The array representation and primary children's understanding and reasoning in multiplication[1]

Patrick Barmby, Tony Harries, Steve Higgins and Jennifer Suggate
School of Education, Durham University, UK

Abstract We examine whether the array representation can support children's understanding and reasoning in multiplication. To begin, we define what we mean by understanding and reasoning. We adopt a 'representational-reasoning' model of understanding, where understanding is seen as connections being made between mental representations of concepts, with reasoning linking together the different parts of the understanding. We examine in detail the implications of this model, drawing upon the wider literature on assessing understanding, multiple representations, self explanations and key developmental understandings. Having also established theoretically why the array representation might support children's understanding and reasoning, we describe the results of a study which looked at children using the array for multiplication calculations. Children worked in pairs on laptop computers, using Flash Macromedia programs with the array representation to carry out multiplication calculations. In using this approach, we were able to record all the actions carried out by children on the computer, using a recording program called Camtasia. The analysis of the obtained audiovisual data identified ways in which the array representation helped children, and also problems that children had with using the array. Based on these results, implications for using the array in the classroom are considered.

Keywords Array · Multiplication · Representations · Reasoning · Understanding

1. Introduction

Many authors (for example Anghileri, 2000; Davydov, 1991; Greer, 1992) have suggested that multiplication is significantly more difficult than addition and subtraction. Nunes and Bryant (1996) highlighted that a commonly held view of multiplication and division is that they are simply "different arithmetic operations … taught after they have learned addition and subtraction" (p.144). However, they stated that this is too limited a view and that in actual fact "multiplication and division represent a significant qualitative change in children's thinking" (p.144). Whilst addition and subtraction can be thought of as the joining of sets, multiplication is about replication. Addition and subtraction are unary operations with each input representing the same kind of element – 3 blocks added to 4 blocks or 3 oranges added to 4 oranges. However, we need to view multiplication as a binary operation with two distinctive inputs (Anghileri, 2000). The first input represents the size of a set (say the number of oranges in a particular set) and the second represents the number of replications of that set (how many sets of oranges). In this way, the two numbers represent distinct elements of the multiplication process.

[1] Paper accepted for publication in *Educational Studies in Mathematics*

The important aspects of multiplication, in relation to whole numbers which primary pupils need to understand and which are discussed by Anghileri (2000) amongst others, are:

- Replication (rather than joining as in addition/subtraction);
- The binary rather than unary nature of multiplication, and the notion of two distinct and different inputs;
- Commutativity for multiplication but not division;
- Distributivity.

However, Dickson *et al*. (1984) drew on earlier research which suggested that only a minority of upper primary children could use commutative properties, and ‘few’ 8 to 9 year olds could draw on distributive properties, in order to solve problems involving multiplicative situations.

In the National Numeracy Strategy (DfEE, 1999), which until 2006 underpinned the teaching of mathematics in primary schools in England and Wales, there seemed to be a clear difference in the approach between the teaching of addition and subtraction and multiplication and division. In teaching addition and subtraction, there was a clear use of representations such as number squares and number lines, with number lines being seen as the most appropriate representation for demonstrating the characteristics of these operations. The learner was encouraged to use these representations to explore these operations. For example, the Year 1 learning objectives for 5 to 6 year old pupils were to understand addition as:

- Combining sets to make a total;
- Steps along a number track (counting on).

Pupils progressed on to the use of number lines in Year 3 (7 to 8 year olds) for informal pencil and paper methods to support, record and explain methods. But when we consider multiplication and division, we found that the approach appeared to be different. The pedagogical processes appeared to be mainly skip counting, learning tables and learning algorithms. For example, the Year 2 learning objective for 6 to 7 year old pupils were to understand multiplication as:

- Repeated addition;
- Describing an array – begin to recognise from arranging arrays that multiplication can be done in any order.

Pupils were encouraged to use arrays or number lines for illustrating the multiplication of single digit numbers (e.g. 5×4). However by Year 4, children simply needed to use the commutative and distributive law, alongside rapid recall of calculations. No visual representation were drawn upon. Therefore, representations were used earlier on, but only for the purpose of illustrating multiplication and rarely for the purpose of supporting calculation. In the recently renewed Primary Framework for literacy and mathematics (DfES, 2006), teachers in England are now being encouraged to use the array representation to illustrate particular aspects of multiplication such as the distributive law. However, we know little about whether and how children are able to access the properties of multiplication within this representation (or any other), and furthermore, whether it helps their ability to carry out calculations. Therefore, the main aim of this paper will be to examine children's reasoning and use of the array representation for multiplication calculations.

2. A model of mathematical understanding

Before we examine the concepts involved in multiplication and the array representation in particular, let us first outline the theoretical model of understanding of mathematical concepts that we will work with. In doing so, we wish to clarify not only our views of 'understanding' and 'reasoning' that make up the subject of this paper, but also the model will point to implications for developing and demonstrating understanding, which in turn will guide us in our research work for how to examine children's understanding of and reasoning within multiplication.

The model for understanding that we have adopted emphasises the importance of the connections between internal or mental representations of a concept. In the literature, we find a host of examples of understanding being defined with respect to these connections. Skemp (1976) described the process of learning relational mathematics as "building up a conceptual structure" (p. 14). Nickerson (1985) also referred to the connections between concepts: "The richer the conceptual context in which one can embed a new fact, the more one can be said to understand the fact" (p. 235-236). Hiebert and Carpenter (1992) specifically defined mathematical understanding as involving the building up of the conceptual 'context' or 'structure' mentioned above:

"The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections." (p. 67)

Sierpinska (1994) identified 'acts of understanding' as being the mental experience associated with linking what is to be understood with the 'basis' for that understanding, and the 'processes of understanding' involving links being made between acts of understanding through reasoning processes.

Central to the model of understanding that we use is this notion of the connections or links that a person makes. The next step is to define what is being linked. Hiebert and Carpenter referred to 'mental representations' of a concept. Davis (1984) defined these mental or internal representations of concepts as:

"Any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in the mind at all, must be represented in some way." (p.203)

Goldin (1998) put forward a variety of internal representations, including verbal/syntactic, imagistic, symbolic, planning/monitoring/controlling and affective representations. The examples of 'bases' being linked in the definition of Sierpinska (1994) were mental representations, mental models, and memories of past experiences. Our understanding associated with a mathematical concept might draw on a variety of possible internal representations, ranging for example from the algorithm associated with carrying out a calculation (Skemp's instrumental understanding) through to links with images or concrete situations exemplifying the algorithms (and therefore a broader relational understanding).

Having put forward *what* is being linked in understanding, we should also clarify *how* these internal links are being made. Sierpinska's definition of understanding already incorporated the notion of *reasoning* linking together parts of understanding. The NCTM's Principles and Standards for School Mathematics (NCTM, 2000) stated that:

"Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and – with different expectations of sophistication – at all grade levels, students should see and expect that mathematics makes sense … Systematic reasoning is a defining feature of mathematics." (p. 56-57)

In fact, we choose to consider reasoning in a broader way, more simply "as the process of drawing conclusions" (Leighton, 2004). The process of being able to link one mental representation to another can then be thought of as the process of concluding why we can link one representation to another. For example, if we have the two representations of 7×3 and 3×7 associated with the concept of multiplication, our understanding might include the link that both of these calculations provide the same answer. How might this link have been established? At a basic level, one might reason that "the answers are the same because we can always swap round numbers in multiplication". A more advanced sequence of reasoning might be "if we represent the calculations as an array, then we can see that the only difference arises in the orientation of the array." In both cases, the two representations are linked through reasoning. We might regard one of the reasons provided as being more developed, but nevertheless, both provide possible ways in which the representations can be linked. This view of reasoning allows us to include the less sophisticated reasoning that we might see from pupils, although the goal of mathematics is to progress towards the systematic reasoning highlighted by the NCTM. We see reasoning as the process by which, both informally and more formally, the links between representations are established. Christou & Papageorgiou (2007) described the process of reasoning as thus:

"Reasoning, in general, involves inferences that are drawn from principles and from evidence, whereby the individual infers new conclusions or evaluates proposed conclusions from what is already known." (p.56)

