PERIODIC POINT DATA DETECTS SUBDYNAMICS IN ENTROPY RANK ONE

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Abstract. A framework for understanding the geometry of continuous actions of $\mathbb{Z}^d$ was developed by Boyle and Lind using the notion of expansive behavior along lower-dimensional subspaces. For algebraic $\mathbb{Z}^d$-actions of entropy rank one, the expansive subdynamics is readily described in terms of Lyapunov exponents. Here we show that periodic point counts for elements of an entropy rank one action determine the expansive subdynamics. Moreover, the finer structure of the non-expansive set is visible in the topological and smooth structure of a set of functions associated to the periodic point data.

1. Introduction

Let β be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space (X, ρ); thus for each $\mathbf{n} \in \mathbb{Z}^d$ there is an associated homeomorphism $\beta^{\mathbf{n}}$, and $\beta^{\mathbf{m}} \circ \beta^{\mathbf{n}} = \beta^{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. Such an action is called *expansive* if there is some $\delta > 0$ with the property that if x, y are distinct points in X then there is some $\mathbf{n}$ for which $\rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) > \delta$. Any such δ is called an expansive constant for the action. Boyle and Lind [1] introduced the following notion, which reveals a rich geometrical structure inside an expansive action. A subset $A \subset \mathbb{R}^d$ is called *expansive for* β, or β is *expansive along* A, if there exist constants $\delta > 0$ and $t > 0$ with the property that

$$\sup_{\mathbf{n}, d(\mathbf{n},A)<t} \rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) \le \delta \implies x = y \text{ for all } x, y \in X$$

where $d(\mathbf{n}, A)$ denotes the distance from the point $\mathbf{n}$ to the set A in the Euclidean metric on $\mathbb{R}^d$. Of particular importance is the behavior along subspaces. Write G_k for the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$; this is a compact $k(d-k)$-dimensional manifold in the usual topology (subspaces are close if their intersections with the unit $(d-1)$-sphere S_{d-1} are close in the Hausdorff topology). Following Boyle and

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Lind, write

$$\mathsf{N}_k(\beta) = \{V \in \mathsf{G}_k \mid V \text{ is not expansive for } \beta\}.$$

The main structural result from [1] is that if X is infinite, then $\mathsf{N}_{d-1}(\beta)$ is a non-empty compact set, and the set $\mathsf{N}_{d-1}(\beta)$ governs all of the non-expansive behavior in the sense that any element of $\mathsf{N}_k(\beta)$ must be a subspace of some element of $\mathsf{N}_{d-1}(\beta)$. For algebraic systems, in which X is a compact metric group and each map $\beta^{\mathbf{n}}$ is a continuous group automorphism, the subdynamical structure was determined by Einsiedler, Lind, Miles and Ward [6], where a finer structure was found inside the set $\mathsf{N}_{d-1}(\beta)$ reflecting the two different ways in which an algebraic dynamical system can fail to be expansive.

A different insight into a topological $\mathbb{Z}^d$ action is a combinatorial one coming from periodic points. Write $F_{\mathbf{n}}(\beta) = \{x \in X \mid \beta^{\mathbf{n}}x = x\}$ for the set of points fixed by the homeomorphism $\beta^{\mathbf{n}}$. The combinatorial data of all these numbers may be thought of as a map

$$\mathbf{n} \mapsto |F_{\mathbf{n}}(\beta)| \in \mathbb{N} \cup \{\infty\},$$

where ∞ denotes the cardinality of an infinite compact group.

Our purpose here is to show that the combinatorial data contained in this map determines the expansive subdynamics for a certain class of systems (Theorem 4.8). These systems are the expansive algebraic systems of entropy rank one. In particular, for these systems the set $F_{\mathbf{n}}(\beta)$ is finite for $\mathbf{n} \neq 0$ except in degenerate situations.

