A representational approach to developing primary ITT students' confidence in their mathematics

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Representations of mathematical concepts play an important role in understanding: both in helping learners understand the to-be-learned material [1] and in facilitating teachers' understanding of pedagogical processes which, in turn, are involved in developing learners' understanding [2,3]. In this paper, we report on work with a cohort of pre-service primary teachers, with the aim of developing their understanding of mathematics, their confidence in their subject knowledge and their confidence in teaching mathematics. This was attempted through the introduction and use of a 'representational approach' to the teaching of the mathematical concepts required of teachers training to teach in primary schools in the UK. We present the results of attitude measures and a follow-up qualitative questionnaire in identifying whether and how the use of this representational approach supported pre-service teachers' understanding and their confidence in teaching mathematics. The results suggest that the representational approach used had a positively significant impact on the attitudes towards studying and teaching mathematics.

Keywords: representations; subject knowledge; primary; pre-service teachers

AMS Subject Classification: 97D40

1. Introduction

1.1 The background

Shulman [4] identified the use of external representations as being part of teachers' pedagogical content knowledge. He defined these external representations as '… analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others' [4, p. 9]. Specifically in mathematics, Ball et al. [5] also highlighted external representations as being part of the 'specialised content knowledge' of mathematics unique to teaching. This specialised knowledge included selecting representations for particular purposes, recognising what is involved in using a particular representation and linking representations to underlying ideas and other representations. Teachers need to be able to draw on a variety of representations as there is 'no single most powerful form of representation' [4, p. 9].

In particular, researchers have highlighted the role that external representations play in the explanations of mathematical concepts by teachers [2,3,6].

> Skilled teachers have a repertoire of such representations available for use when needed to elaborate their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction. [3, p. 352)

Leinhardt et al. [2, p. 108] also identified the skill and knowledge required by teachers in considering the suitability of particular representations, as 'certain representations will take an instructor farther in his or her attempts to explain the to-be-learned material and still remain consistent and useful'. The effective use of representations therefore require that teachers have 'deep understanding' of the topics that they are teaching.

When we speak of representations, we are careful to highlight the distinctions between *internal* and *external* manifestations of representations [15]. Davis [36, p. 203] defined internal representations as ‘Any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in the mind at all, must be represented in some way’. External representations on the other hand are ‘materially instantiated’ entities [13, p. 56]. Alongside the examples given above by Shulman, other examples of external representations are marks on paper, pictures, symbols, sounds, spoken words and computerised objects. As we have seen, these external representations ‘serve to denote or to exemplify’ mathematical concepts [11, p. 2].

External representations also play an important role in the learning of mathematics by students: ‘An important educational goal is for students to learn to use multiple forms of representation in communicating with one another’ [1, p. 363]. More specifically, researchers have outlined the role that external representations play in linking the abstract mathematics to the concrete experiences of learners [6-9].

> Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding. [6, p. 153]

In addition, they can support the working memory of learners [10,11], for example through ‘offloading’ elements of a given computation to externalized representations [12]. Related to the issue of explanation of mathematical concepts highlighted above, representations can be designed in order to constrain interpretation and to highlight particular properties of a mathematical concept [13,14].

More broadly, multiple representations (both external and internal) play an important role in the development of learners’ mathematical understanding: ‘They can be considered as useful tools for constructing understanding and for communicating information and understanding.’ [1, p. 362]. Understanding of a mathematical concept is based on the internal representations of a concept, which are influenced by the external representations of the concept that are presented to learners [16]. Wood [17] stated that conceptual understanding rests on a multiple system of ‘signs’ or representations. Lesh et al. [18] used the definition that a student understands a mathematical concept if he or she could ‘translate’ or move between multiple representations. Hiebert & Carpenter [16] defined mathematical understanding as being a network of internal representations, with more and stronger connections denoting greater understanding.

