Quantum Critical Transport and the Hall Angle in Holographic Models

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We study the Hall conductivity in holographic models where translational invariance is broken by a lattice. We show that generic holographic theories will display a different temperature dependence in the Hall angle as to the dc conductivity. Our results suggest a general mechanism for obtaining an anomalous scaling of the Hall angle in strongly interacting quantum critical systems.

Introduction.—Many interesting experimental systems are governed by quantum criticality [1]. In particular, it has long been suggested that the anomalous scaling seen in the transport properties of the strange metal phase is indicative of quantum critical behavior [2–5]. Unfortunately, performing explicit calculations of strongly interacting transport is an extremely challenging task. As a result, many of the experimental anomalies remain as mysterious as ever.

Particularly puzzling is the scaling of the Hall angle. Simple models of transport, such as the Drude model, suggest that the Hall angle θ_H should be proportional to the dc conductivity σ_{dc}. However, experimental measurements of the Hall angle in the cuprates show a Fermi liquid behavior $\theta_H \sim 1/T^2$, which is hard to reconcile with the ubiquitous linear resistivity of strange metals [6].

Previous theoretical attempts to explain this behavior begin by proposing that transport in these materials is governed by two different relaxation times. In Ref. [7] it is suggested that spinons and holons scatter at different rates, whereas in Ref. [8] it is charge conjugation odd and even quasiparticles that behave independently.

In this Letter, we use holography to study the Hall angle in strongly interacting critical theories. In particular, we obtain simple analytic expressions for the conductivity tensor of a large class of holographic lattice models in the presence of a magnetic field. Our calculations indicate that within these holographic models the Hall angle will generically display a different temperature dependence to the dc conductivity.

Furthermore, these results suggest a general mechanism for obtaining an anomalous scaling in the Hall angle that can apply beyond these holographic models. The essential feature is the existence of two distinct terms that are additive, following an "inverse-Matthiessen" law, in the conductivity. First, there is a "charge-conjugation symmetric" contribution, denoted by σ_{ccs}, which is analogous to that found at relativistic quantum critical points. In addition, there is a contribution to the current from any net charge density. This current is relaxed by the effects of momentum dissipation and results in a second term in the conductivity, σ_{diss}.

In contrast, we find that there is only a single term that controls the scaling of the Hall angle, which is proportional to σ_{diss}. In particular, there is no additive contribution to θ_H analogous to the charge-conjugation symmetric term in σ_{dc}. When momentum relaxation is strong, we therefore find that the Hall angle and dc conductivity can be very different. In this regime, the charge-conjugation symmetric contribution dominates the conductivity, $\sigma_{dc} \sim \sigma_{ccs}$, while the Hall angle continues to scale as $\theta_H \sim \sigma_{diss}$.

Although we have used specific holographic models to identify this mechanism, our arguments can be applied more generally—in particular, we note, to the results of relativistic magnetohydrodynamics studied in Ref. [9]. As such, we end this Letter with a brief discussion of how these ideas could be applied to the experimental phenomenology of the cuprates.

Holographic models.—The holographic correspondence allows us to study certain strongly coupled systems in terms of gravitational physics in anti–de Sitter space. Our goal in this section is to study the Hall angle in a large class of typical holographic models. To this end, we begin by considering an Einstein-Maxwell-Dilaton theory, with additional axion fields that we shall use to break translational invariance in the boundary theory,

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 + \Phi(\phi)((\partial\chi_1)^2 + (\partial\chi_2)^2) \right.$$
$$\left. + V(\phi) - \frac{Z(\phi)}{4}F^2 \right].$$

These models, originally introduced in Ref. [10], can be viewed as the polar decomposition of a theory with two identical complex scalar fields that are assumed to have the same magnitude $\phi_1 = \phi e^{i\chi_1}$, $\phi_2 = \phi e^{i\chi_2}$. If we make the identification $\Phi(\phi) = \phi^2$, this corresponds to the original scalars ϕ_i living in the complex plane. Other forms, in particular, exponentials, can be obtained from scalars taking values in, e.g., hyperbolic space. In addition, we have allowed for an arbitrary potential for the magnitude of

these lattices and a coupling to the electromagnetic field via a dilatonlike factor $Z(\phi)$.

