Reduction & emergence in the fractional quantum Hall state

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Abstract:

We present the fractional quantum Hall (FQH) effect as a candidate emergent phenomenon. Unlike some other putative cases of condensed matter emergence (such as thermal phase transitions), the FQH effect is not based on symmetry breaking. Instead FQH states are part of a distinct class of ordered matter that is defined topologically. Topologically ordered states result from complex long-ranged correlations between their constituent parts, such that the system displays strongly irreducible, qualitatively novel properties.

Keywords: Emergence; condensed matter physics; many-body quantum mechanics; quantum field theory; quantum Hall effect

1. *Introductory remarks*

The *fractional quantum Hall effect* (FQHE) is a type of collective behavior realized in a 2D system of electrons. At low temperatures an interacting ensemble of electrons realizes a fluid-like state. In the presence of a magnetic field applied with a specific magnitude, the longitudinal resistivity becomes exponentially small and Hall resistance of the 2D fluid shows plateaux. Each plateau occurs at a value of resistivity determined by a given value of the so-called filling factor *v*, which describes a ratio of filled to vacant electronic states. In cases when *v* takes an integer number we observe *the integer quantum Hall effec*t (IQHE); in which the Hall resistivity is found to be quantized in units of h/e^2. However, when *v* is a ratio of integers (frequently one with an odd denominator) then we observe the FQHE, which involves the fluid-like electron liquid showing a form of generalized rigidity, along with an unusual spectrum of excitations involving fractional quantum numbers. That is to say, the excitations of a FQH system carry fractions of an electronic charge and are neither bosons nor fermions. Despite their similar names, the FQHE is characteristic of different phases of condensed matter to the IQHE, with this difference arising owing to the role of the interactions between electrons in the former case. Although most phases of condensed matter can be characterized by symmetry considerations, the FQH state is instead characterized by *topological order*. The discovery and theoretical elucidation of the FQHE won Robert Laughlin (1983), Daniel Tsui and Horst Störmer (1982) the 1998 Nobel Prize in Physics.

The aim of this paper is twofold. Firstly, we introduce the rich physics of the FQHE to a philosophical audience. In order to facilitate this, we will go into the physics of the FQHE in some detail. We hope, given the unfamiliarity of topological states of matter to many philosophers of physics, that readers will be understanding if more technical details are included than would usually be the case when discussing a more familiar example (such as superconductivity). Secondly, within physics the FQHE is often considered a paradigmatic case of emergence. We attempt to connect the physicists' conception of emergence to philosophical notions of emergence. We will therefore aim to categorize the ways in which the FQHE can be said to be an emergent phenomenon. Our conclusion is that the presence of topological order in the FQHE is indicative of an intrinsic holism to the FQH system. Because of this, the FQHE bears serious consideration as an example of a metaphysically significant, "strongly" emergent phenomenon.

2. *Ways of characterising emergence*

Condensed matter physics is one of the arenas in which there is a renewed interest in the notion of emergence. The often cited starting point for this resurgence of "New Emergentist" thinking is Anderson's (1972) paper *More is Different*. Anderson claims many condensed matter systems are characterized by novel particular systemic properties. Since then many authors have argued that physics provides examples of emergent phenomena (e.g. see Morrison 2012, Laughlin & Pines 2000, Batterman 2003, Bedau 2008).[1] The senses of emergence argued for by these authors suggest different degrees of metaphysical significance. For example: Bedau argues for weak emergence, which is a failure of predictability due to inherent complexity and contingency. Batterman is primarily concerned with explanatory emergence; where explanations fail to reduce to explanations expressed only in the vocabulary of a lower level description. Morrison is concerned with the ability of higher-level phenomena to bring about new properties of matter, such as spontaneous symmetry breaking within an electromagnetic gauge theory, resulting in superconductivity.