2. Ranks and Subdynamics

The following notions come from [6, Sect. 7]. Let β be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space (X, ρ) as before. The *expansive rank* of β is

$$\operatorname{exprk}(\beta) = \min\{k \mid \mathsf{N}_k(\beta) \neq \mathsf{G}_k\},$$

that is the smallest dimension in which some expansive subspaces are seen. The *entropy rank* of β is

$$\operatorname{entrk}(\beta) = \max\{k \mid \text{there is a rational } k\text{-plane } V \text{ with } h(\beta, V) > 0\},$$

where $h(\beta, V)$ denotes the topological entropy of the $\mathbb{Z}^{\dim(V)}$-action given by restricting β to $V \cap \mathbb{Z}^d$. By [6, Prop. 7.2],

$$\operatorname{entrk}(\beta) \leq \operatorname{exprk}(\beta).$$

Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [10] which we will need. Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \dots, u_d$ with integer coefficients. If X is a compact metrizable abelian group and α is a $\mathbb{Z}^d$-action by continuous automorphisms $\alpha^{\mathbf{n}}$ of X, then the Pontryagin dual group $M = \widehat{X}$ has the structure of a discrete countable R_d-module, obtained by first identifying the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by the monomial $u^{\mathbf{n}} = u_1^{n_1} \dots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, for any countable R_d-module M, there is an associated $\mathbb{Z}^d$-action on a compact group obtained by dualizing the action induced by multiplying by monomials on M. A full account of this correspondence and the resulting theory is given in Schmidt's monograph [19]. An important aspect of this approach is the interpretation of dynamical properties as algebraic properties of M, particularly in terms of the set of associated prime ideals of M, written $\mathrm{Asc}(M)$. We will describe systems as Noetherian if they correspond to Noetherian modules, and in the reverse direction will describe modules as having various dynamical properties if the corresponding system has those properties.

The simplest algebraic systems are those corresponding to cyclic modules $R_d/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset R_d$, and these will be called *prime actions*. This gives a third natural notion of 'rank' to an algebraic $\mathbb{Z}^d$-action. Recall that the *Krull dimension* $\mathrm{kdim}(S)$ of a commutative ring S is the maximum of the lengths r taken over all strictly decreasing chains $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$ of prime ideals in S (see Matsumura [14, Chap. 1§5]). Boyle and Lind [1, Th. 7.5] show that if $\mathfrak{p}$ is a prime ideal generated by g elements, then

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq \mathrm{kdim}(R_d/\mathfrak{p}) \geq d - g$$

and

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq d - g + 1.$$

Moreover, [6, Prop. 7.3] shows that

$$\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p}) \leq \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}})$$

if $\mathfrak{p}$ is non-principal. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset R_d$ is equal to the Krull dimension of R_d localized at $\mathfrak{p}$, equivalently the maximal length r of a strictly decreasing chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r = (0).$$

The *co-height* $\mathrm{coht}(\mathfrak{p})$ of $\mathfrak{p}$ is equal to the Krull dimension of the domain $R_d/\mathfrak{p}$, equivalently it is the maximal length r of a strictly increasing chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r.$$

The domain R_d is universally catenary [7, Prop. 18.9, Cor. 18.10] and hence [7, Th. 13.8] shows that for each $\mathfrak{p} \in R_d$,

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \mathrm{kdim}(R_d) = d + 1.$$

Using associated primes, Einsiedler and Lind [5] provide the following classification of entropy rank one actions for which the associated module M is Noetherian (see Proposition 2.1). When M is not Noetherian, problems arise in relation to finding the set of possible entropy values for general algebraic $\mathbb{Z}^d$-actions; this is closely related to Lehmer's problem and is discussed more fully in [5].

