A Many-to-Many 'Rural Hospital Theorem'*

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Abstract

We show that the full version of the so-called 'rural hospital theorem' generalizes to many-to-many matching problems where agents on both sides of the problem have substitutable and weakly separable preferences. We reinforce our result by showing that when agents' preferences satisfy substitutability, the domain of weakly separable preferences is also maximal for the rural hospital theorem to hold.

Keywords: matching, many-to-many, stability, rural hospital theorem.

JEL–Numbers: C78, D60.

1 Introduction

We study two-sided matching problems. 'Stability' of outcomes in these problems is considered to be the main property that accounts for the success of matching rules. We identify a large and maximal preference domain for which 'underdemanded' institutions (or agents) have the same partners at each stable outcome. Consequently, no stable

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rule can implement possibly desirable changes in the set of partners assigned to such institutions.

Our study is motivated by issues raised in certain centralized labor markets. As an illustration, many countries employ each year a centralized mechanism to assign newly graduated medical students to positions in residency programs. Hospitals in rural areas are typically less preferred than those in urban areas by medical graduates, i.e., they are ranked below urban hospitals in a typical student's preference list. Also, graduates from relatively successful programs are more popular among hospitals, i.e., they are ranked above other students in a typical hospital's preference list. Rural hospitals complain that their positions may not be filled by the stable matching rule in use and that they may not be assigned popular graduates. The 'rural hospital theorem' established in several matching models states that the *number* of medical graduates assigned to a hospital and the *set* of graduates assigned to a hospital in a rural area do not vary across stable outcomes. Even though the theorem's name is a useful reminder of its content and origin, the 'rural hospital theorem' equally applies to other labor markets with similar concerns about the numerical distribution of workers or the composition of the workforce of firms.

We study the 'rural hospital theorem' in the context of many-to-many labor markets, i.e., markets where each agent can engage in multiple partnerships. There are several reasons to focus on *many-to-many* markets instead of many-to-one markets (where each worker can be employed by at most one firm). First, a well-known many-to-many market is the medical labor market in the UK. More specifically, each medical graduate in the UK has to seek two positions (a medical position and a surgical position) to be able to register as a medical doctor. Shallcross (2005) mentioned concerns of doctor shortages in rural areas in the UK. Second, as pointed out by Echenique and Oviedo (2006), even if in a labor market most workers are employed by one firm, the presence of a few workers with multiple employers can make a crucial difference. Precisely, Echenique and Oviedo (2006, Example 2.2) showed that the presence of only one worker with two part-time jobs can already change the stable outcome for all other agents. Thus, the functioning of even 'almost many-to-one' labor markets can only be understood through the study of many-to-many matching models. Third, the literature on many-to-many matching markets has grown in the last couple of years,[1] but there is still a wide gap with respect to many-to-one markets. Fourth, there are important structural differences between many-to-one and many-to-many matching markets, even if all agents have so-called 'responsive' preferences. For instance, Sotomayor (1999) showed that unlike many-to-one markets, in many-to-many markets the set of stable outcomes needs not coincide with the core. Finally, our results are not only novel to the many-to-many framework. Indeed, *the restriction of all our results to the many-to-one framework yields new results and subsumes existing results for that framework* as well.[2]

Next, we describe in more detail the model we study, the existing literature, and our contribution. In a two-sided many-to-many matching problem there are two disjoint

[1] Recent papers on many-to-many matching include, among others, Hatfield and Kominers (2012a,b), Jaramillo et al. (2014), Klaus and Walzl (2009), Kojima and Ünver (2008), Kominers (2012), Konishi and Ünver (2006), Ostrovsky (2008), Sotomayor (2004), and Yazıcı (2012).

[2] See the discussion that precedes Theorem 3 and Remarks 7 and 8.

sets of agents, which we call 'firms' and 'workers.' Each firm (worker) can only form partnerships with workers (firms). Each agent has a preference order over the set of all subsets of partnerships, i.e., subsets of agents in the other set. For each agent, there is a maximal number ('quota') of partnerships the agent can or is willing to be involved in. An outcome of the problem is a 'match' which consists of a collection of partnerships.

A match is 'stable' if no agent prefers to be matched to a proper subset of its current partners, and no set of firms and workers prefer to establish new partnerships only among themselves and possibly break up some existing partnerships.[3] This definition is more stringent than so-called pairwise stability which is another standard solution concept but that only eliminates blocking by firm-worker pairs. Stability proved to be a crucial property in many entry-level labor markets where workers are matched to firms through a clearinghouse. It has been observed that clearinghouses that use stable rules often perform better than those that use rules that do not necessarily produce stable matches.[4] According to Roth (1991, p.422) even the weaker stability concept, pairwise stability, can be of primary importance for many-to-many markets as well.

There are many-to-many problems for which no stable match exists. Certain assumptions on preferences have been identified to guarantee that they do. A firm's preferences are 'substitutable' if whenever a worker is chosen from a group of workers by this firm, she is also chosen from any of the group's subsets to which she belongs.[5] Substitutability of workers' preferences is defined similarly. Substitutability is a standard assumption in the literature and it guarantees the existence of a pairwise stable match.[6] Hatfield and Kominers (2012a) showed that for substitutable preferences, stability and pairwise stability are equivalent.[7] Thus, when preferences are substitutable, the set of stable matches is non-empty and coincides with the set of pairwise stable matches. *With the important exception of Proposition 1, we assume substitutability throughout.*

Taking the requirement of stability as granted, an important question is whether the choice of a particular stable rule affects the numerical distribution of workers; and if not, whether different matches assign different sets of workers to a firm that does not fill all its positions. For instance, in the context of the assignment of medical graduates in the US, the National Resident Matching Program (NRMP) failed to fill the posts of many hospitals in (typically less preferred) rural areas (Sudarshan and Zisook, 1981). However, provided that the preferences satisfy certain conditions, the problem of the rural hospitals cannot be attributed to the particular stable rule used by the NRMP. Indeed, the results obtained by Gale and Sotomayor (1985) and Roth (1984b, 1986) suggest that any other stable rule would yield (R1) the same numerical distribution of medical graduates and would assign (R2) the same medical graduates to each rural

[3]This is an adaptation of the stability definition in Hatfield and Kominers (2012a).

[4]See, for instance, Roth (1991).

[5]Substitutability is an adaptation of the gross substitutability property (Kelso and Crawford, 1982) by Roth (1984a) and Roth and Sotomayor (1990) to matching problems without monetary transfers.

[6]The existence of a pairwise stable match can be shown via an algorithm for strict preferences (Roth, 1984a) and via a non-constructive proof for non-necessarily strict preferences (Sotomayor, 1999). See also Martínez et al. (2004b) for the computation of the full set of pairwise stable matches.

[7]We are thankful to a referee for pointing this out.

hospital that does not fill all its posts. The two results are known as weak and strong versions of the rural hospital theorem.[8]

Both versions of the rural hospital theorem play a functional role in proving many appealing results. For instance, R1 is used to show the lattice structure of the set of stable matches (Martínez et al., 2001) and the group strategy-proofness (for the workers' side) of the worker-optimal stable rule (Martínez et al., 2004a) in a many-to-one model. Ma (2002) studied refinements of Nash equilibrium based on 'truncations at the match point' for the preference revelation game induced by any stable rule. He used R2 to prove that each equilibrium outcome is stable for the true preferences. Pais (2006) studied ordinal Nash equilibria of the preference revelation game induced by any probabilistic stable rule. She used R2 to show that any equilibrium induces a match that is individually rational for the true preferences. Yazıcı (2012) also employed R2 to extend the latter result to many-to-many matching with a more general preference domain. These results show that the relevance of the rural hospital theorem goes beyond its direct interpretation: it is a powerful tool in establishing structural results and analyzing strategic matching games.

The first papers on the rural hospital theorem (e.g., Gale and Sotomayor, 1985; Roth, 1984b, 1986) studied many-to-one matching problems and assumed firms' preferences to be 'responsive.' A firm's preferences over groups of workers are responsive to its preferences over individual workers if (i) for two groups that only differ in one worker, the firm prefers the one with the preferred worker, and (ii) adding an acceptable (unacceptable) worker to a group that does not fill its quota makes the group better (worse). Responsiveness implies substitutability. Several papers have shown R1 and R2 for preference domains that are strictly larger than the domain of responsive preferences.[9] A firm's preferences are 'separable' if condition (ii) above holds. R1 and R2 hold for substitutable and separable preferences (Martínez et al., 2000, Proposition 2). Since responsiveness implies separability, Martínez et al.'s (2000) result subsumes the previous rural hospital theorem results.[10]

Concerning the many-to-many framework, Alkan (2002, Proposition 6) showed that R1 holds for substitutable and 'cardinally monotonic' preferences. A firm's preferences over groups of workers are cardinally monotonic if whenever the group of workers available to the firm expands, it will not employ fewer workers.[11] Since separability implies cardinal monotonicity, Alkan's (2002, Proposition 6) many-to-many result subsumes Martínez et al.'s (2000, Proposition 2a) many-to-one result on R1. On the other hand, R2 has only been shown to hold for responsive preferences (Alkan, 1999, Proposition 2i) and for so-called 'quota-filling' preferences that satisfy separability (Alkan, 2001, Corollary 1). The latter two results on R2 do *not* subsume Martínez et al.'s (2000, Proposi-

[8]Since R2 implies R1, R1 (R2) is often referred to as the weak (strong) rural hospital theorem.