Therefore, a further insight that we can gain is that reasoning can be used to both demonstrate existing understanding (i.e. to explain links between representations) or to develop understanding further (i.e. to establish links between representations). Lawson (1994) highlighted reasoning as being a key part of the constructive process. It is the process of trying to link between our existing understanding of a concept and a range of alternative conceptions or representations that brings about Piaget's processes of assimilation or accommodation. Steinbring (1997), in his epistemological structure of mathematical knowledge, also highlights that "it is necessary to create a 'productive' tension between the sign system and a structural referent context and to develop and maintain this tension. Only the relative differences between these two systems allows students to actively develop meaning by contrasting and comparing the (new) sign/symbol system with the (relatively familiar) reference context" (p. 79). As we try and reason between more representations associated with a concept, our understanding will develop as we encounter difficulties with our reasoning. In summary then, we have the following picture to represent this 'representational-reasoning' model of understanding:

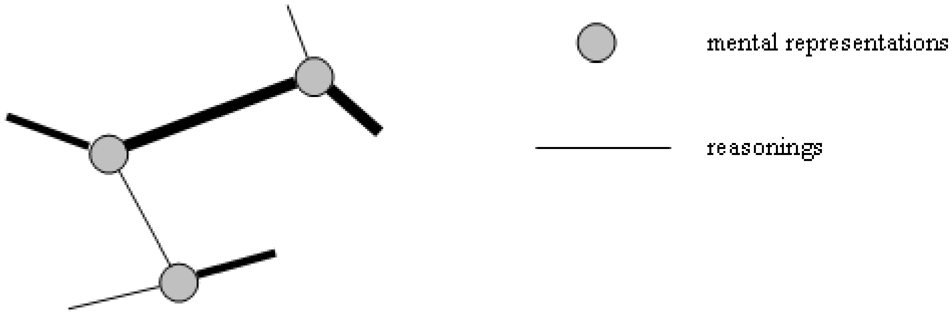

Figure 1: Diagrammatical representation of the 'representational-reasoning' model of understanding

The thicker and thinner lines in the diagram are used to denote different levels of reasoning between representations. For example, as we highlighted above, reasoning between representations may be at a basic level indicative of instrumental understanding, or it might be more advanced, perhaps a product of a deductive process.

3. Implications of the model of understanding

Having described the model of mathematical understanding that we will use, we can also examine the implications of this model for a person's understanding and how we might examine understanding. First of all, from the picture of understanding that we have, there is nothing to prevent us from creating additional links to our understanding of a particular concept. Also, if we have encountered a concept to any degree (even, for example, a word associated with that concept), then there will be some links that we construct (the word might sound like another word we know).

"Understanding is not a dichotomous state, but a continuum … Everyone understands to some degree anything that they know about. It also follows that understanding is never complete; for we can always add more knowledge, another episode, say, or refine an image, or see new links between things we know already." (White and Gunston, 1992, p.6)

The links that we construct might not always be the links that our teachers would have wanted us to construct, being influenced by our diverse range of social experiences (i.e. the social constructivist view of understanding, Ernest, 1994). However, with this model of understanding, we move away from the idea that understanding is something that we have or do not have, and that mathematics is something we know or do not know, that we can or cannot do. Rather, we have differing degrees of understanding. Within this picture, we are more active participants in the construction of understanding.

A second implication of the model is that in order to examine someone's understanding of a mathematical concept, it is important that we examine the connections that a person makes to that concept. Of course, we cannot see these internal connections directly; rather, we must observe the connections that a person can demonstrate and infer their understanding from these. Methods such as mind maps and concept maps have been put forward as ways of assessing understanding (for example, in the field of mathematics education, Williams, 1998, McGowan and Tall, 1999, and Brinkmann, 2003). However, it is not just a case of looking at the number of connections but the quality or strength of the connections as well. With concept maps or mind maps, White and Gunston (1992) argued that the reasons for each link should also be given in the diagrams as well. More commonly, when we wish to examine a student's mathematical understanding, we might ask them to explain their reasoning (i.e. demonstrate the links that they have made through reasoning) either verbally or in a written format. We might use clinical interviews or task-based interviews (e.g. Davis, 1984) to probe the student's understanding (i.e. to see what links they can reason), or open-ended questions. The difficulty lies in accessing the large number of links that might make up the understanding:

"Understanding usually cannot be inferred from a single response on a single task; any individual task can be performed correctly without understanding. A variety of tasks, then, are needed to generate a profile of behavioural evidence." (Hiebert and Carpenter, 1992, p. 89)

We might therefore provide a variety of opportunities for links to be demonstrated in order to get a broader picture of understanding. For example, Niemi (1996), in examining students' understanding of fractions, used a problem solving task where students justified their methods, an open-ended task where they explained fractions more broadly and a task examining the fluency with which students could access visual representations. In these tasks, the open-ended task provided the opportunity to see the possible links that students made, the problem solving task specifically looked at the reasoning used by the students when linking between the problems and their understanding of fractions, and the final task examined which links to visual representations the students could quickly access. Similarly, Lawson and Chinnappan's (2000) study of students' geometrical knowledge utilised free recall of theorems, linking theorems to applications, linking theorems to possible problems, a problem solving task, a graded hinting task, and a timed task for recognising geometrical components. The variety of tasks was designed to elicit the content and also the 'connectedness' of students' geometrical knowledge. The connectedness was measured by the time taken for responses or the hints required for solving a task. Specific links were examined by presenting applications to elicit theorems, and the range of links examined through the open-ended tasks involving free recall or problem solving situations. In both cases, the methodologies were consistent with the picture of understanding that we are using.

A third implication of the model of understanding is that because we are identifying reasoning between representations of a mathematical concept as constituting part of the understanding, then developing the reasoning and developing the representations that we can link to should develop our understanding. We can draw on a number of areas within mathematics education research to support these assertions. First of all, the literature on the use of multiple representations in mathematics education suggests that students who can call upon a broader range of representations possibly have increased understanding of concepts (e.g. Moseley, 2005). Steinbring (1997) goes further and states that access to alternative representations or "reference contexts" is required in order to understand mathematical "signs" or symbols: "Meanings of mathematical concepts emerge in the interplay between sign/symbol systems and reference contexts" (p.50). However, research has also shown that simply being able to access multiple representations does not necessarily lead to improved understanding (Ainsworth, 2006; Seufert, 2003):

> “The combinations of representations that both complement and constrain each other enables learners to deal with the material from different perspectives and with different strategies, and therefore can have synergetic effects on the construction of coherent knowledge structure. However, this synergy does not emerge *per se*. Learners must interconnect the external representations and actively construct a coherent mental representation in order to benefit from the complementing and constraining functions of multiple representations.” (Seufert, 2003, p. 228)

Our picture of understanding is still consistent with these findings if we argue that having access to a variety of representations *and* being able to reason between them contribute to the development of understanding. Simply having access to multiple representations alone does not necessarily result in any deep restructuring of understanding in order to accommodate these new representations.

This notion of reasoning being integral to our model of understanding is also supported by research on students’ use of ‘self-explanations’ in building understanding. For example, Chi *et al*. (1994) showed that eliciting self-explanations from students improved their learning and their understanding of a human biology topic. Große and Renkl (2006) also showed that written self-explanation prompts can be employed to foster learning. However, if the multiple representations vary in only minor, superficial ways, then again, the greater numbers of representations that we can access will not result in any great restructuring of our understanding. Developing this idea further, we could argue that being able to include certain key representations for a concept in our understanding, and be able to reason between this and other representations that we already have, will result in a greater restructuring of our understanding. We can relate this to the notion of ‘key development understandings’ (KDUs), defined by Simon (2006) as “a conceptual advance that is important to the development of a concept. It identifies a qualitative shift in students’ ability to think about and perceive particular mathematical relationships – in other words, a significant change in the assimilatory structures that students have available” (p. 363-364). We also see parallels with the notion of ‘threshold concepts’ put forward by Meyer and Land (2003): “As a consequence of comprehending a threshold concept there may thus be a transformed internal view of subject matter, subject landscape, or even world view” (p. 1).

