TRACE OPERATOR AND THETA SERIES

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1. **Introduction**

Theta series attached to integral positive definite quadratic forms are one of the most important tools for the explicit construction of (Siegel or elliptic) modular forms. The so-called basis problem asks whether a given modular form of some fixed type can be represented as a linear combination of theta series. Here the theta series are usually also restricted to a fixed type, either theta constants of fixed level or theta series of full lattices of fixed level. It is sometimes easier to obtain an answer if one drops this restriction on the type of theta series. In particular, translation of results about theta liftings from adelic representation theory into classical terms typically gives representability as a linear combination of (possibly inhomogeneous) theta series of some not specified level. It appears therefore desirable to investigate the conditions under which such a general linear combination of theta series can be transformed into a linear combination of theta series of the specified type. The present article should be regarded as a first step in this direction. We look at the subspace $\theta^{(n)}(V,M)$ of the space $M_n^k(M,\chi)$ of modular forms for $\Gamma_0^{(n)}(M)$ with the appropriate character χ that is generated by the theta series of degree n of lattices L of level dividing M on the positive definite quadratic space (V,q). We then ask for N dividing M whether

$$\theta^{(n)}(V,M)\cap M_n^k(N,\chi)=\theta^{(n)}(V,N)$$

holds true.

Considering the trace operator, which transforms modular forms for $\Gamma_0^{(n)}(M)$ into modular forms for $\Gamma_0^{(n)}(N)$ we are lead to the reformulation: When can the action of the trace operator on the theta series of a lattice of level M be expressed as the linear combination of theta series of lattices of level N on the same space?

It turns out that in some cases the effect of this action can be explicitly calculated in terms of theta series of level N in a way similar to the well-known formulas for the action of Hecke operators on theta series [An, AnZh, Ei, Fr1, Fr2, Yo]. Like there one has two principal approaches available: The first approach studies the problem for singular modular forms (i.e., high degree of the theta series) and then tries to transfer the results obtained to lower degree theta series by computing commutation relations between

the trace (or Hecke) operators and Siegel's ϕ-operator. A first sketch of this approach has been given by the first named author in [Boe]; previously, Salvati-Manni [SM1, SM2, SM3] had used similar ideas in the case of theta series with characteristics. The other approach starts directly in the given degree by calculating the effect of the operator on the Fourier expansion of the theta series and comparing with certain sums of theta series of lattices containing the given lattice with the help of the (local) arithmetic of quadratic forms. This approach has been worked out by the second named author in his Diplomarbeit [Fu].
The purpose of the present paper is to describe both these approaches. This will be carried out in section 4 and sections 5,6 respectively. It turns out that the cases that cannot be treated are the same ones in both approaches, indicating that these cases are intrinsically difficult.

An adelic approach using the ideas from [Yo] and somewhat similar to the second approach sketched above has been idependently pursued by T. Kume [Ku].

2. **Trace Operator for Modular Forms**

Let $\Gamma \subseteq Sp_n(\mathbb{R})$ be a congruence subgroup, $k \in \frac{1}{2}\mathbb{Z}$ and (ϱ, W) be a polynomial finite dimensional complex representation of $GL_n(\mathbb{C})$. For a multiplier system υ for Γ we denote by $M_n^k(\Gamma, \varrho, \upsilon)$ ($S_n^k(\Gamma, \varrho, \upsilon)$) the space of Siegel modular (cusp) forms of degree n for Γ of type $\varrho \otimes \det^k$; if ϱ can not be factored as $\varrho = \varrho' \otimes \det^r$ with polynomial ϱ' and $0 \neq r \in \frac{1}{2}\mathbb{N}$ we call k the weight of $\varrho \otimes \det^k$. (For details see [Fr3].)

For any function $f : \mathbb{H}_n \longmapsto W$ on the Siegel upper half plane and any $k \in \frac{1}{2}\mathbb{N}$ we write

$$\left(f\big|_{k,\varrho} M\right)(Z) = \left(\det M\right)^{k/2} \det\left(CZ + D\right)^{-k} \varrho\left(CZ + D\right)^{-1} f(MZ)$$

for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G^+Sp_n(\mathbb{R})$.

With the above notation let Γ' be another congruence subgroup containing Γ such that υ can be extended to υ' on Γ'. For $f \in M_n^k(\Gamma, \varrho, \upsilon)$ we define the trace of f as

$$\mathrm{tr}_{\Gamma', \upsilon'}^{\Gamma, \upsilon} f = \frac{1}{[\Gamma' : \Gamma]} \sum_{\gamma \in \Gamma \backslash \Gamma'} (\upsilon')^{-1}(\gamma) f \mid_{k,\varrho} \gamma.$$

(The sum is necessarily finite). Note that the trace depends on the choice of the extension υ' of the multiplier υ (which might not be unique); we will suppress the multipliers υ, υ' in the notation if this can't cause confusion. The trace clearly projects $M_n^k(\Gamma, \varrho, \upsilon)$ onto $M_n^k(\Gamma', \varrho, \upsilon')$.

We will be only concerned with congruence subgroups of the form

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \mid C \equiv 0 \bmod N \right\}$$

and

$$\Gamma_1^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N) \mid \det D \equiv 1 \bmod N \right\}.$$

A complete set of coset representatives of $\Gamma_1^{(n)}(N)\backslash\Gamma_0^{(n)}(N) \simeq \big(\mathbb{Z}/N\mathbb{Z}\big)^\times$ is given by any set of $\alpha_d \in \Gamma_0^{(n)}(N), d \in \big(\mathbb{Z}/N\mathbb{Z}\big)^\times$ such that the lower right block D_d of α_d has $\det D_d = d$.

Moreover, we will assume that υ comes from a Dirichlet character χ mod N which as usual acts on the determinant of the right lower block. We then have $\upsilon = \chi$ for k integral, and for k nonintegral we assume $\Gamma \subseteq \Gamma_0^{(n)}(4)$ so that $\upsilon =: \upsilon_\chi$ is induced from the theta-multiplier, i.e given by

$$\upsilon(\gamma)\det\big(CZ+D\big)^k = \chi(\gamma) j^{(n)}(\gamma, Z) \det\big(CZ+D\big)^{k-1/2} \tag{2.1}$$

with $j^{(n)}(\gamma, Z) = \frac{\vartheta^{(n)}(\gamma Z)}{\vartheta^{(n)}(Z)}$ where $\vartheta^{(n)}(Z) = \sum\limits_{\mathbf{x}\in\mathbb{Z}^n} \exp(2\pi i \operatorname{tr} Z[x])$. For the set of modular forms with respect to $\Gamma_0^{(n)}(N)$ of type $\varrho \otimes \det^k$ with this multiplier υ we write $M_n^k(N, \varrho, \chi)$. Note that for k nonintegral this follows [An] and differs slighty from Shimura's definition [Sh] (for $n = 1$).