By contrast other authors (Humphreys 1997, Teller 1986, Howard 2007, Silberstein & McGeever 1999) have looked towards quantum phenomena such as entanglement as the best candidate for emergence, while Mark Wilson (1993) and Paul Mainwood (2006) have suggested that there are parallels between the emergentism of Anderson and the views expressed by the British Emergentists of the early 20th century such as C. D. Broad (1925). Inspired by these various accounts we will define four ways emergence can be spelled out; each depending on the way the relationship between parts and wholes is considered. This is far from an exhaustive list of the ways emergence can be thought of, but it will provide a simplified framework by which to compare claims about the FQHE.

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[1] Of course there are many other areas where emergence is discussed, such as the philosophy of mind and chemistry (e.g. Gibb 2012, O'Connor & Wong 2005, Hendry 2010).

Emergence 0: (E0) *Failure of inter-theoretic reduction or failure of explanatory reduction*. This is concerned with the properties appearing in two different descriptions of the same composite system. Are there features of system *S* that can only be explained by referring to *S* in terms of a level specific vocabulary, or can all theoretical predictions and explanations ultimately be spelt out in terms of microphysical theories/models/explanatory strategies?

Emergence 1: (E1) *Entity emergentism*. Entities are known to us through their indispensable role in prediction, explanation and manipulation (c.f. Hacking 1988). If a composite system produces novel entities which fulfil this role then these entities are ontologically robust.

Emergence 2: (E2) Novelty of systemic properties of composites compared with the components of those composites in isolation (or other, different, composites). This is the position taken by British emergentists such as Broad and by New Emergentists such as Anderson, Laughlin & Pines.

Emergence 3: (E3) Emergence as a failure of mereological supervenience. This position is concerned with properties of whole systems that are novel relative to the properties of the parts of that composite whilst part of that whole. One possibility is this novelty manifests itself as a new set of causal powers.

These different notions, whilst distinct, are not mutually exclusive, for example, belief in E2 may imply E0 (although E0 certainly does not imply E2). In this paper we leave aside E0 emergence. This is because: 1) we agree with Butterfield & Isham (1999) and Mainwood (2006) that the syntactic form of inter-theory reduction is too flexible a notion to be an interesting sense of emergence. 2) By contrast, although we think that a failure of *explanatory reduction* is an interesting sense of emergence, we want to focus on emergent features potentially unique to topological phases of matter. In short, although we believe FQHE is E0 emergent (in the explanatory sense), we believe that E0 emergence is widespread and discussions of it can be motivated by considering cases involving more familiar physics (e.g. thermal phase transitions). As such we will focus the discussion by comparing the FQHE to the senses of emergence captured by E1, E2, and E3.

3. *The Physics of the FQHE*

In this section we will review the physics of the FQHE before moving on to philosophical discussion in Section 4. Since we suspect the particularities of the FQHE will be unfamiliar to many readers we will spend some time discussing detailed aspects of the physics. At the end of this section (3.7) there will be a brief re-cap which will summarise the key points of the physics.

