}_{\mathcal{PI}}$ is a subset of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$ or $\mathcal{V}_{\mathcal{E}}$.
Step 2. Assume now that $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \subseteq (\mathtt{fa})$ and $r_2 \subseteq (\mathtt{bs})$. This implies that $\mathcal{S}$ is NP-complete or $\mathcal{S}$ is included in one of $\mathcal{HV}_{\mathcal{H}}$, $\mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}$ or $\mathcal{E}_{\mathsf{p}}\mathcal{V}_{\mathcal{E}}$.
Step 3. We note that if $(\mathtt{b}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$ or $(\mathtt{a}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$, then $\mathcal{S}$ is NP-complete or contained in one of the tractable subclasses. Thus, we assume the existence of $r_1, r_2 \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r_1$ and $(\mathtt{a}) \not\subseteq r_2$ and show that $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{SH}}$ or $\mathcal{V}_{\mathcal{EH}}$, or else the previous step applies.
Step 4. Finally, we show that if $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$ or $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{EH}}$, then either $\mathcal{S}$ is NP-complete or is contained in one of the tractable subclasses listed in Theorem 5.

Before the proof, we present a number of derivations that will be frequently used.

Lemma 11 *Assume $r \in \mathcal{S}$ is a non-trivial relation. Then,*

1. *if* $(\mathtt{b}) \not\subseteq r$ *and* $r \cap (\mathtt{sd}) \neq \emptyset$, *then* $(\mathtt{dfa}) \in \mathcal{S}$;
2. *if* $(\mathtt{b}) \not\subseteq r$ *and* $r \cap (\mathtt{sd}) = \emptyset$, *then* $(\mathtt{a}) \in \mathcal{S}$;
3. *if* $(\mathtt{a}) \not\subseteq r$ *and* $r \cap (\mathtt{df}) \neq \emptyset$, *then* $(\mathtt{bsd}) \in \mathcal{S}$;
4. *if* $(\mathtt{a}) \not\subseteq r$ *and* $r \cap (\mathtt{df}) = \emptyset$, *then* $(\mathtt{b}) \in \mathcal{S}$;

Proof. The cases are similar so we only consider the first one: the relation $p(\mathtt{dfa})I$ is derived from $\{qrI, p > q\}$. $\Box$

Lemma 12 *$\mathcal{S}$ is NP-complete or $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$.*

Proof. Suppose that $\mathcal{S}_{\mathcal{PI}}$ is not NP-complete. By Theorem 3, it is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$, $\mathcal{V}_{\mathtt{s}}$, $\mathcal{V}_{\mathtt{f}}$. Assume that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{s}}$. If $(\mathtt{b}) \subseteq r$ for every non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$. Suppose there is a non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r$. Then $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{a}), (\mathtt{dfa})\} \neq \emptyset$ by Lemma 11, a contradiction. The argument is dual when $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{f}}$. $\Box$

In the next three lemmata, we will assume that $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$.

Lemma 13 *Suppose that $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \subseteq (\mathtt{af})$ and $r_2 \subseteq (\mathtt{bs})$. Then either $\mathcal{S}$ is NP-complete or is contained in one of $\mathcal{H}\mathcal{V}_{\mathcal{H}}$, $\mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}$ or $\mathcal{E}_{\mathsf{p}}\mathcal{V}_{\mathcal{E}}$.*

Proof. First note that $\{(\mathtt{a}), (\mathtt{b})\} \subseteq \mathcal{S}_{\mathcal{PI}}$ by Lemma 11. Now, $I(\mathsf{p})J$ is derived from $\{p(\mathtt{a})I, p(\mathtt{b})J\}$. It follows from Theorem 4 that either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or it is contained in one of $\mathcal{H}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$.

Suppose first that we have $(\mathtt{d}) \subseteq r_{\mathtt{d}} \subseteq (\mathtt{dsf})$. By using Lemma 12, we conclude that either $\mathcal{S}_{\mathcal{PI}}$ is NP-complete or $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$. Furthermore, $I(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})J$ is derived from $\{pr_{\mathtt{d}}I, pr_{\mathtt{d}}J\}$. Therefore we have $(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ which now implies that either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{H}$. We conclude that either $\mathcal{S}$ is NP-complete or $\mathcal{S} \subseteq \mathcal{H}\mathcal{V}_{\mathcal{H}}$.