For $M_n^k(N, \varrho, \chi)$ we can remove in the definition of the trace the rather inconvenient condition on the extensibility of χ as follows:

Proposition 2.1. *Let $N|M$ be positive integers and assume $f \in M_n^k(M, \varrho, \chi) \subset M_n^k(\Gamma_1^{(n)}(M), \varrho)$. Then*

(i) *If the conductor of χ does not divide N, then*

$$\operatorname{tr}_{\Gamma_1^{(n)}(N)}^{\Gamma_1^{(n)}(M)} f = 0.$$

(ii) *If the conductor of χ divides N we get*

$$\operatorname{tr}_{\Gamma_1^{(n)}(N)}^{\Gamma_1^{(n)}(M)} f \in M_n^k(N, \varrho, \chi) \qquad \textit{and} \qquad \operatorname{tr}_{\Gamma_1^{(n)}(N)}^{\Gamma_1^{(n)}(M)} f = \operatorname{tr}_{\Gamma_0^{(n)}(N)}^{\Gamma_0^{(n)}(M)} f.$$

Proof. We only do the case of integral weight. For any (arbitrary) level N we have $M_n^k(\Gamma_1^{(n)}(N), \varrho) = \oplus_\psi M_n^k(N, \varrho, \psi)$, where ψ runs through all Dirichlet characters mod N. The projection onto the various factors is given by (using the coset representatives α_d from above) $\operatorname{tr}_\psi := \frac{1}{\varphi(N)} \sum\limits_{\alpha_d} \bar{\psi}(d) \big| \alpha_d$. One then checks

$$\operatorname{tr}_\psi \circ \operatorname{tr}_{\Gamma_1^{(n)}(N)}^{\Gamma_1^{(n)}(M)} = \operatorname{tr}_{\Gamma_0^{(n)}(N),\psi}^{\Gamma_0^{(n)}(M),\tilde{\psi}} \circ \operatorname{tr}_{\tilde{\psi}}$$

for the pullback $\tilde{\psi}$ to $(\mathbb{Z}/M\mathbb{Z})^\times$ of ψ, and the proposition now follows from $\operatorname{tr}_{\tilde{\psi}}(f) = 0$ for $\tilde{\psi} \neq \chi$ on $(\mathbb{Z}/M\mathbb{Z})^\times$ in view of the above direct sum decompositions for both levels M and N. □

This result suggests to simply write $\mathrm{tr}^M_N f := \mathrm{tr}^{\Gamma_1^{(n)}(M)}_{\Gamma_1^{(n)}(N)}$ for $f \in M_n^k(M, \varrho, \chi)$ with $\mathrm{tr}^M_N f = 0$ if χ is not definable mod$N$.

When doing actual computations it is more convenient to consider the trace without its normalizing factor. We will assume this convention for the rest of the paper.

3. Theta Series of Positive Definite Quadratic Forms

We first fix the notation for the rest of the paper.

Let (V, q) be a positive definite quadratic space over $\mathbb{Q}$ of dimension m with attached bilinear form $B(x, y) = q(x+y) - q(x) - q(y)$ and let L be an even lattice on V (i.e. $q(L) \subseteq \mathbb{Z}$) of level M (hence $q(L^{\#})\mathbb{Z} = M^{-1}\mathbb{Z}$, where $L^{\#}$ denotes the dual of L). Fixing a basis of V we will frequently identify V with $\mathbb{Q}^m$ and L with a lattice in there. The associated Gram-matrix with respect to this basis we denote by S. We let $\det L$ be the determinant of L and define the discriminant of L by $\operatorname{disc} L = (-1)^{m/2} \det L$ if m is even and $\operatorname{disc} L = (-1)^{\frac{m-1}{2}} \frac{1}{2} \det L$ if m is odd; hence $\operatorname{disc} L \equiv 0, 1 \ (4)$. Recall that for odd m we have $2 | \det L$ and $4 | \operatorname{level} L$. For an (arbitrary) lattice K in V we write $K^{,a}$ $(V^{,a})$ for the lattice K (space V) together with the quadratic form scaled by a.

For the finite dimensional polynomial representation (ϱ, W) of $GL_n(\mathbb{C})$ we let $P : M_{m,n}(\mathbb{C}) \longmapsto W$ be a polynomial which is pluriharmonic with respect to ϱ; i.e.

$$P(XA) = \varrho({}^tA)P(X) \qquad \text{for all } A \in GL_n(\mathbb{C})$$

and

$$\sum_{i,j}^{m} t_{ij} \frac{\partial P}{\partial x_{ik} \partial x_{jl}} = 0 \qquad \text{for} \qquad 1 \leq k, l \leq n,$$

where $S^{-1} = (t_{ij})$.(For more information see [KV,Fr3]). Considering $V \otimes \mathbb{R} \simeq \mathbb{R}^m$ via some fixed orthonormal basis we interpret P as a function on $(V \otimes \mathbb{R})^n$; such a function is then called a q-pluriharmonic polynomial on $(V \otimes \mathbb{R})^n$.

For these data we form the theta series

$$\vartheta^{(n)}(P; L; Z) = \sum_{\mathbf{x} \in L^n} P(\mathbf{x}) \exp(2\pi i tr(q(\mathbf{x})Z)),$$

where for $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$ the matrix $q(\mathbf{x})$ is given by $q(\mathbf{x})_{ij} = \frac{1}{2}B(x_i, x_j)$. It is well known (see [Fr3,An]) that $\vartheta^{(n)}(P; L; Z)$ is a modular form for $\Gamma_0^{(n)}(M)$ of type $\varrho \otimes \det^{m/2}$ and character χ which (for all m; recall our convention (2.1)) is given by

$$\chi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \left(\frac{\operatorname{disc} L}{\det D}\right).$$

Here the quadratic residue symbol has the same meaning as in [Sh]. Finally note that for the conductor f_χ of χ we have

$$f_\chi = \begin{cases} k(|\operatorname{disc} L|), & \text{if } (-1)^{\left[\frac{m}{2}\right]} k(|\operatorname{disc} L|) \equiv 1 \bmod 4 \\ 4k(|\operatorname{disc} L|), & \text{else,} \end{cases}$$

where $k(s)$ denotes the square free kernel of a positive integer s.

The problem of expressing a modular form by theta series of "appropriate level" (as descibed in the introduction) is connected with the behaviour of theta series under the trace operator by

Remark 3.1. Let $f \in M_n^k(N, \rho, \chi)$ be a linear combination of theta series $\vartheta^{(n)} = \vartheta^{(n)}(P; L; Z)$ of lattices of level M divisible by N with character $\tilde{\chi}$ induced by χ. Assume that for each of the theta series $\vartheta^{(n)}$ involved in the linear combination the trace $\operatorname{tr}_N^M \vartheta^{(n)}$ is a linear combination of theta series of lattices of level dividing N. Then f is a linear combination of theta series of lattices of level dividing N.

Proof. Obvious.

Remark 3.2. It could of course happen that a linear combination of theta series of level M can be expressed by theta series of level N even though the traces of some or all of the individual terms can not be expressed this way. This problem looks rather intractable, and we concentrate our attention in the sequel on the question of expressibility of the trace of an individual theta series of level M by theta series of the lower level N.

4. **Trace of Theta Series**

We fix a prime p dividing $M = \operatorname{level} L$, say $\alpha = \operatorname{ord}_p(M)$, and write $M = Np$. For the completion $L_p = L \otimes \mathbb{Z}_p$ we choose a Jordan decomposition (see [OM])

$$L_p = L_p^{(0)} \perp L_p^{(1)} \perp \cdots \perp L_p^{(\alpha)} \tag{4.1}$$

with $L_p^{(i)}$ p^i-modular; i.e. $\left(L_p^{(i)}\right)^\# = (p^{-i}) L_p^{(i)}$. We call $L_p^{(i)}$ even if $q(L_p^{(i)})\mathbb{Z}_p = (p^i)\mathbb{Z}_p$.