3.1 *The three flavours of the Hall Effect*

This paper is based on a simple experiment shown schematically in Fig. 1. When an electrical current passes through a conductor which lies in a transverse magnetic field, it is found that a voltage develops perpendicular to both the current and the magnetic field directions. This is known as the Hall effect (see Singleton 2001). The effect has a simple explanation based on the physics of classical electromagnetism: the moving electrons constituting the current feel a Lorentz force acting perpendicular to both their velocity and the magnetic field direction. This force deflects the electrons and causes electric charge to build up on the walls of the conductor, as shown in Fig. 1.
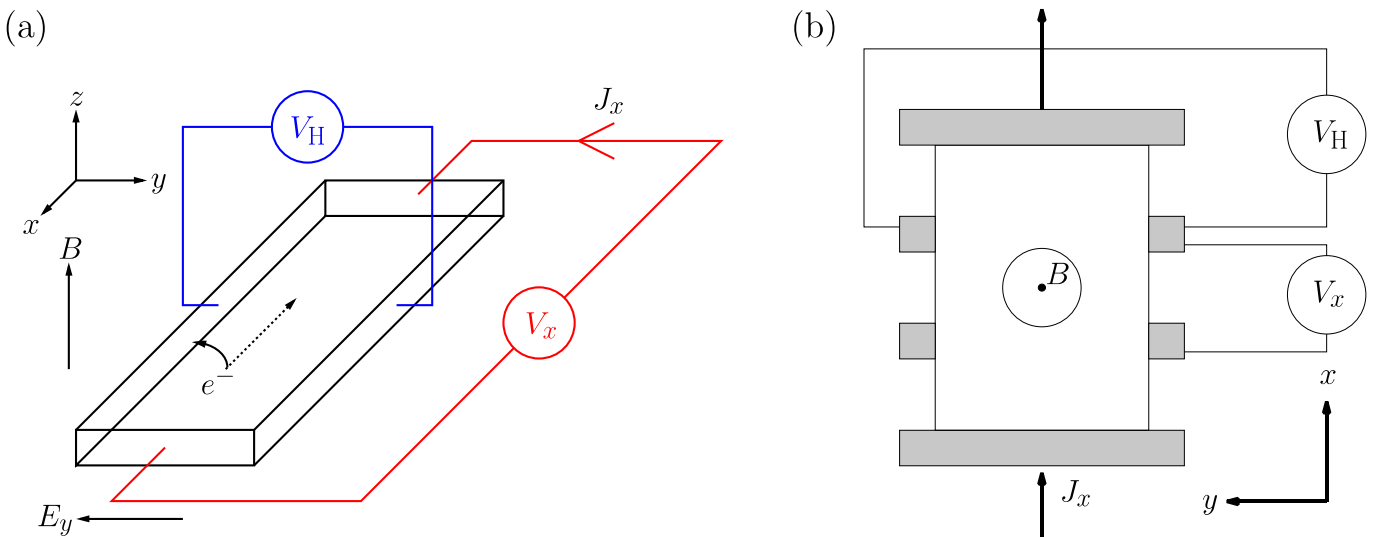

Figure 1: The Hall geometry for the (a) conventional and (b) quantum Hall cases. For the quantum cases, the B-field is perpendicular to the plane containing the 2D electron system

The build-up of charge creates an electric field E_y that opposes the Lorentz force, allowing other electrons to pass though the conductor. However, the result of this build-up of static charge is the transverse Hall voltage. Historically, the Hall effect was important in the development of our understanding of the physics of solid materials. However, in the last forty years it has also been key to the development of our understanding of many-body quantum mechanics. This arose from the possibility to synthesise semiconductors in which electrons are constrained to move in two dimensions (2D) only. These semiconductor structures energetically confine electrons in a deep potential well in one spatial dimension, but do not constrain their motion in the other two. When the Hall effect experiment is repeated on such a system (using the geometry shown in Fig1 (b)), with a magnetic field directed perpendicular to the 2D plane, we measure *quantised* behaviour in the Hall effect, indicative of well-resolved quantum mechanical energy levels. Specifically, we apply the magnetic field B in the *z*-direction and drive a current J_x in the *x*-direction, measuring the transverse (or Hall) resistivity $\rho_{xy} = E_y / J_x$ [Fig. 2(a) and (b)].
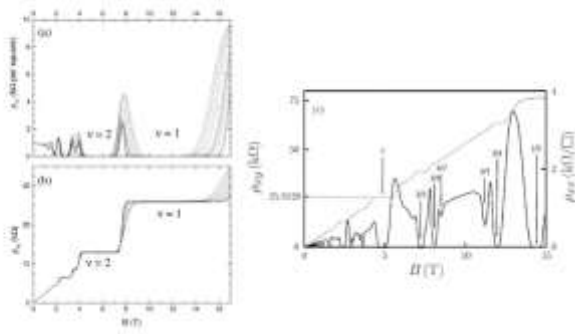
Figure 2: (a) and (b) the quantum Hall effect, with its characteristic plateaux observed for integer filling fractions v. (c) The fractional Hall effect. [Figure reprinted from Lancaster & Blundell 2012, data from Singleton (2001) and P. Gee, D.Phil. Thesis, Clarendon Laboratory, University of Oxford (1997)].