We can now assume that $r_{\mathtt{d}}$ contains $(\mathtt{a})$ or $(\mathtt{b})$ (or both). Suppose we have $(\mathtt{a}) \subseteq r_{\mathtt{d}}$; the second case is dual. It follows that, for every $r \in \mathcal{S}_{\mathcal{PI}}$, $(\mathtt{d}) \subseteq r$ implies $(\mathtt{a}) \subseteq r$. If there exists $r' \in \mathcal{S}_{\mathcal{PI}}$ such that $r' \cap (\mathtt{fa}) = (\mathtt{f})$ then $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{bsd})\} \neq \emptyset$ by Lemma 11 which contradicts the assumption just made. It can now be checked that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$ and we complete the proof by considering two cases.

Case 1. $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{E}}$.
If $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{p}}$ or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{E}_{\mathsf{p}}$ then we get the required result. Otherwise there exist $r_3, r_4 \in \mathcal{S}_{\mathcal{II}}$ such that $r_3 \notin \mathcal{S}_{\mathsf{p}}$ and $r_4 \notin \mathcal{E}_{\mathsf{p}}$, that is, $r_3 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_3$, and $r_4 \cap (\mathsf{pmods}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_3$. Now one can check that the constraint $p(\mathtt{d})y$ is derived from $\{Ir_4J, Jr_3K, p(\mathtt{a})I, p(\mathtt{b})K\}$. Indeed, suppose these constraints are satisfied. Then $p(\mathtt{a})I$, $p(\mathtt{b})K$ imply $I^+ < p < K^-$. Since $(\mathsf{p}) \not\subseteq r_4$ and $(\mathsf{p}) \not\subseteq r_3$, we have $J^- \leq I^+$ and $K^- \leq J^+$. It follows that $J^- < p < J^+$, that is $p(\mathtt{d})J$. On the other hand, if $p(\mathtt{d})J$ then, for any choice of $r_3 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})$ and $r_4 \cap (\mathsf{pmods})$, it is easy to find intervals I and K such that the constraints $\{Ir_4J, Jr_3K, p(\mathtt{a})I, p(\mathtt{b})K\}$ are satisfied. This contradicts the fact that $r_{\mathtt{d}}$ contains $\mathtt{a}$ and/or $\mathtt{b}$.

Case 2. $\mathcal{S}_{\mathcal{PI}} \not\subseteq \mathcal{V}_{\mathcal{E}}$.
It is easy to check that $\mathcal{S}_{\mathcal{PI}}$ contains $r_5 \in \{(\mathtt{sa}), (\mathtt{da}), (\mathtt{sda}), (\mathtt{sfa}), (\mathtt{dfa}), (\mathtt{sdfa})\}$. Then, $p(\mathtt{dfa})I \in \mathcal{S}$ by Lemma 11, and we have $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ because $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{dfa})I, p(\mathtt{b})J\}$. In particular, we obtain that $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{H}$ or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{p}}$. If $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{p}}$ then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}$. Otherwise there is a relation $r_6 \in \mathcal{S}_{\mathcal{II}}$ such that $r_6 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_6$. If $r_6 \cap (\mathsf{mo}) \neq \emptyset$, then $p(\mathtt{d})J$ is derived from $\{Ir_6J, Jr_6K, p(\mathtt{a})I, p(\mathtt{b})K\}$ and we have a contradiction. Otherwise we get $r_7 = r_6 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \subseteq (\mathsf{d}^{-1}\mathsf{f}^{-1})$. Note that $r_7 \in \mathcal{S}_{\mathcal{II}}$. Now one can check that the constraint $p(\mathtt{d})I$ is derived from $\{Ir_7J, p(\mathtt{dfa})I, p(\mathtt{b})J\}$ which leads to a contradiction. $\square$

Assume that $(\mathtt{b}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$ or $(\mathtt{a}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$. By using Lemma 6, we see that either $\mathcal{S}$ is NP-complete (if $\mathcal{S}_{\mathcal{II}}$ is NP-complete) or contained in one of the tractable subclasses $\mathcal{W}\mathcal{V}_{\mathtt{a}}$ or $\mathcal{W}\mathcal{V}_{\mathtt{b}}$ where $\mathcal{W} \in \mathcal{II}_{\mathrm{tr}}$.