1.2 Aims

Representations play an important role in the understanding of learners, and also in the pedagogical processes involved in developing that understanding. For pre-service teachers, who are developing their own understanding and learning how to teach the subject of mathematics, their knowledge of mathematical representations is even more important. However, Turner [19] highlighted that pre-service teachers’ choice and use of representations could be problematic. In this paper therefore, we report on work with a cohort of pre-service primary teachers, with the aim of developing their understanding of mathematics and their confidence in their subject knowledge and their teaching of mathematics. This was attempted through the introduction and use of representations associated with mathematical concepts covered in primary schools. We provide greater detail on this input to pre-service teachers in the section below.

2. Method

2.1 The sample

The main sample of pre-service teachers involved in this work was a cohort of seventy-seven students on a 38-week long postgraduate teaching course (PGCE) at Durham University, in the UK. The course is for those wishing to qualify to teach in primary schools in the UK and who already hold an undergraduate degree. Traditionally, the course population tends to be an academically able group but many tend to have concerns about their mathematical subject knowledge. Only one had an undergraduate degree in mathematics and therefore, the sample comprised mainly 'non-specialists' in mathematics, as the majority had not studied mathematics beyond the GCSE/GCE 'O' level.[1] Consequently, the pre-service teachers in this group could be described as typical of many other pre-service teachers undergoing teacher training in the UK.

In addition, a second cohort of similar students who did not experience the representational approach was also surveyed to provide a comparison group for the attitude measures. This group of students were in the first year of a three-year undergraduate teaching programme and although the undergraduate experience of mathematics differs somewhat from its postgraduate counterpart (e.g. the length of the programme means that students have more time to do mathematics and have more experience in schools) the control group were similar to the intervention group in a number of important respects. First, like the intervention group, they were an academically able group but were not specialists in mathematics and so many of them were also insecure about their mathematical subject knowledge. Second, the control group experienced many of the same mathematical topics in their classes and also received a very similar lecture-then-seminar structure. This meant that the two groups experienced similar amounts of discussion of the mathematical ideas concerned. Third, the two groups were taught by the same tutors and this allowed these important aspects to be controlled for (as far as was practicably possible). All this meant that both the intervention and the control groups experienced similar mathematical content but the intervention group had a much greater emphasis on mathematical representations. In total, sixty-five of the seventy-seven students in the representational group and sixty-nine students in the comparison group completed the pre- and post-measures of attitude.

2.2 Our representational approach

The programme offered in mathematics is well established and helps the students to explore both pedagogy and content within the primary mathematics curriculum through lectures, seminars and workshops involving leading mathematics teachers from the local authority. However, in 2009/10, the usual input to students adopted more of a representational focus, applying Shulman's definition of representation [4] and the research ideas outlined above. In sessions, a variety of representations for a mathematical concept would be introduced to the pre-service teachers. Students would be encouraged to 'explore' what characteristics of a mathematical concept were emphasised by a particular representation; for example, considering the possibility of there being a key representation which was most useful for explaining and understanding the particular key ideas, and considering how the representations could be used to make sense of the various procedures (or algorithms) associated with the mathematical concept. For example, in exploring the big idea of

[1] GCSE is the abbreviation for General Certificate in Secondary Education and is the standard examination taken by pupils at approximately 16-years of age. GCE 'O' level is the abbreviation for General Certificate in Education Ordinary level and was the GCSE's predecessor.

multiplication (and later, division) we presented and asked the students to explore different possible representations of the concept, as illustrated in Figure 1 below.