Proposition 2.1. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action. Then*

(1) *α_M has entropy rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one. Equivalently, for each prime $\mathfrak{p} \in Asc(M)$, $coht(\mathfrak{p}) \leq 1$;*
(2) *α_M has expansive rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one; and*
(3) *if α_M is expansive then α_M has expansive rank one if and only if α_M has entropy rank one.*

Proof. See [5, Prop. 4.4 and 6.1, Th. 7.1 and 7.2]. □

In particular, an expansive rank one action may also be thought of as an expansive entropy rank one action. Further properties of entropy rank one actions are discussed in [5] and [15]. Of particular importance is the observation that if $\mathrm{coht}(\mathfrak{p}) = 1$ then the field of fractions K of the domain $R_d/\mathfrak{p}$ is a *global field* by [5, Prop. 6.1]. Moreover, the places of K, denoted by $\mathcal{P}(K)$, are determined by the ideal $\mathfrak{p}$. From this infinite set of places, we isolate

$$\mathcal{S}_{\mathfrak{p}} = \{w \in \mathcal{P}(K) \mid w \text{ is unbounded on } R_d/\mathfrak{p}\}.$$

Here w being *unbounded* means that $|R_d/\mathfrak{p}|_w$ is an unbounded subset of $\mathbb{R}$. Note that $\mathcal{S}_{\mathfrak{p}}$ contains all the infinite places of K. Furthermore, $\mathcal{S}_{\mathfrak{p}}$ is finite because $R_d/\mathfrak{p}$ is finitely generated.

The description of expansive subdynamics for algebraic $\mathbb{Z}^d$-actions is further refined in [6, Sect. 8] to reflect the two ways in which an algebraic dynamical system can fail to be expansive. It can fail in a way which relates to the Noetherian condition for modules, and this failure will result in a set of directions denoted N^{n}. It can also fail to be expansive in the way a quasihyperbolic toral automorphisms fails to be expansive, by having the higher-rank analogue of an eigenvalue with unit modulus; this failure arising from the varieties of the associated prime ideals results in a set of directions denoted N^{v}.

The Noetherian condition is described as follows. Each $\mathbf{n} \in \mathbb{Z}^d$ defines a half-space

$$H = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{n} \leq 0\} \subset \mathbb{R}^d,$$

which has an associated ring $R_H = \mathbb{Z}[u^{\mathbf{m}} \mid \mathbf{m} \in H \cap \mathbb{Z}^d] \subset R_d$. A module over R_d is also a module over R_H.

Definition 2.2. Let M be a Noetherian R_d-module and let $V \subset \mathbb{R}^d$ be a k-dimensional subspace. Then M is said to be *Noetherian along* V if M is a Noetherian R_H-module for every half-space H containining V. The collection of all k-dimensional subspaces along which M is not Noetherian is denoted $\mathsf{N}_k^{\mathsf{n}}(\alpha_M)$.

For the variety condition, let $\mathfrak{a} \subset R_d$ be any ideal. Write

$$\mathsf{V}(\mathfrak{a}) = \{\mathbf{z} \in (\mathbb{C} \setminus \{0\})^d \mid f(\mathbf{z}) = 0 \text{ for all } f \in \mathfrak{a}\}$$

and define the *amoeba* associated to $\mathfrak{a}$ to be

$$\log|\mathsf{V}(\mathfrak{a})| = \{(\log|z_1|, \ldots, \log|z_d|) \mid \mathbf{z} \in \mathsf{V}(\mathfrak{a})\}.$$

Now let M be a Noetherian R_d module. Then define

$$\mathsf{N}_k^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p} \in \mathrm{Asc}(M)} \{V \in \mathsf{G}_k \mid V^{\perp} \cap \log|V(\mathfrak{p})| \neq \varnothing\}$$

where $V^{\perp}$ denotes the orthogonal complement of V in $\mathbb{R}^d$. The main result in [6, Th. 8.4] says that

$$\mathsf{N}_k(\alpha_M) = \mathsf{N}_k^{\mathsf{n}}(\alpha_M) \cup \mathsf{N}_k^{\mathsf{v}}(\alpha_M)$$

for any Noetherian R_d-module M.