The most striking results is that ρ_{xy} exhibits plateaux as a function of applied magnetic field [Fig. 2(b)]. We also find that the longitudinal resistivity $\rho_{xx} = E_x \, / \, J_x$ becomes vanishingly (or immeasurably) small along the flat sections of the plateaux, taking non-zero values only in the regions between plateaux [Fig. 2(a)]. This observation is known as the integer quantum Hall effect (IQHE) (Singleton 2011).

The explanation for this physics is essentially a quantum mechanical (momentum-) space filling argument. The system comprises mobile electrons, which are fermions and hence are prevented by the exclusion principle from occupying the same states as other electrons. There are only a certain number of states available to the 2D system in which the fermions find themselves and these are further constrained by the additional quantisation caused by the presence of the magnetic field. Classically we expect electric charges in a magnetic field to execute cyclotron orbits; quantum mechanically, the corresponding effect is the quantisation of the electrons' energy into so-called *Landau levels*. In a 2D system these Landau levels are highly degenerate, which is to say that there are many available states for electrons to fill which all have the same energy. Specifically, the degeneracy of each Landau level is proportional to the applied field B, and is given by eB/h, where e is the electronic charge and h is Planck's constant. Changing the value of B, therefore changes the number of states available in each Landau level and hence their occupancy. The filling factor v, introduced above, is the number of completely filled Landau levels.

At a magnetic field where a Landau level is completely filled (i.e. integer v) we can have a situation not unlike that of an insulator: there are no available energy states within easy reach for electrons. The system shows an energy gap between its ground state and first excited state, as would be expected for a system that had completely filled all energetically available quantum states. We might say that attempts to compress the system would therefore fail, as this would usually promote electrons into higher-lying energy levels. No such levels are accessible and so the matter is said to be *incompressible*. As in the case of an insulator, the lack of available states causes the electrical conductivity to be zero (for well-separated Landau levels, at least). Rather unexpectedly, in two dimensions it turns out that conductivity and resistivity are (roughly) proportional and so we also have a vanishing longitudinal resistivity ρ_{xx} for filled Landau levels. Since the zeros of ρ_{xx} correspond to the plateaux in ρ_{xy} then, using the assumption that filling a whole number of Landau levels results in a plateau, allows the correct calculation of the value of ρ_{xy} for each the plateau. We therefore label each plateau by an integer value of v, telling us how many Landau levels are completely filled.

At this point we must consider a subtlety: although this argument gives a reason for the zeros of ρ_{xx} (i.e. completely filled Landau levels) and the values of ρ_{xy} along the plateaux, it does not actually explain the existence of the plateaux themselves and the transitions between them. This is due to the presence of *disorder*. All real materials contain impurities, which will cause small variations (or fluctuations) in the energy of the Landau levels in space, and it is these variations turn out to be responsible for the observed features of the QHE. An idea of the physics at play may be seen through a thought experiment. If we apply a sufficiently large field B [i.e. starting from the right hand side of Figure 2(b)], then we can provide the electrons in the system (of which there are a fixed number, n, per unit area) with a state in the lowest Landau level (LLL) and this level will be completely filled. We have exactly one electron for each state in the first Landau level and v=1. As we decrease the field, there are no longer enough states in the LLL to house all of the electrons and so some will be forced up in energy to the next Landau level. As some electrons are forced into the second Landau level they will fill up the lowest energy states in the fluctuating energy landscape. In two dimensions, these states will reside in isolated valleys in the energy landscape, so that the electrons will form disconnected puddles in space. Since these puddles are isolated from each other the electrons will not play a role in conduction. As a result, the longitudinal resistivity ρ_{xx} will fall to zero and the Hall voltage will remain constant, causing ρ_{xy} to remain at the value it took for v=1, leading to the observed plateau. It is only when the second Landau level is sufficiently filled that the puddles in the energy landscape begin to overlap, that ρ_{xx} becomes measurable and a transition between plateaux is observed.