Understanding is therefore developed through connections made between different internal representations and being able to reason between these. Introducing different external representations to pupils and providing different activities within which pupils can reason would help, we would hope, to develop understanding. We assume here that pupils can reason between these external representations (although we will see later in the results of the study examples where pupils are unable to do so). The representational-reasoning model of understanding has also helped us to identify that if we are to examine children’s understanding of multiplication, we might examine whether they can reason with what we would consider to be key representations for multiplication. One such representation, we would argue, is the array representation of multiplication. In the study that we carried out, we examined whether children could reason with this particular key representation. We hoped that this reasoning might develop pupils’ understanding of multiplication to some extent. In addition, we were interested in whether the array was part of pupils’ understanding of multiplication.

However, before we describe our study, we must put forward our arguments for why the array might be a key representation for multiplication. We move on to this in the next section.

4. The array as a key representation for multiplication

Sfard (1991) highlighted the role that representations (in particular visual representations) play in the way we develop mathematical concepts as learners, moving from an operational or process view of a concept to a structural view (e.g. moving from multiplication as a process to multiplication as a static object, the properties of which can then be examined): "It is the static object-like representation which squeezes the operational information into a compact whole" (p. 26). The question we ask here is which representation for multiplication best conveys the most important properties of multiplication? The representations considered in this paper encourage different ways of viewing the multiplication of whole numbers. For example:

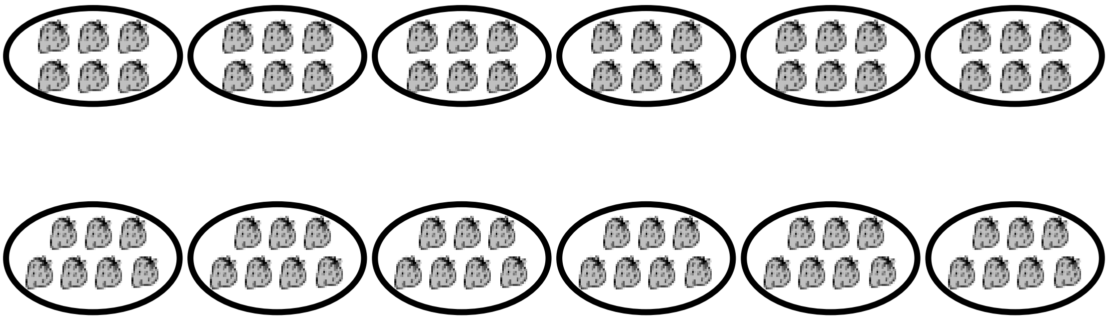

Figure 2: 'Groups of' representation of multiplication, showing 6 × 7 and 7 × 6

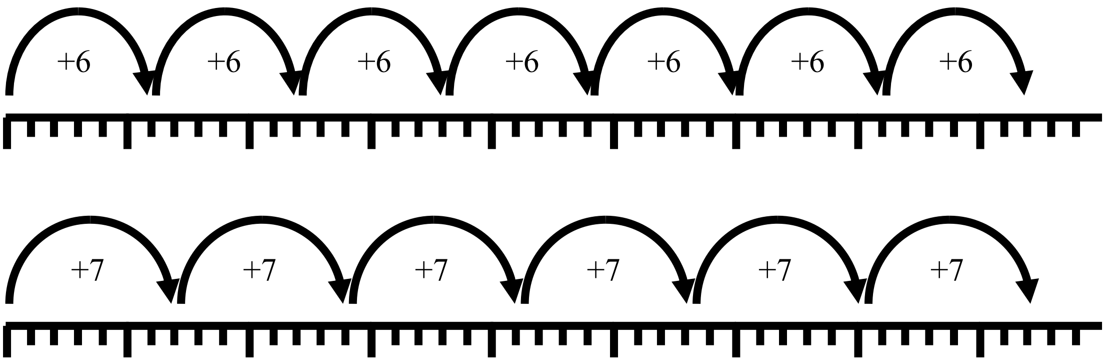

Figure 3: Number line representation of multiplication, showing 6 × 7 and 7 × 6

Figures 2 and 3 show representations that encourage unary thinking and also encourage a repeated addition method of calculation – a method that whilst it may work becomes increasingly inefficient as the value of the numbers increase. These representations which encourage repeated addition are also problematic when the multiplication involves two rational numbers. Both representations illustrate the idea of equal groups, and the number line provides help with the calculation. Neither representation however illustrates the two important aspects of multiplication identified previously, namely commutativity and the distributive characteristic. For example, when the numbers are swapped in the above diagrams, the representations will look quite different. It is not immediately obvious why the commutative law should apply. We contend that the array representation, shown below in Figure 4, encourages pupils to develop their thinking about multiplication as a binary operation with rows and columns representing the two inputs.

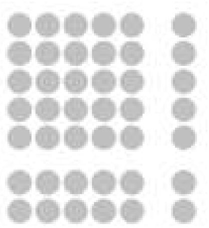

Figure 4: Array representation for 6 × 7 or 7 × 6

Both the commutative and the distributive properties of multiplication are more evident in this representation. There is also a clearer link that can be made to multiplication of fractions. For example $\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$ can be shown using a two-dimensional picture (Figure 5), where we start with the unit, take one fraction of the unit in one direction, and then take a fraction of portion of the unit in the other direction. Both the array and this picture of multiplication of fraction emphasise the binary nature of multiplication.

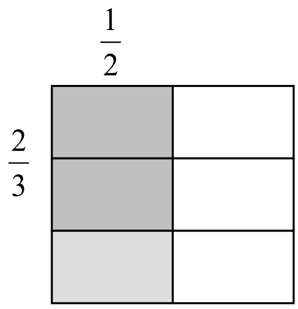

Figure 5: Two-dimensional representation for $\frac{1}{2} \times \frac{2}{3}$

We are therefore interested in exploring ways in which the array could be used as a tool for developing an understanding of multiplication, in the same way that we work with a number line for addition and subtraction. We have included spacings at intervals of five to emphasise the distributive property of multiplication (i.e. 6 × 7 = (5+1) × (5+2) = (5×5) + (5×2) + (1×5) + (1×2)). This allows the pupils to use what Flexer (1986) calls "the power of five" as an aid to calculation. However, we can also see from the array that we could have other spacings. Algebraically, if we have $a + b$ colums and $c + d$ rows, then we would have $(a+b) \times (c+d) = ac + ad + bc + bd$. Extending the array representation to larger numbers, it can also directly lead on to the grid method of multiplication, therefore explaining a written method of carrying out multiplication calculations. Furthermore, as we have just shown, we do not need to restrict the splitting of numbers in the grid method to that of units, tens, hundreds etc. Children may prefer to choose different ways of splitting the numbers to be multiplied, e.g. using intervals of 5. We would therefore argue that the array representation is a key representation for multiplication for primary or elementary school children. Being able to reason with this representation can allow children to reason the important properties of multiplication outlined previously. Some evidence for this is provided by Izsák (2004) who found that using the rectangular array representation did indeed result in classroom strategies using the distributive law and supported the moving on to the grid method. However, individual students in that study still had difficulty in moving from the rectangular array to the grid representation. Steinbring (1997) also provided a case study of children using the distributive law in the context of using an array representation.

5. Methodology

Based on our belief in the importance of the array as a key representation for multiplication, we carried out a classroom study to examine how the array representation could support, or hinder, children's reasoning in multiplication. As we outlined previously, the array representation that we used involved spacings at intervals of five in order to encourage pupils to split up the array, moving towards a recognition of the distributive property of multiplication. The study involved working with pupils in a primary school in the North East of England. More specifically, we worked with a Year 4 class (pupils aged 8 to 9) and a Year 6 class (pupils aged 10 to 11, the final year of primary school in England). Within the school, pupils were set in different classes according to their ability in mathematics, and the pupils that we worked with were all in the upper ability class within that year. There were 20 pupils taking part in the study in the Year 4 class, and 14 pupils in the Year 6 class. These upper ability classes were not specifically requested by the researchers, rather these were the pupils that the school asked us to work with. Therefore, we were clear from the outset that the choices of school and pupils to take part in the study were not representative of the wider cohort in any way. Rather, working with these particular pupils provided us with the opportunity to simply observe the possible use of the array representation with some children. As part of the mathematics curriculum, the pupils would have previously been introduced to the array as a representation of multiplication in Year 2 or Year 3 of their schooling. However, they would not have used it to actively support their multiplication calculations.