Lemma 14 *Suppose there exist $r_1, r_2 \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r_1$ and $(\mathtt{a}) \not\subseteq r_2$. Then, $\mathcal{S}$ is NP-complete, $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{SH}}, \mathcal{V}_{\mathcal{EH}}$, or Lemma 13 applies.*

Proof. $\mathcal{S}$ is NP-complete if $\mathcal{S}_{\mathcal{PI}}$ is not a subset of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$ or $\mathcal{V}_{\mathcal{E}}$ by Lemma 12. Thus, we consider three cases depending on which of these sets $\mathcal{S}_{\mathcal{PI}}$ is included in. The claim obviously holds if $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$ by the definitions of $\mathcal{V}_{\mathcal{SH}}$ and $\mathcal{V}_{\mathcal{EH}}$. Suppose $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$; then $r_2 \subseteq (\mathtt{bs})$. If r_1 can be chosen so that $r_1 \subseteq (\mathtt{sfa})$ and $r_1 \neq (\mathtt{s})$, then we can apply Lemma 13 with r_1 if $(\mathtt{s}) \not\subseteq r_1$ and with $r_1 \cap (\mathtt{dfa})$ otherwise (since $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11). If there is no such r_1 then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{EH}}$. For $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$ the argument is dual. $\square$

By duality, it is sufficient to consider $\mathcal{S}_{\mathcal{PI}}$ with $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$.

Lemma 15 *If $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$ then either $\mathcal{S}$ is NP-complete or is contained in one of the tractable subclasses listed in Theorem 5.*

Proof. We consider three different cases depending on the value of $r_{\mathsf{d}} \cap (\mathtt{ba})$.

Case 1. $r_{\mathsf{d}} \cap (\mathtt{ba}) \in \{(\mathtt{b}), (\mathtt{ba})\}$ (i.e. $(\mathtt{b}) \subseteq r_{\mathsf{d}}$).
In this case we have $(\mathtt{s}) \notin \mathcal{S}_{\mathcal{PI}}$, since otherwise $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11 and $r_{\mathsf{d}} \subseteq (\mathtt{dfa})$. Thus $(\mathtt{b})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result by Lemma 6.

Case 2. $r_{\mathsf{d}} \cap (\mathtt{ba}) = (\mathtt{a})$.
Note that in this case we also have $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$ so $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{SH}}$. We have $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11 since $(\mathtt{d}) \subseteq r_{\mathsf{d}} \in \mathcal{S}_{\mathcal{PI}}$. If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} = \emptyset$ then $(\mathtt{a})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result by Lemma 6. Otherwise we have $(\mathtt{b}) \in \mathcal{S}_{\mathcal{PI}}$ (repeating the argument from the beginning of Lemma 13). Then $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{dfa})I, p(\mathtt{b})J\}$. If $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ then, as follows from Theorem 4, either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or it is contained in one of $\mathcal{H}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}^*$. Thus, if $\mathcal{S}_{\mathcal{II}}$ is not NP-complete then $\mathcal{S}$ is contained in one of the tractable subclasses $\mathcal{H}\mathcal{V}_{\mathcal{H}}$ (since $\mathcal{V}_{\mathcal{SH}} \subseteq \mathcal{V}_{\mathcal{H}}$), $\mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}$, $\mathcal{S}_{\mathsf{o}}\mathcal{V}_{\mathcal{SH}}$, $\mathcal{S}_{\mathsf{d}}\mathcal{V}_{\mathcal{SH}}$, $\mathcal{S}^*\mathcal{V}_{\mathcal{SH}}$.

Case 3. $r_{\mathsf{d}} \cap (\mathtt{ba}) = \emptyset$.
Since $p(\mathtt{d})I$ is derived from $\{q_1 r_{\mathsf{d}} I, q_2 r_{\mathsf{d}} I, q_1 < p < q_2\}$, it follows that $r_{\mathsf{d}} = (\mathtt{d})$. We have $(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ because this relation is derived from $\{p(\mathtt{d})I, p(\mathtt{d})J\}$. In particular, either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or is contained in some maximal tractable subclass of $\mathcal{A}$ other than $\mathcal{S}_{\mathsf{p}}$ and $\mathcal{E}_{\mathsf{p}}$.