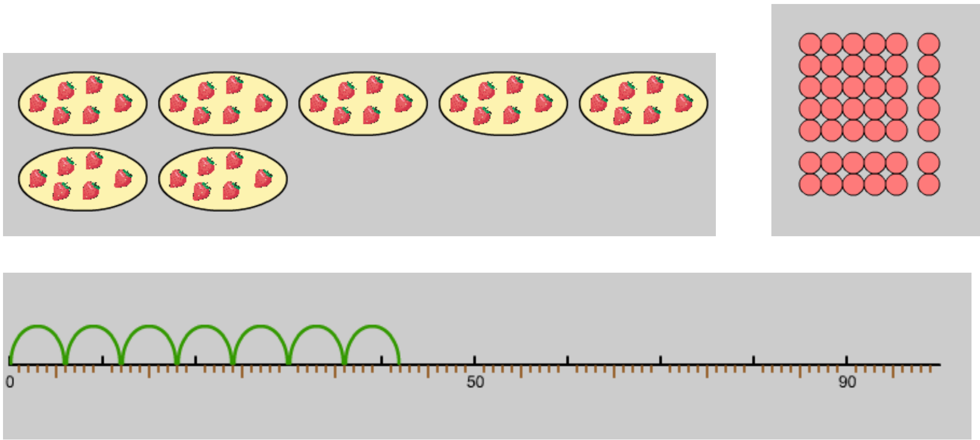

Figure 1. Some of the representations used in exploring multiplication (and division)

Figure 1 shows the equal groups representation, in this case plates of strawberries, which particularly emphasises the relationship between multiplication and repeated addition. Likewise, the number line representation shows the relationship between addition and multiplication. Both of these emphasise the distributive properties of multiplication, e.g. $6 \times 7 = (6 \times 5) + (6 \times 2)$. The array representation emphasises the commutative properties of multiplication, e.g. $6 \times 7 = 7 \times 6$. The array with spaces inserted (shown in Figure 1) further emphasises the distributive properties of multiplication, e.g. $6 \times 7 = (5 \times 5) + (1 \times 5) + (5 \times 2) + (1 \times 2)$.

Within our representational approach, students were asked to consider such questions as:

- What do you notice about the image/representation?
- What are the characteristics of the image/representation?
- Can you explain how this image/representation shows us the binary and/or commutative and/or distributive nature of multiplication?

When working across representations we used questions such as:

- Why do these representations show the same mathematical idea?
- What is the same about the different representations?
- What is different about the representations?
- What are the particular characteristics of the various representations?
- What aspects of the structure of multiplication [or other concept] are emphasised by the representations?
- Can you explain how we move from one representation to another?
- What are the most useful characteristics of a particular representation?

A focus on these questions allowed the students to build up a language which facilitated the discussion about the nature/characteristics of multiplication, or any other specific mathematical concept. In addition, as a medium for exploring ideas on representations, we used a suite of computer programmes (called Danimaths) that we had devised ourselves and which allowed the representations to be explored in a dynamic and interactive way via a computer or interactive whiteboard. The programmes were created as a stimulus and as a scaffold for class discussion.

Alongside the input provided to the pre-service primary teachers, the aim of the study was to examine the impact of this representational approach on the student teachers involved. More specifically, the objectives of the study were:

(1) To measure any change in pre-service teachers' attitudes towards their subject knowledge in mathematics, and also towards teaching the subject;

(2) To gain some qualitative insight into whether the input incorporating representations might impact on teacher attitudes.

Past research has highlighted that there is a link between teachers' beliefs/attitudes and instructional practice, although this link can be complex [20,21,22,23]. Looking specifically at teacher attitudes, Aiken [24] illustrated how teacher attitudes towards mathematics can be particularly important because they can affect their students' attitudes towards the subject. Relich et al. [25] drew on research which related teacher attitudes to student achievement, as did Ernest [26], emphasising though that any possible correlation could be quite weak. However, Ernest [26] also highlighted the importance of teachers' attitudes towards teaching mathematics as being particularly important for student achievement. Elsewhere, Ball [27], Philippou & Christou [28] and Wilkins [29] have highlighted that teachers' attitudes affect classroom practice in teaching mathematics.