For the computation of $\operatorname{tr}_N^M \vartheta^{(n)}(P; L; Z)$ we have to distinguish two basic cases:

- $(N, p) > 1$;
- $(N, p) = 1$.

4.1. **The case $M = Np$ with $p \mid N$.** For the first case we have the following

Theorem 4.1. *With the notations as above let $\alpha =$ord$_p(M) \geq 2$ ($\alpha \geq 3$ for $p = 2$ and m odd). Assume that χ can be defined* mod$N$ *(which is always the case if $p \neq 2$). Then*

(i) *For $p \neq 2$ or if $L_p^{(\alpha-1)}$ is even for $p = 2$ there are even lattices K in V of level dividing N satisfying $L \subset K \subseteq NL^{\#} + L$ such that*
$$\mathrm{tr}_N^M\,\vartheta^{(n)}(P;L;Z) = \sum_K c_K \vartheta^{(n)}(P;K;Z)$$
with rational numbers c_K.

(ii) *If $p = 2$ and $L_p^{(\alpha-1)}$ is not even the same is true with lattices K' in V which contain a lattice K as in i) with index 2. For $\alpha \geq 3$ these lattices are even and have level dividing N.*

Moreover for $P = 1$ we have
$$\mathrm{tr}_N^M\,\vartheta^{(n)}(1;L;Z) \neq 0.$$

Remark 4.2. Locally one gets for the lattices K occuring in the theorem $K_\ell = L_\ell$ for all primes $\ell \neq p$. At the spot p the K_p are contained in
$$L_p^{(0)} \perp L_p^{(1)} \perp \cdots \perp L_p^{(\alpha-1)} \perp \tfrac{1}{p} L_p^{(\alpha)}$$
(if $L_p^{(\alpha-1)}$ is even). In particular we see
$$\alpha - 2 \leq p\text{-adic order of the level of } K \leq \alpha - 1.$$

Remark 4.3. Our proof is essentially based on a calculation of the action of the irregular Hecke-Operator U_p on the theta series $\vartheta^{(n)}(P;L;Z)$. A proof could therefore also be obtained by a generalization of Evdokimov's formula [Ev] for this. As there, an explicit formula can be obtained by making the inclusion-exclusion principle used above explicit, e. g. using a generalized Möbius formula.

Remark 4.4. For P nonconstant $\mathrm{tr}_N^M\,\vartheta^{(n)}(P;L;Z) = 0$ does occur. For example let $n = 1$, $p = 3$ and $L = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 6 & 3 \\ 3 & 6 \end{smallmatrix}\right)$; hence level$\,L = 9$. Put $P(x_1, x_2, x_3, x_4) = (x_2 + i\sqrt{3}x_3)^2$. Then $\vartheta^{(1)}(P;L;z) \in S_1^4(9)$ is nonzero since the first Fourier coefficient is equal to 4, but $\mathrm{tr}_3^9\,\vartheta^{(1)}(P;L;z) = 0$ since $S_1^4(3) = 0$.

Proof of Theorem 4.1. Since $(N,p) > 1$, we have $\left[\Gamma_0^{(n)}(N) : \Gamma_0^{(n)}(Np)\right] = p^{n(n+1)/2}$ (see e.g. [Kl]). Hence, one easily sees that
$$\begin{pmatrix} -I & 0 \\ NS_j & -I \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & NS_j \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$
is a complete system of right coset representatives for $\Gamma_0^{(n)}(Np)\backslash\Gamma_0^{(n)}(N)$, where $S_j, j = 1, \ldots, p^{n(n+1)/2}$ runs through all symmetric $n \times n$ matrices

mod$p$. We have $\upsilon\left(\left(\begin{smallmatrix} -I & 0 \\ NS_j & -I \end{smallmatrix}\right)\right) = (-i)^{mn}$. Moreover, this is compatible with the above product decomposition.

By the theta inversion formula (see [Fr3]), we get

$$\vartheta^{(n)}(P; L; Z) \mid_{k,\varrho} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = (\det L)^{-n/2} i^{\frac{-mn}{2}} \vartheta^{(n)}(P; L^{\#}; Z).$$

Applying the usual character sum argument for the action of the (unscaled) U_p-type Operator $\sum_j \mid_{k,\varrho} \left(\begin{smallmatrix} 1 & NS_j \\ 0 & 1 \end{smallmatrix}\right)$ on $\vartheta^{(n)}(P; L^{\#}; Z)$ gives

$$\sum_j \vartheta^{(n)}(P; L^{\#}; Z) \mid_{k,\varrho} NS_j = p^{n(n+1)/2} \sum_{\substack{\mathbf{x} \in (L^{\#})^n \\ Nq(\mathbf{x}) \text{ integral}}} P(\mathbf{x}) \exp(2\pi i q(\mathbf{x})).$$

(Note $q(\mathbf{x}) \in M_n(\frac{1}{Np}\mathbb{Z})$ for $\mathbf{x} \in (L^{\#})^n$.)

We denote the set over which the summation extends by $L(n,p)$. It can be characterized as the set of all n-tuples $(x_1, \dots, x_n) \in (L^{\#})^n$ which span an even lattice (of usually lower rank) in V. Each element $\mathbf{x} \in L(n,p)$ therefore lies in a maximal integral sublattice of $L^{\#,N}$. Hence $L(n,p) = \cup_{\tilde{K}} \tilde{K}^n$, where $\tilde{K}$ runs through the finite set of sublattices $\tilde{K} \subseteq L^{\#}$ with $q(\tilde{K}) \subseteq N^{-1}\mathbb{Z}$ that are maximal among the sublattices of $L^{\#}$ satisfying this condition. The maximality of such a $\tilde{K}$ implies that

$$\tilde{K} \supseteq \{y \in \frac{1}{N} L \cap L^{\#} \mid Nq(y) \in \mathbb{Z}\} =: \tilde{L}^{\#};$$

this latter set is a sublattice of $L^{\#}$ which equals $\frac{1}{N}L \cap L^{\#}$ if $p \neq 2$ or if $L_p^{(\alpha-1)}$ is even and is (with respect to the scaled form Nq) the even sublattice of this lattice otherwise. More precisely, $\tilde{L}^{\#}$ has Jordan decomposition

$$\tilde{L}_p^{\#} = p^{1-\alpha} \widehat{L_p^{(\alpha-1)}} + p^{2-\alpha} L_p^{(\alpha-2)} + \dots$$

at p, where

$$\widehat{L_p^{(\alpha-1)}} = \{y \in L_p(\alpha - 1) \mid q(y) \in p^{\alpha-1}\mathbb{Z}_p\}.$$

By the inclusion-exclusion principle we then get the following identitiy of characteristic functions:

$$1_{L(n,p)} = \sum_K a_K 1_{(K)^{\#^n}},$$

where $K^{\#}$ runs through all the possible intersections of the lattices $\tilde{K}$ from above; in particular each $K^{\#}$ contains $\tilde{L}^{\#}$.