3. Periodic points

Recall that the dynamical zeta function of a map T is defined formally as

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} |F_n(T)|. \tag{3.1}$$

If $|F_n(T)|$ is finite for all $n \geq 1$ and grows at most exponentially, then (3.1) defines a complex function in some disc. In our setting there is a fixed $\mathbb{Z}^d$-action α, so write $\zeta_{\mathbf{n}}$ for the zeta function of the map $\alpha^{\mathbf{n}}$. Define $\mathsf{Q}(\alpha)$ to be the set of $\mathbf{n} \in \mathbb{Z}^d$ for which $\zeta_{\mathbf{n}}$ is a rational function. Notice that any $\mathbf{n} \in \mathbb{Z}^d$ with the property that $F_j(\alpha^{\mathbf{n}})$ is infinite for some $j \geq 1$ is not a member of $\mathsf{Q}(\alpha)$.

The simplest non-trivial $\mathbb{Z}^2$-action is the '$\times 2, \times 3$' system, and the idea behind what follows is already visible in this example. In non-expansive directions, the periodic orbits for this system exhibit very complex growth properties (see [8] and [9]).

Example 3.1. Consider the $\mathbb{Z}^2$-action α dual to the $\mathbb{Z}^2$-action generated by the commuting maps $\times 2$ and $\times 3$ on $\mathbb{Z}[\frac{1}{6}]$. This is the dynamical system corresponding to the cyclic R_2-module $M = R_2/(u_1-2, u_2-3)$. The set $\mathsf{N}_1(\alpha)$ for this example is shown in Figure 1; it consists of three lines with $\mathsf{N}_1^{\mathsf{n}}(\alpha)$ comprising $2^{n_1} = 1$ and $3^{n_2} = 1$ and $\mathsf{N}_1^{\mathsf{v}}(\alpha)$ being the single irrational line $2^{n_1}3^{n_2} = 1$.

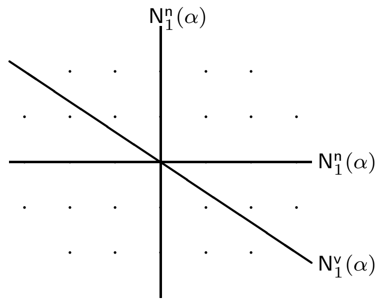

FIGURE 1. The three non-expansive lines for $\times 2, \times 3$.

The map $\mathbf{n} \mapsto |F_{\mathbf{n}}(\alpha)| \in \mathbb{N}$ for the same system is given by

$$|F_{\mathbf{n}}(\alpha)| = |2^{n_1}3^{n_2} - 1||2^{n_1}3^{n_2} - 1|_2|2^{n_1}3^{n_2} - 1|_3. \tag{3.2}$$

Thus, for example, in an expansive direction like $(1,1)$ the formula reduces to $|F_{j(1,1)}| = 6^j - 1$. In a non-expansive direction like $(1,0)$ the ultrametric terms cause more exotic behaviour.

It may be shown (see [9] and [15, Th. 4.7]) that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0\}$$

(the issue here is to show that the zeta functions $\zeta_{(1,0)}$ and $\zeta_{(0,1)}$ in the two rational non-expansive lines are not rational). The question addressed in this paper is the following: does the formula (3.2) determine the subdynamical portrait in Figure 1? As this example shows, the rationality set $\mathsf{Q}(\alpha)$ certainly does not determine $\mathsf{N}_1(\alpha)$, so in particular we are asking if the periodic point data seen along the rational directions can detect the presence of an irrational non-expansive direction.