For simplicity, we wish to consider isotropic solutions in the bulk, and hence, we will assume a background for the metric and gauge field of the form

$$ds^2 = -U dt^2 + U^{-1} dr^2 + e^{2V}(dx^2 + dy^2),$$

$$A = a(r) dt.$$

We assume that the geometry has a regular horizon at $r = r_+$ where the gauge field vanishes $a \sim (r - r_+)$ and $U \sim 4\pi T (r - r_+)$. As the radial coordinate $r \to \infty$, we assume that the metric approaches anti–de Sitter space and that the gauge field approaches a constant, which is interpreted as the chemical potential μ in the boundary.

Associated to the chemical potential is a constant charge density $\mathcal{Q}$, which is identified with the conserved electric flux of the bulk theory $\mathcal{Q} = -e^{2V} Z(\phi) a'$. In order to have a finite conductivity at $\mathcal{Q} \neq 0$, we must break translational invariance. This is done by demanding that the scalar fields are nonvanishing on the boundary $\chi_1 \to kx$, $\chi_2 \to ky$ as $r \to \infty$. This corresponds to introducing oscillatory lattices in the scalar fields ϕ_i.

dc conductivity.—Before proceeding to calculate the Hall conductivity, it will be important to first highlight some features of the dc conductivity of these holographic models. There has recently been a lot of progress in obtaining analytic expressions for the transport properties of holographic theories [11–16]. The key idea [11] is that the dc conductivity does not evolve in the radial direction and, hence, can be expressed solely in terms of horizon data. In particular, for the above holographic models, the resulting expression derived in Ref. [13] is

$$\sigma_{\mathrm{dc}} = \left[Z(\phi) + \frac{4\pi \mathcal{Q}^2}{k^2 \Phi(\phi) s} \right]_{r_+} \tag{1}$$

where $s = e^{2V}/4\pi |_{r_+}$ is the entropy density.

An important observation is the division of the conductivity into two distinct terms. A precise distinction can be made by comparison to the electrothermal conductivity $\bar{\alpha}$, computed in Ref. [16],

$$\bar{\alpha} = \left[\frac{4\pi \mathcal{Q}}{k^2 \Phi(\phi)} \right]_{r_+}, \tag{2}$$

from which we can see that the first term in the dc conductivity $Z(\phi)|_{r_+}$ does not contribute to the electrothermal conductivity. Such a term is already present at $\mathcal{Q} = 0$, where the theory is charge conjugation symmetric. In a weakly coupled system, one can understand this charge conjugation symmetric conductivity as arising from particle-hole pairs moving in opposite directions, as illustrated in Fig. 1. However, we stress that for the holographic
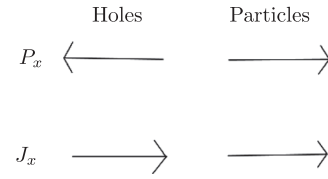

FIG. 1.   At weak coupling, the conductivity of a charge-conjugation symmetric theory can be understood as arising from particle-hole pairs of opposite momenta.

theories discussed here, which are strongly interacting and contain no quasiparticles, this intuition is suggestive at best. Nevertheless, since σ_{dc} is even under charge conjugation symmetry, there is a nonzero conductivity for $\mathcal{Q} = 0$ even at strong coupling.

A more surprising and novel feature of these holographic theories is that this charge-conjugation symmetric conductivity $\sigma_{\mathrm{ccs}} = Z(\phi)|_{r_+}$ remains present even at finite density. For the case of relativistic free fermions, for instance, in graphene, the addition of a chemical potential would introduce a gap for particle-hole creation proportional to μ and we would expect such a term to be exponentially suppressed below this scale. In contrast, for the strongly coupled holographic theories discussed here, σ_{ccs} can have a power-law dependence on T even at finite density [17].

