

Coxeter groups, quiver mutations and geometric manifolds

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ABSTRACT

We construct finite volume hyperbolic manifolds with large symmetry groups. The construction makes use of the presentations of finite Coxeter groups provided by Barot and Marsh and involves mutations of quivers and diagrams defined in the theory of cluster algebras. We generalize our construction by assigning to every quiver or diagram of finite or affine type a CW-complex with a proper action of a finite (or affine) Coxeter group. These CW-complexes undergo mutations agreeing with mutations of quivers and diagrams. We also generalize the construction to quivers and diagrams originating from unpunctured surfaces and orbifolds.

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1. Introduction

In [1] Barot and Marsh provided presentations of finite Weyl groups in terms of arbitrary quivers or diagrams of finite type. In brief, the construction works as follows: one needs to consider the underlying unoriented labeled graph of a quiver (or diagram) as a Coxeter diagram of a (usually infinite) Coxeter group, and then impose some additional relations on this group that can be read off from the quiver/diagram. It is proved in [1] that the resulting group depends on the mutation class of a quiver/diagram only, see Section 3 for the details.

The goal of this paper is to use these presentations to construct hyperbolic manifolds, in particular ones of small volume. The most common way to construct hyperbolic manifolds is by considering finite index torsion-free subgroups of cofinite hyperbolic reflection groups (see e.g. [2, 5, 7, 31]). The approach used in the present paper is completely different: we start with a *symmetry group* of a manifold, more precisely, we are looking for a manifold whose symmetry group contains a given finite Weyl group. In [6] Everitt and Maclachlan describe an algebraic framework for this approach, and in [24] Kolpakov and Slavich explicitly construct an arithmetic hyperbolic 4-manifold with given finite isometry group. We use quite different techniques based on *mutations of quivers and diagrams* defined by Fomin and Zelevinsky in the context of cluster algebras. This technique allows us to obtain manifolds having large symmetry groups and, at the same time, relatively small volume, i.e. two properties of quite opposite nature.

Mutation classes of quivers and diagrams of finite type are indexed by finite Weyl groups. Our interpretation of the result of [1] is that, starting from a quiver or diagram \mathcal{G} of finite type, one can

present the corresponding finite Weyl group W as a quotient of some (usually infinite) Coxeter group $W_0(\mathcal{G})$ by the normal closure $W_C(\mathcal{G})$ of additional relations. If this Coxeter group $W_0(\mathcal{G})$ is hyperbolic, then the Weyl group W acts properly on the quotient of the hyperbolic space by the group $W_C(\mathcal{G})$. *A priori*, this quotient may have singular points. However, this is not the case.

THEOREM 6.2 (Manifold Property). *The group $W_C(\mathcal{G})$ is torsion-free.*

As a corollary, if the Coxeter group $W_0(\mathcal{G})$ is hyperbolic and cofinite, we obtain a hyperbolic manifold of finite volume, and the symmetry group of this manifold contains W . We list the manifolds that we found in this way in Tables 5.1 and 7.1. In dimension 4 we construct a manifold with Euler characteristic 2, i.e. it has the second minimal volume amongst all 4-dimensional manifolds. The symmetry group of this manifold contains the group $\text{Sym}_5 \times \mathbb{Z}_2$ as a subgroup. We note that, since this manifold has small volume and large symmetry group at the same time, it does not appear in [24].

In [14] we generalized the results of [1] to quivers and diagrams of affine type and certain other mutation-finite quivers and diagrams. In particular, every affine Weyl group also admits presentations of a similar type with slightly more complicated groups $W_C(\mathcal{G})$. Using this result, in Section 8.2 we provide the following theorem.

THEOREM 8.1. *The manifold property holds for quivers and diagrams of affine type.*

For quivers and diagrams originating from unpunctured surfaces or orbifolds [15, 11], as well as for exceptional mutation-finite quivers and diagrams [9, 10], the construction in [14] also provides a group assigned to a mutation class. In contrast to the finite and affine cases, this group is not a Coxeter group, however, it also has presentations as a quotient of a Coxeter group W_0 (with more types of generators for W_C required).

Using these presentations, for every quiver or diagram \mathcal{G} of all types discussed above, we construct a CW-complex (actually being a quotient of the Davis complex of a Coxeter group $W_0(\mathcal{G})$) with a proper action of the group $W(\mathcal{G}) = W_0(\mathcal{G})/W_C(\mathcal{G})$, where the group $W(\mathcal{G})$ depends on the mutation class of \mathcal{G} only. We also define mutations of these complexes, the mutations agree with quiver (or diagram) mutations. All the manifolds discussed above are partial cases of these complexes.

The paper is organized as follows. Section 2 contains basic definitions and some essential facts on Coxeter groups and quiver mutations. In Section 3 we recall the construction from [1] assigning to a quiver of finite type a presentation of the corresponding Weyl group. For simplicity, in Sections 3–6 we restrict ourselves to simply-laced Weyl groups (and thus, we work with quivers rather than diagrams), we then consider the general case in Section 7. In Section 4 we describe the construction highlighted above of various actions of a given finite Weyl group. In section 5 we first treat in full details our construction applied to the group A_3 ; then we list other geometric manifolds with Weyl group actions resulted from the algorithm described in Section 4. In particular, Table 5.1 contains the list of hyperbolic manifolds we obtain. In Section 6, after defining the Davis complex of a Coxeter group, we prove the Manifold Property. We also define mutations of quotients of Davis complexes agreeing with quiver mutations. Section 7 is devoted to generalizations of the results to all finite Weyl groups. Finally, in Section 8 we describe how the construction generalizes to affine Weyl groups and some other infinite groups related to quivers and diagrams of finite mutation type.

2. Quiver mutations and presentations of Coxeter groups

2.1. Coxeter groups

We briefly remind some basic properties of Coxeter groups. For the details see [4].

2.1.1. *Definitions* A group W is called a (finitely generated) *Coxeter group* if it has a presentation of the form

$$W = \langle s_1 \dots s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$$

where $m_{ii} = 1$ and $m_{ij} \in \mathbb{N}_{>1} \cup \infty$ for all $i \neq j$. Here $m_{ij} = \infty$ means that there is no relation on $s_i s_j$. A pair (W, S) of a Coxeter group W and its set of generators $S = \{s_1, \dots, s_n\}$ is called a *Coxeter system*. The cardinality n of S is a *rank* of the Coxeter system.

An element of W is said to be a *reflection* if it is conjugated in W to an element of S .

A Coxeter system (W, S) may be depicted by its *Coxeter diagram*:

- the vertices of the Coxeter diagram correspond to the generators s_i ;
- i -th vertex is connected to j -th by an edge labeled m_{ij} with the following exceptions:
 - * if $m_{ij} = 2$ then there is no edge connecting i to j ;
 - * if $m_{ij} = 3$ then the label 3 is usually omitted;
 - * if $m_{ij} = \infty$ the edge is drawn bold.

Given a Coxeter system (W, S) , one can write a symmetric $n \times n$ matrix $M(W, S) = \{M_{ij}\}$ as follows:

$$M_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -\cos \frac{\pi}{m_{ij}} & \text{if } i \neq j \text{ and } m_{ij} \neq \infty; \\ -1 & \text{if } m_{ij} = \infty. \end{cases}$$

An $n \times n$ matrix a_{ij} is *decomposable* if there is a non-trivial partition of the index set as $\{1, \dots, n\} = I \cup J$, so that $a_{ij} = a_{ji} = 0$ whenever $i \in I, j \in J$. A matrix is *indecomposable* if it is not decomposable.

2.1.2. *Actions by reflections* For some Coxeter systems (W, S) the group W has a natural discrete action by reflections on a space of constant curvature (here by an *action by reflections* we mean an action where the reflections of W are represented by orthogonal reflections with respect to hyperplanes).

More precisely, there are three cases of interest:

- $M(W, S)$ is positive definite: then W is a finite group and W acts on a sphere S^{n-1} ;
- $M(W, S)$ is indecomposable and has signature $(n - 1, 0, 1)$: then W acts cocompactly on Euclidean space \mathbb{E}^{n-1} ;
- $M(W, S)$ is indecomposable and has signature $(n - 1, 1)$: then W acts on an $(n - 1)$ -dimensional hyperbolic space \mathbb{H}^{n-1} .

See [34] for the details.

A fundamental domain of such an action is a connected component of the complement to the set of all hyperplanes fixed pointwise by reflections of the group. This fundamental domain is a polytope (which may be of infinite volume) with facets indexed by generators $\{s_i\}_{i=1, \dots, n}$ and all dihedral angles equal to π/m_{ij} (if $m_{ij} = \infty$ the corresponding facets do not intersect). Such polytopes are called *Coxeter polytopes*.

REMARK 2.1. If a Coxeter group W is a finite or affine Weyl group, then the outer normal vectors to the facets of the fundamental polytope can be identified with the simple roots of the corresponding root system.

In the three cases listed above the group W is called *geometric* (note that our class of geometric Coxeter groups is wider than in [4, Chapter 6]). In fact, every Coxeter group acts properly cocompactly by reflections on some specially constructed space [3], we will discuss this action in Section 6.

2.2. Quiver mutations

Let Q be a quiver (i.e. a finite oriented multi-graph) containing no loops and no 2-cycles. If there are q arrows pointing from i -th vertex to the j -th, then we draw one arrow with a weight $w_{ij} = q$.