If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} \neq \emptyset$ then $(\mathtt{b}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11, and $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{d})I, p(\mathtt{b})J\}$. Therefore either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or contained in one of $\mathcal{H}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}^*$. Thus, if $\mathcal{S}_{\mathcal{II}}$ is not NP-complete then $\mathcal{S}$ is contained in one of the tractable subclasses $\mathcal{H}\mathcal{V}_{\mathcal{H}}$, $\mathcal{S}_{\mathsf{o}}\mathcal{V}_{\mathcal{SH}}$, $\mathcal{S}_{\mathsf{d}}\mathcal{V}_{\mathcal{SH}}$, $\mathcal{S}^*\mathcal{V}_{\mathcal{SH}}$.

Otherwise, every non-trivial relation in $\mathcal{S}_{\mathcal{PI}}$ contains $(\mathtt{d})$. If $\mathcal{S}_{\mathcal{II}}$ is included in some tractable subclass except $\mathcal{H}$, the result follows immediately from Lemma 7. If that is not the case, then $\mathcal{S} \subseteq \mathcal{H}\mathcal{V}_{\mathcal{H}}$. $\square$

3.2.2 Case 2: Disequality

We assume now that $(\neq) \in \mathcal{S}_{\mathcal{PP}}$ and $(<) \notin \mathcal{S}_{\mathcal{PP}}$. The proof of this special case contains exactly the same four steps as the proof of the previous case but the proofs themselves are slightly different. We will frequently use the result proved in the previous section so we state it explicitly as a proposition.

Proposition 16 *Let $X \subseteq \mathcal{QA}$ such that $(<) \in X$. Then* QA-SAT(X) *is tractable if and only if X is a included in one of the subclasses listed in Theorem 5. Otherwise,* QA-SAT(X) *is NP-complete.*

Lemma 17 *$\mathcal{S}$ is NP-complete or $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$.*

Proof. Suppose that $\mathcal{S}_{\mathcal{PI}}$ is not NP-complete. Then, by Theorem 3, it is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$, $\mathcal{V}_{\mathtt{s}}$, $\mathcal{V}_{\mathtt{f}}$. Assume that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{s}}$.

If $(\mathtt{b}) \subseteq r_{\mathtt{s}}$ for every non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$. If $(\mathtt{a}) \subseteq r_{\mathtt{s}}$ for every non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$. If $(\mathtt{d}) \subseteq r_{\mathtt{s}}$ for every non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$. Otherwise we have $(\mathtt{s}) \subseteq r_{\mathtt{s}} \subseteq (\mathtt{sf})$. If $(\mathtt{s}) \in \mathcal{S}_{PI}$ then the constraint $p(\mathtt{bdfa})I$ is derived from $\{q(\mathtt{s})I, p \neq q\}$. This contradicts that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{s}}$. If $(\mathtt{sf}) \in \mathcal{S}_{\mathcal{PI}}$ then the constraint $p(\mathtt{bda})I$ is derived from $\{q_1(\mathtt{sf})I, q_2(\mathtt{sf})I, q_1 \neq q_2, p \neq q_1, p \neq q_2\}$ and we have a contradiction once again.

If $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{f}}$ then the argument is dual. $\square$

From now on we will assume that $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$.

Lemma 18 *Suppose that $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \subseteq (\mathtt{af})$ and $r_2 \subseteq (\mathtt{bs})$. Then either $\mathcal{S}$ is NP-complete or is contained in one of $\mathcal{HV}_{\mathcal{H}}$, $\mathcal{S}_p\mathcal{V}_{\mathcal{S}}$ or $\mathcal{E}_p\mathcal{V}_{\mathcal{E}}$.*

Proof. The constraint $p < q$ is derived from $\{pr_2I, qr_1I\}$ and the lemma follows from Proposition 16. $\square$

Assume that $(\mathtt{b}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$ or $(\mathtt{a}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$. By using Lemma 6, we see that either $\mathcal{S}$ is NP-complete (if $\mathcal{S}_{\mathcal{II}}$ is NP-complete) or contained in one of the tractable subclasses $\mathcal{WV}_{\mathtt{a}}$ or $\mathcal{WV}_{\mathtt{b}}$ where $\mathcal{W} \in \mathcal{II}_{\mathrm{tr}}$.