The method that we chose for these classroom observations was to get pairs of pupils to use a Macromedia Flash computer program incorporating the array representation of multiplication. Pupils worked in pairs on separate laptops all at the same time, with the Flash program opened in advance on each computer. The reason for using a computer program for children's work was that it enabled us to use a methodology for examining children's use of the array which we felt to be unobtrusive and a more naturalistic setting for pupils. All the children in each class took part during their normal classroom session, rather than being singled out for observation. During working in pairs and using the Flash program, all the actions carried out by the children on the computer were recorded using Camtasia recording software. As a microphone was connected to each computer, the dialogue of children as they worked on the multiplication problems was also recorded along with their computer actions. Therefore, a rich amount of qualitative audiovisual data, showing the dynamic process of children using the array representation, was obtained. It was made clear to all the pupils at the start of each session that their work on the computer and what they were saying was being recorded. However, we found from observing their conversations that pupils generally felt comfortable and uninhibited whilst carrying out their work. The number of studies in mathematics education that have utilised this methodology of using Camtasia recording software is still limited (one example is Davison, 2003). In the discussion part of this paper, we will further examine the possible advantages and disadvantages of this methodology.

In total, we worked with the classes over two different sessions for each class. In each of the sessions, pupils were asked to use slightly different programs so that slightly different aspects of their reasoning in multiplication might be accessed. The screenshots for the computer programs used in each of the sessions are shown in Figures 6 and 7. In the first session with each class, children were given a series of multiplication calculations, one after the after, and they had to type in the number of rows and columns that the array needed before working out the answer. The specific calculations that they were given were (in order) 5×7, 2×4, 5×6, 6×8, 7×9, 8×5, 9×6, 9×9, 11×5, 12×5, 18×9 12×7, 11×9, 13×15, 13×14 and 14×12. Therefore, there was a general progression of moving from multiplication with single digit numbers to multiplication with two-digit numbers. The calculations were kept below 20×15 due to the constraints of the array used. In the second session with each class, children again were given a series of calculations: (in order) 11×6, 14×7, 12×13, 18×9, 14×12, 13×15, 13×17 and 18×14. This time, children were asked to move a right-angle to highlight the array that represented a particular calculation, before again putting in the rows and columns and then the answer. During each session, the pairs of children worked on as many calculations as they could during the one-hour lesson. Therefore, different pairs of children completed different numbers of calculations. However, our focus was not on the numbers of calculations that they completed, but rather how they used the array in their calculations.

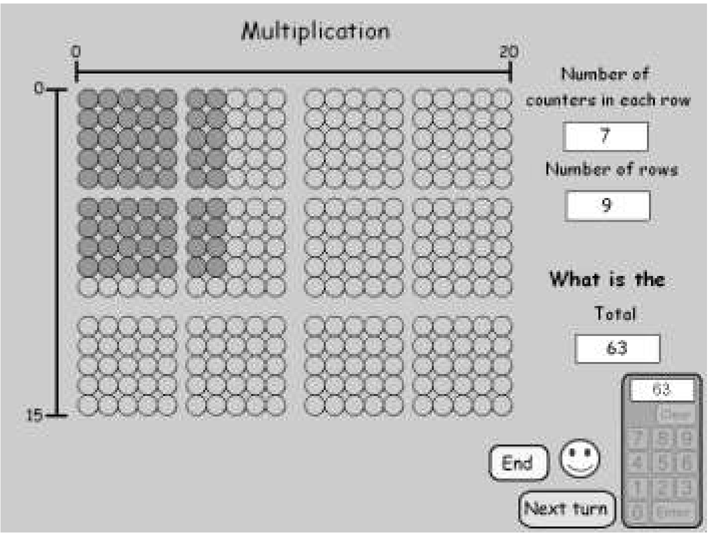

Figure 6: Screenshot of the program used in the first session

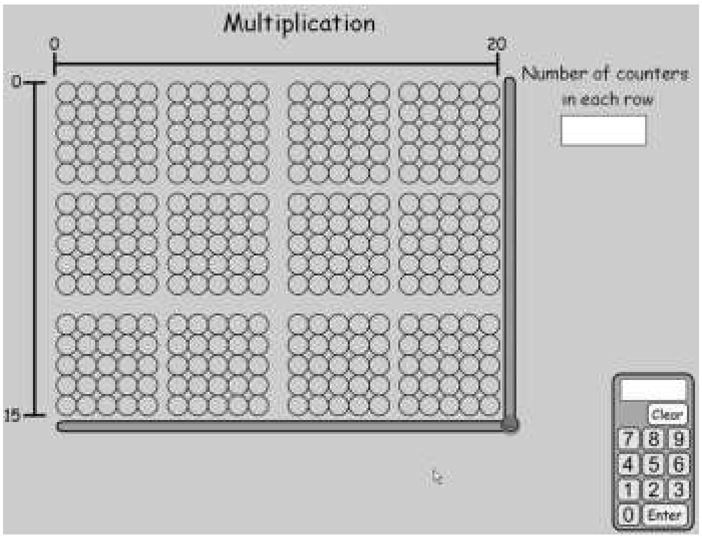

Figure 7: Screenshot of the program used in the second session

In analysing the recordings obtained during the classroom sessions, we looked for ways in which the array supported or hindered children's reasoning during multiplication calculations. In a previous paper (Harries & Barmby, 2007), we highlighted particular examples (by two pairs of children) of how the array might be used. In the following sections, we have looked in greater detail at the recordings and have categorised our observations, drawing on pupils' conversations as examples of their reasoning.

6. The array supporting children's multiplication calculations

We observed that the array provided a representation from which children could simply 'count' the result of the calculation, as well as a range of other calculation strategies. Some children simply counted up in 1s or small groups (e.g. 2s or 5s), and other children began to use the structure of the array to separate out the counting into different sections. Taking the cohort of Year 4 and Year 6 pupils as a whole, we identified the following ways in which the pupils used the array for calculations:

1. Counting in ones;
2. Counting in groups of 2s, 3s, 4s etc. depending on the actual calculation;
3. Counting in 5s;

4. Picking out the 25s and then counting in ones;
5. Picking out the 25s and then counting in groups;
6. Picking out the 25s, then counting in 5s;
7. Picking out the 25s, then do vertical 5s and horizontal 5s, then the last group;

8. Completing the 25s by rearranging the array and then adding the rest;

9. Completing a larger calculation and then taking away an appropriate amount.

We suggest four separate groupings to the calculation strategies. First of all, we have the simple *counting strategies* (1 to 3) based on counting in ones or small groups. Secondly, we have the *distributive strategies* (4 to 7) based on the distributive properties of the array leading to groups of 25 being picked out first. We have *rearranging strategies* (8) where parts of the array were moved to make the calculation easier. Finally, we have *completing strategies* (9) where pupils completed a calculation for a larger array, and then subtracted an appropriate amount. We give examples below of each of these groupings of calculation strategies.

6.1 Counting strategies (strategies 1 to 3)

The first example is from a pair of Year 4 pupils in the second of the classroom sessions, moving the right angle to calculate 14 × 7.

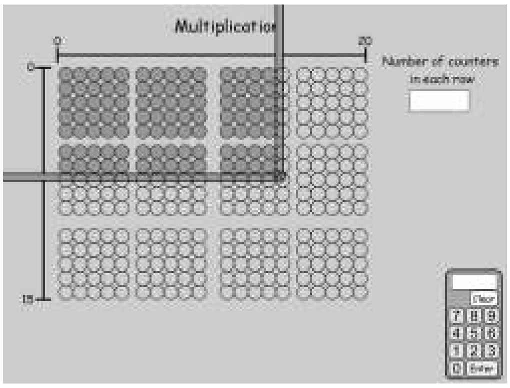

Figure 8: Counting strategy for 14 × 7

Pupil 1: 14 times 7, this is easy as pie! Down a bit, down a bit, down a bit. That's it, that's it, that's it. Yeah, that's good. 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85…
Pupil 2: 90, 95…
Pupil 1: No!
Together: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70 (both stop).
Pupil 1: And that's 4 … 70 … 74, 78 … 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98. So the last question was 66, and this one is 98. That was so easy!