2.2.1. *Definition of a quiver mutation* For every vertex k of the quiver Q one can define an involutive operation μ_k called *mutation of Q in direction k* . This operation produces a new quiver denoted by $\mu_k(Q)$ which can be obtained from Q in the following way (see [16]):

- orientations of all arrows incident to the vertex k are reversed;
- for every pair of vertices (i, j) such that Q contains arrows directed from i to k and from k to j the weight of the arrow joining i and j changes as described in Figure 2.1.

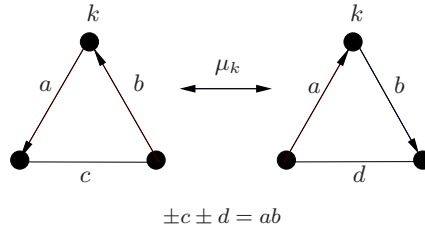


FIGURE 2.1. Mutations of quivers. The sign before c (resp., d) is positive if the three vertices form an oriented cycle, and negative otherwise. Either c or d may vanish. If ab is equal to zero then neither the value of c nor orientation of the corresponding arrow changes.

Given a quiver Q , its *mutation class* is a set of all quivers obtained from Q by all sequences of iterated mutations. All quivers from one mutation class are called *mutation-equivalent*.

Quivers without loops and 2-cycles are in one-to-one correspondence with integer skew-symmetric matrices $B = \{b_{ij}\}$, where b_{ij} is the number of arrows from i -th vertex to j -th one. In terms of the matrix B the mutation μ_k can be written as $\mu_k(B) = B'$, where

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

This transformation is called a *matrix mutation*.

REMARK 2.2. The procedure of matrix mutation is well-defined for more general case of *skew-symmetrizable* integer matrices. Matrix mutations for non-skew-symmetric matrices correspond to *diagram mutations* (see Section 7 for the definitions and generalizations of the results to the case of diagram mutations).

2.2.2. *Finite type* A quiver is of *finite type* if it is mutation-equivalent to an orientation of a simply-laced Dynkin diagram. All orientations of a given Dynkin diagram are mutation-equivalent, so any quiver of finite type is of one of the following mutation types: A_n, D_n, E_6, E_7, E_8 .

2.2.3. *Finite mutation type* A quiver Q is of *finite mutation type* (or *mutation-finite*) if there are finitely many quivers mutation-equivalent to Q . It is shown in [9] that each quiver of finite mutation type

is either of order 2, or a quiver arising from a triangulated surface (see [15] for details), or mutation-equivalent to one of the 11 exceptional quivers.

In this paper we deal with quivers of finite mutation type only.

3. Presentations of Coxeter groups arising from quiver mutations

In [1] Barot and Marsh show that for each quiver of finite type there is a way to explicitly construct the corresponding Weyl group.

3.1. Construction of the group by a quiver

Let Q be an orientation of a simply-laced Dynkin diagram with n nodes, let W be the corresponding finite Coxeter group, and Q_1 be any quiver mutation-equivalent to Q . Denote by $W(Q_1)$ the group generated by n generators s_i with the following relations:

(R1) $s_i^2 = e$ for all $i = 1, \dots, n$;

(R2) $(s_i s_j)^{m_{ij}} = e$ for all i, j , where

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an arrow.} \end{cases}$$

(R3) (**cycle relations**) for every chordless oriented cycle \mathcal{C} given by

$$i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{d-1} \rightarrow i_0$$

we take the relation

$$(s_{i_0} s_{i_1} \dots s_{i_{d-2}} s_{i_{d-1}} s_{i_{d-2}} \dots s_{i_1})^2 = e,$$

Note that for a cycle of length d we have a choice of d relations of type (R3), we take any one of them.

It is shown in [1, Theorem A] that the group $W(Q_1)$ does not depend on the choice of a quiver in the mutation class of Q . In particular, it is isomorphic to the initial Coxeter group W .

REMARK 3.1. The results of [1] are proved in more general settings, we come to the general case in Section 7.

3.2. Generators of $W(Q_1)$

Generators of $W(Q_1)$ satisfying relations (R1)–(R3) can be expressed in terms of standard generators of W . It is shown in [1] that these generators of $W(Q_1)$ can be found inductively in the following way:

1. for $Q_1 = Q$ the generators $\{s_i\}$ are standard Coxeter generators (indeed, Dynkin diagrams contain no cycles, so, we will get no cycle relations and obtain standard presentation of the Coxeter group).
2. Let $\{s_i\}$ be the generators of $W(Q_1)$, and let $Q_2 = \mu_k(Q_1)$. Then the generators $\{t_i\}$ of $W(Q_2)$ satisfying relations (R1)–(R3) are

$$t_i = \begin{cases} s_k s_i s_k & \text{if there is an arrow from } i \text{ to } k \text{ in } Q_1; \\ s_i & \text{otherwise.} \end{cases}$$

4. Construction of various actions of W

As before, let Q be an orientation of a Dynkin diagram, and let W be the corresponding Weyl group. Choose any quiver Q_1 mutation-equivalent to Q , and denote by $W_0(Q_1)$ the group defined by relations of type (R1) and (R2) only (note that $W_0(Q_1)$ is a Coxeter group).

Let C_1, \dots, C_p be a collection of all cycle relations for Q_1 (we take one relation for each chordless oriented cycle). Denote by $W_C(Q_1)$ the normal closure of the union of the elements C_1, \dots, C_p in $W_0(Q_1)$. By the definition of $W(Q_1)$ we have $W(Q_1) = W_0(Q_1)/W_C(Q_1)$. Theorem A from [1] says that $W(Q_1)$ is isomorphic to $W = W(Q)$ which, in particular, means that $W_C(Q_1)$ has finite index in $W_0(Q_1)$.

The group $W_0(Q_1)$ depends on the quiver Q_1 chosen in the mutation class of Q . Moreover, depending on the choice of Q_1 , the group $W_0(Q_1)$ may be finite or infinite. As a consequence, it may act naturally by reflections on different spaces (sphere, Euclidean space, hyperbolic space or, in general, the Davis complex of the group $W_0(Q_1)$, see Section 6). Since $W = W(Q_1) = W_0(Q_1)/W_C(Q_1)$, this results in actions of the initial Coxeter group W on different spaces.

In Section 6 we show that the group $W_C(Q_1)$ is torsion-free (we call this *Manifold Property*). In particular, this implies that if the Coxeter group $W_0(Q_1)$ acts properly on a space $\mathbb{X} = \mathbb{E}^n$ or \mathbb{H}^n by isometries with a fundamental domain of finite volume, then the quotient $\mathbb{X}/W_C(Q_1)$ is a finite volume manifold with a symmetry group containing W . In the next section we show examples of the construction of hyperbolic manifolds in this way, as well as of actions of Weyl groups of type D on flat tori.

5. Examples of manifolds constructed via quiver mutations

5.1. Action of A_3 on the flat 2-torus

We now present the first (and the easiest) example showing how different actions of the same Coxeter group may be produced via mutations of quivers.

5.1.1. Action on the sphere We start with the quiver Q of type A_3 (see Fig. 5.1(a),(b)). The corresponding Dynkin diagram of type A_3 determines a finite Coxeter group W acting on the 2-dimensional sphere by reflections:

$$W = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

The fundamental domain of this action is a spherical triangle with angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2})$, see Fig. 5.1(c) for the stereographic projection image of the action and the fundamental domain.

5.1.2. Action on the flat 2-torus Applying the mutation μ_2 to the quiver Q , we obtain a quiver Q_1 (see Fig. 5.2(a)). The group

$$W_0(Q_1) = \langle t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3 = e \rangle$$

is the group generated by reflections in the sides of a regular triangle in the Euclidean plane (the corresponding Dynkin diagram is of type \tilde{A}_2), see Fig. 5.2(b). Note that the plane is tessellated by infinitely many copies of the fundamental domain, each copy F_g labeled by an element $g \in W_0(Q_1)$. To see the action of the initial group $A_3 = W = W(Q_1)$ we need to take a quotient by the cycle relation

$$(t_1 t_2 t_3 t_2)^2 = e,$$

which means that in the tessellated plane we identify F_e with the fundamental triangle labeled by the $(t_1 t_2 t_3 t_2)^2$.

Now, let us see what the transformation $T_1 = (t_1 t_2 t_3 t_2)^2$ means geometrically. The element t_1 acts on the plane as the reflection in the side l_1 of F_e , $t_2 t_3 t_2$ is the reflection with respect to the line l_{232} parallel to l_1 and passing through the vertex of F_e not contained in l_1 . Denote by d the distance between l_1 and l_{232} . Then the transformation $(t_1 t_2 t_3 t_2)$ is a translation of the plane by the distance $2d$ in the direction orthogonal to l_1 . Hence, the element $T_1 = (t_1 t_2 t_3 t_2)^2$ is a translation by $4d$ in the direction orthogonal to l_1 .