Hence

$$\sum_{\mathbf{x} \in L(n,p)} P(\mathbf{x}) \exp(2\pi i q(\mathbf{x})) = \sum_K a_K \vartheta^{(n)}(P; K^{\#}; Z),$$

and another application of the theta inversion formula gives the assertion. $\square$

4.2. **The case M=Np with N coprime to p.** The case level $L = Np$ with $(N,p) = 1$ is more delicate. Whereas in the other case the existence of lower level lattices was (almost) always guaranteed, we have in this case the following easy, but fundamental observation:

Lemma 4.5. *Let (L,q) be an even lattice on the quadratic space V of level Np with $(N,p) = 1$. Then the following statements are pairwise equivalent:*

(i) *$s_p(V) = 1$.*

(ii) *V carries (even) lattices of level N.*

(iii) *If $L_p = L_p^{(0)} \perp L_p^{(1)}$ denotes the Jordan splitting at the spot p, then $L_p^{(1)}$ is an orthogonal sum of hyperbolic planes.*

Here $s_p(V)$ is the Witt-invariant of the completion V_p, normalized as in [Sch] (in particular $s_p(V) = 1$ if V_p is an orthogonal sum of hyperbolic planes).

Proof. (i) $\Leftrightarrow$ (iii) follows from an easy (and for $p = 2$ tedious) calculation. It is well known that V_p carries an even unimodular lattice if and only if the discriminant group $L_p^{\#}/L_p$ is an orthogonal sum of hyperbolic planes, which (by Hensel's lemma, see e.g. [Kn, Satz 14.2, Ki, Ch. 5.4]) is again equivalent to $L_p^{(1)}$ being an orthogonal sum of hyperbolic planes. Hence (ii) $\Leftrightarrow$ (iii) follows. $\square$

Theorem 4.6. *Let L be an even lattice of* level Np *such that the p-part of* $\det L$ *is a square, say* $\det_p L = p^{2t}$ *(i.e. the character $\chi = \left(\frac{\operatorname{disc} L}{\cdot}\right)$ is definable* $\bmod N$*!). Then*

(i) *If $s_p(V) = 1$ then there is a rational number c such that*

$$tr_N^{Np}\vartheta^{(n)}(P;L;Z) = c\sum_K \vartheta^{(n)}(P;K;Z),$$

where the sum goes over all maximal even lattices K on V of level N such that $L \subset K \subset L^{\#}$. Moreover, for $P = 1$ the trace does not vanish.

(ii) *If $s_p(V) = -1$ and $n \geq t$ then*

$$tr_N^{Np}\vartheta^{(n)}(P;L;Z) = 0.$$

For $P = 1$ these two conditions are also necessary for the vanishing.

Remark 4.7. If P is not constant then the converse for the vanishing statements in the theorem is not true. For example, let $n = 1$ and consider the quadratic extension $K = \mathbb{Q}(\sqrt{-p})$, where $p \equiv 3$ (4) and $p \neq 3$. The ring of integers $\mathcal{O}_K$ form an even lattice in K with the quadratic form $q(w) = |w|^2$. We let $P(w) = w^2$ which is q-harmonic of weight 2. Then $\vartheta^{(1)}(P;\mathcal{O}_K;z) \in S_1^3\big(p, \left(\frac{-p}{\cdot}\right)\big)$ is nonzero since the first Fourier coefficient is 2.

(i) Consider $\mathcal{O}_K^4$ and put $\tilde{P}(w_1,\dots,w_4) = P(w_1)$. Note $s_p(\mathcal{O}_K^4) = 1$. Then $\vartheta^{(1)}(\tilde{P};\mathcal{O}_K^4;z) = \vartheta^{(1)}(P;\mathcal{O}_K;z)\vartheta^{(1)}(\mathcal{O}_K;z)^3 \in S_1^6(p)$ and $tr_1^p\vartheta^{(1)}(\tilde{P};\mathcal{O}_K^4;z) = 0$ since $S_1^6(1) = 0$.

(ii) Now assume moreover $p \equiv 3\ (8)$ and let L be the lattice $\mathcal{O}_K^3 \perp 2\otimes\mathcal{O}_K^3$; hence level $L = 2p$ and $\det L = 4p^4$ (so $n = 1 < t$). Check $s_p(L) = -1$. With $\tilde{P}$ as in (i) we then get the nonzero theta series $\vartheta^{(1)}(\tilde{P};L;z) = \vartheta^{(1)}(P;\mathcal{O}_K;z)\vartheta^{(1)}(\mathcal{O}_K;z)^2\vartheta^{(1)}(\mathcal{O}_K;2z) \in S_1^6(2p)$, but have $tr_1^p\vartheta^{(1)}(\tilde{P};L;z) = 0$ as $S_1^6(2) = 0$.

Proof of Theorem 4.6. For the proof (which for $n > 1$ was not yet contained in [Fu]) we use ideas from [BS, Section7,8]. Some of the computations arising are (in spite of the different setup used) similar to those in [Ku], a preprint version of which was communicated to us in December of 1995.

We first construct a system of right coset representatives for $\Gamma_0^{(n)}(Np)\backslash\Gamma_0^{(n)}(N)$.

For the finite field $\mathbb{F}_p$ we let $P = \left\{\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right)\right\} \subset Sp_n(\mathbb{F}_p)$ be the Siegel parabolic and define for $0 \le j \le n$

$$\omega_j = \omega_j(p) = \begin{pmatrix} 1_{n-j} & 0 & 0_{n-j} & 0 \\ 0 & 0_j & 0 & -1_j \\ 0_{n-j} & 0 & 1_{n-j} & 0 \\ 0 & 1_j & 0 & 0_j \end{pmatrix}.$$

Then there is the Bruhat decomposition

$$Sp_n(\mathbb{F}_p) = \coprod_{j=0}^{n} P\omega_j P,$$

and the double coset $P\omega_jP$ consists precisely of the set of elements $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in Sp_n(\mathbb{F}_p)$ with $\operatorname{rank}(C) = j$. Using the Levi decomposition $P = MN$ with Levi factor

$$M = \left\{m(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix} \mid A \in GL_n(\mathbb{F}_p)\right\}$$

and unipotent radical

$$N = \left\{n(B) = \begin{pmatrix} 1 & B \\ 0 & 1\end{pmatrix} \mid B \in M_n(\mathbb{F}_p) \text{ symmetric}\right\},$$

we easily see that

$$\left\{\omega_j\, n(B_j)\, m(A) \mid B_j \in M_j(\mathbb{F}_p) \text{ symmetric, } A \in P_{n,j}(\mathbb{F}_p)\backslash GL_n(\mathbb{F}_p)\right\}$$

is a complete set of right coset representatives for $P\backslash P\omega_jP$. Here M_j is naturally embedded into M_n by $B_j \longmapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & B_j\end{smallmatrix}\right)$ and $P_{n,j} = \left\{g \in GL_n \mid g = \left(\begin{smallmatrix} * & * \\ 0_{j,n-j} & *\end{smallmatrix}\right)\right\}$ is the standard parabolic subgroup of GL_n.

Since $(N,p) = 1$, we can lift these coset representatives to representatives of $\Gamma_0^{(n)}(Np)\backslash\Gamma_0^{(n)}(N)$ via strong approximation (and we identify the lifts with their image mod$p$). Hence ω_j satisfies the congruences

$$\omega_j \equiv 1_{2n} \bmod N \quad \text{and} \quad \omega_j \equiv \begin{pmatrix} 1_{n-j} & 0 & 0_{n-j} & 0 \\ 0 & 0_j & 0 & -1_j \\ 0_{n-j} & 0 & 1_{n-j} & 0 \\ 0 & 1_j & 0 & 0_j \end{pmatrix} \bmod p.$$

More precisely, one first gets (see e.g. [BS], Lemma 8.1)

$$\Gamma_0^{(n)}(N) = \coprod_{j=0}^{n} \Gamma_0^{(n)}(Np)\omega_j\Gamma_0^{(n)},$$

where $\Gamma_0^{(n)} = Sp_n(\mathbb{Z}) \cap P(\mathbb{Q})$. One then checks that every right coset representative $\omega_j\, n(B)\, m(A)$ of $\Gamma_0^{(n)}(Np)\backslash\Gamma_0^{(n)}(N)$ defines one of $P\backslash P\omega_j P$ in the above form.