[9]For the reader's convenience, we refer to the Venn diagram of Figure 1 (in Section 3) which depicts the inclusion relations among the preference domains we discuss.

[10]Kojima (2012) also introduced the domain of separable preferences with so-called affirmative action constraints. This domain is a strict superset of the domain of separable preferences but a strict subset of the domain of cardinally monotonic preferences. Kojima (2012) showed that on his domain an appropriately adjusted version of R2 holds.

[11]Cardinal monotonicity is called size monotonicity and law of aggregate demand in Alkan and Demange (2003) and Hatfield and Milgrom (2005), respectively.

tion 2b) many-to-one result on R2.

The contribution of our paper is twofold. We first introduce a new preference domain called weak separability by relaxing separability. We prove that the strong rural hospital theorem, i.e., R2, holds on the domain of substitutable and weakly separable preferences (Theorem 3). Thus, our result generalizes the results of Martínez et al. (2000, Proposition 2b) for many-to-one matching and Alkan (1999, Proposition 2i and 2001, Corollary 1) for many-to-many matching. Our short proof is based on a strong structural result regarding the set of stable matches due to Roth (1984a).

Our second contribution shows that the two largest domains for R1 and R2 discussed above are in fact maximal (in a sense made precise below). First, we provide a maximal domain result that complements Alkan's (2002, Proposition 6) result regarding R1 and cardinal monotonicity. Precisely, if some agent's preferences do not satisfy cardinal monotonicity then we construct substitutable and cardinally monotonic preferences for the other agents such that R1 fails (Proposition 1). Second, we provide a maximal domain result that complements our Theorem 3 regarding R2 and weak separability. Precisely, if some agent has substitutable preferences that are not weakly separable then we construct substitutable and weakly separable preferences for the other agents such that R2 fails (Proposition 2). In fact, our two maximality results are stronger in two ways: 1) the constructed preferences are responsive and 2) the proofs are also effective for the many-to-one framework (and yield novel results in that framework as well).

Concerning many-to-one matching with contracts, Hatfield and Milgrom (2005, Theorems 8 and 9) proved R1 for substitutable and cardinally monotonic preferences and established a maximality result. Hatfield and Kojima (2010) introduced a weaker condition than substitutability called bilateral substitutability.[12] Hatfield and Kojima (2010, Theorem 6) extended R1 to bilaterally substitutable and cardinally monotonic preferences for many-to-one matching with contracts. Hatfield and Kominers (2012a) studied many-to-many matching with contracts where multiple contracts can be signed between any firm-worker pair. Hatfield and Kominers (2012a, Section 4.2) also obtained R1 for substitutable and cardinally monotonic preferences. In matching without contracts, substitutability and bilateral substitutability are equivalent. Thus, in that framework, Hatfield and Milgrom (2005, Theorem 9), Hatfield and Kojima (2010, Theorem 6), and Hatfield and Kominers (2012a, Section 4.2) boil down to the earlier mentioned result of Alkan (2002, Proposition 6) for many-to-one and many-to-many matching. In Remark 8 we show that in the framework without contracts, Hatfield and Milgrom (2005, Theorem 9) does not imply nor is implied by our Proposition 1.

Hatfield and Kominers (2012b) studied matching in networks with bilateral contracts: agents trade goods via contracts and each agent may be both a seller and a buyer of a good. Their Theorem 8 shows that a 'generalized version of R1' holds if preferences satisfy 'same-side and cross-side substitutability' and two laws of aggregate demand and supply. In terms of two-sided many-to-many matching without contracts, their result boils down to Alkan (2002, Proposition 6). Hatfield and Kominers (2012b, Theorem 9) also proved a maximality result similar to Proposition 1. More precisely, if

[12] Consider a firm that faces a set of contracts Y and two contracts x and z. Substitutability requires that if x is chosen in $Y \cup \{x, z\}$ then x is still chosen in $Y \cup \{x\}$. Bilateral substitutability only requires the same property for contracts x and z whose associated workers are not involved in contracts in Y.

some agent's preferences violate the law of aggregate demand or supply but do satisfy same-side and cross-side substitutability, then there are same-side and cross-side substitutable preferences for the other agents satisfying the laws of aggregate demand and supply such that the generalized version of R1 fails. In Remark 8 we show that in terms of two-sided many-to-many matching without contracts Hatfield and Kominers (2012b, Theorem 9) does not imply our Proposition 1.

In Section 2, we present the model. In Section 3, we formally introduce and relate the aforementioned preference domains. In Section 4, we state and prove our results on the rural hospital theorem.

2 Model

There are two disjoint and finite sets of agents: a set of **firms** F and a set of **workers** W. Let $A = F \cup W$ denote the set of agents. Generic elements of F, W, and A are denoted by f, w, and a, respectively. The set of (possible) partners of agent a is $S_a \equiv W$ if $a \in F$, and $S_a \equiv F$ if $a \in W$. The preferences of agent a are given by a linear order P_a over 2^{S_a}.[13] Let $\mathcal{P}_a$ denote the collection of all possible preferences for a. Since we fix the set of agents, a **many-to-many matching market** is given by a preference profile, i.e., a tuple $P = (P_a)_{a \in A}$. For each $a \in A$, let R_a denote the 'at least as desirable as' relation associated with P_a, i.e., for each pair $b, c \in S_a$, $b \, R_a \, c$ if and only if $b = c$ or $b \, P_a \, c$. For each agent a with preferences P_a, let $\mathsf{Ch}(., P_a)$ be the induced choice function on 2^{S_a}. In other words, for each $S \subseteq S_a$, $\boldsymbol{\mathsf{Ch}(S, P_a)}$ is agent a's most preferred subset of S according to P_a. One easily verifies that Ch satisfies consistency (Alkan, 2002),[14] i.e., for each pair $S, T \subseteq S_a$,

$$\mathsf{Ch}(S, P_a) \subseteq T \subseteq S \implies \mathsf{Ch}(T, P_a) = \mathsf{Ch}(S, P_a). \tag{1}$$

A set of agents $S \subseteq S_a$ is **acceptable** to agent a at P if $S \; R_a \; \emptyset$. For each agent $a \in A$, let its '**quota**' be the smallest integer $q_a \geq 0$ such that for each $S \subseteq S_a$ with $|S| > q_a$, S is not acceptable to a.[15] If each agent on (at least) one side of the market has quota at most 1, then the market is **many-to-one**. If all agents have quota at most 1, then the market is **one-to-one**.

A **match** μ is a mapping from A into $2^F \cup 2^W$ such that for each pair $a, a' \in A$, $\mu(a) \in 2^{S_a}$ and $[a \in \mu(a') \Leftrightarrow a' \in \mu(a)]$. Let $\mathcal{M}$ denote the set of all matches.

Next, we adapt the definition of stability due to Hatfield and Kominers (2012a, Section 2.2) to our model. Match μ is **blocked by an agent** $a \in A$ at P if $\mathsf{Ch}(\mu(a), P_a) \neq \mu(a)$. Match μ is **blocked by a set of firms and workers** $F' \cup W'$ at P, where $\emptyset \neq F' \subseteq F$ and $\emptyset \neq W' \subseteq W$, if there is a match μ' such that for each $a \in F' \cup W'$,

(b1). $\emptyset \neq (\mu'(a) \backslash \mu(a)) \subseteq F' \cup W'$;
(b2). $\mu'(a) \subseteq \mathsf{Ch}(\mu'(a) \cup \mu(a), P_a)$.

[13] In other words, P_a is transitive, antisymmetric (strict), and total.

[14] This condition is earlier used by Blair (1988). See also Aygün and Sönmez (2012, 2013).

[15] The interpretation is that agent a can definitely not work for/hire more than q_a agents from the other side of the problem. Note that for each $q'_a \geq |S_a|$, $[S \subseteq S_a$ with $|S| > q'_a$ implies $\emptyset \, P_a \, S]$ is vacuously true. Hence, $q_a \leq |S_a|$.

Loosely speaking, the agents in $F' \cup W'$ are strictly better off by establishing new partnerships only among themselves and possibly breaking up some existing partnerships. A match is **stable** at P if it is not blocked by any agent or any set of firms and workers at P. Let $\boldsymbol{\Sigma(P)}$ denote the set of stable matches at P.