Clearly, $e = t_2(t_1 t_2 t_3 t_2)^2 t_2 = (t_2 t_1 t_2 t_3)^2$ in W , and $e = t_3(t_1 t_2 t_3 t_2)^2 t_3 = t_3(t_1 t_3 t_2 t_3)^2 t_3 = (t_3 t_1 t_3 t_2)^2$ in W . This implies that we also need to take a quotient of the plane by the actions of the elements

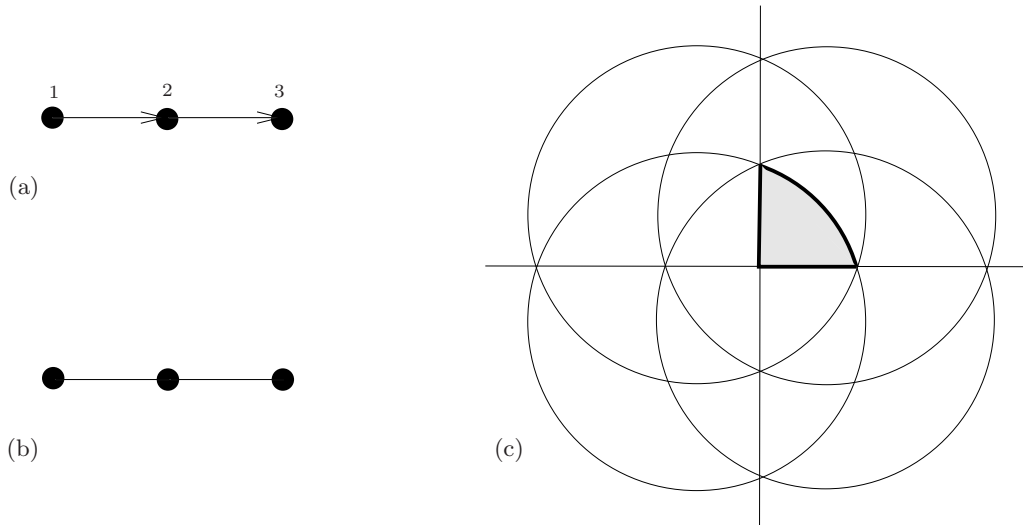


FIGURE 5.1. (a) A quiver of type A_3 ; (b) Dynkin diagram of type A_3 ; (c) action of A_3 on the sphere (in stereographic projection): a sphere is tiled by 24 copies of (spherical) triangles with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$.

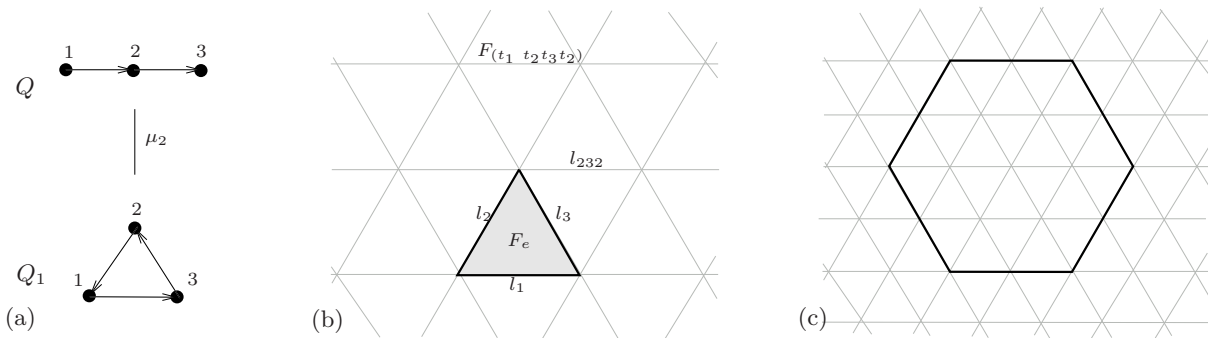


FIGURE 5.2. (a) mutation μ_2 ; (b) action of $W_0(Q_1) = \tilde{A}_2$ on the plane; (c) action of $W = W(Q_1) = A_3$ on the torus: 24 regular triangles tile the hexagon whose opposite sides are identified.

$T_3 = (t_2 t_1 t_2 t_3)^2$ and $T_2 = (t_3 t_1 t_3 t_2)^2$, i.e. by the translations by $4d$ in directions orthogonal to the lines l_2 and l_3 .

The quotient of the plane by the group $W_C(Q_1)$ generated by three translations T_1, T_2 and T_3 is a flat torus, see Fig. 5.2(c). Thus, we obtain an action of the quotient group $A_3 = W = W(Q_1)$ on the torus: the fundamental domain of this action is a regular triangle, 24 copies of this triangle tile the torus (as 24 copies tile the hexagon).

REMARK 5.1. The action described above is not an action by (topological) reflections: the fixed sets of the generators t_i are connected, so they do not divide the torus into two connected components (cf. [4, Chapter 10]).

5.1.3. *From the sphere to the torus: ramified covering* Now, we will show how the action of A_3 on the torus is related to the initial action on the sphere.

Recall from Section 3.2 that the presentation for $W(Q_1)$ is obtained from the presentation for $W(Q)$ by the following change of generators

$$(s_1, s_2, s_3) \rightarrow (t_1, t_2, t_3) \quad \text{where} \quad t_1 = s_2 s_1 s_2, \quad t_2 = s_2, \quad t_3 = s_3.$$

In the above sections we considered the action of the s -generators on the sphere and of the t -generators on the plane and the torus. However, the t -generators may be also considered directly on the sphere.

Namely, let s_1, s_2, s_3 be the reflections with respect to the corresponding sides l_1, l_2, l_3 of the fundamental spherical triangle Δ_s , see Fig. 5.3(a). Then the elements t_2 and t_3 are reflections with respect to l_2 and l_3 respectively. The element t_1 is also a reflection as it is conjugate to the reflection s_1 ; more precisely, t_1 is a reflection with respect to the (spherical) line l_{212} (see Fig. 5.3(b)).

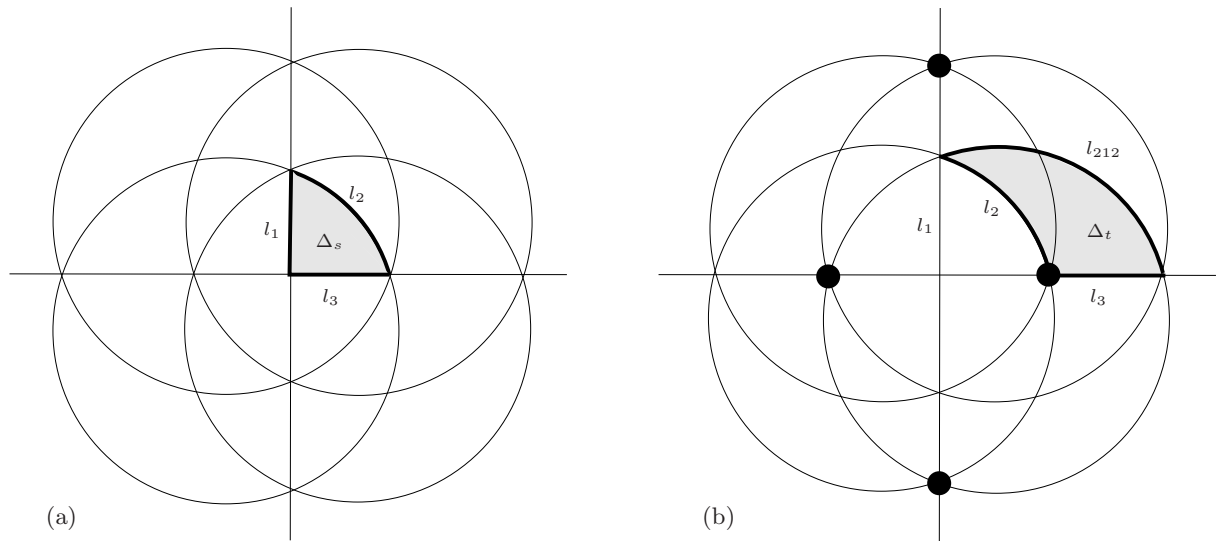


FIGURE 5.3. Construction of the ramified covering

Consider the triangle Δ_t bounded by l_{212}, l_2 , and l_3 (there are 8 possibilities to choose such a triangle, see Remark 5.2 for the justification of our choice).

In contrast to the triangle Δ_s bounded by l_1, l_2, l_3 , the triangle Δ_t is not a fundamental triangle for the action: Δ_t has angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3})$, and the group generated by t_2 and t_3 has order 6, but the whole neighborhood of the vertex $p \in l_2 \cap l_3$ is tiled by 3 copies of Δ_t only.

To resolve this inconsistency we consider a 2-sheet covering of the sphere branched in p and in all W -images of p (the four bold points in Fig. 5.3(b)). Clearly, the covering space is a torus (moreover, one can easily see that the covering is given by Weierstrass \wp -function). It is also not difficult to see that the action of A_3 on this covering torus is exactly the same as its action on the torus in Section 5.1.2: the covering torus consists of 24 triangles, and every vertex is incident to exactly six triangles.

REMARK 5.2 On the choice of the triangle Δ_t . Consider the outer normals v_1, v_2, v_3 to the sides of Δ_s (cf. Remark 2.1). To get the outer normals u_1, u_2, u_3 to the sides of Δ_t we use the following rule:

$$u_i = \begin{cases} -v_i & \text{if } i = k; \\ s_k(v_i) & \text{if there is an arrow from } i \text{ to } k \text{ in } Q; \\ v_i & \text{otherwise.} \end{cases}$$

Here $s_k(v_i)$ is the vector obtained from v_i by the reflection s_k .

5.1.4. *Back from the torus to the sphere* Now we will obtain the action of A_3 on the sphere from its action on the torus.