Note that for the "lifted" $m(A)$, $A \in GL_n(\mathbb{Z})$, we can assume $\det A = 1$. This implies, since $\omega_j \equiv 0 \bmod N$, that we picked coset representatives which are trivial on the multiplier system.

For the action of the "partial Atkin-Lehner involution" ω_j on theta series we need

Lemma 4.8 ([BS]§8)**.** *With our notation as above we let $L^{\#;p} = L^{\#} \cap \mathbb{Z}\big[\frac{1}{p}\big]L$ be the lattice dualized only at the spot p. We put*

$$\vartheta^{(n-j,j)}(P; L, L^{\#;p}; Z) = \sum_{\mathbf{x} \in L^{n-j} \times (L^{\#;p})^j} P(\mathbf{x}) \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z)).$$

Then

$$\vartheta^{(n)}(P; L; Z)\big|_{\varrho,m/2}\omega_j = \big(\gamma_p s_p(V)\big)^j (\det{}_p L)^{-j/2} \vartheta^{(n-j,j)}(P; L, L^{\#;p}; Z),$$

where $s_p(V)$ is the Witt-invariant of the completion V_p normalized as in [Sch] and γ_p depends only on $\det L\big(\mathbb{Q}_p^\times\big)^2$.

Proof. For $P = 1$ this is [BS], Lemma 8.2. The general case is proven in the same manner. □

Remark 4.9. The constant γ_p is computed in [Fu]. For $\det_p = p^{2t}$ (that is the case we are interested in) one gets $\gamma_p = 1$.

We now apply the partial $U_p^{(j)}$-operator $\sum_{T \in \mathrm{Sym}_r(\mathbb{F}_p)} \left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$ on $\vartheta^{(n-j,j)}(P; L, L^{\#;p}; Z)$. Since L_p has level p, we have $q(\mathbf{x}) \in M_n(\frac{1}{p}\mathbb{Z})$ and therefore

$$\vartheta^{(n-j,j)}(P; L, L^{\#;p}; Z)\big|_{\varrho,m/2}U_p^{(j)} = p^{j(j+1)/2} \sum_{\substack{\mathbf{x} \in L^{n-j} \times (L^{\#;p})^j \\ q(\mathbf{x}) \text{ integral}}} P(\mathbf{x}) \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z)).$$

We denote the set over which the summation extends by $L(n, j, p)$.

Applying $m(A)$ for $A \in GL_n(\mathbb{Z})$ finally gives

(4.2)

$$\begin{aligned}
&\vartheta^{(n)}(P;L;Z)\big|_{\varrho,m/2}\omega_j\, U_p^{(j)}\, m(A) \\
&\qquad = s_p(V)^j p^{-jt} p^{j(j+1)/2} \varrho(A^t) \sum_{\mathbf{x}\in L(n,j,p)} P(\mathbf{x}) \exp(2\pi i\, \mathrm{tr}(q(\mathbf{x})AZA^t)) \\
&\qquad = s_p(V)^j p^{-jt} p^{j(j+1)/2} \sum_{\mathbf{x}\in L(n,j,p)A} P(\mathbf{x}) \exp(2\pi i\, \mathrm{tr}(q(\mathbf{x})Z)),
\end{aligned}$$

using $P(\mathbf{x}A) = \varrho(A^t)P(\mathbf{x})$. Note that for $A \in Gl_n(\mathbb{Z})$ and $\mathbf{x} \in L(n,j,p)$ we have $\mathbf{x}A \in L(n,p) = L(n,n,p) = \big\{\mathbf{x} \in (L^{\#;p})^n \mid q(\mathbf{x}) \text{ integral}\big\}$.

In the following we will determine for a given $\mathbf{y} = (y_1,\dots,y_n) \in L(n,p)$ the number of $A \in P_{n,j}(\mathbb{F}_p)\backslash GL_n(\mathbb{F}_p)$ such that

$$\mathbf{y}A \in L(n,j,p). \tag{4.3}$$

So choose $\mathbf{y} \in L(n,p)$ arbitrarily and let r be the rank of $\mathbf{y}$ in the discriminant lattice $L^{\#;p}/L \simeq L_p^{\#}/L_p$ over $\mathbb{F}_p$. Since GL_n acts from the right on the column vectors of $\mathbf{y}$, we can assume $\mathbf{y} \in L(n,r,p) \subseteq L^{n-r} \times (L^{\#;p})^r$. For $r > j$ there are obviously no $A \in Gl_n(\mathbb{F}_p)$ such that (4.3) holds, whereas for $r \le j$ we see

$$\mathbf{y}A \in L(j,p) \quad \Longleftrightarrow \quad A \in P_{n,r}P_{n,j}$$

As $A \in P_{n,j}(\mathbb{F}_p)\backslash GL_n(\mathbb{F}_p)$ we conclude that the number of A with (4.3) is equal to

$$\Big|P_{n,j}\backslash P_{n,r}P_{n,j}\Big| = \Big|P_{n,j}\cap P_{n,r}\backslash P_{n,r}\Big| = \Big|P_{n-r,n-j}(\mathbb{F}_p)\backslash GL_{n-r}(\mathbb{F}_p)\Big| = \binom{n-r}{n-j}_p,$$

where $\binom{s}{t}_p$ is the number of t-dimensional subspaces of $\mathbb{F}_p^s$.

Hence, taking our explicit system of coset representatives, using (4.2) and therefore writing

$$\mathrm{tr}_N^{Np}\, \vartheta^{(n)}(P;L;Z) = \sum_{\mathbf{y}\in L(n,p)} P(\mathbf{y})a(\mathbf{y}) \exp(2\pi i\, \mathrm{tr}(q(\mathbf{y})Z)), \tag{4.4}$$

we get for the coefficient $a(\mathbf{y})$ for $\mathbf{y} \in L(n,p)$ such that $\mathrm{rank}_p(\mathbf{y}) = r$ (note $\mathrm{rank}_p(Y) \le \min(m,n)$)

$$\begin{aligned}
a(\mathbf{y}) &= \sum_{j=r}^{n} s_p(V)^j p^{-jt} p^{j(j+1)/2} \binom{n-r}{n-j}_p \\
&= \big(s_p(V)p^{-t}\big)^r p^{r(r+1)/2} \prod_{j=1}^{n-r} \big(1 + s_p(V)p^{r-t+j}\big) \qquad (4.5)
\end{aligned}$$

by a combinatorial identity attributed to Cauchy ([GR],p.252-54).

Thus the trace vanishes if $s_p(V) = -1$ and $n \geq t$. The converse for $P = 1$ follows from looking at the 0-th Fourier coefficient.

We now consider $\overline{L^{\#;p}} = L^{\#;p}/L \simeq L_p^{\#}/L_p$ as space over $\mathbb{F}_p$ with quadratic form $\bar{q} = p \cdot q \bmod p$. $\overline{L^{\#;p}}$ is regular of dimension $2t = \mathbb{Z}_p$-rank $L_p^{(1)}$. We are only interested in $\overline{L^{\#;p}}$ split (i.e hyperbolic); i.e. $s_p(V) = 1$ (see Lemma 4.5), although the following considerations are certainly valid for $\overline{L^{\#;p}}$ nonsplit as well.