Remark 1. Note that when $F' = \{f\}$ is a singleton and $W' \neq \emptyset$, blocking by $F' \cup W'$ is equivalent to the following two conditions: (1) for each $w \in W', f \notin \mu(w)$ and $f \in \mathsf{Ch}(\mu(w) \cup \{f\}, P_w)$ and (2) $W' \subseteq \mathsf{Ch}(\mu(f) \cup W', P_f)$. In one-to-one matching, the standard definition of stability of a match requires that neither an individual agent nor a set $F' \cup W'$ with $|F'| = 1$ and $|W'| = 1$ blocks the match. This is usually referred as pairwise stability. In many-to-one matching, the definition of stability of a match eliminates blocking by agents or by sets of firms and workers $F' \cup W'$ with $|F'| = 1$ and $|W'| \geq 1$. This is referred as many-to-one stability. Clearly, stability implies many-to-one stability and many-to-one stability implies pairwise stability. $\diamond$

Remark 2. For later reference we note that any $a \in A$ with $\mu(a) = \mathsf{Ch}(S_a, P_a)$ cannot be part of a blocking set. $\diamond$

Remark 3. Since unacceptable sets of partners cannot be part of a stable match, it is sufficient to describe each agent's ranking of acceptable sets of possible partners. For instance,

$$P_f : \{w_1, w_2\}, \{w_1\}, \{w_3\}, \{w_2\}, \emptyset$$

indicates that $\{w_1, w_2\}\, P_f\, \{w_1\}\, P_f\, \{w_3\}\, P_f\, \{w_2\}\, P_f\, \emptyset$ and that all other sets of possible partners are not acceptable to f. $\diamond$

Unless explicitly noticed otherwise, we make the following (standard) assumption on each agent a's preferences P_a.

Substitutability:
For each $b, c \in S \subseteq S_a$ with $b \neq c$, $[\, b \in \mathsf{Ch}(S, P_a) \implies b \in \mathsf{Ch}(S \backslash \{c\}, P_a)\,]$.

Remark 4. For each profile of substitutable preferences P, $\Sigma(P)$ is non-empty (see, e.g., Roth, 1984a, Theorem 1).[16] Moreover, stability coincides with many-to-one stability and pairwise stability (Hatfield and Kominers, 2012a, Proposition 1).[17,18] $\diamond$

It follows from Remark 4 that without loss of generality we can focus on pairwise stability, i.e., only consider possible blockings by individual agents and firm-worker pairs. In particular, the terms 'stability' and 'pairwise stability' can be employed interchangeably.[19]

[16] Martínez et al. (2004b) provided an algorithm to calculate all stable matchings when preferences are substitutable.

[17] We thank a referee for pointing us to Proposition 1 in Hatfield and Kominers (2012a).

[18] Unlike in many-to-one matching with substitutable preferences, pairwise stability is not equivalent to core-stability in many-to-many matching. Indeed, no logical relation exists between the two concepts (Blair, 1988).

[19] The only exception is Proposition 1 since there one of the preference relations may not satisfy substitutability.

Theorem 2 in Roth (1984a) shows that there is a firm-optimal stable match μ_F (which all firms find at least as desirable as any other stable match), and likewise a worker-optimal stable match μ_W. In fact, the set of stable matches satisfies the following properties which will be key in the proof of our first main result.

Theorem 1 (Roth, 1984a, Theorem 2)**.** *Let P be substitutable. Let $\mu \in \Sigma(P)$. For each $f \in F$ and each $w \in W$,*
(i). $\mathsf{Ch}(\mu_F(f) \cup \mu(f), P_f) = \mu_F(f)$;
(ii). $\mathsf{Ch}(\mu_W(w) \cup \mu(w), P_w) = \mu_W(w)$.

3 Preference Domains

The matching literature has studied the following preference domains. Let $a \in A$.

Responsiveness:[20,21]
For each $S \subseteq S_a$ with $|S| < q_a$ and for each $b, c \in (S_a \backslash S) \cup \{\emptyset\}$,
$[\, b\ P_a\ c \iff (S \cup \{b\})\ P_a\ (S \cup \{c\})\,]$.

Separability:[22]
For each $S \subseteq S_a$ with $|S| < q_a$ and for each $b \in S_a \backslash S$, $[b\ P_a\ \emptyset \iff (S \cup \{b\})\ P_a\ S]$.

Cardinal monotonicity:[23]
For each pair $S', S \subseteq S_a$, $[\, S' \subseteq S \implies |\mathsf{Ch}(S', P_a)| \leq |\mathsf{Ch}(S, P_a)|\,]$.

Quota-filling:[24]
P_a is substitutable and for each $S \subseteq S_a$ with $|S| \geq k_a \equiv \max\{|\mathsf{Ch}(T, P_a)| : T \subseteq S_a\}$, $|\mathsf{Ch}(S, P_a)| = k_a$.

We introduce a new preference domain by relaxing separability in the following way.

Weak separability:
For each $S \subseteq S_a$ with $|S| < q_a$ and $\mathsf{Ch}(S, P_a) = S$, and for each $b \in S_a \backslash S$,
$[\, b\ P_a\ \emptyset \implies (S \cup \{b\})\ P_a\ S\,]$.

Next, we give an example of a type of preference relations that is captured by relaxing separability to weak separability.

Example 1. (Substitutable and weakly separable but not separable.)
The following substitutable preference relation P_f satisfies weak separability but not separability:

$$P_f : \{w_1, w_2\}, \{w_1\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_2\}, \emptyset. \tag{2}$$

One can think of the preferences in (2) as obtained from some production function. Workers w_1 and w_2 are 'productive' workers (as individuals and as a group), but worker w_3 is a nuisance that especially wreaks havoc on the productive workers' synergy. $\diamond$

[20] Responsiveness was introduced by Roth (1985).
[21] With a slight abuse of notation we sometimes write x for a singleton $\{x\}$.
[22] Separability was introduced by Martínez et al. (2000).
[23] Cardinal monotonicity was introduced by Alkan (2002).
[24] Quota-filling was introduced by Alkan (2001). Note that here we do explicitly mention substitutability, since unlike the other preference domains mentioned in this section, it is an integral part of the definition. Also note that $k_a \leq q_a$ and possibly $k_a \neq q_a$.

Figure 1 depicts all inclusion relations among the preference domains. For instance, responsiveness implies separability, cardinal monotonicity, and substitutability, but not quota-filling. The inclusion relations follow from Lemmas 1, 3, and 4 below. For each of the 13 numbered nodes in Figure 1 we provide in Example 2 a preference relation that pertains to the associated subdomain.

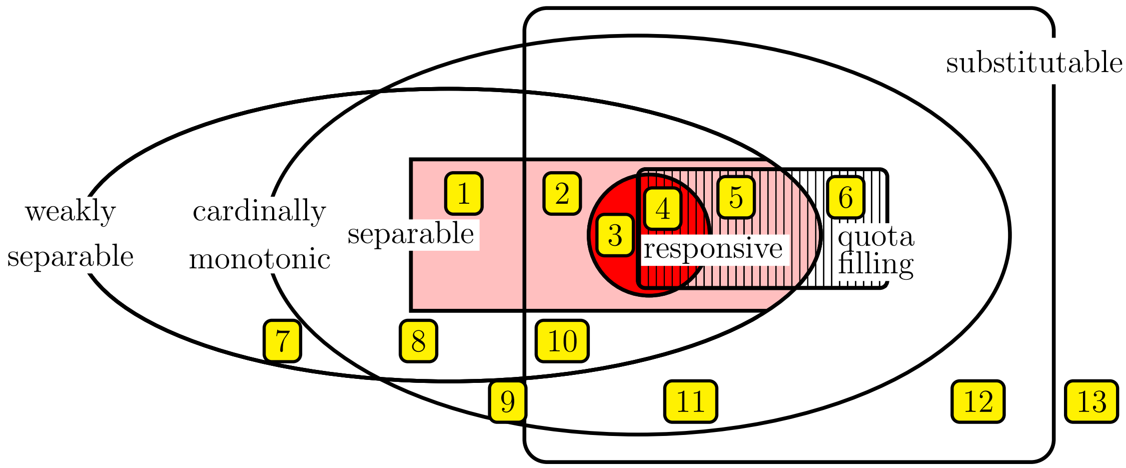

Figure 1: Inclusion relations among preference domains. Note: the domain of

- responsive preferences is depicted by the circle in the center;
- quota-filling preferences is depicted by the rectangle with vertical lines;
- separable preferences is depicted by the largest shaded area (which includes the circle).