Let Δ_t be the fundamental triangle for the action of A_3 on the flat torus (see Fig. 5.4(a)). Let π be the canonical projection from Euclidean plane to the torus (corresponding to the action of $W_C(Q_1)$ on the plane), denote by $\tilde{\Delta}_t$ any preimage of Δ_t under π . The generators t_1, t_2, t_3 can be thought as reflections with respect to the sides of $\tilde{\Delta}_t$. The generators s_i can be obtained from the generators t_i as follows:

$$s_1 = t_2 t_1 t_2, \quad s_2 = t_2, \quad s_3 = t_3.$$

These are reflections with respect to the sides of the dashed domain $\tilde{\Delta}_s$ (see Fig. 5.4(b)), where $\tilde{\Delta}_s$ is an unbounded strip on the Euclidean plane.

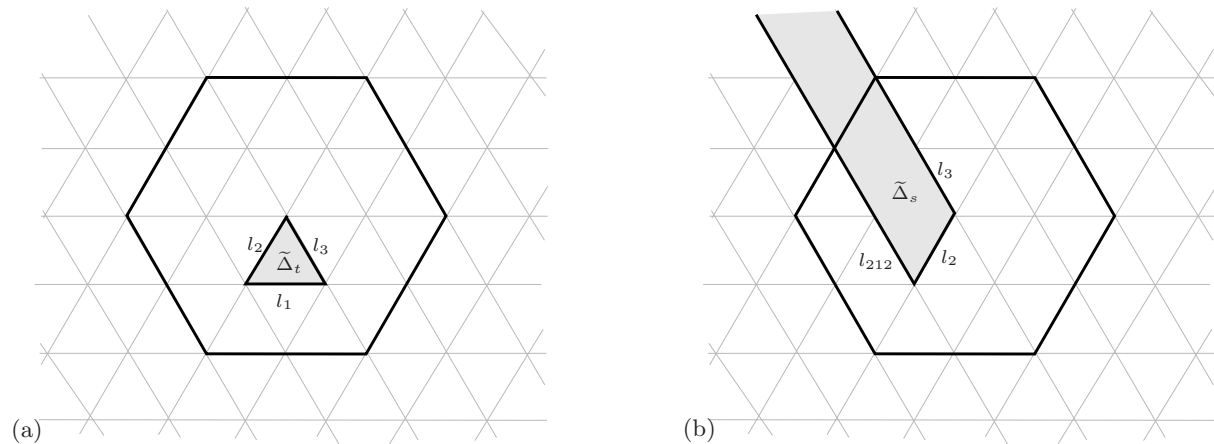


FIGURE 5.4. *Back from the flat torus to the sphere*

We now glue a manifold from 24 copies of the strip (according to the group action, as we glued 24 triangles above), denote by X the space obtained via this gluing. Note that due to the cycle relations four strips attached one to another along the “long” (unbounded) sides will result in an (unbounded) cylinder. The manifold X consists of exactly $6 = 24 : 4$ cylinders. Compactifying X by adding a limit point at the end of each of the 6 cylinders, we get exactly the tiling shown in Fig. 5.1(c).

5.2. *Actions of D_n on the flat $(n - 1)$ -torus*

The next series of examples is provided by the finite Coxeter group $W = D_n, n > 3$ (see Fig. 5.5(a) for the Dynkin diagram), which naturally acts on $(n - 1)$ -dimensional sphere (for $n = 3$ the Dynkin diagram and Weyl group D_3 coincides with A_3 , so the example from the previous section can be considered as a partial case). It is easy to check that the quiver Q of type D_n (see Fig. 5.5(b)) is mutation-equivalent to the oriented cycle Q_1 shown in Fig. 5.5(c) (in contrast to A_3 -example, one needs more than one mutation for this).

Note that the group $W_0(Q_1)$ (i.e. Coxeter group without cycle relations) is the affine Coxeter group \tilde{A}_{n-1} , thus, a group acting on the Euclidean space \mathbb{E}^{n-1} . A fundamental domain of this action is a Coxeter simplex Δ with the Coxeter diagram \tilde{A}_{n-1} . The generators for this action are the reflections t_1, \dots, t_n with respect to the facets Π_1, \dots, Π_n of this fundamental simplex.

Our next aim is to take a quotient by the cycle relation

$$(t_1 t_2 t_3 \dots t_{n-1} t_n t_{n-1} \dots t_3 t_2)^2 = e.$$

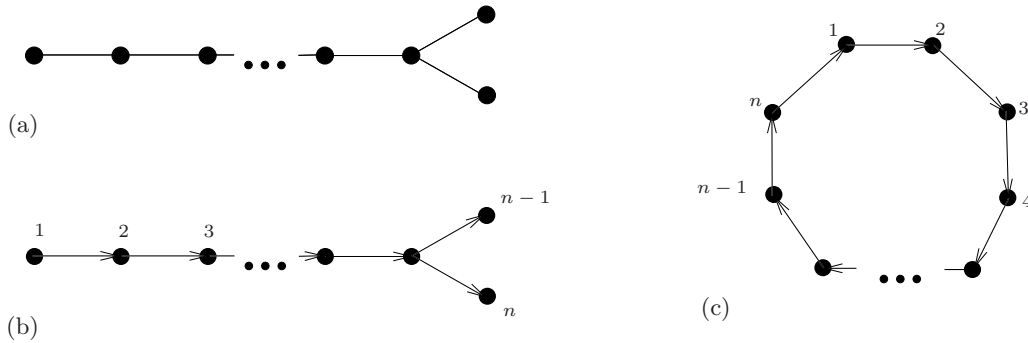


FIGURE 5.5. Type D_n : (a) Dynkin diagram; (b) acyclic quiver of type D_n ; (c) oriented cycle.

The element $(t_2 t_3 \dots t_{n-1} t_n t_{n-1} \dots t_3 t_2)$ is a conjugate of t_n , so it is a reflection with respect to the hyperplane Π parallel to Π_1 and passing through the vertex of Δ opposite to Π_1 . If d is the distance between the hyperplanes Π and Π_1 , then the element $(t_1 t_2 t_3 \dots t_{n-1} t_n t_{n-1} \dots t_3 t_2)$ is a translation by $2d$ in the direction orthogonal to Π_1 , and its square $T_1 = (t_1 t_2 t_3 \dots t_{n-1} t_n t_{n-1} \dots t_3 t_2)^2$ is a translation by $4d$. So, we need to take the quotient by the action of this translation.

The element T_1 has $n - 1$ conjugates of type $T_k = (t_k t_{k+1} \dots t_n t_1 \dots t_{k-2} t_{k-1} t_{k-2} \dots t_1 t_n \dots t_{k+1})^2$, $(k = 2, \dots, n)$. They all are translations by $4d$ in the direction orthogonal to Π_k . The group generated by T_1, \dots, T_n is a free abelian group of rank $n - 1$. Hence, taking a quotient of E^{n-1} by the translations T_1, \dots, T_n we get an $(n - 1)$ -dimensional torus \mathbb{T}^{n-1} . The action of $W_0(Q_1)$ on E^{n-1} turns into a faithful action of $D_n = W(Q_1)$ on the torus \mathbb{T}^{n-1} .

REMARK 5.3. The Coxeter diagram \tilde{A}_n is the unique affine Dynkin diagram containing a cycle. This implies that our construction gives no further examples of actions on non-trivial quotients of Euclidean space.

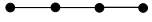
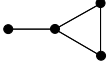
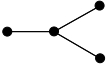

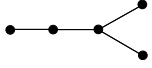

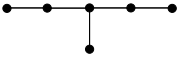

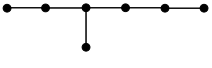




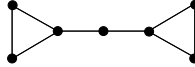
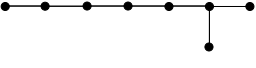
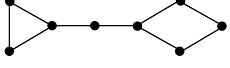
5.3. Actions on hyperbolic manifolds

We now consider the case when the group $W_0(Q_1)$ acts on a hyperbolic space, and hence, the group $W = W(Q_1)$ acts on its quotient $X = \mathbb{H}^n / W_C(Q_1)$ (recall that $W_C(Q_1)$ is the normal closure of all cycle relations). As it is shown below (Theorem 6.2, Corollary 6.3), in this case X is always a manifold. In this section we list all known actions on hyperbolic manifolds obtained in this way (Table 5.1).

NOTATION. In Table 5.1 we omit orientations of arrows of quivers Q and Q_1 . Here arrows of Q may be oriented in arbitrary way, and the only requirement on the orientations of arrows of Q_1 is that all chordless cycles are oriented.

The underlying graphs of Q and Q_1 are the Coxeter diagrams of fundamental polytopes of the action of W and $W_0(Q_1)$ on the sphere and the hyperbolic space respectively. All the volumes except the last two are computed in [20], the Euler characteristic (and thus the volume) of the fundamental polytope of the action of D_8 was computed by the program `CoxIter` written by Rafael Guglielmetti. The volume of the fundamental polytope of the action of A_7 is not known to the authors. The Euler characteristic is computed for all even-dimensional manifolds. The precise value of the volume of X for four-dimensional X is equal to $4\pi^2\chi(X)/3$, and for six-dimensional X equals $-8\pi^3\chi(X)/15$.

TABLE 5.1. Actions on hyperbolic manifolds.