The above effect of electrons being trapped in the potential landscape is known as *localization* and, while a detailed discussion of the mechanism for localization is beyond the scope of this article, it is notable that disorder has a special dual role in the quantum Hall effect: it is necessary for the observation of the effect in the components of resistivity, but despite sample-dependent disorder, the resistance in any step is constant to better than a few parts in 10^8. In fact, the quantization of the $\rho_{xy} = (1/v)(h/e^2)$ has led to this becoming the international standard of resistance.

In summary, most of the salient features of the IQHE may be accounted for within this framework, without the need to consider the repulsive interactions between the mobile, charge-carrying electrons in the system.

The subject of this paper is what happens to the 2D system when the applied magnetic field is increased to very large values. Under these circumstances all electrons should be confined within the first Landau level. However, here we find something more unexpected: additional Hall plateaux appear at fractional filling factors, corresponding to $v = p / q$, where p and q are integers with no common factors (q is found to be an odd number for most states). Examples of fractional values of v include v = 1/3, 2/5, 3/7, 2/3, 3/5, 4/7... [Fig. 2(c)]. This is the fractional quantum Hall effect (FQHE) [13, 14, 12]. In these cases, despite the first Landau level being only partly filled, the system shows an energy gap between its ground state and first excited state, as would be expected for a system that had completely filled all available quantum states. This is the same as in the IQHE described above and so the matter is also said to be *incompressible*.

Perhaps surprisingly, the explanation of the FQHE we will give is rather different to that of the IQHE. The description of the IQHE given above has relied on an argument involving filling single-particle energy levels, that does not need to invoke the repulsive electrostatic interactions between the electrons. In contrast to the IQHE, the FQHE occurs when a Landau level is partly filled. The incompressibility of the state relies on the electron-electron interactions and these interactions are indispensible in any description of this physics. One way of seeing the role of electron-electron interactions is that the partially filled Landau level will have an enormous degeneracy (i.e. a huge number of possible states with the same energy) making it susceptible to dramatic macroscopic restructuring by (possibly small) interactions. Moreover, we will see that, unlike most phases of ordered matter, the FQH fluid can be described using a *topological gauge theory*. We will also see that the fractional quantum Hall fluid is home to exotic anyonic excitations which have fractions of an electronic charge and which exhibit fractional statistics.[2]

3.2 *Order in the FQH states*

Many phases of matter show some form of order, which allows different phases to be distinguished by the nature of that ordering. Order usually accompanies some form of *generalized rigidity* (Anderson 1997), which reflects the fact that the deformation of an ordered state is usually resisted by forces that appear upon ordering, but which are absent in the disordered state. From this point of view each FQH state (distinguished by a different fractional v) should be regarded as a distinct phase of matter. That is to say, we should speak of the fractional quantum Hall *effects* and distinguish between the phase of matter described, for example, by v = 1/3 and that described by some other fractional value of v. The FQH state is, however, a different form of order to that usually found at temperatures below a thermal phase transition. The FQH state is often called topologically ordered. Topological orders are general properties of many-body quantum states at zero temperature, which have a non-zero energy gap between the ground state and excited states (Wen 2004)

Most ordered phases occur on lowering the temperature of a system, and thus forcing it through a thermal phase transition that occurs at a critical temperature T_c. In contrast to a thermal phase transition, a *quantum phase transition* (Sachdev 2011) takes place at $T = 0$ and is caused by the change of another parameter such as applied magnetic field or pressure. It is a quantum phase transition that separates each different FQH state. (It is worth noting that this doesn't imply the FQHE is only observed at the inaccessible temperature T=0, although it is observed exclusively at low (usually milliKelvin) temperatures.) Moreover, although most phase transitions involve a lowering of the symmetry of a state, the FQH states all have the same symmetry, despite having different values of v (Wen 2004).