When $\mathcal{Q} \neq 0$ we also have to consider the second term Eq. (1). For a translationally invariant theory, this term would diverge, but is rendered finite in our models by the presence of the lattice. As we outlined in the Introduction, we will refer to this term, which is associated with momentum dissipation, as σ_{diss}. The key point of Eq. (1) is that the finite density conductivity consists of two terms added together—that is, they follow an inverse-Matthiessen law

$$\sigma_{\mathrm{dc}} = \sigma_{\mathrm{ccs}} + \sigma_{\mathrm{diss}}. \tag{3}$$

In particular, we reiterate that within these holographic models both of these terms can remain at low energies, even in the presence of a chemical potential.

Hall conductivity.—Having explained the salient features of the dc conductivity, we now wish to generalize the techniques of Ref. [13] to calculate the Hall conductivity. In Lorentz invariant theories, the conductivity in the presence of a magnetic field is constrained to obey the simple form $\sigma_{xx} = 0$, $\sigma_{xy} = \mathcal{Q}/B$, which was originally reproduced from holography in Ref. [18].

Calculating the Hall conductivity in theories without translational invariance is more complicated. To do this, we consider the same class of models as before, but we add a magnetic field $A_y = Bx$ to the background. In order to calculate the conductivity, we perturb the background solution by a constant electric field $A_x = -E_x t$ as performed in Ref. [13]. The bulk equations then force us to turn on other fields, for which a consistent ansatz is

$$A_{x_i} = -E_i t + \delta a_{x_i}(r),$$
$$g_{tx_i} = e^{2V} \delta h_{tx_i}(r),$$
$$g_{rx_i} = e^{2V} \delta h_{rx_i}(r),$$
$$\chi_i = kx_i + \delta\chi_i(r),$$

where i runs over (1,2) and we of course mean that $x_1 = x, x_2 = y$. As is well known in these calculations, the trick is to find quantities that are independent of the bulk radial coordinate. These are provided for us by the perturbed Maxwell equations, which give us the two constant fluxes

$$J_x = -Z(\phi)U\delta a'_x + Q\delta h_{tx} - BZ(\phi)U\delta h_{ry},$$
$$J_y = -Z(\phi)U\delta a'_y + Q\delta h_{ty} + BZ(\phi)U\delta h_{rx} \quad (4)$$

from which we can evaluate the conductivities via the ratios

$$\sigma_{xx} = \frac{J_x}{E_x} \qquad \sigma_{xy} = \frac{J_y}{E_x}.$$

Since J_x and J_y are constants, we can calculate these ratios anywhere in the bulk. The simplest place to do this is at the horizon, where the constraints of regularity are enough to determine the conductivity. That is, we demand the smooth behavior

$$\delta a_{x_i} = -\frac{E_i}{4\pi T}\ln(r - r+) + \mathcal{O}(r - r_+),$$
$$\delta\chi_i = \mathcal{O}((r - r_+)^0),$$
$$\delta h_{tx_i} = U\delta h_{rx_i} + \mathcal{O}(r - r_+). \quad (5)$$

The quickest way to evaluate the conductivity is to plug these requirements into the $t - x$ component of Einstein's equations. This results in a pair of simultaneous equations for the value of δh_{tx_i} at the horizon,

$$(B^2 Z(\phi) + e^{2V}k^2\Phi(\phi))\delta h_{tx} - BZ(\phi)e^{2V}a'\delta h_{ty}$$
$$= -e^{2V}Z(\phi)a'E_x,$$
$$(B^2 Z(\phi) + e^{2V}k^2\Phi(\phi))\delta h_{ty} + BZ(\phi)e^{2V}a'\delta h_{tx}$$
$$= BZ(\phi)E_x.$$

Inverting these equations gives the values of the graviton at the horizon, from which we can proceed to extract the Hall conductivity

$$\sigma_{xx} = \left.\frac{e^{2V}k^2\Phi(B^2Z^2 + Q^2 + Ze^{2V}k^2\Phi)}{(B^2Z + e^{2V}k^2\Phi)^2 + B^2Q^2}\right|_{r_+},$$
$$\sigma_{xy} = \left.\frac{BQ(B^2Z^2 + Q^2 + 2Ze^{2V}k^2\Phi)}{(B^2Z + e^{2V}k^2\Phi)^2 + B^2Q^2}\right|_{r_+}. \quad (6)$$

We are now able to turn to the question of ultimate interest, which is to calculate the Hall angle $\theta_H = \sigma_{xy}/\sigma_{xx}$ for holographic theories in a magnetic field. Whereas transport properties in a magnetic field can be unfamiliar, the Hall angle is especially simple. In many ways, it behaves like the familiar dc conductivity—in the absence of a lattice it is an infinite delta function that will now be resolved via momentum dissipation into a Drude peak.