W	Q	Q_1	$ W $	$\dim X$	vol X approx.	number of cusps	$\chi(X)$
A_4			$5!$	3	$ W \cdot 0.084578$	5	
D_4			$2^3 \cdot 4!$	3	$ W \cdot 0.422892$	16	
D_5			$2^4 \cdot 5!$	4	$ W \cdot 0.013707$	10	2
E_6			$2^7 \cdot 3^4 \cdot 5$	5	$ W \cdot 0.002074$	27	
E_7			$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	6	$ W \cdot 2.962092 \times 10^{-4}$	126	-52
E_8			$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	7	$ W \cdot 4.110677 \times 10^{-5}$	2160	
A_7			$8!$	5		70	
D_8			$2^7 \cdot 8!$	6	$ W \cdot 0.002665$	1120	-832

REMARK 5.4. The manifold X obtained in the case $W = A_4$ is the manifold M_5 studied in [25], it has minimal known volume amongst 5-cusped hyperbolic 3-manifolds. It can be obtained via a ramified 2-fold covering over the 3-sphere \mathbb{S}^3 branching along 1-skeleton of a 4-simplex. This manifold can be also obtained as the complement of the five-chain link in the 3-sphere.

REMARK 5.5. The symmetry group of X always contains W . Further, if the underlying graph of the quiver Q_1 has symmetries, then the fundamental polytope of the action of $W(Q_1)$ on X also has these symmetries, and thus the symmetry group of X also contains a semidirect product of W and the group of symmetries of the graph. In particular, one can note from Table 5.1 that the underlying graph of Q_1 always has an additional symmetry of order 2, therefore we actually get the group $W \rtimes \mathbb{Z}_2$ acting on X . Moreover, in the cases $W = D_4, A_7$ or D_8 the symmetry group of the fundamental polytope is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which results in the action of the group $W \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ on the manifold X .

REMARK 5.6.

1. In all cases the fundamental domain of the action $W(Q_1)$ on X is a non-compact hyperbolic polytope, i.e. the manifold X has cusps.
2. The combinatorial type of the fundamental domain in all but the last two cases is a simplex. In the last two cases it is a pyramid over a product of two simplices.
3. We do not know how to compute the volume of the 5-dimensional hyperbolic pyramid which is used to construct the manifold whose symmetry group contains $A_7 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ (the second last line of Table 5.1). At the same time, one can note that this pyramid consists of four copies of a pyramid with Coxeter diagram shown on Figure 5.6 (the pyramid in the Table 5.1 can be obtained from one on Fig. 5.6 by two consequent reflections in the facets corresponding to the leaves of the diagram). This smaller pyramid belongs to *Napier cycles*, a class of polytopes classified by Im Hof [19]. In particular, it is a simply truncated orthoscheme, and thus there is a hope to compute its volume using the technique developed by Kellerhals [23].



FIGURE 5.6. A 5-dimensional hyperbolic Coxeter pyramid tessellating the manifold with symmetry group containing $A_7 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ (see Table 5.1 and Remark 5.6)

REMARK 5.7. The list in Table 5.1 is complete in the following sense. Any two nodes of any quiver of finite type are connected by at most one arrow, therefore all dihedral angles of the corresponding fundamental polytope of the group action are either $\pi/2$ or $\pi/3$. All hyperbolic Coxeter polytopes satisfying such property are classified by Prokhorov in [30]. One can see that the only Coxeter diagrams whose orientations are quivers of finite type are those listed in Table 5.1.

On the other hand, our approach allows to construct more hyperbolic manifolds. In [28, 29], Parsons introduced a notion of a *companion basis* of a finite root system. In our terms, all companion bases (where every vector is considered up to plus/minus sign) can be described as follows. Take a quiver Q_1 of finite type, construct the corresponding group $W_0(Q_1)$, and construct the isomorphism between $W_0(Q_1)$ and the Weyl group $W = W(Q_1)$ by iterated changes of generators (see Section 3.2). Denote by g_1, \dots, g_n the images of generating reflections of $W_0(Q_1)$ in $W(Q_1)$. Now, take the root system of W , and an n -tuple of roots corresponding to reflections g_1, \dots, g_n . This n -tuple forms a companion basis. Thus, all the manifolds constructed above originate from companion bases. Below we present an example of a hyperbolic manifold obtained from a generating set which does not correspond to a companion basis.

As is mentioned in [1], not every generating set of reflections of W corresponds to a companion basis. The easiest example is the quadruple of generators (t_1, t_2, t_3, t_4) of $W = A_4$ given by roots $e_1 - e_5, e_2 - e_5, e_3 - e_5, e_4 - e_5$ respectively, where the root system A_4 consists of vectors $\{e_i - e_j\}_{1 \leq i, j \leq 5}$ in \mathcal{R}^5 with standard basis $\{e_i\}_{1 \leq i \leq 5}$. The reason is the following: every two of these four roots form an angle $2\pi/3$, so if there were a corresponding quiver, then its underlying graph should have been the complete graph on four vertices. However, it is easy to see that any such quiver is mutation-infinite, so it cannot be mutation-equivalent to any orientation of Dynkin diagram A_4 .

Now we may note that the Coxeter group

$$W_0 = \langle u_1, u_2, u_3, u_4 \mid u_i^2 = (u_i u_j)^3 = e \rangle$$

with Coxeter diagram shown on Fig. 5.7 is generated by reflections in the facets of an ideal hyperbolic 3-dimensional simplex. Further, since $\{t_1, t_2, t_3, t_4\}$ generate $W = A_4$ and satisfy the same relations as $\{u_i\}$ above, we can write W as a quotient of W_0 by a normal subgroup W_C which is the normal closure of additional relations. It is easy to see that we can take W_C as the normal closure of elements $(u_1 u_2 u_3 u_2)^2, (u_1 u_3 u_4 u_3)^2$ and $(u_2 u_3 u_4 u_3)^2$. Now, one can check that W_C is torsion-free, so the quotient of \mathbb{H}^3 by the action of W_C will be a finite volume hyperbolic manifold (it has 20 cusps and symmetry group containing $A_5 \rtimes A_4$, where A_4 is the symmetry group of the Coxeter diagram).

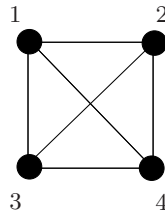


FIGURE 5.7. A 3-dimensional ideal hyperbolic simplex

6. Actions on quotients of Davis complex

In this section we first recall the construction of Davis complex $\Sigma(W, S)$ for each Coxeter system (W, S) .

Then, given a Dynkin quiver Q and a quiver Q_1 from the same mutation class, we consider the Davis complex $\Sigma(W_0(Q_1))$ for the group $W_0(Q_1)$ together with the action of the Coxeter group $W = W(Q)$ on the corresponding quotient space $X = \Sigma(W_0(Q_1))/W_C(Q_1)$ and prove the Manifold Property for X (which states that $W_C(Q_1)$ is torsion-free). In particular, we prove that all quotients listed in Table 5.1 are hyperbolic manifolds.

6.1. Davis complex

For any Coxeter system (W, S) there exists a contractible piecewise Euclidean cell complex $\Sigma(W, S)$ (called *Davis complex*) on which W acts discretely, properly and cocompactly. The construction (based on results of Vinberg [33]) was introduced by Davis [3]. In [26] Moussong proved that this complex yields a natural complete piecewise Euclidean metric which is $CAT(0)$. We first give a brief description of this complex following [27], and then follow [13] to define convex polytopes in $\Sigma(W, S)$.

6.1.1. *Construction of $\Sigma(W, S)$.* For a finite group W the complex $\Sigma(W, S)$ is just one cell, which is obtained as a convex hull K of the W -orbit of a suitable point p in the standard linear representation of W as a group generated by reflections. The point p is chosen in such a way that its stabilizer in W is

trivial and all the edges of K are of length 1. The faces of K are naturally identified with Davis complexes of the subgroups of W conjugate to standard parabolic subgroups.

If W is infinite, the complex $\Sigma(W, S)$ is built up of the Davis complexes of maximal finite subgroups of W glued together along their faces corresponding to common finite subgroups. The 1-skeleton of $\Sigma(W, S)$ considered as a combinatorial graph is isomorphic to the Cayley graph of W with respect to the generating set S .

The action of W on $\Sigma(W, S)$ is generated by reflections. The walls in $\Sigma(W, S)$ are the fixed points sets of reflections in W . The intersection of a wall α with cells of $\Sigma(W, S)$ supplies α with a structure of a piecewise Euclidean cell complex with finitely many isometry types of cells. Walls are totally geodesic: any geodesic joining two points of α lies entirely in α . Since Σ is contractible and $CAT(0)$, any two points of Σ can be joined by a unique geodesic.

Any wall divides $\Sigma(W, S)$ into two connected components. All the walls decompose $\Sigma(W, S)$ into connected components which are compact sets called *chambers*. Any chamber is a fundamental domain of the W -action on $\Sigma(W, S)$. The set of all chambers with appropriate adjacency relation is isomorphic to the Cayley graph of W with respect to S .

In what follows, if W and S are fixed, we write $\Sigma(W)$ or Σ instead of $\Sigma(W, S)$.

REMARK 6.1. If the group W acts cocompactly by reflections on spherical, Euclidean or hyperbolic space \mathbb{X} , then the Davis complex Σ is quasi-isometric to \mathbb{X} .

6.1.2. *Convex polytopes in Σ* For any wall α of Σ we denote by α^+ and α^- the closures of the connected components of $\Sigma \setminus \alpha$, we call these components *halfspaces*.