Example 3.2. The zero-dimensional analog of the $\times 2, \times 3$ system is Ledrappier's example [11], which is the action α corresponding to the module $M = R_2/(2, 1 + u_1 + u_2)$. Using the local structure of X_M in terms of completions of the function field $\mathbb{F}_2(t)$ and the periodic point formula from [2] we have

$$|F_{(n_1,n_2)}(\alpha)| = |t^{n_1}(1+t)^{n_2}-1|_\infty |t^{n_1}(1+t)^{n_2}-1|_t |t^{n_1}(1+t)^{n_2}-1|_{1+t},$$

the three absolute values being given by

$$|r(t)|_t = 2^{-\operatorname{ord}_t(r(t))}, |r(t)|_\infty = |r(t^{-1})|_t \text{ and } |r(t)|_{1+t} = 2^{-\operatorname{ord}_{1+t}(r(t))}$$

where $r(t) \in \mathbb{F}_2(t)$ (see [2], [5] or [20] for the details). The set $\mathsf{N}_1(\alpha)$ for this example is shown in Figure 2; $\mathsf{N}_1^{\mathsf{n}}(\alpha)$ comprises the lines

$$n_1 = 0,\ n_2 = 0 \text{ and } n_1 + n_2 = 0,$$

while $\mathsf{N}_1^{\mathsf{v}}(\alpha)$ is automatically empty since the associated prime ideal has an empty variety.

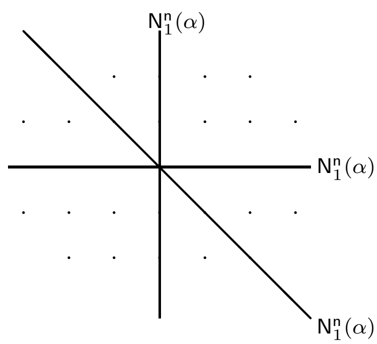

Figure 2. The three non-expansive lines for Ledrappier's example.

Once again the zeta function is known to be irrational in the non-expansive directions, and in this example $\mathsf{Q}(\alpha)$ does indeed detect all the non-expansive behavior. In each of the expansive regions, the periodic point formula simplifies significantly. For example, in the expansive region $n_1 < 0$, $n_2 > 0$, $n_1 + n_2 > 0$ we have

$$|t^{n_1}(1+t)^{n_2}-1|_t = 2^{-n_1},\ |t^{n_1}(1+t)^{n_2}-1|_\infty = 2^{n_1+n_2}$$

and

$$|t^{n_1}(1+t)^{n_2}-1|_t = 1,$$

giving $|F_{(n_1,n_2)}(\alpha)| = 2^{n_2}$.

The formula in non-expansive directions may be found similarly, though the resulting expression is a little more involved. For example, in [2, Ex. 8.5] it is shown that $|F_{(n,0)}(\alpha)| = 2^{n-2^{\operatorname{ord}_2(n)}}$.

An alternative way to compute the number of periodic points, better adapted to more complicated situations, may be found in [15, Lem. 4.8].

To convert the periodic point data into a form which exposes the expansive subdynamics, we introduce a normalized encoding of the rational zeta functions arising from elements of the action.

Definition 3.3. Given a rational function $h \in \mathbb{C}(z)$, denote the set of poles and zeros of h by $\Psi(h) \subset \mathbb{C}$. Let α be an algebraic $\mathbb{Z}^d$-action of entropy rank one, and define

$$\Omega_\alpha = \left\{ (\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid z \in \Psi(\zeta_{\mathbf{n}}), \mathbf{n} \in \mathsf{Q}(\alpha) \right\} \subset \mathsf{S}_{d-1} \times \mathbb{R}$$

where $\hat{\mathbf{n}}$ denotes the unit vector in the direction of $\mathbf{n}$.

In order to exhibit the relationship between Ω_α and $\mathsf{N}(\alpha)$ we need a 'formula' for $|F_{\mathbf{n}}(\alpha)|$, and this has been found by Miles [15] using the structure of entropy rank one systems from [5].

A *character* is a continuous homomorphism from an abelian group into $\mathbb{C}^\times$. We will be particularly interested in characters of the form $\chi : \mathbb{Z}^d \to \mathbb{C}^\times$. A *real character* is one with real image. By a *list* we mean a finite sequence of the form $L = \langle \chi_1, \dots, \chi_n \rangle$ which allows for multiplicities. The notation

$$\chi_L = \chi_1 \chi_2 \cdots \chi_n$$

is used to denote the product over all elements of L, with the understanding that $\chi_\varnothing \equiv 1$.