Lemma 19 *Suppose there exist $r_1, r_2 \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r_1$ and $(\mathtt{a}) \not\subseteq r_2$. Then, $\mathcal{S}$ is NP-complete, $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{SH}}, \mathcal{V}_{\mathcal{EH}}$, or Lemma 18 applies.*

Proof. $\mathcal{S}$ is NP-complete if $\mathcal{S}_{\mathcal{PI}}$ is not a subset of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$ or $\mathcal{V}_{\mathcal{E}}$ by Lemma 17. Thus, we consider three cases depending on which of these sets $\mathcal{S}_{\mathcal{PI}}$ is included in. The claim obviously holds if $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$ by the definitions of $\mathcal{V}_{\mathcal{SH}}$ and $\mathcal{V}_{\mathcal{EH}}$.

Suppose $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$; then $r_2 \subseteq (\mathtt{bs})$. If r_1 can be chosen so that $r_1 \subseteq (\mathtt{sfa})$ and $r_1 \neq (\mathtt{s})$ then we can apply Lemma 18. Indeed we can use Lemma 18 with r_1 if $(\mathtt{s}) \not\subseteq r_1$; otherwise either $(\mathtt{b}) \in \mathcal{S}_{\mathcal{PI}}$ and $p < q$ is derived from $\{p(\mathtt{b})I, qr_1I\}$ (and we can apply Proposition 16), or else $(\mathtt{s}) \in \mathcal{S}_{\mathcal{PI}}$ and $pr_1 \cap (\mathtt{sfa})I$ is derived from $\{p(\mathtt{sfa})I, q(\mathtt{s})I, p \neq q\}$. If there is no such r_1 then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$. For $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$ the argument is dual. □

By duality, it remains to consider only $\mathcal{S}_{\mathcal{PI}}$ with $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$.

Lemma 20 *If $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$ then either $\mathcal{S}$ is NP-complete or is contained in one of the tractable subalgebras listed in Theorem 5..*

Proof. We distinguish three cases.

Case 1. $(\mathtt{b}) \subseteq r_{\mathsf{d}}$.
If $(\mathtt{s}) \notin \mathcal{S}_{\mathcal{PI}}$ then $(\mathtt{b})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result from Lemma 6.

Assume instead that $(\mathtt{s}) \in \mathcal{S}_{\mathcal{PI}}$. Then the relations $(\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$, $(\equiv \mathsf{ss}^{-1})$ are derived from $\{p(\mathtt{s})I, q(\mathtt{s})J, p \neq q\}$ and $\{p(\mathtt{s})I, p(\mathtt{s})J\}$, respectively. Therefore either $S_{\mathcal{II}}$ is NP-complete or is contained in one of $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}^*$, $\mathcal{H}$ by Theorem 4.

If $(\mathtt{ba}) \subseteq r_{\mathsf{d}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}} \cap \mathcal{V}_{\mathcal{S}}$, and we get the required result. Suppose now that $(\mathtt{ba}) \cap r_{\mathsf{d}} = (\mathtt{b})$. Consider the constraint IrJ derived from

$$\{pr_{\mathsf{d}}I, p(\mathtt{s})J, qr_{\mathsf{d}}J, q(\mathtt{s})I, p \neq q\}.$$

It can be checked that r is equal to $(\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$ if $(\mathtt{f}) \subseteq r_{\mathsf{d}}$ and to $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$ otherwise. In either case we conclude that $S_{\mathcal{II}}$ is NP-complete or else is contained in one of $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}^*$, $\mathcal{H}$. The result follows.

Case 2. $r_{\mathsf{d}} \cap (\mathtt{ba}) = (\mathtt{a})$.
Note that in this case we also have $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$. If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} = \emptyset$ then $(\mathtt{a})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result. Otherwise the constraint $p < q$ is derived from $\{prI, qr_{\mathsf{d}}I, p \neq q\}$ where r is one of $(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})$. Now the result follows from Lemma 16.