A *convex polytope* $P \subset \Sigma$ is an intersection of finitely many halfspaces $P = \bigcap_{i=1}^n \alpha_i^+$, such that P is not contained in any wall. Clearly, any convex polytope $P \subset \Sigma$ is a union of closed chambers.

In what follows by writing $P = \bigcap_{i=1}^n \alpha_i^+$ we assume that the collection of walls α_i is *minimal*: for any $j = 1, \dots, n$ we have $P \neq \bigcap_{i \neq j} \alpha_i^+$. A *facet* of P is an intersection $P \cap \alpha_i$ for some $i \leq n$. For any $I \subset \{1, \dots, n\}$ a set $f = \bigcap_{i \in I} \alpha_i \cap P$ is called a *face* of P if it is not empty.

We can easily define a dihedral angle formed by two walls: if $\alpha_i \cap \alpha_j \neq \emptyset$ there exists a maximal cell C of Σ intersecting $\alpha_i \cap \alpha_j$. We define the angle $\angle(\alpha_i, \alpha_j)$ to be equal to the corresponding Euclidean angle formed by $\alpha_i \cap C$ and $\alpha_j \cap C$. Clearly, any dihedral angle formed by two intersecting walls in Σ is equal to $k\pi/m$ for some positive integers k and m . A convex polytope P is called *acute-angled* if each of the dihedral angles of P does not exceed $\pi/2$.

A convex polytope P is called a *Coxeter polytope* if all its dihedral angles are integer submultiples of π . In particular, a fundamental domain of any reflection subgroup of W is a Coxeter polytope in Σ . Conversely, any Coxeter polytope in Σ is a fundamental chamber for the subgroup of W generated by reflections in its walls (see e.g. [4, Theorem 4.9.2]).

6.2. Manifold Property

Let us briefly recall the procedure described in Section 4:

1. Start with a Dynkin quiver Q with corresponding Weyl group W . Let Q_1 be any quiver mutation-equivalent to Q .
2. Consider the Coxeter group $W_0(Q_1)$ defined by the relations (R1) and (R2) (its Coxeter diagram coincides with the underlying graph of the quiver Q_1).
3. Let $\Sigma(W_0(Q_1))$ be the Davis complex of the group $W_0(Q_1)$. By construction, $W_0(Q_1)$ acts on $\Sigma(W_0(Q_1))$ by reflections.

4. Let C_1, \dots, C_p be the collection of all defining cycle relations for Q_1 , define $W_C(Q_1)$ to be the normal closure of all these relations in $W_0(Q_1)$.
5. Define $W(Q_1) = W_0(Q_1)/W_C(Q_1)$. As is shown in [1], $W(Q_1)$ is isomorphic to W .
6. Since $W_C(Q_1) \subset W_0(Q_1)$, $W_C(Q_1)$ acts on $\Sigma(W_0(Q_1))$. Consider $X(Q_1) = \Sigma(W_0(Q_1))/W_C(Q_1)$. Then $W = W(Q_1)$ acts properly on $X(Q_1)$.

So, for each quiver in the mutation class we have assigned an action of W on the space $X(Q_1)$ (the latter heavily depends on the quiver Q_1).

THEOREM 6.2 (Manifold property). *The group $W_C(Q_1)$ is torsion-free.*

COROLLARY 6.3. *If $\Sigma(W_0(Q_1))$ is a manifold then $X(Q_1)$ is also a manifold.*

Let F be a fundamental domain for the action of $W(Q_1)$ on $\Sigma(W_0(Q_1))$. A *star* $\text{Star}(f)$ of a face f of F is the union of all fundamental domains F_i of Σ containing f :

$$\text{Star}(f) = \bigcup_{f \in F_i} F_i.$$

We will prove the following statement which is equivalent to the Manifold Property (cf. [6, Corollary 2]).

THEOREM 6.4. *Let F be a fundamental domain for the action of $W(Q_1)$ on $\Sigma(W_0(Q_1))$. Then for any face f of F the interior $(\text{Star}(f))^0$ of the star of f in $X(Q_1)$ is isometric to the interior of the star of f in $\Sigma(W_0(Q_1))$.*

Equivalently, if a face f of F is contained in q fundamental domains of the action of $W_0(Q_1)$ on $\Sigma(W_0(Q_1))$, then f is also contained in q fundamental domains of the action of $W(Q_1)$ on $X(Q_1)$.

To prove the Manifold Property we will use the following observation.

LEMMA 6.5. *Let \tilde{t}_i and \tilde{t}_j be two generating reflections of $W_0(Q_1)$, and let t_i and t_j be their images in $W = W(Q_1)$ under the canonical projection $W_0(Q_1) \rightarrow W(Q_1) = W_0(Q_1)/W_C(Q_1)$. Then the orders of the elements $\tilde{t}_i\tilde{t}_j$ and t_it_j coincide.*

Proof. The dihedral group generated by t_i and t_j is a quotient of the dihedral group generated by \tilde{t}_i and \tilde{t}_j . The order of the latter group equals 6 or 4, therefore, if we assume that the two groups are distinct, the order of the former equals two, i.e. $t_i = t_j$, which is impossible since a Coxeter group W of rank n cannot be generated by less than n reflections (see e.g. [13, Lemma 2.1]). □

Proof of Theorem 6.4. Let f be a face of F . Denote by $I \subset \{1, \dots, n\}$ the index set such that f is the intersection of the walls fixed by $\{\tilde{t}_i\}_{i \in I}$. To prove Theorem 6.4 (and thus the Manifold Property) it is sufficient to prove that the group $\tilde{T}_I \subset W_0(Q_1)$ generated by $\{\tilde{t}_i\}_{i \in I}$ has the same order as the group $T_I \subset W(Q_1) = W$ generated by $\{t_i\}_{i \in I}$.

Denote $k = |I|$. The group \tilde{T}_I can be considered as a group generated by reflections in the facets of a spherical Coxeter $(k-1)$ -simplex \tilde{P} with dihedral angles π/m_{ij} , where m_{ij} is the order of $\tilde{t}_i\tilde{t}_j$. Furthermore, since every $t_i \in W = W(Q)$ is a reflection on the $X(Q) = \mathbb{S}^{n-1}$, the group T_I can also be considered as a group generated by reflections in the facets of a spherical $(k-1)$ -simplex P . The simplex P may not be Coxeter, however, by Lemma 6.5, its dihedral angles are equal to $\pi p_{ij}/m_{ij}$ for some p_{ij}

being coprime with m_{ij} . Since $m_{ij} = 2$ or 3 , this implies that either $p_{ij} = 1$, or $p_{ij}/m_{ij} = 2/3$. According to [8], this means that reflections in the facets of \tilde{P} and P generate isomorphic groups. \square

6.3. Mutation of complexes

In this section we describe how to obtain $X(Q_2)$ from $X(Q_1)$, where $Q_2 = \mu_k(Q_1)$. The construction generalizes the examples shown in Sections 5.1.2 and 5.1.4.

Step 1. Let s_1, \dots, s_n be the generators of W corresponding to Q_1 (i.e., if $\{s_i\}$ with relations (R1) and (R2) are generating reflections of $W_0(Q_1)$ composing a Coxeter system, and the $\{s_i\}$ with relations (R1)–(R3) are their images under canonical homomorphism $W_0(Q_1) \rightarrow W_0(Q_1)/W_C(Q_1)$), let t_1, \dots, t_n be the generators corresponding to Q_2 . As it is shown in [1], t_i can be obtained from s_i as follows

$$t_i = \begin{cases} s_k s_i s_k & \text{if there is an arrow from } i \text{ to } k \text{ in } Q_1; \\ s_i & \text{otherwise.} \end{cases}$$

Denote by π_j , $j = 1, 2$, the canonical projection $\Sigma(W_0(Q_j)) \rightarrow X(Q_j)$. According to the construction, a fundamental chamber $F_j \subset X(Q_j)$ of the action of W on $X(Q_j)$ is an image of a fundamental chamber $\tilde{F}_j \subset \Sigma(W_0(Q_j))$ under the projection. The elements s_i act on $\Sigma(W_0(Q_1))$ by reflections in the facets of \tilde{F}_1 , let walls $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \subset \Sigma(W_0(Q_1))$ be the mirrors of the reflections s_1, \dots, s_n . Similarly, the elements t_i are reflections in the facets of \tilde{F}_2 , let walls $\tilde{\beta}_1, \dots, \tilde{\beta}_n \subset \Sigma(W_0(Q_2))$ be the mirrors of the reflections t_1, \dots, t_n .

Step 2. Consider the action of the element t_i on $\Sigma(W_0(Q_1))$. It is a reflection with respect to a wall $\tilde{\gamma}_i$ satisfying

$$\tilde{\gamma}_i = \begin{cases} s_k(\tilde{\alpha}_i) & \text{if there is an arrow from } i \text{ to } k \text{ in } Q_1; \\ \tilde{\alpha}_i & \text{otherwise.} \end{cases}$$

Step 3. The fundamental chamber \tilde{F}_1 is an intersection of n halfspaces bounded by walls $\{\tilde{\alpha}_i\}$. Denote these halfspaces by $\tilde{\alpha}_i^+$, i.e. $\tilde{F}_1 = \bigcap \tilde{\alpha}_i^+$. Note that the dihedral angles of a polytope \tilde{F}_1 equal π/m_{ij} , where m_{ij} is the order of $s_i s_j$ in $W_0(Q_1)$. In particular, as all $m_{ij} = 2$ or 3 , every two facets of \tilde{F}_1 have a non-empty intersection.