Let $\mathfrak{p} \subset R_d$ be a prime ideal with $\operatorname{coht}(\mathfrak{p}) = 1$ and let K be the field of fractions of $R_d/\mathfrak{p}$. Assume that $\operatorname{char}(R_d/\mathfrak{p}) = 0$, so all the infinite places are archimedean. These infinite places are uniquely determined by the embeddings of $R_d/\mathfrak{p}$ into $\mathbb{C}$. A point $z \in V_{\mathbb{C}}(\mathfrak{p})$ determines a ring homomorphism into $\mathbb{C}$ via the substitution map $f + \mathfrak{p} \mapsto f(z)$. The map is injective because $R_d/\mathfrak{p}$ has Krull dimension 1. Each $z \in V_{\mathbb{C}}(\mathfrak{p})$ induces a character on $\mathbb{Z}^d$ in an obvious way; there are finitely many such characters and the coordinates of these are all algebraic numbers. More generally, any place w of a domain of the form $R_d/\mathfrak{p}$ induces a real character on $\mathbb{Z}^d$ via the map

$$(n_1, \dots, n_d) \mapsto (|\overline{u}_i|_w^{n_1}, \dots, |\overline{u}_d|_w^{n_d}),$$

where $\overline{u}_i$ denotes the image of u_i in $R_d/\mathfrak{p}$, $i = 1 \dots d$. This will always be our method of constructing characters using non-archimedean places.

Using the construction of characters given above, for a prime $\mathbb{Z}^d$-action $\alpha_{R_d/\mathfrak{p}}$ with $\operatorname{coht}(\mathfrak{p}) = 1$, let $\mathcal{W}(R_d/\mathfrak{p})$ be the list of characters induced by the non-archimedean $v \in \mathcal{S}_{\mathfrak{p}}$ and let $\mathcal{V}(R_d/\mathfrak{p})$ be the list of characters induced by the distinct complex embeddings of $R_d/\mathfrak{p}$. Note that $\mathcal{V}(R_d/\mathfrak{p}) = \varnothing$ when $\operatorname{char}(R_d/\mathfrak{p}) > 0$.

Now suppose that α_M is a Noetherian entropy rank one action. The module M admits a *prime filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{3.3}$$

where for each k, $1 \leq k \leq n$ we have $M_k/M_{k-1} \cong R_d/\mathfrak{q}_k$ for a prime ideal $\mathfrak{q}_k \subset R_d$ which is either an associated prime of M or which contains an associated prime of M. Lemma 8.2 of [5] shows that each minimal element of $\mathrm{Asc}(M)$ always appears in such a filtration with a fixed multiplicity $m(\mathfrak{p})$, so $m(\mathfrak{p})$ is well-defined for all $\mathfrak{p} \in \mathrm{Asc}(M)$ with $\mathrm{coht}(\mathfrak{p}) = 1$. Set

$$\mathcal{W}(M) = \bigsqcup \mathcal{W}(R_d/\mathfrak{p}),$$

$$\mathcal{V}(M) = \bigsqcup \mathcal{V}(R_d/\mathfrak{p}),$$

where the union of lists is taken over all $\mathfrak{p} \in \mathrm{Asc}(M)$ with $\mathrm{coht}(\mathfrak{p}) = 1$, ensuring that each prime $\mathfrak{p}$ appears with the appropriate multiplicity $m(\mathfrak{p})$.

If M has torsion-free rank one, then $\mathcal{V}(M)$ has a particularly simple form.