Case 3. $r_{\mathsf{d}} \cap (\mathtt{ba}) = \emptyset$.
We have $(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ because this relation is derived from $\{pr_{\mathsf{d}}I, pr_{\mathsf{d}}J\}$. In particular, either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or is contained in some maximal tractable subalgebra of $\mathcal{A}$ other than $\mathcal{S}_{\mathsf{p}}$ and $\mathcal{E}_{\mathsf{p}}$.

If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} \neq \emptyset$ then the constraint $p < q$ is derived from $\{prI, qr_{\mathsf{d}}I, p \neq q\}$ where r is one of $(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})$. Now the result follows from Lemma 16.

Finally, If every non-trivial relation in $\mathcal{S}_{\mathcal{PI}}$ contains $(\mathtt{d})$ then the result follows immediately from Lemma 7. □

3.2.3 Case 3: Equality

In the final part of the proof, we assume that $\mathcal{S}_{\mathcal{PP}} \subseteq \{(=),(\leq),(\geq)\}$. If $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \cap r_2 = \emptyset$ then the constraint between p and q derived from $\{pr_1 I, qr_2 I\}$ is one of $\neq$, $<$, $>$, which contradicts the fact that $\mathcal{S}_{\mathcal{PP}} \subseteq \{=,\leq,\geq\}$. It follows that the intersection of all non-trivial relations in $\mathcal{S}_{\mathcal{PI}}$ is non-trivial and we denote this relation by r'. We consider four different cases.

Case 1. $r' \cap (\mathtt{ba}) \neq \emptyset$.
The result follows immediately from Lemma 6.

Case 2. $(\mathtt{d}) \subseteq r' \subseteq (\mathtt{sdf})$.
$I(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})J$ is derived from $\{pr'I, pr'J\}$ which implies that $\mathcal{S}_{\mathcal{II}} \not\subseteq \mathcal{S}_{\mathsf{p}}$ and $\mathcal{S}_{\mathcal{II}} \not\subseteq \mathcal{E}_{\mathsf{p}}$. So, if $\mathcal{S}_{\mathcal{II}}$ is NP-complete, then $\mathcal{S}$ is NP-complete. Otherwise, $\mathcal{S}$ is tractable by Lemma 7.

Case 3. $r' = (\mathtt{sf})$.
$I(\equiv \mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})J$ is derived from $\{pr'I, pr'J\}$. It follows from Theorem 4 that either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or is contained in one of $\mathcal{A}_{\equiv}$, $\mathcal{A}_i (1 \leq i \leq 4)$, $\mathcal{B}_i (1 \leq i \leq 4)$. In the latter case $\mathcal{S}$ is contained in one of the tractable subclasses $\mathcal{WV}'_{\mathtt{s}}$ or $\mathcal{WV}'_{\mathtt{f}}$

Case 4. $r' = (\mathtt{s})$ or $r' = (\mathtt{f})$.
Suppose that $r' = (\mathtt{s})$; the case $r' = (\mathtt{f})$ is dual. $I(\equiv \mathsf{ss}^{-1})J$ is derived from $\{pr'I, pr'J\}$. Moreover, $r \cap (\equiv \mathsf{ss}^{-1}) \neq \emptyset$ for each non-trivial $r \in \mathcal{S}_{\mathcal{II}}$, since otherwise the constraint between p and q derived from $\{p(\mathtt{s})I, q(\mathtt{s})J, IrJ\}$ belongs to $\{\neq,<,>\}$ which contradicts that $\mathcal{S}$ is closed under derivations. We conclude the proof by showing that every subalgebra $\mathcal{S}_{\mathcal{II}}$ in Allen's algebra satisfying the conditions above either is NP-complete or is contained in one of $\mathcal{E}^*, \mathcal{A}_{\equiv}$, $\mathcal{A}_i, 1 \leq i \leq 4$. By Lemma 9, this implies that $\mathcal{S}$ is either NP-complete or tractable.