Step 4. Consider the domain $H \subset \Sigma(W_0(Q_1))$ defined as follows:

H is the intersection of n halfspaces $H = \bigcap_{i=1}^n \tilde{\gamma}_i^+$, where the walls $\tilde{\gamma}_i$ are defined in Step 2 and the half-spaces $\tilde{\gamma}_i^+$, $i \neq k$, are chosen by

$$\tilde{\gamma}_i^+ = \begin{cases} \tilde{\alpha}_k^- & \text{if } i = k, \\ s_k(\tilde{\alpha}_i^+), & \text{if there is an arrow from } i \text{ to } k \text{ in } Q, \\ \tilde{\alpha}_i^+ & \text{otherwise} \end{cases}$$

(cf. Remark 5.2).

Note that H is well-defined, i.e. H indeed has exactly n facets. This follows from the fact that the facet $\tilde{\alpha}_k \cap \tilde{F}_1$ of \tilde{F}_1 (which is also a facet of H) has exactly $n - 1$ facets $\tilde{\alpha}_k \cap \tilde{\alpha}_i \cap \tilde{F}_1$, and by construction of H all these faces of \tilde{F}_1 are also faces of H .

Step 5. The domain H is a convex (possibly, non-compact) polytope in $\Sigma(W_0(Q_1))$ with n facets. The reflections with respect to the facets of H are precisely t_i , they generate the group $W_0(Q_1)$, however, in general, H is not a fundamental domain of the action of $W_0(Q_1)$ on $\Sigma(W_0(Q_1))$.

Step 6. Now, we will use H to construct a new cell complex Y (such that the group W acts on Y with fundamental chamber H).

Consider $|W|$ copies of the polytope H , labeled by the elements of W (denote by H_w the polytope corresponding to $w \in W$).

Glue the polytope H_w to the polytope H_{wt_i} along the facets $\tilde{\gamma}_i$ of H_w and H_{wt_i} so that the reflection in the common facet swaps the polytopes (and takes the facet $\tilde{\gamma}_j$ of H_w to the facet of H_{wt_i} having the same number). Denote by Y the space obtained by the gluing. Note that by “reflection” here we mean a map from one polytope to another, this may not extend to a reflection on the whole Y (cf. Remark 5.1).

Step 7. Denote by m_{ij} the order of the subgroup $\langle t_i, t_j \rangle \subset W$ (note that the Coxeter group

$$\langle t_1, \dots, t_n \mid (t_i t_j)^{m_{ij}} = e \rangle$$

is the group $W_0(Q_2)$). Consider the following completion Y' of Y .

- * First, we complete the fundamental domain H (obtaining a completed fundamental chamber H') in the following way. For every pair $\{i, j\} \subset \{1, \dots, n\}$ we add an $(n - 2)$ -dimensional face $\tilde{\gamma}_i \cap \tilde{\gamma}_j$ if it is not a face of H yet; we define the dihedral angle composed by facets $\tilde{\gamma}_i \cap H$ and $\tilde{\gamma}_j \cap H$ of H' as π/m_{ij} . Then, for each finite subgroup $\langle t_i \mid i \in I \rangle \subset W_0(Q_2)$ (where $I \subset \{1, \dots, n\}$ is an index set) we add an $(n - |I|)$ -dimensional face $\bigcap_{i \in I} \tilde{\gamma}_i$ if it is not a face of H yet. The procedure is well-defined since any subgroup of a finite group is finite. The resulting polytope H' can be identified with a fundamental chamber of $\Sigma(W_0(Q_2))$, i.e. with \tilde{F}_2 .
- * Now, we do this for each $H_w, w \in W$ (a new face $\bigcap_{i \in I} \tilde{\gamma}_i$ of H_w is identified with the similar face of H_{wt_j} if $j \in I$). The procedure results in a new complex Y' .

Step 8. As H' can be identified with \tilde{F}_2 , and Y' is obtained from copies of H' by gluings along the action of W (see Step 6), the complex Y' can be identified with $X(Q_2) = \Sigma(W_0(Q_2))/W_C(Q_2)$.

7. General case

In all the considerations above we started with a Dynkin quiver Q . In this section we consider mutations of *diagrams* instead of quivers, which allows us to deal with non-simply-laced Dynkin diagrams.

7.1. Mutations of diagrams

Recall that quivers are in one-to-one correspondence with skew-symmetric integer matrices. An integer $n \times n$ matrix B is *skew-symmetrizable* if there exists a diagonal $n \times n$ matrix $D = \{d_1, \dots, d_n\}$ with positive integer entries such that the product BD is skew-symmetric, i.e., $b_{ij}d_j = -b_{ji}d_i$.

Given a skew-symmetrizable matrix B one obtains a *diagram* $\mathcal{G} = \mathcal{G}(B)$ in the following way: \mathcal{G} is an oriented labeled graph with vertex set $1, \dots, n$, where vertex i is connected to vertex j by an arrow labeled by $|b_{ij}b_{ji}|$ if $b_{ij} \neq 0$.

One diagram may correspond to several distinct matrices. Any diagram \mathcal{G} constructed by a skew-symmetrizable matrix satisfies the following property: for any chordless cycle in \mathcal{G} the product of labels along this cycle is a perfect square (cf. [21, Exercise 2.1]).

Since skew-symmetric matrices are also skew-symmetrizable, we may understand quivers as a partial case of diagrams: to make a diagram out of a quiver one needs to square all the labels.

A *mutation* μ_k of a diagram \mathcal{G} is defined similarly to the mutation of quivers [17]:

- orientations of all arrows incident to a vertex k are reversed;
- for every pair of vertices (i, j) such that \mathcal{G} contains arrows directed from i to k and from k to j the weight of the arrow joining i and j changes as described in Figure 7.1.

As for quivers, a diagram is *of finite type* if it is mutation-equivalent to an orientation of a Dynkin diagram.

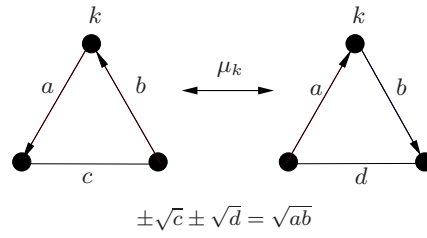


FIGURE 7.1. Mutations of diagrams. The sign before \sqrt{c} (resp., \sqrt{d}) is positive if the three vertices form an oriented cycle, and negative otherwise. Either c or d may vanish. If ab is equal to zero then neither value of c nor orientation of the corresponding arrow does change.

7.2. Generalization of the construction to non-simply-laced diagrams of finite type

The construction is generalized straightforwardly, but the definition of the group defined by a diagram is a bit longer than the one in the simply-laced case.

Let \mathcal{G} be an orientation of a Dynkin diagram with n nodes, let W be the corresponding finite Weyl group, and let \mathcal{G}_1 be any diagram mutation-equivalent to \mathcal{G} . Denote by $W(\mathcal{G}_1)$ the group generated by n generators s_i with the following relations:

- (R1) $s_i^2 = e$ for all $i = 1, \dots, n$;
- (R2) $(s_i s_j)^{m_{ij}} = e$ for all i, j , not joined by an edge labeled by 4, where

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge.} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge labeled by 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge labeled by 3.} \end{cases}$$

- (R3) (cycle relation) for every chordless oriented cycle \mathcal{C} given by

$$i_0 \xrightarrow{w_{i_0 i_1}} i_1 \xrightarrow{w_{i_1 i_2}} \dots \xrightarrow{w_{i_{d-2} i_{d-1}}} i_{d-1} \xrightarrow{w_{i_{d-1} i_0}} i_0$$

and for every $l = 0, \dots, d - 1$ we compute $t(l) = \left(\prod_{j=l}^{l+d-2} \sqrt{w_{i_j i_{j+1}}} - \sqrt{w_{i_{l+d-1} i_l}} \right)^2$, where the indices are considered modulo d ; now for every l such that $t(l) < 4$, we take the relation

$$(s_{i_l} s_{i_{l+1}} \dots s_{i_{l+d-2}} s_{i_{l+d-1}} s_{i_{l+d-2}} \dots s_{i_{l+1}})^{m(l)} = e,$$

where

$$m(l) = \begin{cases} 2 & \text{if } t(l) = 0; \\ 3 & \text{if } t(l) = 1; \\ 4 & \text{if } t(l) = 2; \\ 6 & \text{if } t(l) = 3 \end{cases}$$

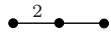



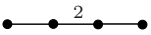

(this form of cycle relations was introduced by Seven in [32]).

According to [1], all the cycle relations for a given chordless cycle follow from one with $m(l) = 2$ (such a relation always exists). Thus, as for quivers, we may take exactly one defining relation per cycle.

It is shown in [1] that the group $W(\mathcal{G}_1)$ does not depend on the choice of a diagram in the mutation class of \mathcal{G} . In particular, it is isomorphic to the initial Coxeter group W .

The further part of the construction generalizes to the diagram settings verbatim. We obtain three manifolds, see Table 7.1

TABLE 7.1. *Actions on hyperbolic manifolds, non-simply-laced case.*

W	\mathcal{G}	\mathcal{G}_1	$ W $	$\dim(X)$	vol X approx.	number of cusps	$\chi(X)$ ($\dim X$ even)
B_3			$2^3 \cdot 3!$	2	8π	compact	-4
B_4			$2^4 \cdot 4!$	3	$ W \cdot 0.211446$	16	
F_4			$2^7 \cdot 3^2$	3	$ W \cdot 0.222228$	compact	

REMARK 7.1. The manifold X constructed for the group B_4 coincides with the manifold constructed in Section 5.3 for the group D_4 (see Table 5.1).