In this example and in many others, the pupils tended to mix the simple counting strategies. Here, they began by counting in 5s (strategy 3), recognising the horizontal groupings of 5 on the left-hand side of the diagram. They then counted in horizontal groups of 4 (strategy 2) on the right-hand side before completing the calculation by counting in 1s (strategy 1).

6.2 Distributive strategies (strategies 4 to 7)

Using the array representation with spacings, we did find examples of pupils recognising the groupings or multiples of 25 in their calculations. The first example is a pair of Year 6 pupils working on 18 × 9.

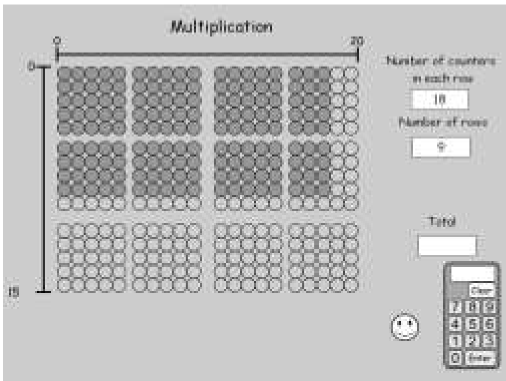

Figure 9: Distributive strategy for 18 × 9

Pupil 1: That's 25, 25, 25 … That's 1, 2… 25 add 25 add 25 that's our 75.
Pupil 2: 4 times 5 is 20. I've got 2 times 20, so that's 40.
Pupil 1: No actually it's 3 times 20 instead. That's 60.
Pupil 2: I've got 3 times 4 equals 12, so that's 24. So 75, 60, 24. I'm not writing them down, so that's 159.
(159 entered and sad face appears)
Pupil 1: Well it's not 159.
Pupil 2: Oh yeah, I counted both as 3 times 4 but that's 3 times 5. So it's 25, 25, 25, 60, 15, 12 … That's 162.

Here, the pupils picked out the groups of 25, then groups of 5, then groups of 3 (strategy 7), although initially they made an error which they corrected after a prompt. Thus, symbolically, we have the calculation (5+5+5+3) × (5+4). We of course do not claim that the pupils would have been able to use this symbolic representation, but the beginning of distributivity was there.

The second example below is from a Year 4 pair working on the calculation 6 × 8.

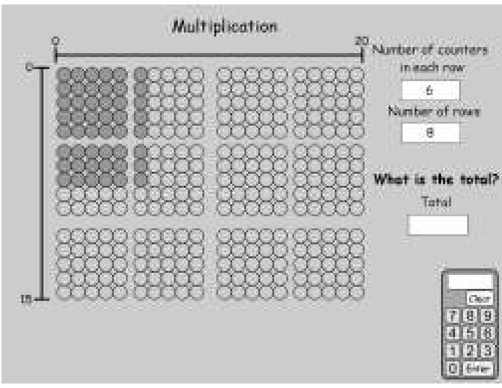

Figure 10: Distributive strategy for 6 × 8

Pupil 1: 6…
Pupil 2: Now enter…
Pupil 1: Enter… Do you want to do it?
Pupil 2: Alright…
Pupil 1: And then you can do … you can type in the working out.
Pupil 2: Alright… 8… 8…
Pupil 1: Bring it down…
Pupil 2: I like it up… Right, so 25 and then 5 is 30, then 40, 45, 48, it's 48.
Pupil 1: 25, 30, 31, 32, 33, 34, 35…
Pupil 2: 25 add 5 is 30, then 45, 48.
Pupil 1: Right. So…

Pupil 2: It’s 48.

This shows the pupils working in slightly different ways within the sphere of distributivity. One of the pupils separated the calculation into four parts and worked through them sequentially counting the appropriate groups (strategy 7). The other pupil again started from the 25 group again but then used a mixture of counting in fives and ones (strategies 4 and 6). So again, both pupils were using their interpretation of the representation to drive their method of working.

6.3 Rearranging and completing strategies (strategies 8 and 9)

A strategy which many Year 6 pupils used was ‘rearranging to complete the 25s and then adding the rest’. Pupils mentally completed as many 25s as they could by moving other parts of the array and then added on what was left to complete the calculation.

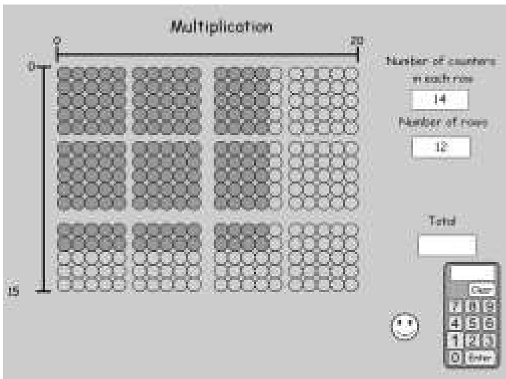

Figure 11: Completing the 25s and then adding the rest for 14 × 12

Pupil 1: Right, that’s 100, then take those (indicates two lots of 3 groups of 5 in the top right of array, to be moved to the bottom left of the array) away and that’s 50.
Pupil 2: Then it’s 10 and 8.
Pupil 1: So that’s 168.

This rearranging strategy (strategy 8) is clearly efficient, but it is also a strategy which undoes the multiplication picture within this representation (i.e. when parts of the array are moved, we no longer have the rectangular array representing the calculation). This would cause difficulties for much larger arrays where more parts of the array would need to be moved. Alternatively, completing strategies (strategy 9) where pupils completed a larger calculation and then took away an appropriate amount were very seldom used by pupils. However, there was one example of a pair of Year 6 pupils using it for the calculation of 18 × 9.

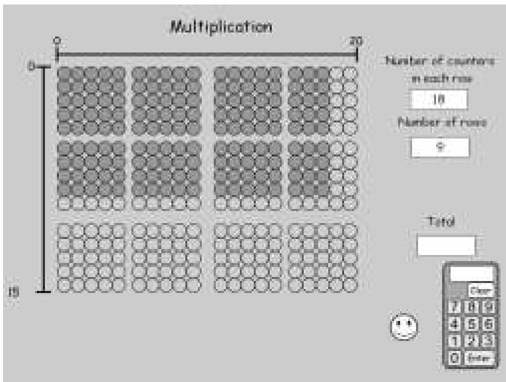

Figure 12: Completing a larger calculation for 18 × 9

Pupil 1: 18 times 10 is 180.
Pupil 2: So it's 174, no 172.
Pupil 1: No, it's 162. You just take away 18.
Pupil 2: Oh yes, that's right, 162.

Symbolically, we could say that the pupils are calculating (10×18) – (1×18). A second example of this completing strategy was with the same pupils; one pupil wished to use the rearrangement strategy, the other the completing strategy for 18 × 14.

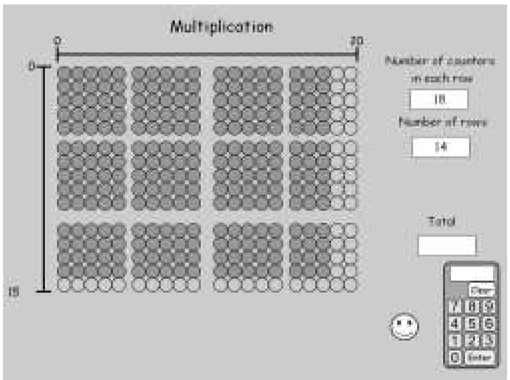

Figure 13: Completing and rearranging strategies for 18 × 14

Pupil 1: Let's make it bigger…
Pupil 2: No, I don't want to, I want to make it smaller. See, that's 100 and 50. Then take those three (indicates the three groups of 5 in top right of array) and put them there (indicates bottom left)…
Pupil 1: I think it's 252…
Pupil 2: That's 200, 225. Then that's 1, 2, 3…. 240. Then there's 3, 6, 9…
Pupil 1: I'm going to put in the answer (puts in 252).
Pupil 2: How did you work that out?
Pupil 1: I just did 14 times 20 and took off 2 fourteens.