We let W be a totally isotropic subspace of dimension r and embed W into a hyperbolic space H_W by Witt's Theorem. Every maximal totally isotropic subspace U containing W now defines a maximal totally isotropic subspace $H_W^{\perp} \cap U$ in $H_W^{\perp}$ which has Witt index $t-r$. On the other hand gives every maximal totally isotropic subspace of $H_W^{\perp}$ rise to a maximal totally isotropic subspace of $\overline{L^{\#;p}}$ containing W.

Hence every r-dimensional totally isotropic subspace of $\overline{L^{\#;p}} \simeq H_t$, the hyperbolic space of dimension $2t$, is contained in the same number of maximal totally isotropic subspaces of $\overline{L^{\#;p}}$. We denote that number by $\alpha(r, H_t)$. We have

$$\alpha(r, H_t) = \alpha(0, H_{t-r})$$

by the preceding discussion. Note that $\alpha(0, H_s)$ is the number of maximal totally isotropic subspaces of H_s. Using the explicit formulas for the number of isotropic vectors in a regular space over $\mathbb{F}_p$ (see e.g. [Ki], Lemma 1.3.1) one easily computes

$$\alpha(0, H_s) = \prod_{j=0}^{r-1} (p^j + 1). \tag{4.6}$$

Moreover, by using Hensel's Lemma, one sees that there is a 1-1 correspondence between the maximal totally isotropic subspaces of $\overline{L^{\#;p}}$ and even lattices K containing L, maximal under the condition $K \subset L^{\#;p}$. These lattices have level N by construction (see Lemma 4.5).

Consider finally

$$f(Z) = \sum_K \vartheta^{(n)}(P; K; Z),$$

where the sum goes over all these lattices. From the above discussion we get

$$f(Z) = \sum_{\mathbf{y} \in L(n,p)} P(\mathbf{y}) b(\mathbf{y}) \exp(2\pi i \operatorname{tr}(q(\mathbf{y})Z))$$

with $b(\mathbf{y}) = \alpha(r, \overline{L^{\#;p}})$ if rank $\mathbf{y} = r$. But now one easily checks that by (4.5) and (4.6) the quotient $\frac{a(\mathbf{y})}{b(\mathbf{y})}$ is independent of r (consider the cases $t \leq n$ and $t > n$ separately).

This completes the proof of the theorem. $\square$

Remark 4.10. An analogous representation theoretic problem is: Let π be an irreducible cuspidal automorphic representation of $Sp_n(\mathbb{A})$ containing a vector fixed under the compact group $K^{(n)}_{\mathbb{A},f}(N)$ of $Sp_n(\mathbb{A})$ that arises as the adelic version of $\Gamma^{(n)}_0(N)$. Let V be a regular quadratic space over $\mathbb{Q}$ with in integral lattice L of level Np ($p \not| N$) on it and let φ be an automorphic form on $O_{\mathbb{A}}(V)$ that is right invariant under the finite part $O_{\mathbb{A},f}(L)$ of the adelic group of units of L and whose theta lifting $\theta^{(n)}_L(\varphi)$ to $Sp_n(\mathbb{A})$ with respect to the characteristic function of L (and a suitable test function at ∞) is nonzero and in π (with φ generating an irreducible automorphic representation τ of $O_{\mathbb{A}}(V)$). Assume that V admits lattices of level N. Is it then true that π contains the lift $\theta^{(n)}_K(\varphi) \neq 0$ of an automorphic form ψ on $O_{\mathbb{A}}(V)$ right invariant under $O_{\mathbb{A},f}(K)$ for some lattice K of level N?

If N is squarefree this question can be answered positively as follows: By [Mo] τ is the theta lifting of π, hence from [Aub] it follows that τ contains a function ψ right invariant under $O_{\mathbb{A}}(K)$ for some lattice K of level dividing N. Again using results from [Mo] it can then be shown that $\theta^{(n)}_K(\psi) \neq 0$ (the details will appear in [SP]). If $N = 1$ or $n = 1$, the space of $K^{(n)}_{\mathbb{A},f}$-invariant vectors of π is known to be 1-dimensional, and we can even deduce that $\theta^{(n)}_K$ is proportional to $\theta^{(n)}_L(\psi)$, retrieving our classical result from above. The general situation seems to be more difficult.

Remark 4.11. The given proofs worked for the half-integral weight case as well. However, if one does not want to worry about the symplectic theta multiplier at all, one can consider the lattice $\tilde{L} = L \perp (2)$ instead (see also [Br]). One gets (up to a possible normalization factor) $\mathrm{tr}^{Np}_N \vartheta^{(n)}(P; \tilde{L}; Z) = \vartheta^{(n)}(Z)\, \mathrm{tr}^{Np}_N \vartheta^{(n)}(P; L; Z)$. On the other hand, a careful analysis of the given proofs shows that all formulas for $\mathrm{tr}^{Np}_N \vartheta^{(n)}(P; \tilde{L}; Z)$ preserve the orthogonal decomposition of $\tilde{L}$. More precisely, for $p \neq 2$ $\tilde{L}_p$ differs from L_p in the Jordan decomposition only in the unimodular part which is unaffected by the trace. $p = 2$ can only occur in our first basic case $(N, p) > 1$ since the level of even lattices of odd dimension is divisible by 4. Moreover we assumed in that case $ord_p(\text{level of } L) \geq 3$. Now increasing the dimension changes the 2-modular component, but this is irrelevant as only higher Jordan components are affected by the trace. In any case, we conclude that the appearing lattices $\tilde{K}$ split as $K \perp (2)$ with K being of lower level.

5. Siegel's ϕ-operator and traces

We recall the definition of Siegel's ϕ -operator: For a modular form $f \in M^k_n(M, \rho, v)$ and $Z \in \mathbb{H}_{n-1}$ it is given by

$$(\phi f)(Z) = \lim_{\lambda \to \infty} f(\begin{pmatrix} i\lambda & 0 \\ 0 & Z \end{pmatrix})$$

Then ϕf is a modular form in $M^k_{n-1,}(M,\rho',v')$, for the precise description of ρ' and v' we refer to [Fr3]. In particular, we have

$$\phi(\vartheta^{(n)}(P;L)) = \vartheta^{(n-1)}(P_0;L)$$

with $P_0(X) := P(\mathbf{0},X)$, $X \in V^{n-1}$ and $\mathbf{0} \in V$.
The commutation rules for Hecke operators ("away" from the level) and the ϕ-operator are well known [Zha, Fr2]. In this section we try to establish a similar commutation law for the trace operator and ϕ within the realm of Siegel modular forms; the special case of theta series will turn out to be somewhat simpler, see section 6.