REMARK 7.2. In contrast to the simply-laced case, we do not know whether the list of hyperbolic manifolds (Table 7.1) that can be obtain by our construction is complete: there is no known classification of finite volume hyperbolic Coxeter polytopes with angles $\pi/2, \pi/3, \pi/4$, so potentially the construction may provide other examples of finite volume hyperbolic manifolds.

REMARK 7.3. The proof of the Manifold Property works for diagrams, but we need to update the proof of Lemma 6.5.

Let us proceed by induction on the number of mutations required to obtain \mathcal{G}_1 from \mathcal{G} . Assume that the lemma holds for a diagram \mathcal{G}_2 mutation-equivalent to \mathcal{G} , and we want to deduce the lemma for $\mathcal{G}_1 = \mu_k(\mathcal{G}_2)$. In the notation of Lemma 6.5 (after substituting Q by \mathcal{G}) this means the following: if we denote by $\tilde{u}_i, \tilde{u}_j, \tilde{u}_k \in W_0(\mathcal{G}_2)$ the reflections corresponding to nodes i, j, k of \mathcal{G}_2 and by $u_i, u_j, u_k \in W(\mathcal{G}_2) = W$ their projections to W , we want to deduce the equality of orders of elements $\tilde{t}_i \tilde{t}_j \in W_0(\mathcal{G}_1)$ and $t_i t_j \in W(\mathcal{G}_1) = W$ from the equality of the orders of $\tilde{u}_i \tilde{u}_j \in W_0(\mathcal{G}_2)$ and $u_i u_j \in W(\mathcal{G}_2) = W$.

If one of i, j (say, i) equals k the statement is obvious: in this case the label of the arrow between i and j is the same in \mathcal{G}_1 and \mathcal{G}_2 (and thus the orders of $\tilde{t}_i \tilde{t}_j \in W_0(\mathcal{G}_1)$ and $\tilde{u}_i \tilde{u}_j \in W_0(\mathcal{G}_2)$ are the same), and $t_i t_j = (u_i u_j)^{\pm 1}$ (since either $t_j = u_j$ or $t_j = u_i u_j u_i$), so the orders of $t_i t_j$ and $u_i u_j$ are also the same.

If the other case we can restrict our consideration to the subdiagrams of \mathcal{G}_1 and \mathcal{G}_2 spanned by three vertices i, j, k . A subdiagram of a diagram of finite type also has finite type [17], so if it is not simply-laced then \mathcal{G}_1 (and \mathcal{G}_2) is mutation-equivalent to one of the three diagrams $B_3, B_2 + A_1$ and $G_2 + A_1$. Now, a short straightforward check verifies the lemma.

7.3. *Example: action of B_3 on the 2-sphere and a hyperbolic surface of genus 3.*

Consider the diagram $\mathcal{G} = B_3$ (see Fig. 7.3(a)). The corresponding Dynkin diagram of type B_3 determines a finite Coxeter group W acting on the 2-dimensional sphere by reflections:

$$W = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_1 s_3)^2 = e \rangle$$

The fundamental domain of this action is a spherical triangle with angles $(\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2})$, see Fig. 7.3(b) for the action and a fundamental domain in stereographic projection.

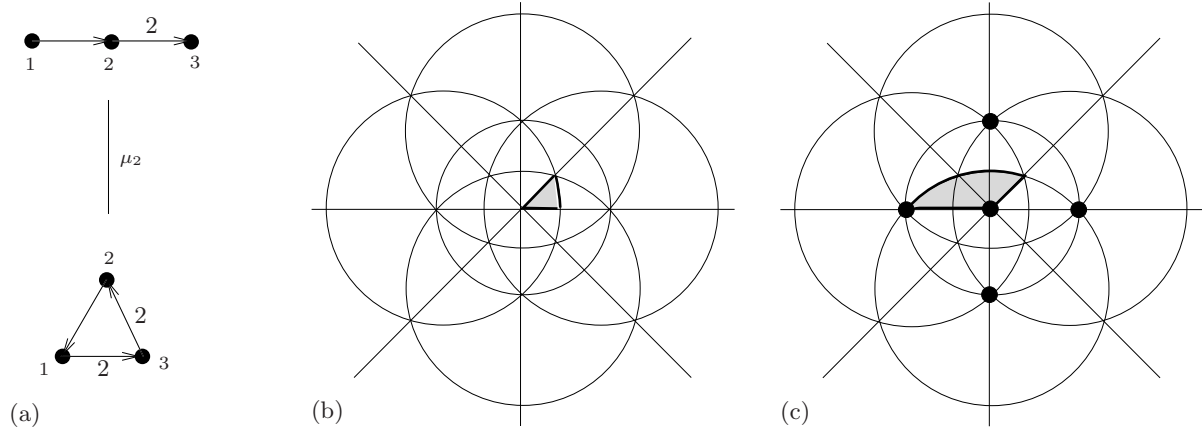


FIGURE 7.2. (a) A diagram of type B_3 and its mutation μ_2 ; (b) action of B_3 on the sphere, in s -generators (stereographic projection); (c) action of B_3 on the sphere in t -generators: bold points are the branching points for the covering (one more branching point at infinity)

Now, apply the mutation μ_2 to the diagram \mathcal{G} obtaining the diagram \mathcal{G}_1 (see Fig. 7.3(a)). The corresponding group

$$W_0(\mathcal{G}_1) = \langle t_1, t_2, t_3 \mid t_i^2 = (t_1 t_2)^3 = (t_2 t_3)^4 = (t_1 t_3)^4 = e \rangle$$

acts on the hyperbolic plane by reflections, and the fundamental domain of this action is a triangle F with angles $(\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4})$. We need to take a quotient of this action by the normal closure of the element $(t_1 t_2 t_3 t_2)^2 \in W_0(\mathcal{G}_1)$. As we expect from Manifold Property, we will get some finite volume hyperbolic surface S tiled by 48 images of the triangle F . It is easy to see that S is a genus 3 surface.

Similarly to Section 5.1.3 describing A_3 case, one can see that S can be obtained as a ramified degree 4 covering of the sphere branching at six points composing the orbit of the “central point” (see Fig. 7.3(c)).

8. Infinite groups

8.1. Group constructed by a diagram

The result of [14] allows to generalize the construction to infinite groups. More precisely, it is shown in [14] that to a diagram \mathcal{G} of finite mutation type that either is exceptional (see [9, 10]) or arises from a triangulation of a unpunctured surface or orbifold (see [15, 11]) one can assign a Coxeter group $W_0(\mathcal{G})$ (using exactly the same rule as above).

Then one takes a quotient group imposing relations of the following types:

- (R3) cycle relations;
- (R4) additional affine relations;
- (R5) additional handle relations;
- (R6) additional X_5 -relations.

All these relations are of the type $(r_i r_j)^m = e$, where r_i and r_j are some conjugates of the generating reflections, see [14] for details.

As before, we define $W_C(\mathcal{G})$ as the normal closure of all relations of types (R3)-(R6). It is shown in [14] that the group $W(\mathcal{G}) = W_0(\mathcal{G})/W_C(\mathcal{G})$ does not depend on the choice of \mathcal{G} in a given mutation class.

This implies that we can repeat verbatim the construction of actions on factors of the Davis complex $\Sigma(W_0(\mathcal{G}))$, so that we get an action of an (infinite) group $W = W_0(\mathcal{G})/W_C(\mathcal{G})$ on the quotient space $X(\mathcal{G}) = \Sigma(W_0(\mathcal{G}))/W_C(\mathcal{G})$. Furthermore, as before, a mutation of \mathcal{G} induces a mutation of $X(\mathcal{G})$.

8.2. Diagrams of affine type

A diagram \mathcal{G} is called a diagram of *affine type* if \mathcal{G} is mutation-equivalent to an orientation of an affine Dynkin diagram different from an oriented cycle. In particular, if \mathcal{G} is of affine type, the group $W = W(\mathcal{G})$ is an affine Coxeter group.

Mutation classes of diagrams of affine type may be rather complicated (see [18] for the detailed description of mutation classes of non-exceptional diagrams of finite and affine type). In particular, a diagram \mathcal{G} in the mutation class may contain one arrow labeled by 4 (this corresponds to two reflections in a Coxeter system of $W_0(\mathcal{G})$ generating an infinite dihedral group, or, equivalently, to two non-intersecting facets of a fundamental chamber of $\Sigma(W_0(\mathcal{G}))$).

Nevertheless, applying considerations from Section 6.3 and Remark 7.3, we obtain the Manifold Property for diagrams of affine type.

THEOREM 8.1. *If \mathcal{G} is a diagram of affine type then the group $W_C(\mathcal{G})$ is torsion-free.*

REMARK 8.2. We are currently unable to drop the assumption that \mathcal{G} is of affine type. Our proof of Manifold Property uses an induction on the number of mutations one needs to perform to obtain a diagram without oriented cycles (such diagrams are called *acyclic*, and diagrams from their mutation classes are *mutation-acyclic*). If \mathcal{G} is not of finite or affine type, then either \mathcal{G} is not mutation-acyclic, or it is not mutation-finite (see [15, 9, 10]).

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