Lemma 1. *Let $a \in A$.*
(i). *If P_a is responsive, then it is substitutable.*
(ii). *If P_a is responsive, then it is separable.*
(iii). *If P_a is separable, then it is weakly separable.*
(iv). *If P_a is quota-filling, then it is substitutable.*

Proof. Item (i) is well-known (see, e.g., Roth and Sotomayor, 1990, p.173). Items (ii) and (iii) are immediate. Item (iv) holds by definition of quota-filling. □

Lemma 2. *Let $a \in A$. Assume P_a is quota-filling. Then, for each $S \subseteq S_a$ with $|S| \leq k_a$, $\mathsf{Ch}(S, P_a) = S$.*

Proof. If $|S| = k_a$, then since P_a is quota-filling, $\mathsf{Ch}(S, P_a) = S$. Let $|S| < k_a$. Suppose $\mathsf{Ch}(S, P_a) \neq S$. Then, $\mathsf{Ch}(S, P_a) \subsetneq S$. Let $s \in S \backslash \mathsf{Ch}(S, P_a)$. By definition of k_a there exists $K \subseteq S_a$ such that $K \supseteq S$ and $|K| = k_a$. By quota-filling, $\mathsf{Ch}(K, P_a) = K$. So, $s \in \mathsf{Ch}(K, P_a)$. By Lemma 1(iv), $s \in \mathsf{Ch}(S, P_a)$, which is a contradiction. □

The third lemma will also be useful to prove our first main result in Section 4.

Lemma 3. *Let $a \in A$.*
(i). *If P_a is substitutable and weakly separable, then it is cardinally monotonic.*
(ii). *If P_a is separable, then it is cardinally monotonic.*
(iii). *If P_a is quota-filling, then it is cardinally monotonic.*

Proof. Let P_a violate cardinal monotonicity. Let $S', S \subseteq S_a$ be such that $S' \subseteq S$. Suppose that $|\mathsf{Ch}(S', P_a)| > |\mathsf{Ch}(S, P_a)|$. Then, $\mathsf{Ch}(S', P_a) \setminus \mathsf{Ch}(S, P_a) \neq \emptyset$. Let $s' \in \mathsf{Ch}(S', P_a) \setminus \mathsf{Ch}(S, P_a)$. Note $s' \in \mathsf{Ch}(S', P_a) \subseteq S' \subseteq S$.

(i). Let P_a satisfy weak separability. We show that it violates substitutability. Since $s' \in S$, it follows from the definition of Ch that $\mathsf{Ch}(S, P_a)\ P_a\ (\mathsf{Ch}(S, P_a) \cup \{s'\})$. Then, from $|\mathsf{Ch}(S, P_a)| < |\mathsf{Ch}(S', P_a)| \leq q_a$, $\mathsf{Ch}(\mathsf{Ch}(S, P_a), P_a) = \mathsf{Ch}(S, P_a)$, and weak separability, it follows that $\emptyset\ P_a\ s'$. Thus, $s' \in \mathsf{Ch}(S', P_a)$ but $\mathsf{Ch}(\{s'\}, P_a) \neq \{s'\}$. Hence, P_a violates substitutability.

(ii). Suppose $s'\, P_a\, \emptyset$. Note that $|\mathsf{Ch}(S, P_a)| < q_a$ and $\mathsf{Ch}(S, P_a)\, P_a\, (\mathsf{Ch}(S, P_a) \cup \{s'\})$. Then, P_a violates separability. Suppose now that $\emptyset\, P_a\, s'$. Note that $|\mathsf{Ch}(S', P_a) \backslash \{s'\}| < q_a$ and $\mathsf{Ch}(S', P_a)\, P_a\, (\mathsf{Ch}(S', P_a) \backslash \{s'\})$. Then, P_a violates separability.

(iii). Note that $|\mathsf{Ch}(S, P_a)| < |\mathsf{Ch}(S', P_a)| \leq k_a$. Suppose that $|S| \geq k_a$. Then, P_a is not quota-filling. Suppose now that $|S| < k_a$. Since $s' \in S$ and $s' \notin \mathsf{Ch}(S, P_a)$, we have $\mathsf{Ch}(S, P_a) \subsetneq S$. Hence, by Lemma 2, P_a is not quota-filling. □

Lemma 4. *Let $a \in A$. If P_a is quota-filling and weakly separable, then it is separable.*

Proof. If for all $S \subseteq S_a$, $\mathsf{Ch}(S, P_a) = \emptyset$, then $q_a = 0$ and separability is satisfied trivially. Suppose for some $S \subseteq S_a$, $\mathsf{Ch}(S, P_a) \neq \emptyset$. Then, $k_a \geq 1$. Let $S \subseteq S_a$ with $|S| < q_a$. Let $b \in S_a \backslash S$.

Suppose that $(S \cup \{b\})\, P_a\, S$. From Lemma 2, $\mathsf{Ch}(\{b\}, P_a) = \{b\}$. Thus, $b\, P_a\, \emptyset$.

Suppose now that $b\, P_a\, \emptyset$. It is sufficient to show that $\mathsf{Ch}(S, P_a) = S$. (Because then, by weak separability, $(S \cup \{b\})\, P_a\, S$.) Assume $\mathsf{Ch}(S, P_a) \neq S$. Then, $\mathsf{Ch}(S, P_a) \subsetneq S$. Let $K = \mathsf{Ch}(S, P_a)$. Note that $K = \mathsf{Ch}(K, P_a)$ and $|K| < |S| < q_a$. Let $s \in S \backslash K$. From Lemma 2 and $k_a \geq 1$, it follows that $\mathsf{Ch}(\{s\}, P_a) = \{s\}$. Thus, $s\, P_a\, \emptyset$. Then, from weak separability, $(K \cup \{s\})\, P_a\, K$, which contradicts $\mathsf{Ch}(S, P_a) = K$. □

Finally, for each 'non-empty area' in Figure 1 it is not difficult to construct preferences that are indeed in the associated domains (and not in the other domains).

Example 2. For each of the 13 numbered nodes in Figure 1 we provide, without loss of generality, a preference relation P_f of a firm f such that P_f pertains (only) to the associated domains.

1. $\{w_1, w_2\}, \{w_3, w_4\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \emptyset$.
2. $\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_3, w_4\}, \{w_2, w_4\}, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \emptyset$ and there is at least one other worker, say w_5.
3. $\{w_1, w_2\}, \{w_1\}, \{w_2\}, \emptyset, \{w_1, w_2, w_3\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}$ and there are no other workers.
4. $\{w_1, w_2\}, \{w_1\}, \{w_2\}, \emptyset$ and there are no other workers.

5. $\{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_4\}, \{w_2, w_4\}, \{w_3, w_4\}, \{w_2\}, \{w_3\}, \{w_4\}, \{w_1\}, \emptyset$
   and there are no other workers.
6. $\{w_1\}, \{w_1, w_2\}, \{w_2\}, \emptyset$
   and there are no other workers.
7. $\{w_1, w_4\}, \{w_2, w_3, w_4\}, \{w_4\}, \emptyset$.
8. $\{w_1, w_2, w_3, w_4\}, \{w_1\}, \{w_1, w_2, w_3\}, \emptyset$.
9. $\{w_2, w_3, w_4\}, \{w_1, w_4\}, \{w_2\}, \{w_4\}, \emptyset$.
10. $\{w_1, w_2\}, \{w_1\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_2\}, \emptyset$ (see Example 1).
11. $\{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\}, \emptyset, \{w_1, w_2\}$.
12. $\{w_1\}, \{w_2, w_4\}, \{w_2, w_3\}, \{w_4\}, \{w_3\}, \{w_2\}, \emptyset$.
13. $\{w_1, w_4\}, \{w_2, w_3, w_4\}, \{w_2\}, \{w_4\}, \emptyset$.

◇

4 Rural Hospital Theorem

The matching literature established the next two results for different domains of *substitutable* preferences in different models (one-to-one, many-to-one, and many-to-many matching).

Rural Hospital Theorem. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in A$,
R1. $|\mu(a)| = |\mu'(a)|$;
R2. $|\mu(a)| < q_a \implies \mu(a) = \mu'(a)$.

It is easy to verify that R2 implies R1. For many-to-many matching, R1 holds for substitutable and cardinally monotonic preferences. We state this result for later reference.

Theorem 2 (**Weak Rural Hospital Theorem.** Alkan, 2002, Proposition 6)**.** *For each profile P of substitutable and cardinally monotonic preferences, each agent is assigned to the same number of partners across stable matches at P.*

The literature has established R2 for

(i). many-to-*one* with substitutable and separable preferences (Martínez et al., 2000);
(ii). many-to-many with responsive preferences (Alkan, 1999);
(iii). many-to-many with quota-filling and separable preferences (Alkan, 2001).[25]

There are no logical relations among the three results (i), (ii), and (iii). But our first main result, which we present next, does subsume all three results. More precisely, we show that R2 holds for the domain of substitutable and weakly separable preferences. The proof uses the structural result stated in Theorem 1.

Theorem 3 (**Strong Rural Hospital Theorem**)**.** *For each profile P of substitutable and weakly separable preferences and each pair $\mu, \mu' \in \Sigma(P)$, if an agent does not fill her/its quota at μ, then she/it is assigned to the same set of partners at μ and μ'.*

[25]See Appendix B for further details on Alkan's (2001) result.

Proof. From Theorem 2 and Lemma 3(i) it follows that R1 holds for P. Hence, it suffices to show that for each $f \in F$ with $|\mu_F(f)| < q_f$, $\mu(f) \subseteq \mu_F(f)$. (Similar arguments can be used to show that for each $w \in W$ with $|\mu_W(w)| < q_w$, $\mu(w) \subseteq \mu_W(w)$.)