The holographic results above imply that the Hall angle takes the somewhat clumsy form

$$\theta_H = \frac{BQ}{e^{2V}k^2\Phi}\left[\frac{B^2Z^2 + Q^2 + 2Ze^{2V}k^2\Phi}{B^2Z^2 + Q^2 + Ze^{2V}k^2\Phi}\right]\bigg|_{r_+}. \quad (7)$$

While this formula is complicated, we can extract the physics by noticing that for all geometries the quantity in square brackets is simply a number bounded between 1 and 2. We can, therefore, deduce that the scaling of the Hall angle is predominantly controlled by the overall factor outside the brackets, which may be written as

$$\theta_H \sim \left.\frac{4\pi BQ}{k^2\Phi(\phi)s}\right|_{r_+}. \quad (8)$$

Furthermore, for small magnetic fields the thermodynamic and lattice factors appearing in Eqs. (1) and (8) must agree, and so we may write

$$\theta_H \sim \frac{B}{Q}\sigma_{\text{diss}}. \quad (9)$$

The central point of this Letter is the observation that, in contrast to the dc conductivity, there is only a single contribution to the Hall angle.

In particular, there is no additive contribution to the Hall angle analogous to the charge-conjugation symmetric conductivity in σ_{dc}. In fact, at least at weak coupling, it is easy to see why this should be the case. Recall that this current was carried by particle-hole pairs moving in opposite directions. Upon adding a magnetic field, these pairs are deflected in the same direction, and hence, they do not contribute to the Hall conductivity σ_{xy} (Fig. 2). This
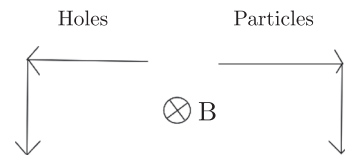

FIG. 2. In the presence of a magnetic field, the particle-hole pairs responsible for σ_{ccs} are deflected in the same direction. They, therefore, cannot carry a Hall current.

simple observation continues to hold in strongly coupled theories; σ_{xy} is odd under charge conjugation symmetry and, hence, must vanish when $\mathcal{Q} = 0$.

Motivated by the experimental results, our goal is to understand how we can obtain different scalings in the Hall angle and dc conductivity. It is easy to reproduce the original puzzle of the Hall angle. For geometries where the lattice is very small, $\sigma_{\mathrm{diss}} \gg 1$, then the dc conductivity is dominated by the second term in Eq. (1) and so scales in the same manner as the Hall angle.

This result should not be a surprise. In this regime, the correct framework to describe strongly coupled transport is the memory matrix [9,19,20]. Within this framework, every operator that has a projection onto the momentum operator, such as the electric and Hall currents, is controlled by the momentum relaxation rate. The physics is dominated by this single time scale, and hence, the Hall angle and dc conductivity must agree.

However, a simple resolution of this puzzle is equally clear. Outside of this momentum dissipation dominated regime, we must also consider the charge-conjugation symmetric contribution to σ_{dc}. For strong momentum dissipation, this dominates the dc conductivity, $\sigma_{\mathrm{dc}} \sim \sigma_{\mathrm{ccs}}$, and so within the holographic models this is given by the dilaton factor. Conversely, the Hall angle is still given by $\theta_H \sim \sigma_{\mathrm{diss}}$ and so is determined by the profile of the lattice according to Eq. (8). While in this Letter we do not consider any specific model, there is not generically a simple relationship between the dilaton factor and the lattice profile. As such, we can deduce that general holographic models will have very different scalings in the Hall angle as to the dc conductivity.