Another way of viewing the order in a particular FQH state (Wen 2004) is to consider a picture of semiclassical electron motion in a magnetic field. This reveals that interactions force the many electrons in the fluid to adopt patterns of motion that characterise the FQH state. This picture follows from the (fully quantum mechanical) Laughlin wavefunctions (Laughlin 1983), which encode the quantum mechanical dynamics of the states mathematically. An example is the v=1/3 wavefunction $\Psi=\Pi_{i<j}(z_i-z_j)^3\exp(-\Sigma|z_i|^2/4l_B^2)$, where z is the complex coordinate and l_B is the magnetic length. We stress that these wavefunctions are not the result of an exact diagonalization of the microphysical Hamiltonian; rather they were discovered as an ansatz, intended to capture some of the observed behaviour of the FQH states. It is also worth noting that they are many-body wavefunctions, describing correlations of all of the electrons in the material in one wavefunction. A semiclassical interpretation of these wavefunctions (Wen 2004) suggests that these wavefunctions are a good ansatz as (i) they guarantee the electrons are in the LLL, (ii) they preserve Fermi statistics for all electrons as they exchange places and (iii) they minimize interaction energies very effectively. Indeed a quantum mechanical analysis shows that such a wavefunction does indeed describe an incompressible state with fractional excitations.

3.3. *Topology and FQH states*

So far we have described the FQH fluid as 'topological', without providing a link between the FQHE and mathematical topology. In this section we discuss this link. Topology describes the study of the overall shapes of objects and spaces, irrespective of the details of their exact geometry. In topological terms a doughnut is identical to a coffee mug, in that both may be continuously deformed into an identical torus. (This equivalence under smooth deformations is known as a diffeomorphism.) Topology is therefore blind to the details of lengths and time-interval, which are mathematically described by a spacetime metric tensor $g_{\mu\nu}$ Topology enters our discussion of the FQHE in two places. The first is in the determination of the exchange statistics of identical particles in two dimensions, which relies on considering the overall shapes of paths in a two-dimensional plane. The second is in the field theoretical formulation of the FQHE, which is a mathematical theory that does not involve the components of the metric $g_{\mu\nu}$ and is therefore blind to geometry and consequently described as topological.

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[2] A detailed comparison of the IQHE and FQHE is beyond the scope of this article. We note however that, although the simple account of IQH physics given here does not rely on electron-electron interactions, these are invoked in explaining some more subtle features of the state of matter. Most crucially, it is the special topological features of the FQH fluid, not possessed by the IQH fluid, that are our main concern in discussing emergence in this context and which lead to the fractional excitations.
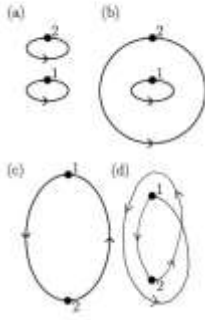
Figure 3: Particle exchange of identical particles in two dimensions. [Figure reprinted from Lancaster & Blundell 2012.]

The bosonic and fermionic statistics of identical particles follow from their exchange properties. The topology of the two-dimensional space in which the FQH effects occur requires that particles are exchanged by moving them around within a plane. We shall see that this feature allows more possibilities over and above the existence of Bose and Fermi particles. The crux of the matter is that the manner in which particles must slide around the two-dimensional plane may be described by a path in space and that these paths, usually parametrized by the angle φ through which one particle is moved around another, have topological properties. Although the details of the exact route the particle paths have taken are unimportant (since these can be altered by smoothly and continuously changing them, as topology allows), there are some key features of the paths that cannot be removed without introducing singularities into the paths (a move that is strictly forbidden by topology). Specifically, moving one particle around another allows us to label a path with a winding number, which counts how many times one particle has passed around the other. Such a number is *topologically protected* in 2D. However, it is not protected in this way in 3D where all paths can be made equivalent, since the extra dimension allows paths within the plane to be effectively lifted over each other and then contracted into a point.