Let $f \in F$ with $|\mu_F(f)| < q_f$. Suppose $\mu(f) \not\subseteq \mu_F(f)$. Then, there is $w \in \mu(f)$ with $w \notin \mu_F(f)$. Since $\mu \in \Sigma(P)$, $\mathsf{Ch}(\mu(f), P_f) = \mu(f)$. Since $w \in \mu(f)$, it follows from substitutability that $\mathsf{Ch}(\{w\}, P_f) = \{w\}$. So, $w\, P_f\, \emptyset$. Hence, from $|\mu_F(f)| < q_f$, $\mathsf{Ch}(\mu_F(f), P_f) = \mu_F(f)$, and weak separability, it follows that

$$(\mu_F(f) \cup \{w\})\, P_f\, \mu_F(f) \overset{\text{Th.1(i)}}{=} Ch(\mu_F(f) \cup \mu(f), P_f), \tag{3}$$

which, since $w \in \mu(f)$, contradicts the consistency of Ch. More precisely, (3) shows that (1) is violated for $S = \mu_F(f) \cup \mu(f)$ and $T = \mu_F(f) \cup \{w\}$. Hence, $\mu(f) \subseteq \mu_F(f)$. ☐

Corollary 1. *Let Ψ be a solution concept for many-to-many matching such that for each profile of substitutable preferences P, $\Psi(P) \subseteq \Sigma(P)$ (see, e.g., Remark 5). Then, for each profile of substitutable and cardinally monotonic preferences P, the counterpart of R1 defined for matches in $\Psi(P)$ holds. Similarly, for each profile of substitutable and weakly separable preferences P, the counterpart of R2 defined for matches in $\Psi(P)$ holds.*

Remark 5. Theorem 3 holds for stronger solution concepts, e.g., setwise stability (Sotomayor, 1999), strong setwise stability, and weak setwise stability (Klaus and Walzl, 2009, Theorem 1). Moreover, since we assume substitutability for all our results but Proposition 1, (pairwise) stability in fact coincides with some of these and other solution concepts as well (Echenique and Oviedo, 2006, Section 6; Klaus and Walzl, 2009, Theorem 2). ◇

We complement Theorems 2 and 3 with two maximality results. Recall that we assume throughout that preferences are substitutable. Let R be a property for preference profiles. A domain D of substitutable preferences is maximal for property R if (m1) each profile that consists of preferences in D satisfies R and (m2) whenever *some* agent's preferences are not in D, there are preferences in D for all other agents such that the resulting preference profile does not satisfy R. Formally, a **domain $\boldsymbol{D}$ of substitutable preferences is maximal for property R** if

(m1). whenever for each $a \in A$, $P_a \in D$, preference profile $(P_a)_{a \in A}$ satisfies property R, and

(m2). whenever for some $a \in A$, $P_a \notin D$, there are $(P_b)_{b \in A \setminus a}$ such that for each $b \in A \setminus a$, $P_b \in D$, and preference profile $(P_a)_{a \in A}$ does not satisfy property R.

With respect to Theorem 2, we prove that cardinal monotonicity is maximal for R1. In fact, we prove a stronger result: if some agent's preferences are not cardinally monotonic,[26] then there are *responsive* preferences for the other agents such that R1 fails (Proposition 1).

[26]Also note that we do *not* assume that this agent's preferences satisfy substitutability. Obviously, this strengthens the maximality result and also implies that we cannot recur to pairwise stability as it may no longer coincide with stability.

Proposition 1 (**Maximality of Weak Rural Hospital Theorem**). *Suppose* $|F|, |W| \geq 2$. *Suppose some firm or some worker has preferences that are not cardinally monotonic. Then, there are responsive preferences for the other agents such that R1 fails. Thus, cardinal monotonicity is maximal for R1.*

Proof. Assume, without loss of generality, $P_{f_1} \in \mathcal{P}_{f_1}$ violates cardinal monotonicity. Then, for some $S' \subseteq S \subseteq W$, $|\mathsf{Ch}(S', P_{f_1})| > |\mathsf{Ch}(S, P_{f_1})|$. Hence,

$$\mathsf{Ch}(S', P_{f_1}) \backslash \mathsf{Ch}(S, P_{f_1}) \neq \emptyset. \tag{4}$$

We first show that $\mathsf{Ch}(S, P_{f_1}) \backslash \mathsf{Ch}(S', P_{f_1}) \neq \emptyset$. Assume otherwise. Then, $\mathsf{Ch}(S, P_{f_1}) \subseteq \mathsf{Ch}(S', P_{f_1})$. Since $\mathsf{Ch}(S', P_{f_1}) \subseteq S'$, we have $\mathsf{Ch}(S, P_{f_1}) \subseteq S'$. Hence, from (1), $\mathsf{Ch}(S', P_{f_1}) = \mathsf{Ch}(S, P_{f_1})$, which contradicts $|\mathsf{Ch}(S', P_{f_1})| > |\mathsf{Ch}(S, P_{f_1})|$. So, indeed,

$$\mathsf{Ch}(S, P_{f_1}) \backslash \mathsf{Ch}(S', P_{f_1}) \neq \emptyset. \tag{5}$$

For the construction of a preference profile of the other agents, we introduce the partition $\{W^A, W^B, W^C\}$ of the workers in $\mathsf{Ch}(S, P_{f_1}) \cup \mathsf{Ch}(S', P_{f_1})$ by defining

$$\begin{aligned} W^A &\equiv \mathsf{Ch}(S, P_{f_1}) \cap \mathsf{Ch}(S', P_{f_1}), \\ W^B &\equiv \mathsf{Ch}(S, P_{f_1}) \setminus \mathsf{Ch}(S', P_{f_1}), \text{ and} \\ W^C &\equiv \mathsf{Ch}(S', P_{f_1}) \setminus \mathsf{Ch}(S, P_{f_1}). \end{aligned}$$

From (5) and (4) it follows that $W^B \neq \emptyset$ and $W^C \neq \emptyset$, respectively. In particular, $|W^C| \geq 1$. (It is possible that $W^A = \emptyset$.) Figure 2 depicts the partition.

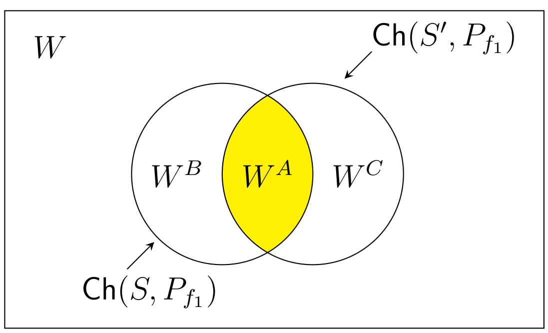

Figure 2: The partition $\{W^A, W^B, W^C\}$ of the workers in $\mathsf{Ch}(S, P_{f_1}) \cup \mathsf{Ch}(S', P_{f_1})$.

We choose the preferences of the agents distinct from f_1 as follows. Since $|F| \geq 2$, we can fix some $f_2 \in F$, $f_2 \neq f_1$. Then, we choose the preferences of f_2 and those of any firm in $F \backslash \{f_1, f_2\}$ as follows.

- P_{f_2} is responsive with $q_{f_2} \equiv |W^C|$ and for each $w^C \in W^C$, each $w^B \in W^B$, and each $w \in W \backslash (W^B \cup W^C)$,

$$\{w^C\} \; P_{f_2} \; \{w^B\} \; P_{f_2} \; \emptyset \; P_{f_2} \; \{w\};$$

- $P_f : \emptyset$ for each $f \in F \backslash \{f_1, f_2\}$.

Note that by responsiveness and construction of P_{f_2},

$$\mathsf{Ch}(W, P_{f_2}) = W^C.$$

The preferences of the workers are given by Table 1. In particular, each worker in $W^A \cup W^B \cup W^C$ has quota 1 and any other worker has quota 0. Obviously, for each $a \in A \backslash \{f_1\}$, P_a is responsive.

Workers			
$w^A \in W^A$	$w^B \in W^B$	$w^C \in W^C$	$w \in W \backslash (W^A \cup W^B \cup W^C)$
$\boxed{\boldsymbol{f_1}}$	$\boldsymbol{f_2}$	$\boldsymbol{f_1}$	$\boxed{\boldsymbol{\emptyset}}$
$\emptyset$	$\boxed{f_1}$	$\boxed{f_2}$	
	$\emptyset$	$\emptyset$	

Table 1: Preferences of the workers

To show that R1 fails for preference profile P we consider the two matches μ_1 and μ_2 provided in Table 2. In Table 1, match μ_1 is indicated by boldface font and match μ_2 by boxes.

	f_1	f_2	$f \in F \backslash \{f_1, f_2\}$
$\boldsymbol{\mu_1}$	$W^A \cup W^C$	W^B	$\emptyset$
$\boxed{\mu_2}$	$W^A \cup W^B$	W^C	$\emptyset$

Table 2: Matches μ_1 and μ_2

From

$$|W^A \cup W^C| = |\mathsf{Ch}(S', P_{f_1})| > |\mathsf{Ch}(S, P_{f_1})| = |W^A \cup W^B|, \tag{6}$$

it follows that $|\mu_1(f_1)| \neq |\mu_2(f_1)|$. Therefore, to complete the proof that R1 fails for P, it suffices to show that $\mu_1, \mu_2 \in \Sigma(P)$.