Lemma 3.4. *Let M be a Noetherian R_d-module of torsion-free rank one, and suppose α_M has entropy rank one. Then $\mathcal{V}(M)$ contains one element.*

Proof. Consider a prime filtration of M of the form (3.3). Since M has torsion-free rank one, there is at least one associated prime $\mathfrak{p}$ such that $\mathrm{char}(R_d/\mathfrak{p}) = 0$. Let $k \leq n$ be the least integer such that $\mathrm{char}(R_d/\mathfrak{q}_k) = 0$. Then $\mathrm{coht}(\mathfrak{q}_k) = 1$ and $\mathfrak{q}_k \in \mathrm{Asc}(M)$. Suppose $k < n$. Since $M_{k+1}/M_k \cong R_d/\mathfrak{q}_{k+1}$, there exists $a \in M_{k+1} \setminus M_k$ such that any element of M_{k+1} can be written in the form $x + fa$ for some $x \in M_k$ and $f \in R_d$ with $fa \in M_k$ if and only if $f \in \mathfrak{q}_{k+1}$. However, both M_k and M have torsion-free rank one so there exists $c \in \mathbb{Z}$ such that $ca \in M_k$. Therefore, $c \in \mathfrak{q}_{k+1}$ and $\mathrm{char}(R_d/\mathfrak{q}_{k+1}) > 0$. In a similar way, it follows that $\mathrm{char}(R_d/\mathfrak{q}_j) > 0$ for all $j > k$. Hence $\mathfrak{q}_k$ is the only prime with $\mathrm{char}(R_d/\mathfrak{q}_k) = 0$; moreover $m(\mathfrak{q}_k) = 1$. So

$$\mathcal{V}(M) = \mathcal{V}(R_d/\mathfrak{q}_k).$$

If $k = n$ then again $\mathcal{V}(M) = \mathcal{V}(R_d/\mathfrak{q}_k)$. Finally, $R_d/\mathfrak{q}_k$ is isomorphic to a subring of $\mathbb{Q}$, so $\mathcal{V}(M)$ contains one character induced by the single infinite place of $\mathbb{Q}$. $\square$

Any character $\chi : \mathbb{Z}^d \to \mathbb{C}^\times$ induces a real character χ^* on $\mathbb{R}^d$, by setting

$$\chi^*(\kappa \mathbf{e}_i) = |\chi(\mathbf{e}_i)|^\kappa$$

where $\kappa \in \mathbb{R}$ and $\mathbf{e}_i$ is the standard i-th basis vector in $\mathbb{Z}^d$, $i = 1 \dots d$. Applying this construction to an element of $\mathcal{W}(M)$ yields a genuine extension, but the same is not necessarily true for elements of $\mathcal{V}(M)$.

Proposition 3.5. *Suppose α_M is an algebraic $\mathbb{Z}^d$-action of expansive rank one. Then $\mathsf{N}_{d-1}(\alpha_M)$ consists precisely of the finite set of hyperplanes defined by the equations*

$$\chi^*(\mathbf{n}) = 1$$

where $\chi \in \mathcal{V}(M) \cup \mathcal{W}(M)$. Furthermore, $\mathsf{N}^{\mathsf{v}}_{d-1}(\alpha_M)$ is determined by those characters in $\mathcal{V}(M)$ and $\mathsf{N}^{\mathsf{n}}_{d-1}(\alpha_M)$ by those characters in $\mathcal{W}(M)$.

Proof. This is a combination of [16, Th. 4.3.10] and [6, Th. 8.4]. □

By expressing $\mathsf{N}_1(\alpha_M)$ in terms of the intersection of non-expansive lines with S_{d-1} and referring to [1, Th. 3.6], we also find the following description of the expansive subdynamics.

Corollary 3.6. *If α_M is an algebraic $\mathbb{Z}^d$-action of expansive rank one then*

$$\mathsf{N}^{\mathsf{v}}_1(\alpha) = \bigcup_{\chi \in \mathcal{V}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\},$$

$$\mathsf{N}^{\mathsf{n}}_1(\alpha) = \bigcup_{\chi \in \mathcal{W}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\}$$

and the set of expansive directions is dense in S_{d-1}.