To exchange two identical particles in 2D, we may simply move one particle around the other by an angle ϕ of π (and then translate the whole system if necessary), so we end up with a system that is the same as the one with which we started. In this process we have swapped the particles $x_1 \to x_2$ and $x_2 \to x_1$. We could also loop one particle around the other by an angle $\phi = 2\pi$, such that $x_1 \to x_1$ and $x_2 \to x_2$. In fact, the first process could include winding the first particle around the second by an angle $\phi = (2p + 1)\pi$ with p an integer (including zero) [examples of this are shown in Fig. 3(c) and (d)]; while the second involves an angle $2p\pi$ [Fig. 3(a) and (b)]. Each value of the winding number p describes a topologically distinct process. This process of winding particles around each other is known technically as braiding.

In terms of quantum mechanics, we note that, during the exchange process, when we move one around the other by some angle ϕ, the contribution this rotation makes to the wave function $\Phi(\phi)$ should be pure phase and have the property $\Phi(\phi_1 + \phi_2) = \Phi(\phi_1)\Phi(\phi_2)$. This implies $\Phi = e^{i\eta\phi}$, where η is some numerical parameter. As described above, exchanging two particles can be achieved by rotating one about the other by an angle π and then translating the system, resulting in a multiplicative contribution to the wave function of $e^{i\eta\pi}$. To recover Bose statistics we need only assume η is an even integer; while for Fermi statistics we take η to be an odd integer. However, there is nothing that necessitates η being an integer at all, which opens up the possibility of anyon statistics, which are those described by fractional values of η. The prediction of a fractional value of η falls neatly out of the mathematical formulation of the FQHE.

3.4. *The problem with a microscopic description of correlated electron physics*

It might be naively expected that a description of the FQHE would follow from writing down a Hamiltonian for the electrons in a two-dimensional plane, subject to a magnetic field. Although this approach should contain all of the physics, it has proven impossible to deduce FQH states using only the information in this microphysical Hamiltonian. It is well known that a theory of a macroscopic number of strongly interacting electrons is a very difficult one, which appears not to admit an exact solution. However, the FQH fluid has special properties preventing, in principle, even those approximate methods used for conventional many-electron physics, which have proved useful in elucidating the behaviour of metals.

The problem is seen if we imagine being able to (temporarily) turn off electron-electron interactions. Then, for $v < 1$, there are more states in the lowest Landau level than particles and, crucially, these states are degenerate in energy. This causes the system as a whole to have a macroscopic degeneracy. For the $v = 1/5$ state, for example, we expect five states in the lowest Landau level per electron. The interactions between electrons, when turned back on, will break this vast degeneracy, but from the microphysics alone it is impossible to track how the degeneracy will be broken.

In fact, it is this degeneracy that prevents our treating the interaction as a perturbation (Murthy & Shankar 2003), which we would gradually turn back on, as one would do for a calculation in the conventional many-body problem using perturbation theory (Lancaster & Blundell 2014). Moreover, since perturbation analysis is impossible we cannot look to asymptotic behaviour to pick out macrostates. Another conventional many-body route to treating the interacting problem, the Hartree-Fock mean-field theory, also fails rather dramatically for the FQH fluid, as it predicts that the 2D electron gas in a strong magnetic field leads to the Wigner crystal state with broken translational symmetry.

3.5 The Chern-Simons theory

Topological theories do not rely on the clocks and rulers implicit in ordinary theories. These clocks and rulers are represented by the metric tensor $g_{\mu\nu}$, which is an object that tells us how to combine vectors together to extract length and time intervals. Topological theories are based on *gauge fields*. These fields, which also form the basis of theories such as the standard model of particle physics, for example, guarantee that a system has a *local symmetry* (Lancaster & Blundell 2014). The origin of the gauge field a_μ, which features in the explanation of the FQHE, is the

electron-electron interactions. This can be demonstrated using Hamiltonian theories of the effect (Fradkin 2013, Murthy & Shankar 2003). (In fact, a_μ is often casualy called a dynamically generated emergent field in the physics literature. (Wen 2004)) In the effective Lagrangian picture a_μ is assumed to capture the low-energy properties of the experiment, which only probes the long-time, low-energy phenomena of the conductor. As with other gauge theories, the details of how the gauge field is generated and its physical interpretation are open to question. However, we will show later that a picture of quantum entanglement provides a more direct means of classifying topological theories.