We first show that $\mu_1 \in \Sigma(P)$. Since $\mu_1(f_1) = W^A \cup W^C = \mathsf{Ch}(S', P_{f_1})$, it follows from (1) that $\mathsf{Ch}(\mu_1(f_1), P_{f_1}) = \mu_1(f_1)$, i.e., μ_1 is not blocked by f_1. From (6) and the fact that W^A, W^B, and W^C are mutually disjoint sets, we have

$$|W^C| > |W^B|.$$

Hence, $|\mu_1(f_2)| = |W^B| < |W^C| = q_{f_2}$. Then, by responsiveness of P_{f_2}, $\mathsf{Ch}(\mu_1(f_2), P_{f_2}) = \mu_1(f_2)$ as well. One easily verifies that no other individual agent blocks μ_1 either. In fact, since each worker is matched to her most preferred subset of firms at μ_1, it follows from Remark 2 that no worker is part of a blocking set for μ_1. Hence, $\mu_1 \in \Sigma(P)$.

Finally, we show that $\mu_2 \in \Sigma(P)$. Using similar arguments as before, it is easy to see that no individual agent blocks μ_2. Now, assume that a set of firms and workers $F' \cup W'$

with $F', W' \neq \emptyset$ blocks μ_2. Since each $f \in F \setminus \{f_1\}$ and each $w \in W \setminus (W^B \cup W^C)$ is matched to its/her most preferred subset of partners at μ_2, it follows from Remark 2 that $F' = \{f_1\}$ and $W' \subseteq (W^B \cup W^C)$.

From Remark 1, blocking by $F' = \{f_1\}$ and $W' \subseteq (W^B \cup W^C)$ is equivalent to the following: (1*) for each $w \in W'$, $f_1 \notin \mu_2(w)$ and $f_1 \in \mathsf{Ch}(\mu_2(w) \cup \{f_1\}, P_w)$ and (2*) $W' \subseteq \mathsf{Ch}(\mu_2(f_1) \cup W', P_{f_1})$.

From (2*) and $\mu_2(f_1) = W^A \cup W^B = \mathsf{Ch}(S, P_{f_1})$,

$$W' \subseteq \mathsf{Ch}([\mathsf{Ch}(S, P_{f_1}) \cup W'], P_{f_1}). \tag{7}$$

By $W' \subseteq (W^B \cup W^C)$, $W^B \subseteq \mu_2(f_1)$ and (1*), we have $W' \subseteq W^C$. From $W' \subseteq W^C$ and $W^C \subseteq S' \subseteq S$, it follows that $W' \subseteq S$. By $\mathsf{Ch}(S, P_{f_1}) \subseteq S$ and $W' \subseteq S$, we have $[\mathsf{Ch}(S, P_{f_1}) \cup W'] \subseteq S$. Hence, from (1) with $T = [\mathsf{Ch}(S, P_{f_1}) \cup W']$,

$$\mathsf{Ch}([\mathsf{Ch}(S, P_{f_1}) \cup W'], P_{f_1}) = \mathsf{Ch}(S, P_{f_1}). \tag{8}$$

By (7), (8) and $W' \subseteq W^C$, we have $W^C \cap \mathsf{Ch}(S, P_{f_1}) \neq \emptyset$, which contradicts the definition of W^C. Therefore, there is no set of firms and workers that blocks μ_2. Hence, $\mu_2 \in \Sigma(P)$. □

Next, we complement Theorem 3 by proving that weak separability is maximal for R2. In this case we also prove a stronger result: if some agent has substitutable preferences that are not weakly separable, then there are *responsive* preferences for the other agents such that R2 fails (Proposition 2).

Proposition 2 (Maximality of Strong Rural Hospital Theorem). *Suppose $|F|$, $|W| \geq 2$. Suppose some firm or some worker has preferences that are substitutable but not weakly separable. Then, there are responsive preferences for the other agents such that R2 fails. Thus, weak separability is maximal for R2.*

Proof. Assume, without loss of generality, $P_{f_1} \in \mathcal{P}_{f_1}$ satisfies substitutability but violates weak separability. Suppose that P_{f_1} violates cardinal monotonicity. Then, by Proposition 1, there is a profile of responsive preferences for the other agents such that R1 fails. Since R2 implies R1, R2 fails too.

Now suppose that P_{f_1} satisfies cardinal monotonicity. Since P_{f_1} violates weak separability, there are $S \subseteq W$ and $w_1 \notin S$ with $|S| < q_{f_1}$, $\mathsf{Ch}(S, P_{f_1}) = S$, $w_1 \, P_{f_1} \, \emptyset$, and $S \, P_{f_1} \, (S \cup \{w_1\})$.

Case I: $w_1 \in \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$.

Since $S \, P_{f_1} \, (S \cup \{w_1\})$, we have $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \neq (S \cup \{w_1\})$. In particular, $|\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})| \leq |S|$. On the other hand, by cardinal monotonicity, $|\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})| \geq |\mathsf{Ch}(S, P_{f_1})| = |S|$. Hence, $|\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})| = |S|$. Note that $w_1 \in \mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \subseteq (S \cup \{w_1\})$ and $w_1 \notin S$. Hence, by setting $B = \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$, $C = S$, and $b = w_1$ in Lemma 5 in Appendix A it follows that

$$|\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \setminus S| = |S \setminus \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})| = 1$$

and $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})\backslash S = \{w_1\}$. For the construction of a preference profile of the other agents, we first define

$$\begin{aligned} \{w_2\} &\equiv S \setminus \mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \text{ and} \\ W^A &\equiv S \cap \mathsf{Ch}(S \cup \{w_1\}, P_{f_1}). \end{aligned}$$

Figure 3 provides a graphical summary.

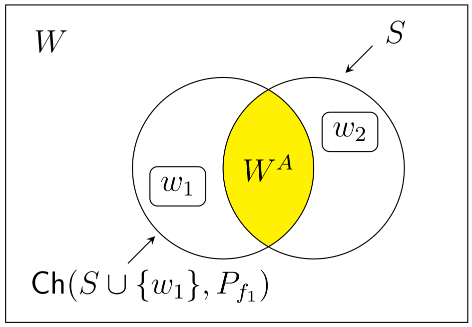

Figure 3: The sets of workers S and $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$ in Case I

Since $|F| \geq 2$, we can fix some $f_2 \in F$, $f_2 \neq f_1$. Then, we choose the preferences of the agents different from f_1 as in Table 3. Obviously, for each $a \in A\backslash\{f_1\}$, P_a is responsive.

Firms		Workers			
f_2	$f \in F\backslash\{f_1, f_2\}$	w_1	w_2	$w \in W^A$	$w \in W\backslash(\{w_1, w_2\} \cup W^A)$
$\boxed{w_2}$	$\boxed{\boldsymbol{\emptyset}}$	$\boldsymbol{f_2}$	$\boldsymbol{f_1}$	$\boxed{\boldsymbol{f_1}}$	$\boxed{\boldsymbol{\emptyset}}$
$\boldsymbol{w_1}$		$\boxed{f_1}$	$\boxed{f_2}$	$\emptyset$	
$\emptyset$		$\emptyset$	$\emptyset$		

Table 3: Preferences of the agents distinct from f_1 in Case I

To show that R2 fails for preference profile P we consider the two matches μ_1 and μ_2 in Table 4. In Table 3, match μ_1 is indicated by boldface font and match μ_2 by boxes.

	f_1	f_2	$f \in F\backslash\{f_1, f_2\}$
$\boldsymbol{\mu_1}$	S	$\{w_1\}$	$\emptyset$
$\boxed{\mu_2}$	$\mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$	$\{w_2\}$	$\emptyset$

Table 4: Matches μ_1 and μ_2 in Case I

We now complete the proof of Case I. Since $|\mu_1(f_1)| = |S| < q_{f_1}$ and $\mu_1(f_1) \neq \mu_2(f_1)$, it only remains to prove that $\mu_1, \mu_2 \in \Sigma(P)$. By Remark 4, it suffices to show that no individual agent nor firm-worker pair blocks μ_1 or μ_2.

We first show that $\mu_1 \in \Sigma(P)$. By assumption, $\mathsf{Ch}(S, P_{f_1}) = S$. Hence, f_1 does not block μ_1. One easily verifies that no other agent blocks μ_1 either. In fact, since each worker is matched to her most preferred subset of firms at μ_1, no worker is part of a blocking pair for μ_1. Hence, $\mu_1 \in \Sigma(P)$.