Here, the pupils were clearly discussing how the representation assisted the calculation. It showed the two pupils working in different ways – in the second case, the multiplication structure was maintained whereas in the first, the multiplication structure was undone as in the example of Figure 11. The completing strategy would have the advantage of still being efficient to use for much larger arrays, simply needing to complete to the next multiple of five.

6.4 Comparing the strategies for Year 4 and Year 6 pupils

Although the classes and the children concerned were not chosen as representative samples of Year 4 or Year 6 pupils in any way, it was still of interest to examine whether the different groups of children seemed to favour different calculation strategies. We used the broad categories of counting, distributive, rearranging and completing and examined the proportions of children that drew on each of these.

Table 1: Proportion of strategies used by Year 4 pupils

	Pupil pairing										
Strategy	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Proportion**
Counting	✓	✓		✓	✓	✓	✓	✓	✓	✓	90%
Distributive	✓	✓	✓	✓	✓		✓			✓	70%
Rearranging											0%
Completing											0%

Table 2: Proportion of strategies used by Year 6 pupils

	Pupil pairing							**Proportion**
Strategy	**1**	**2**	**3**	**4**	**5**	**6**	**7**	
Counting	✓							14%
Distributive	✓	✓	✓	✓	✓	✓	✓	100%
Rearranging	✓	✓	✓	✓	✓	✓	✓	100%
Completing					✓			14%

Tables 1 and 2 show the different strategies employed by the pairs of pupils, looking at all the calculations that they completed (with different pairs completing different numbers of calculations). As can be seen in the tables, almost all of the Year 4 pupils used some form of the counting strategies with the array, with most of them using a distributive strategy as well. There was no evidence of Year 4 pupils using the rearranging or completing strategies. In contrast, the Year 6 pupils rarely used the purely counting strategies. Rather, they all used the distributive and rearranging strategies. One pair, as we saw previously, used the completing strategy.

7. Difficulties that children had with using the array

In addition to being able to observe the calculation strategies used by pupils in using the array, we also came across particular examples of problems that children had. First of all, one of the concerns that we had as we observed the children using the array was the over-reliance on counting in ones to carry out the calculation. This corresponds to stategies 1 and 4 in the previous section. On one level, this was because counting was simply an inefficient method, but also, counting could lead to children losing track in their counting and getting an incorrect answer. An example from two Year 4 pupils is given below, where one pupil is trying to count too much, because with the array, "you just count them".

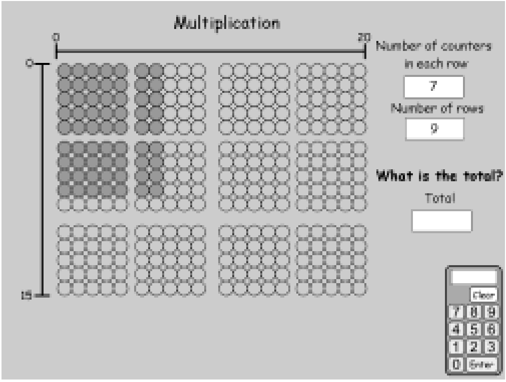

Figure 14: Encouraging with counting for 7 × 9

Pupil 1: Number five, seven times nine. It doesn't matter, you just count them, don't you?
Pupil 2: Yeah, this is mint! This is a class idea.
Pupil 1: Yeah … seven times nine … seven times nine … seven times nine, enter… Right then, so it's 25 … 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48…
Pupil 2: Right, that's five, that's the five, so how much is that?
Pupil 1: Right, there's a 25 each there…
Pupil 2: Yeah, but there's only 25 there and not there...
Pupil 1: Oh yes, so 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45… umm 45 … 46 …
Pupil 2: 55! (emphasising the block of ten)
Pupil 1: 55… 56, 57 58, 59, 60, 61, 62, 63!
Pupil 2: 63, upsadaisy, dropped my pen…

In the end, the pair of pupils did get the correct answer. However, we would want to try and move the pupils on from the over use of simple and often laborious counting strategies to identifying more groups within the array.

Perhaps the most interesting example of problems that children had with using the array was that of not being able to represent the calculation in an array form. For example, in the illustrative dialogue below, two Year 4 pupils are calculating 11 × 6 in the second of the classroom sessions. In fact, they already know the answer to be 66 from their knowledge of multiplication tables. However, they are then unable to move the right-angle in order to show the calculation.

Pupil 1: To work out 66 … 11 … So hang on, 50, 100… On no, we need 50 … and then, come here … 50, then we need to get 66.
Pupil 2: Hang on … right, hang on, hang on, hang on…
Pupil 1: We need to get 66 in there. I don't know how we are going to do it though.
Pupil 2: OK, right, hang on, we need 50, 66...
Pupil 1: We need a 50 block... But we need a 66.
Pupil 2: Right, hang on, hang on, hang on. We've got 25 and 25 is 50. Then we need a 6…
Pupil 1: How are we going to get that?
Pupil 2: We need a 66…
Pupil 1: How?
Pupil 2: I don't know.

The pupils in this case did not associate the dimensions of '11' and '6' with the dimensions of the required array. This was despite the fact that they had used a similar program in a previous session, where they typed in the two numbers being multiplied to produce an array of highlighted counters. Eventually, by accident, they placed the right-angle in the correct position and complete the question by writing '11' for the number of rows, '6' for the number of columns, then '66' for the answer. They then tried 14 × 7. They began by trying to calculate the answer first (incorrectly getting 104), then tried to show 104 on the array. They again made no association between the array and the multiplication sum. After about eight minutes of moving the right-angle around the array, one of the researchers intervened.

With this particular problem faced by the pupils, it was again interesting to note the proportions of the pupils that experienced this. In the Year 4 cohort, 6 out of the 10 participating pairs had a problem with associating the two-dimensional structure of the array with the calculation. With the Year 6 pupils however, none of the pairs had this problem.

8. Discussion

Taking an overview of our findings from the classroom observations, what we found was that the array representation for multiplication did indeed support calculation strategies. This ranged from simple counting strategies, through to identifying groups within the array (possibly recognising the distributive properties of the array) and moving parts of the array around or completing the array to make the calculation easier. These findings are, to a degree, in agreement with the case study of Steinbring (2007), where elementary school children in Germany were able to draw upon distributive and completing strategies in their use of an array representation. Therefore, at one level, we would argue that the array is a useful representation for children to use for calculations. The array, or at least the activities associated with the array that we have outlined in this study, was also perceived positively by the pupils as well. A number of comments came through from the pupil transcripts that using the array was ‘easy’ or ‘class/mint’ (colloquialisms for very good).

On another level, we also identified some potential difficulties with using the array representation. We saw that the array could encourage the over-use of inefficient counting strategies, in particular for the younger pupils, although this was less prevalent for the older pupils. This echoes the issues highlighted by Dickson *et al.* (1984) and raised earlier in the paper that 8 to 9 year olds (corresponding to Year 4) would be unlikely to use distributive properties in multiplicative situations. We also observed an interesting lack of understanding about the structure of the array and what it represented, in that quite a number of the Year 4 pupils could not represent a calculation in the two dimensions of the array. There seemed to be a lack of understanding of the binary nature of multiplication, identified as one of the important ideas within multiplication. In this case, the array was a useful tool for diagnosing this lack of understanding amongst pupils. In addition, despite the potential benefits that we have identified for the array above, we must recognise that pupils did not necessary make the link between the grouping strategies that they used, and the fact that they were drawing specifically on the distributive law. For this to be done, we would need to follow up in any teaching sequence the use of the array with discussion about why the distributive law applies to multiplication.