4. Main Results

To begin this section, we return to the examples in Section 3

Example 4.1. Let α be the $\mathbb{Z}^2$-action corresponding to the R_2-module

$$M = R_2/(u_1 - 2, u_2 - 3)$$

discussed in Example 3.1. Recall that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0\}.$$

Here, $\mathsf{Q}(\alpha)$ consists precisely of those $\mathbf{n} \in \mathbb{Z}^2$ for which $\alpha_M^{\mathbf{n}}$ is expansive, but this need not always be the case (see [15, Ex. 4.3] for an example). Notice that expansiveness of the elements $\alpha_M^{\mathbf{n}}$ of the action can only ever detect non-expansiveness in rational directions, so the irrational line in $\mathsf{N}_1(\alpha)$ will be missed. For $\mathbf{n} = (n_1, n_2) \in \mathsf{Q}(\alpha)$, using the periodic point formula from [15],

$$|F_j(\alpha_M^{\mathbf{n}})| = |2^{jn_1}3^{jn_2} - 1|_\infty |2^{jn_1}3^{jn_2} - 1|_2 |2^{jn_1}3^{jn_2} - 1|_3.$$

It follows that

$$\zeta_{\mathbf{n}}(z) = (1 - g(\mathbf{n})z)^{\lambda_1}(1 - g(\mathbf{n})2^{n_1}3^{n_2}z)^{\lambda_1},$$

where $\lambda_1, \lambda_2 \in \{-1, 1\}$ and

$$g(\mathbf{n}) = |2^{n_1} - 1|_2|3^{n_2} - 1|_3.$$

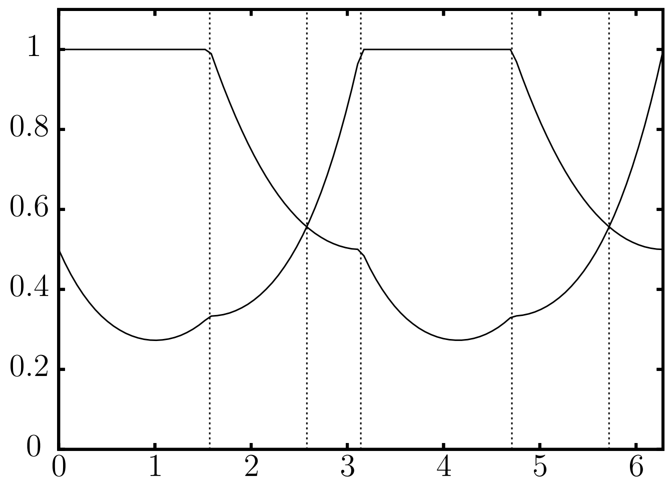

FIGURE 3. $\overline{\Omega}_\alpha$ for the $\times 2, \times 3$ system in Example 4.1.

The resulting directional pole and zero data $\overline{\Omega}_\alpha$, realized as a subset of $[0, 2\pi) \times \mathbb{R}$, is shown in Figure 3. Non-expansive directions are marked with a dashed line.

Example 4.2. Let α be the $\mathbb{Z}^2$-action corresponding to the R_2-module

$$M = R_2/(2, 1 + u_1 + u_2)$$

discussed in Example 3.2 Recall that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0 \text{ and } n_1 + n_2 \neq 0\}.$$

For $\mathbf{n} = (n_1, n_2) \in \mathsf{Q}(\alpha)$, using the periodic point formula from [15],

$$\zeta_{\mathbf{n}}(z) = (1 - g(\mathbf{n})z)^{-1}$$

where

$$g(\mathbf{n}) = |t^{n_1}(1+t)^{n_2} - 1|_{1+t}|t^{n_1}(1+t)^{n_2} - 1|_t|t^{n_1}(1+t)^{n_2} - 1|_\infty.$$

The resulting directional pole and zero data $\overline{\Omega}_\alpha$, realized as a subset of $[0, 2\pi) \times \mathbb{R}$, is shown in Figure 4. Non-expansive directions are marked with a dashed line.