Finally, we show that $\mu_2 \in \Sigma(P)$. Note that no agent blocks μ_2. Now, assume that a firm-worker pair $(\hat{f}, \hat{w})$ blocks μ_2. Since each $f \in F \backslash \{f_1\}$ is matched to its most preferred subset of workers at μ_2, it follows that $\hat{f} = f_1$. The only worker w for which $f_1 \notin \mu_2(w)$ and $f_1 \in \mathsf{Ch}(\mu_2(w) \cup \{f_1\}, P_w)$ is $w = w_2$. Hence, by Remark 1, $\hat{w} = w_2$. Then, since (f_1, w_2) blocks μ_2,

$$w_2 \in \mathsf{Ch}(\mu_2(f_1) \cup \{w_2\}, P_{f_1}). \tag{9}$$

On the other hand, since $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \subseteq \mu_2(f_1) \cup \{w_2\} \subseteq S \cup \{w_1\}$, (1) applied to $\mu_2(f_1) \cup \{w_2\}$ and $S \cup \{w_1\}$ yields $\mathsf{Ch}(\mu_2(f_1) \cup \{w_2\}, P_{f_1}) = \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$. Since $w_2 \notin \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$, we have $w_2 \notin \mathsf{Ch}(\mu_2(f_1) \cup \{w_2\}, P_{f_1})$, which contradicts (9). Hence, there is no blocking pair for μ_2. Hence, $\mu_2 \in \Sigma(P)$.

Case II: $w_1 \notin \mathsf{Ch}(S \cup \{w_1\}, P_{f_1})$.
From $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) \subseteq S$ and (1) applied to S and $S \cup \{w_1\}$,

$$\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) = \mathsf{Ch}(S, P_{f_1}) = S. \tag{10}$$

Let $S' \subseteq S$ be such that

(a) $\mathsf{Ch}(S' \cup \{w_1\}, P_{f_1}) = S' \cup \{w_1\}$ and

(b) $S' \subsetneq S'' \subseteq S \implies \mathsf{Ch}(S'' \cup \{w_1\}, P_{f_1}) \neq S'' \cup \{w_1\}$.

Since $w_1 \, P_{f_1} \, \emptyset$, we have $\mathsf{Ch}(\{w_1\}, P_{f_1}) = \{w_1\}$. Hence, there exists a subset of S that satisfies (a), namely the empty set. Hence, S' is well-defined. From $w_1 \notin S$ and (10), it follows that $S' \neq S$. Thus, $S' \subsetneq S$. This implies that $S \backslash S' \neq \emptyset$. Let $w_2 \in S \backslash S'$. Figure 4 depicts the sets S and S' together with w_1 and w_2.

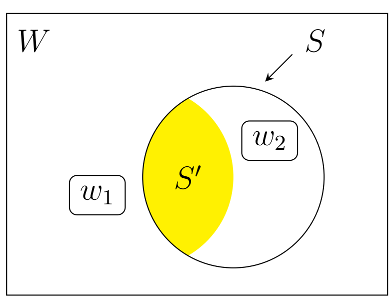

Figure 4: The sets S and S' and workers w_1 and w_2 in Case II

From substitutability of P_{f_1} and $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) = S$ it follows that

$$\mathsf{Ch}(S' \cup \{w_2\}, P_{f_1}) = S' \cup \{w_2\}. \tag{11}$$

Next, we show that $w_1 \notin \mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1})$. Assume by contradiction that

$$w_1 \in \mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}). \tag{12}$$

From substitutability of P_{f_1}, $\mathsf{Ch}(S \cup \{w_1\}, P_{f_1}) = S$, and $S' \cup \{w_2\} \subseteq S$ it follows that

$$S' \cup \{w_2\} \subseteq \mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}). \tag{13}$$

Then, from $\mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}) \subseteq S' \cup \{w_1, w_2\}$, (12), and (13) it follows that $\mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}) = S' \cup \{w_1, w_2\}$, which is a contradiction to (b) with $S'' = S' \cup \{w_2\}$. Hence,

$$w_1 \notin \mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}). \tag{14}$$

Since $|F| \geq 2$, we can fix some $f_2 \in F$, $f_2 \neq f_1$. Then, we choose the preferences of the agents different from f_1 as in Table 5. Obviously, for each $a \in A \backslash \{f_1\}$, P_a is responsive.

Firms		Workers			
f_2	$f \in F\backslash\{f_1, f_2\}$	w_1	w_2	$w \in S'$	$w \in W\backslash(\{w_1, w_2\} \cup S')$
$\boldsymbol{w_1}$	$\boxed{\boldsymbol{\emptyset}}$	$\boxed{f_1}$	$\boxed{f_2}$	$\boxed{\boldsymbol{f_1}}$	$\boxed{\boldsymbol{\emptyset}}$
$\boxed{w_2}$		$\boldsymbol{f_2}$	$\boldsymbol{f_1}$	$\emptyset$	
$\emptyset$		$\emptyset$	$\emptyset$		

Table 5: Preferences of the agents distinct from f_1 in Case II

To show that R2 fails for preference profile P we consider the two matches μ_1 and μ_2 in Table 6. In Table 5, match μ_1 is indicated by boldface font and match μ_2 by boxes.

	f_1	f_2	$f \in F\backslash\{f_1, f_2\}$
$\boldsymbol{\mu_1}$	$S' \cup \{w_2\}$	$\{w_1\}$	$\emptyset$
$\boxed{\mu_2}$	$S' \cup \{w_1\}$	$\{w_2\}$	$\emptyset$

Table 6: Matches μ_1 and μ_2 in Case II

We now complete the proof of Case II. Since $|\mu_1(f_1)| = |S' \cup \{w_2\}| \leq |S| < q_{f_1}$ and $\mu_1(f_1) \neq \mu_2(f_1)$, it only remains to prove that $\mu_1, \mu_2 \in \Sigma(P)$. By Remark 4 it is sufficient to discard blocking by individual agents and firm-worker pairs.

We first show that $\mu_1 \in \Sigma(P)$. From (11), f_1 does not block μ_1. One easily verifies that no other individual agent blocks μ_1 either. Now, assume that a firm-worker pair $(\hat{f}, \hat{w})$ blocks μ_1. Since each $f \in F\backslash\{f_1\}$ is matched to its most preferred subset of workers at μ_1, it follows that $\hat{f} = f_1$. The only worker w for which $f_1 \notin \mu_1(w)$ and

$f_1 \in \mathsf{Ch}(\mu_1(w) \cup \{f_1\}, P_w)$ is $w = w_1$. Hence, by Remark 1, $\hat{w} = w_1$. Then, since (f_1, w_1) blocks μ_1, we have $w_1 \in \mathsf{Ch}(\mu_1(f_1) \cup \{w_1\}, P_{f_1})$. Then, from $\mu_1(f_1) = S' \cup \{w_2\}$,

$$w_1 \in \mathsf{Ch}(S' \cup \{w_1, w_2\}, P_{f_1}),$$

which contradicts (14). Hence, there is no blocking pair for μ_1. Hence, $\mu_1 \in \Sigma(P)$.

Finally, we show that $\mu_2 \in \Sigma(P)$. From (a), f_1 does not block μ_2. One easily verifies that no other individual agent blocks μ_2 either. In fact, since each worker is matched to her most preferred subset of firms at μ_2, no worker is part of a blocking pair for μ_2. Hence, $\mu_2 \in \Sigma(P)$. $\square$

We conclude with three remarks about Propositions 1 and 2.

Remark 6 (Cardinalities $|F|$ and $|W|$). In Propositions 1 and 2 we assume that $|F|, |W| \geq 2$. If $|F| = 1$ or $|W| = 1$, then the conclusions in Propositions 1 and 2 need not hold. To see this, suppose, without loss of generality, that $|F| = 1$ (the case $|W| = 1$ is symmetric). Let f be the unique firm.

Consider Proposition 1.

- Suppose $|W| \geq 3$. Let the firm's preferences be given by $P_f : \{w_1\}, \{w_2, w_3\}, \{w_2\}, \emptyset$. Then, P_f is not cardinally monotonic. Consider any possible preferences of the other agents (i.e., all workers). Since there is a unique firm, all workers' preferences are responsive. Moreover, it is not difficult to prove that R1 does *not* fail. Therefore, the conclusion in Proposition 1 that R1 fails is not true.

- Suppose $|W| \leq 2$. Then, one can easily show that any possible preferences of the firm and the worker(s) are cardinally monotonic. Hence, the assumption that 'some firm or some worker has preferences that are not cardinally monotonic' (in Proposition 1) cannot be fulfilled.

Next, consider Proposition 2.

- Suppose $|W| \geq 2$. Let the firm's preferences be given by $P_f : \{w_1\}, \{w_2\}, \{w_1, w_2\}, \emptyset$. Then, P_f is substitutable but not weakly separable. Consider any possible preferences of the other agents (i.e., all workers). Since there is a unique firm, all workers' preferences are responsive. Moreover, it is not difficult to prove that R2 does *not* fail. Therefore, the conclusion in Proposition 2 that R2 fails is not true.