Again we have to treat the two "basic cases" separately. Our computations are based on the standard embedding of $Sp_1(\mathbb{R}) \times Sp_{n-1}(\mathbb{R})$ into $Sp_n(\mathbb{R})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\uparrow} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\downarrow} := \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}$$

and we shall use the rules

$$\phi\left(f \mid_{k,\rho} \gamma^{\downarrow}\right) = (\phi f) \mid_{k,\rho'} \gamma, \qquad (\gamma \in Sp_{n-1}(\mathbb{R})) \tag{5.1}$$

and

$$\phi(f \mid_{k,\rho} \begin{pmatrix} 1 & a_2 & b_1 & b_2 \\ 0 & 1_{n-1} & b_3 & 0 \\ \multicolumn{2}{c}{\mathbf{0}_n} & 1 & 0 \\ & & -a_2^t & 1_{n-1} \end{pmatrix}) = \phi(f) \tag{5.2}$$

for all real symplectic matrices of the type above. (To obtain the second rule, one can e.g. compare the Fourier expansions on both sides.)

5.1. The case $M = Np$ with $p \mid N$.

Proposition 5.1. *If $p \mid N$ we have for any $f \in M^k_n(Np,\rho,v)$*

$$\phi(tr^{Np}_N(f) \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}) = p^n(tr^{Np}_N \phi(f \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow})$$

We remark here that $f \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}$ is not in $M^k_n(M,\rho,v)$ but its image under ϕ is again in $M^k_{n-1}(M,\rho',v')$.

Proof. To describe the trace in this case, we use the same system of coset representatives for $\Gamma^{(n)}_0(Np)\backslash\Gamma^{(n)}_0(N)$ as in the proof of Theorem 4.1. For any symmetric n-rowed matrix $S = \begin{pmatrix} s_1 & s_2 \\ s_2^t & S_4 \end{pmatrix}$ with S_4 symmetric and

$(n-1)$-rowed we use

$$\begin{pmatrix} 1_n & 0_n \\ S & 1_n \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}\begin{pmatrix} 1 & s_2 & -s_1 & 0 \\ 0 & 1_{n-1} & 0 & 0_{n-1} \\ \multicolumn{2}{c}{0_n} & 1 & 0 \\ & & -s_2^t & 1_{n-1} \end{pmatrix}\begin{pmatrix} 1_{n-1} & 0_{n-1} \\ S_4 & 1_{n-1} \end{pmatrix}^{\downarrow}$$

From this we obtain, using the rules for the ϕ-operator mentioned earlier,

$$\phi(tr_N^{Np}(f) \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}) =$$

$$\phi(\sum_{S \bmod p} f \mid_{k,\rho} \begin{pmatrix} 1_n & 0_n \\ NS & 1_n \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}) =$$

$$p^n \sum_{S_4 \bmod p} (\phi(f \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mid_{k,\rho_0} \begin{pmatrix} 1_{n-1} & 0_{n-1} \\ NS_4 & 1_{n-1} \end{pmatrix}$$

The proposition follows by observing that our multiplier systems v and v' are trivial on matrices of type $\begin{pmatrix} 1 & 0 \\ NS & 1 \end{pmatrix}$ with S symmetric and integral. $\square$

Remark 5.2. The proposition above is not the only possible version of such a commutation rule, e.g. there is a similar law for the map

$$f \longmapsto \phi(f \mid_{k,\rho} \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}) \mid_{k,\rho'} \begin{pmatrix} 0_{n-1} & -1_{n-1} \\ 1_{n-1} & 0_{n-1} \end{pmatrix}$$

5.2. **The case M=Np with N coprime to p.** The group-theoretic background for the "second case" (i.e. $M = Np$ with N coprime to p) is the following description of $\Gamma_0^{(n)}(M)\backslash\Gamma_0^{(n)}(N)$, which is in some sense compatible with the ϕ-operator.

Lemma 5.3. *A complete set of representatives for $P_n\backslash Sp_n(\mathbb{F}_p)$ is given by*

$$\{g^{\downarrow} \mid g \in P_{n-1}\backslash Sp_{n-1}(\mathbb{F}_p)\}$$

and

$$\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}^{\uparrow}\begin{pmatrix} 1 & s & \multicolumn{2}{c}{\multirow{2}{*}{0_n}} \\ 0 & 1_{n-1} & & \\ \multicolumn{2}{c}{\multirow{2}{*}{0_n}} & 1 & 0 \\ & & s^t & 1_{n-1} \end{pmatrix} g^{\downarrow}\}$$

with $s \in \mathbb{F}_p^{n-1}, l \in \mathbb{F}_p, g \in P_{n-1}\backslash Sp_{n-1}(\mathbb{F}_p)\}$

Proof. By direct computation, we see that the matrices above are indeed pairwise inequivalent. On the other hand ([Kl])

$$[Sp_n(\mathbb{F}_p) : P_n] = p^{\frac{n(n+1)}{2}} \prod_{\nu=1}^{n}(1 + p^{-\nu}) = (1 + p^n)[Sp_{n-1}(\mathbb{F}_p) : P_{n-1}]$$

A more conceptual proof would start from a double coset decompostion

$$P_n \backslash Sp_n(\mathbb{F}_p)/P_{n,n-1}$$

where $P_{n,n-1}$ denotes the standard parabolic with Levi factor $GL_1 \times Sp_{n-1}$; there are two such double cosets with 1_{2n} and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}$ as representatives.

□

We choose an element $\omega = \omega_p \in \Gamma_0^{(n)}(N)$ satisfying $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}$ mod p, then we obtain as a set of representatives for $\Gamma_0^{(n)}(M)\backslash\Gamma_0^{(n)}(N)$:

$$\{g^{\downarrow} \mid g \in \Gamma_0^{(n-1)}(M)\backslash\Gamma_0^{(n-1)}(N)\}$$

$$\{\omega \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}^{\uparrow} \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1_{n-1} \end{matrix} & 0_n \\ 0_n & \begin{matrix} 1 & 0 \\ s^t & 1_{n-1} \end{matrix} \end{pmatrix} g^{\downarrow}\}$$

with $l \in \mathbf{Z}$ mod $p, s \in \mathbf{Z}^{n-1}$ mod $p, g \in \Gamma_0^{(n-1)}(M)\backslash\Gamma_0^{(n-1)}(N)$. Again by the rules (5.1) and (5.2) we get for a modular form $f \in M_n^k(M,\rho,v)$

Proposition 5.4.

$$\phi(tr_N^M(f)) = tr_N^M(\phi f) + v(\omega)p^n tr_N^M \phi(f \mid_{k,\rho} \omega)$$

6. Application to theta series

The commutation rules of propositions 5.1 and 5.4 become simpler when applied to theta series:

Proposition 6.1. *Suppose that $\vartheta^{(n)}(P,L) \in M_n^k(M,\rho,\chi)$ with $M = Np$ and $p \mid N$. Then*

(6.1)

$$\phi(tr_N^{Np}\vartheta^{(n)}(P,L) \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}) = i^{\frac{-m}{2}} p^n \det(L)^{\frac{-1}{2}} \times tr_N^{Np}(\vartheta^{(n-1)}(P_0,L))$$

Proof. To prove this, we write the polynomial $P(\mathbf{x})$ as sum of its homogeneous components with respect to its "first" variable $\mathbf{x_1}$:

$$P(\mathbf{x}) = P(\mathbf{x}_1,\mathbf{x}_2) = \sum P_{(i)}(\mathbf{x}_1,\mathbf{x}_2), \qquad \mathbf{x}_1 \in V, \quad \mathbf{x}_2 \in V^{n-1}$$