In terms of actually examining teachers' attitudes towards the subject, Ernest [26] identified the two components of teachers' attitudes towards mathematics and towards teaching mathematics. Relich et al. [25] similarly identified two dimensions of pre-service teachers' attitudes; the attitudes of the pre-service teachers towards studying mathematics and their attitudes towards teaching mathematics. We adopted a similar approach and the attitudes of the pre-service teachers in this cohort were measured at the beginning and end of their 38-week course of study using a questionnaire. We developed measures for attitudes towards studying mathematics and towards teaching mathematics. The two attitude measures consisted of 15 and 8 questionnaire items respectively, with responses to the items elicited on a 5-point Likert Scale of *strongly agree* through to *strongly disagree*. Previous examination of the psychometric properties of these measures revealed Cronbach alpha reliabilities of 0.91 and 0.89 for the attitudes towards studying and teaching mathematics measures respectively. The unidimensionality of these measures were examined through exploratory factor analysis, and the validity of these measures was examined through Rasch analysis and also through interviews with pre-service teachers. For this study, we were able to draw on these measures and survey the attitudes of pre-service teachers in both the representational and control groups at the beginning and end of their respective year of study. The average score for each student came out of the Rasch analysis.

In addition to examining any potential impact of the representational input on teachers' attitudes, the other objective was to gain some insight into possible reasons why the input might impact on teacher attitudes, and also whether there may be any subsequent impact in pre-service teachers' classroom practice. In order to do so, the representational group of pre-service teachers were asked at the end of the course to reflect on their own learning as part of the course and asked to provide written answers to open-ended questions on a questionnaire. The questions asked were:

- State a topic or concept in maths in which you feel you have deepened or modified your understanding;

- Describe an incident or event which helped you learn.

The following further closed question was also asked on the questionnaire:

- Which of the following were important to your learning:

    - Wrestling with a problem;
    - Talking to others about their understanding (tutors, peers, children);
    - Using some of the computer programs;
    - Teaching a topic.

The next section sets out the results from both the quantitative and qualitative aspects of the instruments.

3. Results

The results are discussed in two parts. First we explore the quantitative data from the attitude measures which show the impact of the respective teaching input on the attitudes of the pre-service teachers. Figures 2 and 3 show the change in the average measures of attitudes for the input (representational) group and the control group over the course of the year. These measures of attitude were obtained from the Rasch analysis carried out on the questionnaire data.