- Suppose $|W| \leq 1$. Then, any possible preferences of the firm and the worker are weakly separable. Hence, the assumption that 'some firm or some worker has preferences that are not weakly separable' (in Proposition 2) cannot be fulfilled.

$\diamond$

Remark 7 (Maximality for Many-to-One). In the proof of Propositions 1 and 2 we fix the preferences of a firm and construct preferences of the other agents. The construction is such that all workers have a quota of at most 1. Hence, the proofs are also effective for the many-to-one framework. Therefore, the two maximality results also hold for the many-to-one framework. In fact, the restriction of Propositions 1 and 2 to the many-to-one framework are novel for that framework as well. ◇

Remark 8 (Existing Maximality Results of the Weak Rural Hospital Theorem). There exist two maximality results of the weak rural hospital theorem that are related to Proposition 1: Hatfield and Milgrom (2005, Theorem 9) and Hatfield and Kominers (2012b, Theorem 9). Below we discuss these related results and in particular show that neither of them implies Proposition 1.

Hatfield and Milgrom (2005) studied many-to-one matching with contracts. They proved R1 for substitutable and cardinally monotonic preferences (Theorem 8). Moreover, they showed that cardinal monotonicity is a maximal domain for R1. More precisely, if some agent's preferences violate cardinal monotonicity but do satisfy substitutability, then there are preferences with a quota of at most 1 for all other agents such that R1 fails (Theorem 9). The latter result differs from our Proposition 1 in two respects. First, in contrast to Hatfield and Milgrom (2005, Theorem 9), we do not make nor need the assumption that preferences are substitutable. Second, in contrast to Hatfield and Milgrom (2005, Theorem 9), in our construction (only) one of the other agents has preferences with a quota larger than 1. Therefore, in the framework without contracts, Proposition 1 does not imply nor is implied by Hatfield and Milgrom (2005, Theorem 9).

In the many-to-many matching model of Hatfield and Kominers (2012b), agents trade goods through contracts that specify a seller, a buyer, and terms of exchange. An essential feature of their model is that an agent may be a buyer in some contracts while a seller in some others. The restriction of Theorem 9 in Hatfield and Kominers (2012b) to the two-sided many-to-many model without contracts boils down to the following. If some agent's preferences violate cardinal monotonicity but do satisfy substitutability, then there are substitutable and cardinally monotonic preferences for the other agents such that R1 fails. Proposition 1 is distinct from Theorem 9 in at least three respects. First, Theorem 9 assumes that the set of contracts is exhaustive, i.e., any two agents can be part of a contract. Clearly, exhaustiveness does not hold in two-sided models such as ours, as the definition of a match does not permit partnerships between agents on the same side of the market. Second, in each case in the proof of Theorem 9, the failure of R1 for the agent whose preferences violate cardinal monotonicity relies on the assumption that this agent's preferences are substitutable. However, Proposition 1 does not make this assumption. Third, in contrast to Theorem 9, Proposition 1 shows that R1 can even fail when the other agents' preferences have a very simple structure (i.e., they are responsive). In fact, the proof of Proposition 1 only requires that one of the other agents has quota 2 (all other agents have a quota of at most 1). Therefore, in the framework without contracts, Proposition 1 is not implied by Hatfield and Milgrom (2005, Theorem 9). ◇

A Lemma for Proof of Proposition 2

Below we state and prove the lemma that is used in the proof of Proposition 2.

Lemma 5. *Let B and C be sets such that $|B| = |C|$ and for some $b \notin C$, $b \in B \subseteq C \cup \{b\}$. Then, $|B \backslash C| = |C \backslash B| = 1$ and $B \backslash C = \{b\}$.*

Proof. Since $B \subseteq C \cup \{b\}$, $B \backslash \{b\} \subseteq C$. So, $(B \backslash \{b\}) \backslash C = \emptyset$. Hence, since $b \in B \backslash C$, we have $B \backslash C = \{b\}$. In particular, $|B \backslash C| = 1$.

We now show that $|C \backslash B| = 1$. Note that $\{B \backslash C, C \backslash B, B \cap C\}$ is a partition of $B \cup C$. Hence,

$$|B \cup C| = |B \backslash C| + |C \backslash B| + |B \cap C|. \tag{15}$$

Since $B \subseteq C \cup \{b\}$ and $b \in B \backslash C$, we have $B \cup C = C \cup \{b\}$ and $B \cap C = B \backslash \{b\}$. Then, equation (15) yields

$$|C \cup \{b\}| = |B \backslash C| + |C \backslash B| + |B \backslash \{b\}|. \tag{16}$$

Given that $|B \backslash C| = 1$, equation (16) can be written as

$$|C| + 1 = 1 + |C \backslash B| + |B \backslash \{b\}|.$$

So, $|C| = |C \backslash B| + |B \backslash \{b\}|$. Or equivalently, $|C| - |B \backslash \{b\}| = |C \backslash B|$. Hence,

$$|C| - (|B| - 1) = |C \backslash B|. \tag{17}$$

From equation (17) and $|B| = |C|$, it follows that $|C \backslash B| = 1$. □

B Quota-filling Preferences

In this Appendix we discuss how Alkan (2001, Corollary 1) relates to R2. For convenience, we first state and prove a lemma.

Lemma 6. *Let $a \in A$. If P_a is quota-filling and separable, then $q_a = k_a$.*

Proof. Suppose $k_a = 0$. Then, a finds all sets of agents $S \subseteq S_a$ unacceptable. Hence, $q_a = 0$. So, $q_a = k_a$.

Let $k_a \geq 1$. Suppose by contradiction that $q_a \neq k_a$. Then, since $q_a \geq k_a$, we have $q_a > k_a$. By definition of q_a, there exists some $S \subseteq S_a$ such that $|S| = q_a$ and $S \, P_a \, \emptyset$. Let $T = \mathsf{Ch}(S, P_a)$. Since P_a is quota-filling and $|S| \geq k_a$, we have $|T| = k_a$. Then, since $|S| = q_a > k_a = |T|$, we have $S \backslash T \neq \emptyset$. Let $b \in S \backslash T$.

Suppose $b \, P_a \, \emptyset$. Then, from separability and $|T| < q_a$, it follows that $(T \cup \{b\}) \, P_a \, T$, which is in contradiction with $T = \mathsf{Ch}(S, P_a)$. Hence, $\emptyset \, P_a \, b$. Since $k_a \geq 1$, Lemma 2 yields $\mathsf{Ch}(\{b\}, P_a) = \{b\}$, which is in contradiction with $\emptyset \, P_a \, b$. Hence, $q_a = k_a$. □

Corollary 1 in Alkan (2001) shows that for each profile of quota-filling preferences P and each pair $\mu, \mu' \in \Sigma(P)$,

R2*. $|\mu(a)| < k_a \implies \mu(a) = \mu'(a)$.

From Lemma 6 it follows that on the domain of quota-filling and separable preferences, for each agent a, $k_a = q_a$, and hence R2* coincides with R2. However, as item 6 in Example 2 shows, if for some agent a, P_a is not separable, then possibly $k_a \neq q_a$. In fact, the following example shows that in this case R2 does not necessarily hold.

Example 3. There are two firms and three workers and their preferences P are given in Table 7. Obviously, for each $a \in A \backslash \{f_1\}$, P_a is quota-filling and separable. Also, P_{f_1} is quota-filling but not separable.

Firms		Workers		
f_1	f_2	w_1	w_2	w_3
$\boldsymbol{\{w_1, w_2\}}$	$\boldsymbol{w_3}$	$\boxed{f_2}$	$\boxed{\boldsymbol{f_1}}$	$\boxed{f_1}$
$\{w_1, w_3\}$	$\boxed{w_1}$	$\boldsymbol{f_1}$	f_2	$\boldsymbol{f_2}$
$\boxed{\{w_2, w_3\}}$	w_2	$\emptyset$	$\emptyset$	$\emptyset$
$\{w_1, w_2, w_3\}$	$\emptyset$			
$\{w_1\}$				
$\{w_2\}$				
$\{w_3\}$				
$\emptyset$				

Table 7: Preferences in Example 3

Consider the two matches μ_1 and μ_2 in Table 8. In Table 7, match μ_1 is indicated by boldface font and match μ_2 by boxes. One easily verifies that μ_1 and μ_2 are stable at P. Notice that $q_{f_1} = 3$, $|\mu_1(f_1)| = |\mu_2(f_1)|$, and $\mu_1(f_1) \neq \mu_2(f_1)$. Hence, R2 does not hold for P. ◇

	f_1	f_2
$\boldsymbol{\mu_1}$	$\{w_1, w_2\}$	$\{w_3\}$
$\boxed{\mu_2}$	$\{w_2, w_3\}$	$\{w_1\}$

Table 8: Matches μ_1 and μ_2 in Example 3

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