In order to make sense of these difficulties that were experienced by pupils, we need to place the difficulties in the context of the tasks that the children were asked to do. Looking at the first of the difficulties of children using inefficient calculation strategies with the array, it must be recognised that children were not required to be efficient, so they were not incorrect in using strategies such as simple counting. However, this still does not avoid the fact that these strategies could be problematic. What is therefore required is that pupils recognise the benefits and the drawbacks of the different strategies, and this can be led by the teacher who has an overview of the different strategies. Related to this, in terms of understanding the mathematical properties of the array, is the second difficulty of the pupils not recognising the array representation as a representation for multiplication. We have already highlighted why the array is felt to be a key representation for multiplication; that it provides a way of accessing the important theoretical properties of multiplication. Wittmann (2005) highlights the role played by mathematical representations in building mathematical knowledge:

> "Contrary to real objects or models of real situations which are charged with various constraints mathematical objects allow for unlimited operations and for establishing theoretical knowledge which is more applicable than knowledge directly derived from mathematizing real situations." (p. 19)

Because of its structure, the array provides a clearer theoretical link to the important properties of multiplication than would a 'real-life' situation for multiplication. However, Wittmann also highlights the abstract nature of these representations, that "compared with the abstract objects these representations are more concrete than the mathematical objects which they represent, and compared with the real objects which they model they are more abstract" (p.18). There is therefore a possible conflict here – that useful mathematical representations need to be more abstract, but this in turn can mean that pupils do not recognise this more abstract representation. Steinbring (1997) also highlights the possibly changing role of representations in the development of a mathematical concept:

> "To develop mathematical meaning in the interplay between a reference context and a sign system implies a start from a referential domain, which is seemingly familiar or which is supposed to be known at least in some basic aspects, and then to transfer possible meanings on to a new, still meaningless sign system; in a next step, this allows for a flexible switching between sign system and reference context, by exchanging the roles between sign system and reference context." (p. 54-55)

In utilising these more abstract representations, we must recognise that when children are first introduced to these representations, they must become part of their understanding of the mathematical concept. Initially then, the representation is itself a "symbol/sign" that the pupil must make sense of by making links to more familiar reference contexts, whether these are real-life situations or symbolic expressions. If the array is to be used to introduce pupils to the key ideas of multiplication, then we need to encourage recognition of what the array actually represents in multiplication, and how its structure can be used to efficiently carry out calculations. This would suggest a more careful use of the array (or any other representation) in a teaching sequence in order to incorporate these ideas. It is interesting first of all to look at the suggested teaching progression provided in the National Numeracy Strategy (DfEE, 1999) that English primary schools were using at the time of the study (Figure 15).

In Year 2 of primary school, the Strategy suggested that multiplication is described using an array, emphasising the binary nature of multiplication and also making clear the commutative nature of the operation. This is reinforced in Year 3 before moving on to using the grid method for calculation in Year 4. At the time of the classroom observations, we might have expected pupils to be familiar with the array, and it is therefore surprising that quite a number of the Year 4 pupils had difficulty with representing calculations as arrays. This lack of understanding is concerning as these pupils will then have been introduced to the grid method of calculation in Year 4. Not understanding the binary nature of multiplication would seem to be a hindrance to understanding the structure of the grid method. We would suggest that the use of the array representation is promoted more as a key representation for multiplication, and pupils are given the opportunity to recognise the array as a representation for multiplication. Even though Year 4 children know multiplication facts such as 11 × 6 = 66, being able to recognise and then represent the calculation through the array will provide more opportunities for pupils to construct their understanding of multiplication.

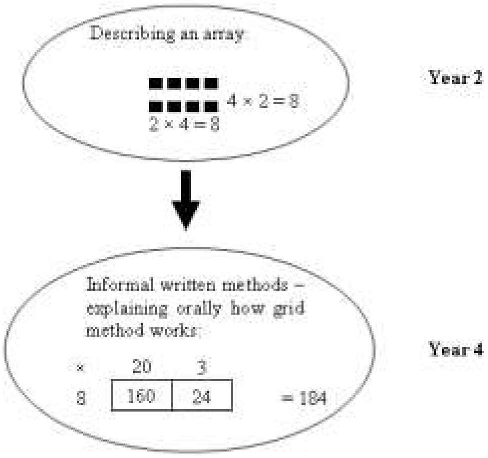

Figure 15: Teaching progression for understanding multiplication in the National Numeracy Strategy

We have also observed the limitations of the array for some pupils in terms of an over-dependence on simple counting strategies. Our study has shown that primary pupils can recognise the groupings within the arrays and that we can move them on towards distributive strategies, as highlighted in the new Primary Strategy document here in England (DfES, 2006). Therefore, children need to be encouraged to take on these distributive strategies rather than the simple counting strategies. We would suggest that a way of doing this would be to use the grid method, or at least to work towards this more efficient representation alongside the use of the array, so that children can clearly make the links between calculations, the array and the grid method. We would suggest a teaching sequence as shown in Figure 16. Having been introduced to multiplication situations and multiplication facts, starting at the top of the diagram, the array is introduced alongside other representations for multiplication. Children can then start to use the array as a calculation tool. The groupings within the array are then identified to make the use of the array more efficient, and then we can move the children on to the grid method. Here though, the groupings for the grid method are not necessarily tens and units. The array may suggest other groupings, say fives and units, and this may simplify the use of the grid method for children. It emphasises that the groupings do not have to be in tens and units, and so children can have greater insight and understanding of using the grid. The children can then be moved on to tens and units with the grid as a more efficient method, and to aid understanding of column multiplication methods emphasising the separation of numbers into units, tens etc. At each stage, we must once again be aware that the relations between the representations need to be constructed by the pupils, and not to take for granted that the specific connections will be made by the pupils. Of course, this model is based mainly on our findings from this particular study. However, we suggest that the teaching sequence clearly shows the links between different representations for multiplication (e.g. array to grid method) and may be used more broadly to help develop children's understanding of the topic.

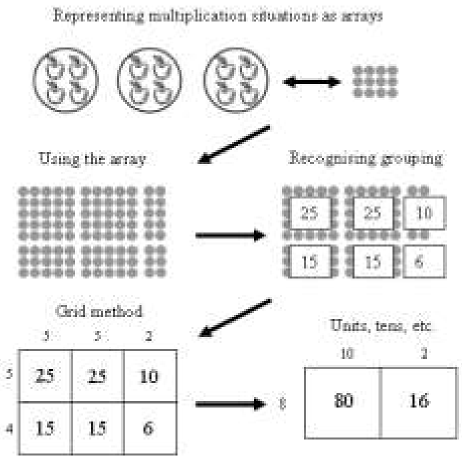

Figure 16: Suggested teaching sequence incorporating the array and the grid method

Looking at this study more broadly, we have been able to observe the reasoning of children using the array, and to obtain some indication of their understanding, because of the methodology that we used recording children's work on the computer. As this kind of methodology has not been widely reported in the mathematics education literature, it is worthwhile to discuss here some of the advantages and disadvantages that we faced using the method.

In line with the model of understanding that we adopted, what we wanted to be able to do was to gain access to pupils' reasoning as they used the array, to be able to see the kind of links they were able or unable to make. There were a number of ways that we could have done this. We could have carried out direct observations of pupils using the computer programs, or used 'talk-aloud' procedures or task-based interviews to gain access to children's reasoning. Alternatively, we could have used video methods to record children's work. Powell *et al.* (2003) outlined two of the advantages of video methods as the 'density' and 'permanence' of the recorded data. That is, the ability to record a great deal of information about a given situation (including audio and visual information) and the ability to return to the recorded situation again and again (unlike live situations). The present methodology also has these advantages. In addition, we felt that the methodology used had a number of other advantages. Firstly, we did not have to concern ourselves with the quality of the image due to issues such as lighting, a problem faced by video methods. What we also found to be particularly valuable in our study was the ability to record the 'natural' conversation of the children as they worked, which may not have been so readily available if we had used video methods or task-based interviews. For example, Davis (1984) highlighted the difficulties with task-based interviews that interviewers might unconsciously encourage pupils to succeed in the task given. In addition, although task-based interviews allow us to probe more deeply into children's understanding, this probing may result in the children reflecting the researcher's ideas rather than their own. In the examples of the dialogue that we have included in this paper, we have tried to convey the informal nature of the conversation that we obtained, and there were many other examples that we could have chosen, including singing and conversations about sport!

Listening to the children's conversation, we felt that the method provided a setting where children did not feel constrained in what they said, and provided a more natural setting within which children were doing mathematics.