Discussion.—We have shown that, within a large class of holographic lattice models, one will generically find different scalings in the Hall angle and the dc conductivity. Ultimately, this was made possible by the presence of a charge conjugation symmetric contribution to the dc conductivity that was absent from the Hall angle. We emphasized that at weak coupling one would expect such a term to be exponentially suppressed. The novel step in our argument is the realization that holography provides examples of strongly interacting systems where this is no longer the case. Hence, despite charge-conjugation symmetry being badly broken by the chemical potential, one can find a regime, of strong momentum dissipation, where σ_{ccs} dominates the low-energy conductivity. Since the Hall angle is independent of σ_{ccs}, it is then simple to obtain anomalous scaling.

The existence of σ_{ccs} and, hence, this method of obtaining an anomalous Hall scaling is more general than these specific lattice models. For instance, in Ref. [9] the transport properties of Lorentz invariant conformal field theories deformed by a net charge density $\mathcal{Q}$ and magnetic field B were studied using hydrodynamics. The dc and Hall conductivities calculated in Ref. [9] take the same form as

the holographic lattice results of Eqs. (1) and (6) [21]. Although it was not appreciated at the time, our general arguments are equally applicable to these models, and it is possible to describe an anomalous Hall angle within relativistic hydrodynamics.

More recently, an anomalous scaling of the Hall angle was identified in probe brane models of holographic transport [22]. Although the conductivity of these models takes a more complicated form than Eq. (3), it is still possible to identify a charge-conjugation symmetric contribution to σ_{dc}. The mechanism introduced here can be used to provide a microscopic understanding of these results.

Given the general nature of these ideas, we end this Letter with a brief discussion of the implications of our work for the experimental phenomenology of the cuprates. Recall that we have two contributions to the conductivity, a charge-conjugation symmetric term σ_{ccs} and another from explicit charge density relaxed by some momentum dissipation σ_{diss}. These are additive in the dc conductivity following an inverse Mathiessen rule [23]

$$\sigma_{\mathrm{dc}} = \sigma_{\mathrm{ccs}} + \sigma_{\mathrm{diss}}. \tag{10}$$

Conversely, the temperature scaling of the Hall angle is always

$$\theta_H \sim \frac{B}{\mathcal{Q}} \sigma_{\mathrm{diss}}. \tag{11}$$

In order to obtain an anomalous Hall scaling, we needed the dc conductivity to be dominated by the charge-conjugation symmetric term. Therefore, the linear resistivity of the cuprates would arise from a scaling $\sigma_{\mathrm{ccs}} \sim 1/T$, whereas we could reproduce the scaling of the Hall angle provided we have that $\sigma_{\mathrm{diss}} \sim 1/T^2$. This should be contrasted with the many attempts to use holography to obtain a linear resistivity that assume that σ_{diss} dominates the conductivity [24,25]. Our results suggest it would be difficult to accommodate the Hall angle anomaly within this approach. Rather, our ideas are more in line with the recent suggestion that a linear resistivity could arise from universal incoherent transport [26].

After having made a combination of these two contributions, the resistivity ρ predicted by this approach would take the general form

$$\rho \sim T^2/(W + T) \tag{12}$$

where W is a model-dependent energy scale. In our picture, the charge conjugation symmetric conductivity dominates for $T \gg W$ and we are left with the linear resistivity. Conversely, for $T \ll W$, momentum relaxation is very weak, and the resistivity passes back over to a Fermi-liquid-like T^2 law. Although arising from a completely different model, a similar form for the resistivity was recently presented in Refs. [27,28]. In particular, it was

emphasized that this crossover is remarkably similar to the behavior of the overdoped cuprates and was found in Ref. [28] to provide a good fit to experimental data. Furthermore, the existence of both T and T^2 contributions to the resistivity has also been seen in Ref. [29]—albeit while fitted to a more conventional Mathiessen law.

Finally, we comment that the dichotomy in scalings will be evident in more general transport properties. In particular, the thermoelectric current is also observed to show the same anomalous behavior as the Hall angle [8]. It is clear that our picture reproduces this result—as is explicit in Eq. (2), the thermoelectric conductivity only receives a contribution from net charge and so, like the Hall angle, is proportional to σ_{diss}.

While it is hard to make any concrete comparisons with experiment due to the qualitative nature of this work, it is encouraging that these simple features hold up. It would of course be of great interest to develop specific models realizing this scenario so that a more detailed comparison with experimental phenomenology could be performed.

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