Lemma 21 *Assume that* $(\equiv ss^{-1}) \in \mathcal{S}_{\mathcal{II}}$. *If* $r \cap (\equiv ss^{-1}) \neq \emptyset$ *for every non-trivial* $r \in \mathcal{S}_{\mathcal{II}}$ *then either* QA-SAT$(\mathcal{S}_{\mathcal{II}})$ *is NP-complete or* $\mathcal{S}_{\mathcal{II}}$ *is contained in one of* $\mathcal{E}^*$, $\mathcal{A}_{\equiv}$, $\mathcal{A}_i$, $1 \leq i \leq 4$.

Proof. The proof consists of two cases.

Case 1. There is a non-trivial $r_1 \in \mathcal{S}_{\mathcal{II}}$ such that $r_1 \cap (\mathsf{ss}^{-1}) = \emptyset$.
Then $(\equiv) \subseteq r_1$. If every element r in $\mathcal{S}_{\mathcal{II}}$ satisfies $(\equiv) \subseteq r$ then $\mathcal{S} \subseteq \mathcal{A}_{\equiv}$. Otherwise there is $r_2 \in \mathcal{S}_{\mathcal{II}}$ such that $(\equiv) \not\subseteq r_2$. Note that, since $\mathcal{S}_{\mathcal{II}}$ is closed under derivation, it is also closed under intersection. We have $r_2 \cap (\equiv \mathsf{ss}^{-1}) \in \mathcal{S}$ where $r_2 \cap (\equiv \mathsf{ss}^{-1})$ is one of (s), (s^{-1}), (ss^{-1}). We may without loss of generality assume that $r_2 \in \{(\mathsf{s}), (\mathsf{ss}^{-1})\}$. It is not hard to

check that if $r_1 \not\subseteq (\equiv \mathsf{ff}^{-1})$ then one of the following derivations gives a non-trivial relation r' between I and K such that $r' \cap (\equiv \mathsf{ss}^{-1}) = \emptyset$:

$$\{Ir_2J, Jr_1K, Ir_1K\},\ \{Jr_2I, Jr_1K, Ir_1K\}.$$

We can therefore assume that $r_1 \subseteq (\equiv \mathsf{ff}^{-1})$. If $(\mathsf{s}) \notin \mathcal{S}$ then, for every $r \in \mathcal{S}$, $r \cap (\mathsf{ss}^{-1}) \neq \emptyset$ implies $(\mathsf{ss}^{-1}) \subseteq r$, and so $\mathcal{S} \subseteq \mathcal{E}^*$. Let $(\mathsf{s}) \in \mathcal{S}_{\mathcal{II}}$. It can be verified that the relation (pmods) between I and L is derived from

$$\{Ir_1J, Kr_1J, K(\mathsf{s})L\}.$$

Thus (s) is contained in each of r_{p}, r_{m}, r_{o}, r_{d}, and we conclude that $\mathcal{S} \subseteq \mathcal{E}^*$.

<u>Case 2.</u> $r \cap (\mathsf{ss}^{-1}) \neq \emptyset$ for every non-trivial $r \in \mathcal{S}_{\mathcal{II}}$.
Assume that QA-SAT($\mathcal{S}_{\mathcal{II}}$) is not NP-complete. Then $\mathcal{S}_{\mathcal{II}}$ is contained in one of 18 subclasses from Table 3. We now show that if $\mathcal{S}_{\mathcal{II}}$ is contained in one of 12 subclasses from Table 3 not listed in this lemma then it is also contained in one of those listed. Note that all relations r_{p}, r_{m}, r_{o}, r_{d}, and r_{f} have non-empty intersection with (ss^{-1}).

If $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{p}}$ then $\mathcal{S}_{\mathcal{II}}$ is contained in $\mathcal{A}_1$ or $\mathcal{A}_2$ depending on whether r_{p} contains (s^{-1}) or (s). The argument is similar if $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{d}}$ or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}_{\mathsf{o}}$.