This corresponds to decomposing the representation $\rho_{|GL_1\times GL_{n-1}}$ into its irreducible components as representations of GL_1:

$$\rho_{|GL_1\times GL_{n-1}} = \bigoplus_i \rho_i^{(1)} \otimes \rho_i^{(n-1)}$$

with $\rho_i^{(1)}(x) = x^i$, $x \in \mathbb{C}^\times$ and $\rho_2^{(n-1)}$ is some representation of GL_{n-1}. For $\mathbf{x_2}$ fixed the function

$$\tau \longmapsto \sum_{\mathbf{x}\in L} P_{(i)}(\mathbf{x}_1, \mathbf{x}_2) exp(2\pi i q(\mathbf{x_1})\tau), \qquad \tau \in \mathbb{H}_1$$

is then a cusp form for $i > 0$ (and does not contribute to the right hand side of (6.1)); for $i = 0$ we have $P_{(0)}(\mathbf{x}_1, \mathbf{x}_2) = P_0(\mathbf{x}_2)$ and
(6.2)

$$\phi(\vartheta^{(n)}(P, L) \mid_{k,\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow} = \phi(\vartheta^{(1)}(L) \mid_{\frac{m}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \cdot \vartheta^{(n-1)}(P_0, L))$$

Using the standard inversion formula for $\vartheta^{(1)}(L)$ we obtain the proposition from (6.2) and proposition 5.1.

□

We now turn to the ”second basic case”: Combining proposition 5.1 and Lemma 4.2 (and choosing $\omega \equiv 1_{2n} \bmod N$) we obtain

Proposition 6.2. *Suppose that $\vartheta^{(n)}(P, L) \in M_n^k(M, \rho, \chi)$ and $M = Np$ with N coprime to p. Then*

$$\phi(tr_N^{Np}\vartheta^{(n)}(P, L)) = (1 + s_p(V)p^n det(L)_p^{-\frac{1}{2}})tr_N^{Np}\vartheta^{(n-1)}(L, P_0)$$

By a standard procedure we can now recover essentially the same results as in Section 4.
We just mention the main steps:
(*) The theory of singular modular forms [Fr3] asserts that $tr_N^{Np}\vartheta^{(n)}(P, L)$ is indeed (some) linear combination of theta series of appropriate level if the weight r is smaller than $\frac{n}{2}$, in particular, this trace is zero,if the ambient quadratic space does not allow lattices of level dividing N.
(**) For $\vartheta^{(n-1)}(P_0, L)$ there always exists $\vartheta^{(n)}(P, L)$ with $\phi(\vartheta^{(n)}(P, L)) = \vartheta^{(n-1)}(P_0, L)$, see [Fr3]. By the commutation laws above we can get an (explicit) expression for $tr_N^{Np}(\vartheta^{(n-1)}(P_0, L))$ if $tr_N^{Np}(\vartheta^{(n)}(P, L))$ admits such an (explicit) expression unless $1 + s_p(V)p^n det(L)_p^{-\frac{1}{2}} = 0$
(***) The statement ”some linear combination of theta series of appropriate level” in (*) can be made explicit by reconsidering the first nonsingular case ”n=2r” more carefully.

Remark: For theta series with characteristics such commutation laws were established by Salvati-Manni [SM1, SM2, SM3] (The idea to use the trace operator, the theory of singular modular forms and some commutation rules with respect to the ϕ-operator is actually due to him.) Roughly speaking, our results given here can be recovered from [SM3] if the quadratic space in question is rationally equivalent to

$$\perp_1^m < 1 >$$

References

[An] A. Andrianov, *Quadratic Forms and Hecke Operators*, Grundlehren der math. Wiss. **286**, Berlin Heidelberg (1987).

[AnZh] A. N. Andrianov, V. G. Zhuravlev: *Modular Forms and Hecke Operators*, Translations of Mathematical Monographs **145**, AMS 1995

[Aub] A.-M. Aubert: *Correspondance de Howe et sous-groupes parahoriques*, J. f. d. reine u. angew. Math. **392**, 176-186 (1988).

[BS] S. Böcherer, R. Schulze-Pillot, *Siegel Modular Forms and Theta Series attached to Quaternion Algebras*, Nagoya Math. J. **121**, 35-96 (1991).

[Boe] S.Böcherer, Traces of theta series. Proceedings of the Kinosaki Conference on automorphic forms 1993, 101-105

[Br] D. Brümmer, *Eine Spurbildung an Thetareihen*, Dissertation, Münster (1970)

[Ei] M. Eichler: *Quadratische Formen und orthogonale Gruppen*, 2. Aufl., Springer Verlag 1974

[Ev] S. Evdokimov, *Action of the irregular Hecke operator of index p on the theta series of a quadratic form*, J. Sov. Math. **38**, 2078-2081 (1987).

[Fr1] E. Freitag, *Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren*, Math. Zeit.**156**, 141-155 (1977).

[Fr2] —-, *Die Wirkung von Heckeoperatoren auf Thetareihen mit harmonischen Koeffizienten*, Math. Ann. **258**, 419-440 (1982)

[Fr3] —-, *Singular Modular Forms and Theta relations*, Lecture Notes Math. **1487**, Berlin Heidelberg (1991).

[Fu] J. Funke, *Spuroperator und Thetareihen quadratischer Formen*, Diplomarbeit, Köln (1994).

[GR] J. Goldman, G.-C. Rota, *On the foundations of combinatorial theory IV*, Studies in applied Math. **49**, 239-258 (1970).

[KV] M. Kashiwara, M. Vergne, *On the Segal-Shale-Weil representation and harmonic polynomials*, Inv. Math. **44**, 1-47 (1978).

[Ki] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge (1993).

[Kl] H. Klingen, *Eine Bemerkung über Kongruenzuntergruppen der Modulgruppe n-ten Grades*, Arch. d. Math. **10**, 113-122 (1959).

[Kn] M. Kneser, *Quadratische Formen*, Vorlesungsausarbeitung (Lecture notes) Göttingen 1974

[Ku] T. Kume, *Calculation of traces of theta series by means of the Weil representation*, J. Math. Kyoto Univ. **38–3**(1998), 453–473.

[Mo] C. Moeglin: *Quelques propriétés de base de séries theta*, J. of Lie Theory **7** (1997), 231-238

[OM] O. O'Meara, *Introduction to quadratic forms*, Grundlehren der math. Wiss. **117**, Berlin Göttingen (1963).

[SM1] R.Salvati-Manni, *Thetanullwerte and stable modular forms.* Am.J.Math.**111**, 435-455 (1989)

[SM2] R.Salvati-Manni, *Thetanullwerte and stable modular forms II.* Am.J.Math.**113**, 733-756 (1991)

[SM3] R.Salvati-Manni, *Thetanullwerte and stable modular forms for Hecke groups.* Math.Z.**216**, 529-539(1994)

[Sch] W. Scharlau *Quadratic and Hermitian Forms*, Grundlehren der math. Wiss. **270**, Berlin Heidelberg (1985).

[SP] R. Schulze-Pillot, *Theta liftings- a comparison between classical and representation theoretic results*, Preprint 1998, to appear in Proceedings of the Symposium on Automorphic Forms, RIMS Kyoto 1998

[Sh] G. Shimura, *On modular forms of half integral weight*, Ann. Math **97**, 440-481 (1973).

[Yo] H. Yoshida *The action of Hecke operators on theta series*, in: Algebraic and topological theories – to the memory of Dr. T. Miyata –, 197-238 (1985).

[Zha] N.A.Zharkovskaya, *The Siegel operator and Hecke operators*. Funct.Anal.Appl.8, 113-120(1974)

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