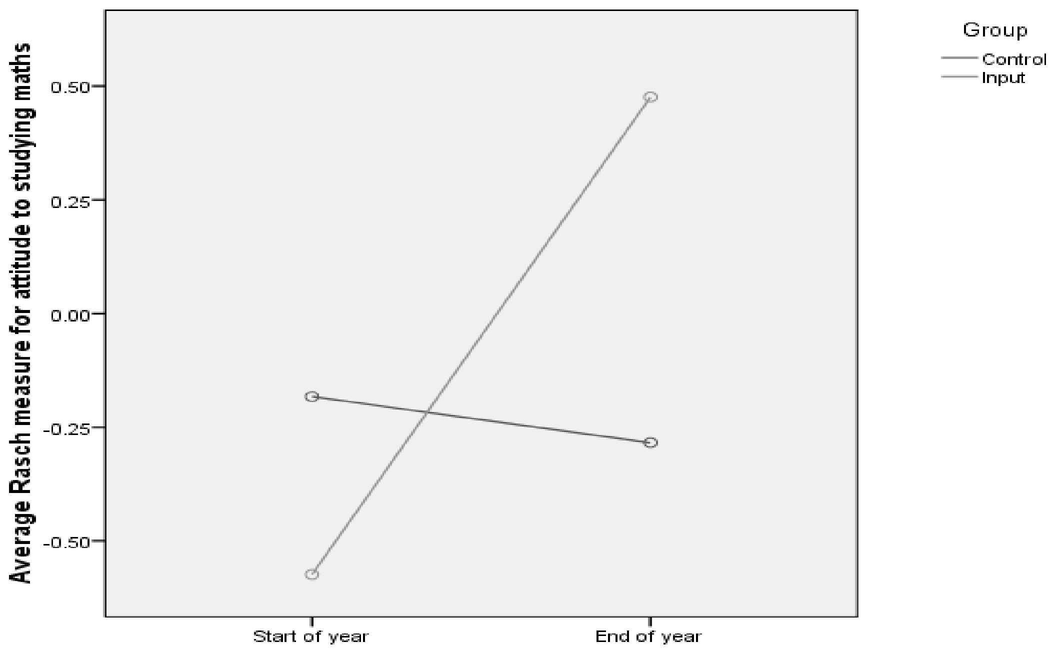

Figure 2. Change in the average attitude towards studying mathematics

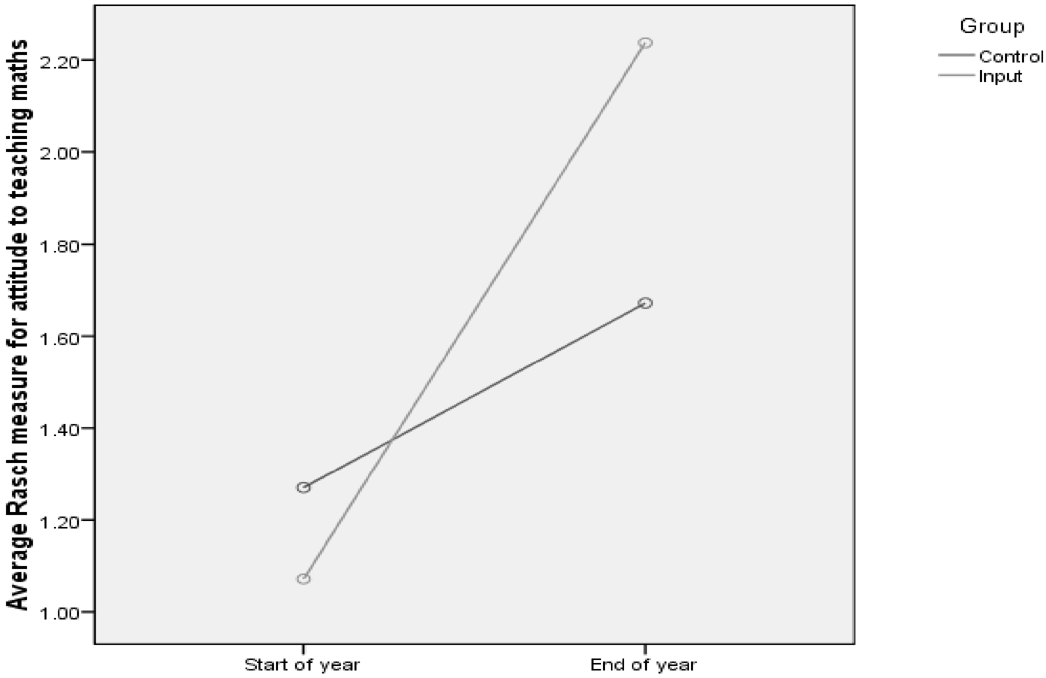

Figure 3. Change in the average attitude towards teaching mathematics

The average measures of attitude were calculated in terms of the average responses given by each person to each given item, where the responses were scored 1 to 5 on the 5-point Likert scale (5 being the most positive response and 1 being the most negative). Therefore, the scores from the attitude measure could range from 1 to 5. As can be seen from the graphs, the average attitudes for the representational group had a greater increase over the year than for the control group for both measures. Analysing the data using repeated measures ANOVA (Tables 1 and 2) showed that the attitude measures for both groups improved significantly over the year. This analysis also showed that the interactions between the group and both attitude measures were also significant. That is, the interaction between which group the students were in (input/control) and start/end of the year for the attitude towards studying mathematics measure was significant, $F(1,131) = 19.01$, $p < 0.001$, and the interaction between group and start/end of year for the attitude towards teaching mathematics measure was also significant, $F(1,131) = 6.78$, $p = 0.010$.