Let $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{E}_{\mathsf{p}}$. If $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{p}}$ then it follows that $(\mathsf{ss}^{-1}) \subseteq r$ whenever $r \cap (\mathsf{pmod}) \neq \emptyset$ or $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{d}^{-1}) \neq \emptyset$. Then $\mathcal{S}_{\mathcal{II}}$ is contained in $\mathcal{A}_3$ or $\mathcal{A}_4$ depending on whether r_{f} contains (s) or (s^{-1}), and the same holds if $(\mathsf{s}) \subseteq r_{\mathsf{p}}$. The argument is similar if $\mathcal{S}_{\mathcal{II}}$ is contained in one of $\mathcal{E}_{\mathsf{d}}$, $\mathcal{E}_{\mathsf{o}}$, $\mathcal{B}_1$, $\mathcal{B}_2$ If $\mathcal{S}_{\mathcal{II}}$ is contained in $\mathcal{B}_3$ or $\mathcal{B}_4$ then one can show (as above) that $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{A}_1$ or $\mathcal{S} \subseteq \mathcal{A}_2$.

It is obvious that if $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}^*$ then $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{A}_{\equiv}$.

Finally, assume that $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{H}$. It follows from condition 3) of $\mathcal{H}$ that $r_{\mathsf{o}} \subseteq r_{\mathsf{p}}$ and $r_{\mathsf{o}} \subseteq r_{\mathsf{m}}$. We consider four subcases:

<u>Subcase 1:</u> $(\mathsf{s}) \subseteq r_{\mathsf{o}}$ and $(\mathsf{s}) \subseteq r_{\mathsf{d}}$.
Then, $\mathcal{S}_{\mathcal{II}}$ is contained in $\mathcal{A}_3$ or $\mathcal{A}_4$ depending on whether r_{f} contains (s) or (s^{-1}).

<u>Subcase 2:</u> $(\mathsf{s}) \subseteq r_{\mathsf{o}}$ and $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{d}}$.
If $(\mathsf{s}) \subseteq r_{\mathsf{f}}$ then, by condition 1) of $\mathcal{H}$, we have $(\mathsf{d}) \subseteq r_{\mathsf{f}}$, and, consequently, $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$. So, in any case we have $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$. It is easy to verify that $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{A}_2$.

<u>Subcase 3:</u> $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{o}}$ and $(\mathsf{s}) \subseteq r_{\mathsf{d}}$.
If $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$ then, by condition 2) of $\mathcal{H}$, we have $(\mathsf{o}^{-1}) \subseteq r_{\mathsf{f}}$, and, consequently, $(\mathsf{s}) \subseteq r_{\mathsf{f}}$. So, in any case we have $(\mathsf{s}) \subseteq r_{\mathsf{f}}$, and, hence, $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{A}_1$.

<u>Subcase 4:</u> $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{o}}$ and $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{d}}$.

By applying condition 2) of $\mathcal{H}$ to r_{d} we get that $(\mathsf{o}^{-1}) \subseteq r_{\mathsf{d}}$, and, therefore, $(\mathsf{s}) \subseteq r_{\mathsf{d}}$. Then apply condition 1) of $\mathcal{H}$ to r_{o} and obtain that $(\mathsf{d}^{-1}) \subseteq r_{\mathsf{o}}$, and, consequently, $(\mathsf{ss}^{-1}) \subseteq r_{\mathsf{o}}$. Once again, we conclude that $\mathcal{S}_{\mathcal{II}}$ is contained in $\mathcal{A}_3$ or $\mathcal{A}_4$ depending on whether r_{f} contains (s) or (s^{-1}). $\square$

4 Conclusions

We have studied the computational complexity of the Qualitative Algebra which is a temporal formalism that combines the point algebra, the point-interval algebra and Allen's interval algebra. We have identified all tractable fragments by using combinatorial techniques and this method has made it possible to avoid the use of computer-assisted enumeration techniques. The tractable fragments have a clear description which allows one to easily incorporate the checking for these cases into general-purpose temporal constraint solvers. To the best of our knowledge, this is the first time a temporal constraint language able to represent different temporal entities (points and intervals) has been completely classified with respect to tractability. We have also proved that all other fragments are NP-complete.

There are several possible ways to continue this work. One continuation is to study the complexity of QA extended by metric constraints – for instance, Meiri [16] suggests one such extension. Investigations of such formalisms can probably be carried out using methods similar to those found in [13]. Another interesting future research directions is to see if these results can be used for improving heuristics or constraint solvers for temporal reasoning.

Acknowledgements

This research was partially supported by the UK EPSRC grant GR/R29598 and the Swedish Research Council (VR) grant 221-2000-361.

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