This natural feel was also promoted by the fact that pupils simply worked in pairs without necessarily the presence of a teacher or researcher. In turn, because children were all working in pairs at the same time, this meant that a large amount of classroom data could be collected quickly for many children. Because the computers used by the children themselves were the recording tools, as long as all pupils had access to a computer, then recording of what the children were doing and saying could be done. This is in contrast to a video method where the camera would have to set up and manned for each pairing, resulting in more time-intensive data collection. However, the use of the computer-based recording still provided us with the visual information of what the children were doing on the screen. In some ways, this is an added advantage of the method in that the visual recording specifically focuses in detail on the problem that the children are working on, rather than recording visual information more 'at a distance'.

There were of course some disadvantages to the method. Although the method captured the informal nature of pupils' reasoning, this could mean at times that the children were 'off-task' for amounts of time whilst carrying out the work. However, we feel that if we wish to indeed observe the informal reasoning of children, then this is a disadvantage that we are prepared to accept. A second disadvantage was that as opposed to traditional video methods or observations/interviews with researchers, we could not observe the broader picture of how children were reasoning. Examples would be the social (but not necessarily audible) interaction occurring between the children and the gestures that children were making at the screen (i.e. pointing with their finger). To remove this disadvantage would require the additional use of video or the presence of a researcher, and we would have to consider the balance between observing these additional actions and the possibility that a less informal interaction of the pupils would be observed. Also, to a degree, Powell *et al.* (2003) still recognises that video methods are problematic in that the researcher is still selective in what to record. Although videoing provides a broader picture of the situation in which the children are working, it cannot capture the full experience of the children, especially if the researcher needs to focus on details that they feel to be of particular interest. In some ways, the present methodology removes this problem of selectivity in that the researcher cannot choose what to record – one concentrates on the work done on the computer screen and what is said. In addition, the ethical problem of recording children physically as opposed to just their spoken word is avoided.

To conclude this discussion, we examine how we can develop this research work in the future. Having carried out this study, there are a number areas that we would wish to investigate further. Firstly, in our study, we have focussed upon two particular groups of children, and we could examine children's understanding of the array representation amongst children with a broader range of abilities. Secondly, we have put forward a suggested teaching sequence incorporating the array representation and progressing on to the grid method. We may wish to develop this teaching sequence further with teachers and examine the impact of the implementation of such a sequence on primary pupils. This may include looking more at pupils' understanding of the grid method and other written methods for multiplication as well as the array.

We could use comparisons with a control group to highlight the possible impact of this sequence. Thirdly, we can look at the wider understanding that children have of multiplication. Having adopted a clear theoretical basis for what we mean by understanding and reasoning in mathematics, and examined the implications for the definitions that we have used, it is clear that we need to use a variety of methods that encourage children's reasoning in order to gain insight into their understanding of multiplication (we provided earlier example studies from Niemi, 1996, and Lawson and Chinnappan, 2000). This might include open-ended written questions, task-based interviews, or further use of the computer-based recording software that we have used in this study. We can therefore build on the approach that we have taken here in our future work on examining children's understanding of multiplication.

Acknowledgements

The authors would like to thank the referees who looked at this paper, for their constructive comments and their suggestions of related references.

References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16 (3)*, 183-198.

Anghileri, J. (2000). *Teaching number sense*. London: Continuum.

Brinkmann, A. (2003). Graphical Knowledge Display – Mind Mapping and Concept Mapping as Efficient Tools in Mathematics Education. *Mathematics Education Review, 16*, 35-48.

Chi, M. T. H., De Leeuw, N., Chiu, M-H. & LaVancher, C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science, 18*, 439-477.

Christou, C. & Papageorgiou, E. (2007). A framework of mathematics inductive reasoning. *Learning and Instruction, 17(1)*, 55-66.

Davydov, V. V. (1991). 'A psychological analysis of multiplication'. In L. P. Steffe, (Ed.), *Psychological abilities of primary school children in learning mathematics. Soviet Studies in Mathematics Education Series, Volume 6* (pp. 1-85). Reston, VA: NCTM.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Approach to Mathematics Education*. London: Croom Helm.

Davison, I. (2003). Using an interactive whiteboard to facilitate pupil understanding of quadrilateral definitions. *Proceedings of the British Society for Research into Learning Mathematics, 23(1)*, 13-18

DfEE (1999). *The National Numeracy Strategy: Framework for teaching mathematics from Reception to Year 6.* Sudbury: DfEE Publications.

DfES (2006). *Primary framework for literacy and mathematics: Core position papers underpinning the renewal of guidance for teaching literacy and mathematics.* Norwich: DfES.

Dickson, L., Brown, M. & Gibson, O. (1984). *Children learning mathematics: A teacher's guide to recent research*. London: Cassell Educational Ltd.

Ernest, P. (1994). Social constructivism and the psychology of mathematics education. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and Mathematics Education* (pp. 62-72). London: The Falmer Press.

Flexer, R. J. (1986). The power of five: The step before the power of ten. *The Arithmetic Teacher*, *34(3)*, 5-9.

Goldin, G. A. (1998). Representational Systems, Learning and Problem Solving in Mathematics. *Journal of Mathematical Behaviour, 17(2)*, 137-165.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Große, C. S. & Renkl, A. (2006). Effects of multiple solution methods in mathematics learning. *Learning and Instruction, 16*, 122-138.

Harries, T. & Barmby, P. (2007). 'Representing and understanding multiplication'. *Research in Mathematics Education*, 9, 33-45.

Hiebert, J. & Carpenter, T. P. (1992). Learning and Teaching with Understanding. In Grouws, D. A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 65-97). New York: Macmillan.

Izsák, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practice and individual student learning. *Mathematical Thinking and Learning, 6(1)*, 37-79.

Lawson, A. E. (1994). Research on the acquisition of science knowledge: Epistemological foundations of cognition. In Gabel, D. (Ed.) *Handbook of Research on Science Teaching and Learning* (pp. 131-176). New York: Macmillan.

Lawson, M. J. & Chinnappan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education*, *31(1)*, 26-43.

Leighton, J. P. (2004). Defining and Describing Reason. In Leighton, J. P. & Sternberg, R. J. (Eds.) *The Nature of Reasoning* (pp. 3-11). Cambridge: Cambridge University Press.

McGowan, M. & Tall, D. (1999). Concept Maps and Schematic Diagrams as Devices for the Growth of Mathematical Knowledge. In Zaslavsky, O. (Ed.) *Proceedings of the 23rd Conference of PME, Haifa, Israel, July 1999 (Volume 3)*, 281-288.

Meyer, J. H. F. and Land, R. (2003) Threshold Concepts and Troublesome Knowledge: Linkages to ways of thinking and practising within the disciplines. *Enhancing Teaching-Learning Environments in Undergraduate Courses Project, Occasional report 4.* [Accessed 27th July 2007 at http://www.ed.ac.uk/ etl/publications.html]

Moseley, B. (2005). Students' early mathematical representation knowledge: The effects of emphasising single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics*, *60*, 37-69.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nickerson, R. S. (1985). Understanding Understanding. *American Journal of Education*, *93(2)*, 201-239.

Niemi, D. (1996). Assessing Conceptual Understanding in Mathematics: Representations, Problem Solutions, Justifications, and Explanations. *Journal of Educational Research, 89(6)*, 351-364.

Nunes T., & Bryant P. (1996). *Children doing Mathematics*. Oxford: Blackwell Publishers.

Powell, A. B., Francisco, J. M., Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behaviour, 22*, 405-435.

Seufert, T. (2003). Supporting coherence formation in learning from multiple representations. *Learning and Instruction, 13*, 227-237.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.

Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8(4)*, 359-371.

Skemp, R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, *77*, 20-26.

Steinbring, H. (1997). Epistemological investigation of classroom interaction in elementary mathematics teaching. *Educational Studies in Mathematics*, 32(1), 49-92.

Thompson, I. (1999). *Issues in teaching numeracy in primary schools*. Oxford: Open University Press.

White, R. & Gunston, R. (1992). *Probing Understanding*. London: The Falmer Press

Williams, C. G. (1998). Using Concept Maps to assess Conceptual Knowledge of Function. *Journal of Research in Mathematics Education*, *29(4)*, 414-421.

Wittmann, E. Ch. (2005). Mathematics as the Science of Patterns – A Guideline for Developing Mathematics Education from Early Childhood to Adulthood. *Plenary Lecture at International Colloquium 'Mathematical learning from Early Childhood to Adulthood (Belgium, Mons).*