Table 1. ANOVA results for attitude towards studying mathematics

Interaction	Type III Sum of Squares	df	Mean Square	F	Sig.
Start/End of year	14.931	1	14.931	12.895	0.000
Start/End * Group	22.008	1	22.008	19.008	0.000
Error(Start/End)	151.675	131	1.158		

Table 2. ANOVA results for attitude towards teaching mathematics

Interaction	Type III Sum of Squares	df	Mean Square	F	Sig.
Start/End of year	40.818	1	40.818	28.488	0.000
Start/End * Group	9.709	1	9.709	6.776	0.010
Error(Start/End)	187.700	131	1.433		

In order to gain a greater insight into why the pre-service teachers had become more confident in both their understanding of and their teaching of mathematics, we can now examine the teachers' responses to the questions examining their reflections on their learning during the course. We present these responses both in terms of categorising responses and presenting the frequencies of responses, and we also provide examples of written responses from individual teachers in order to exemplify particular issues. Looking firstly at the responses to the question '*State a topic or concept in maths in which you feel you have deepened or modified your understanding'*, we counted the number of times teachers mentioned different topics, as shown graphically in Figure 4 below.

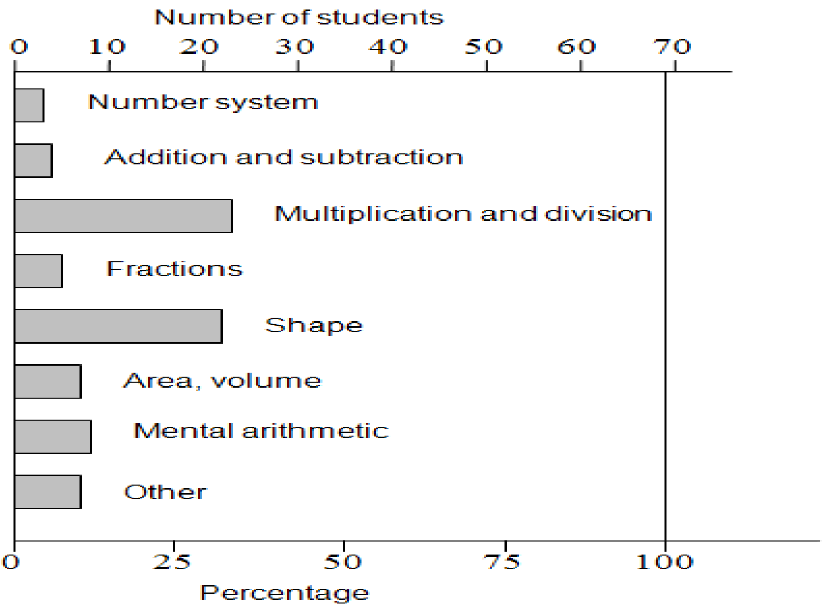

Figure 4.Topics mentioned by students in which they deepened their understanding

A variety of topics were highlighted by students, however two in particular were mentioned by about 30% of the students: multiplication and division, and shape. In terms of particular incidents or events that helped teachers learn during the course, the teachers' responses were categorised into the following categories and counted (Figure 5).

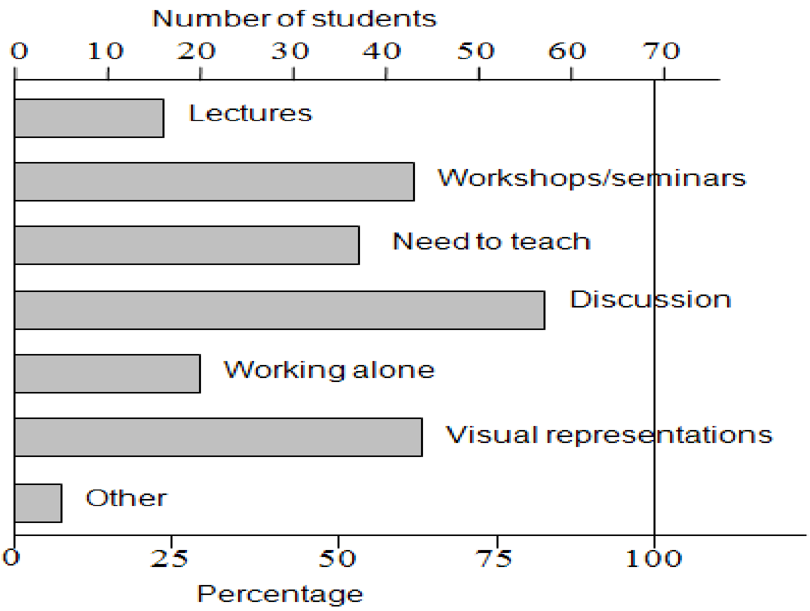

Fig. 5 Events or activities reported to have supported students own learning

The lectures were considered helpful by sixteen students (23%) and the workshops and seminars by forty-three students (62%) (workshops were sessions led by external professionals such as Local Authority advisors and seminars were led by university lecturers).

> I found the workshops with the visiting teachers very useful as they provided a good range of activities which were adaptable to all abilities.

> The lecture by [name] and the seminar by [name] helped me see how to build 3-D shapes with children and how to differentiate between 2-D and 3-D shapes.

Thirty-seven (54%) students referred to the need to actually teach a topic in helping them learn the specifics of the to-be-learned material.

> Teaching grid multiplication to my Year 4 class and using Danimaths programs to do so improved and deepened my understanding of multiplying big numbers.

However, the most common response was the value of discussion, mentioned by forty-nine (71%) of the sample. The following were typical of the value students attached to the opportunities afforded by discussion.

> Discussion with university tutors and the class teacher helped to develop both my subject knowledge and knowledge of how to teach it in an engaging way.

> Talking to others to develop my understanding. For example speaking to (lecturers) in individual (or small group) sessions and talking to peers to share and compare different strategies for tackling mental maths problems.

It appears that the work in the university was useful in this respect, not only in providing opportunities for students to talk to staff, but also by encouraging discussion among the students themselves.

Visual representations were seen as second in importance to the discussion. Forty-three (62%) of the students referred to the value of the visual representations in helping with their learning in mathematics.

> Visual representations helped me understand multiplication and fractions.
>
> Visual representations accompanied by explanations and talking to others helped develop my understanding.

The use of concrete materials as well, particularly the seminar on shape where the practical experiences of handling solid shapes was provided, was highlighted as beneficial. Visual representations delivered through the Danimaths programme was frequently highlighted by the pre-service teachers.

> Particularly one-to one tuition with (a lecturer) where Danimaths was used to show pictorial representations of problems. I particularly liked the arrays and found the number line very useful on teaching practice.
>
> When we looked at the Danimaths program in a lecture and workshop using slicers when multiplying and dividing fractions, it made the systems we had learnt at school to solve the problems relevant and for the first time I understood the concept
>
> Representations of fractions very useful. (The lecturer) went through it with a group of us. We had discussion about it and were able to have a go for ourselves.

As can be seen, the use of the visual representations could be used in conjunction with other activities, for example discussion with others or even in the teaching of topics. Finally, nineteen (27%) said that working alone was important in their learning, although seventeen of these responses also once again included the usefulness of discussion with others.

> Talking to others about their understanding helped deepen my own understanding of subjects and I learnt about methods others used to come to an answer. Attempting problems myself allowed me to identify gaps in my own knowledge and recognise when I had achieved success.

4. Discussion

This study investigated a ‘representational approach’ as a way of improving pre-service elementary teachers’ understanding of mathematical concepts and their confidence in studying and teaching the subject. Those in the first cohort had significantly higher scores on the attitude and confidence measures. The main difference between the instruction of the first and the second cohort was the use of representations. As illustrated in the Results, students in the first cohort often referred directly to the usefulness of the representations and the approach. Given that the second cohort had more recent instruction in mathematics and had spent more time teaching it in schools, this difference is even more remarkable. We could see no cause of the difference other than the emphasis on representations.

Despite both the representational and control groups being self-selected we believe that the pre-service teachers described here are typical of many other pre-service teachers in the UK. For instance, most had not studied mathematics beyond GCSE/GCE ‘O’ level and therefore other teacher educators are likely to be able to find similar results with their own pre-service teachers [30].

We also identified the important roles that discussion and visual representations played in developing these attitudes to mathematics. In order to explain why these elements might be important for pre-service teachers, we first examine the issue of developing pre-service teachers' understanding of mathematics. Ball [31, p. 458] emphasised the importance of understanding the subject for teachers: 'Teachers should understand the subject in sufficient depth to be able to represent it appropriately and in multiple ways'. We can conceptualise 'understanding' in terms of connections made between (internalised) representations of mathematical concepts through reasoning processes [16,32]. Therefore, developing the range of representations (in the case of this study, visual representations) that pre-service teachers have available to them is likely to develop their understanding of a mathematical concept. However, increasing the range of representations for teachers is not enough in itself – teachers also need to develop the connections that they have between representations, for example the connection between visual representations and symbolic representations or algorithms.

It is with regards to this development of 'connections' between representations that discussion can play a role. From her own work on discussion and learning mathematics Hoyles [33] highlighted three aspects of discussion: articulating ideas brings about reflection on those ideas; discussion involves framing ideas in a way that will be accepted by others; and listening to others modifies your own thoughts. Interpreting these ideas in terms of our view of understanding, in discussing our mathematical ideas, we modify the representations that we have and the connections that we have made, both through our own reflection and as a result of articulating our understanding, and also through comparisons with other people's understanding. Therefore, we see the importance of representations for understanding and the process of discussion in developing that understanding, as reciprocal processes.

For pre-service teachers however, representations of mathematical concepts have an additional importance. As highlighted in the introduction, representations are important for the explanation of mathematical concepts in the classroom as well [2,3]. In terms of our view of understanding, we are developing pupil understanding through the introduction of representations from which they can reason to symbolic or procedural representations. Therefore, representations have the dual role as tools for developing teachers' own understandings, and also tools for explanation in developing pupils' understanding. It is for these reasons, based on the research and the qualitative comments made by the pre-service teachers, that we see why visual representations and a discussion-based approach might develop pre-service teachers' attitudes towards studying and teaching mathematics.

5. Conclusion

The findings discussed here suggest that a 'representational approach' with pre-service teachers can be effective in developing their attitudes towards the subject of mathematics. This is not to say that using representations is likely to be a panacea or is itself without its limitations. Some representations can help in some circumstances and mislead in others. Cobb et al. [34, p. 2] remind us that 'meanings given to these representations are the product of students' interpretive activity'. Consequently, teachers need to have an awareness of many different representations and be able to build on children's existing understanding by presenting the most suitable representation to children at their particular level of understanding. Furthermore, as instructors on university courses, we in turn need to be aware of the different levels of understanding held by different pre-service teachers and in different areas of mathematics. It is noticeable in this study that the impact of representations concerning fractions on the pre-service teachers' knowledge was perceived to be much weaker and this is perhaps not surprising given that research has shown that 'fractions' is a common area of difficulty for pre-service teachers (31,35]. Therefore, further work is required in using a representational approach to develop pre-service teachers' knowledge and confidence in particular areas of mathematics.

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