The topological theory that explains the FQHE is given by the Chern-Simons (CS) theory. Physically, we may think of this as a mathematical means of tying flux (or, more correctly, vorticity) of the a_μ field to the particles making up the current J_μ. The consequence of the topological nature of this theory is seen if we try to extract a Hamiltonian H from the CS term of this theory. We run into trouble since this requires us to take a functional derivative of the action $S = \int d^4x \, \mathcal{L}$ with respect to the components of $g_{\mu\nu}$. As the CS term in the Lagrangian does not contain $g_{\mu\nu}$ then we obtain *H = 0*. The solution to this Hamiltonian is that all energy levels have *E = 0*. However, a serious complication is that we don't know how many states we have with zero energy and, in fact, evaluating this degeneracy relies on our knowledge of the topology of the manifold in which the theory is defined.

It may be shown (Wen 2004, Fradkin 2013, Lancaster & Blundell 2012), using the machinery of quantum field theory, that a CS Lagrangian coupled to the electromagnetic field accounts for the FQHE. Using this theory the main features of the FQHE for the odd-denominator values of *v* may be accounted for, along with the prediction (via the braiding arguments outlined above) that the quasiparticles that exist in the FQH fluid are anyonic, with fractional statistics and charge. It is worth noting that the argument that forces 1/*v* to be an odd integer, and which therefore explains the formation of a subset of the FQH states and the statistics of their quasiparticles, follows from forcing the theory to be consistent with the existence of conventional electrons with fermionic statistics in the system (implying that the electrons are, in some sense, still a building block of the system). Of course, accounting for its features should not be confused with explaining the FQHE. It is an open question whether the FQHE can be explained by this approach (see below).

3.6 Multiple realisability and topological computation

Since it is the topological properties of the FQH system that characterise it (such as the winding numbers for the particle paths, or the Chern-Simons Lagrangian), FQH systems are formally stable against perturbations, such as those introduced by impurities in the conductor. This stability reflects the extreme multiple realisability of the states. This multiple realisability goes far beyond many standard examples from physics, such as superconductivity, or phase changes. This is because FQHE is not only multiply realisable with respect to changes in the microphysical makeup (such as the material and its individual level of disorder), but also with respect to changes in the space-time metric. It is well known that in quantum mechanics we can describe the evolution of the system by acting on the wavefunction with operators that represent physical interactions.

In the case of a topological phase of matter, invariance under a diffeomorphism means the only thing a local interaction of the system can do is keep the system the same. Any local perturbation, such as a photon striking an electron, couples to local operators and cannot change the state of the system. This stability is indicative of the whole-system dynamics which prevent local perturbations destroying the system. Again, it is important to stress that this is not the usual insensitivity to perturbations encountered in standard physics. In these standard cases, a local perturbation is epistemically irrelevant for some phenomenon defined on a particular length scale. In topological phases of matter this stability is much stronger and is grounded in the non-local topological order of the system, not simply the epistemic irrelevance of perturbations. This difference can be illustrated by thinking about a potential application of the FQHE for topological quantum computation (see Nayak et al. 2008 for a comprehensive review).

Topological computers build logic gates out of the different possible braiding relations of quasiparticles and use those gates to execute programs. If realisable, topological computers would represent a significant improvement over standard classical and even quantum computing. Classical computers have definite states whereas a quantum computer can use the indeterminacy of quantum mechanics to use many different states in parallel to perform calculations. However, one significant challenge to realising quantum computers is the problem of error correction. As a computer performs operations small errors creep in due to quantum decoherence, and these must be corrected if the calculation is to be performed accurately. A topological computer, by contrast, has no such problems. Since we are braiding quasiparticles around one another to perform operations the system only depends on things that affect the topology of space, and no local perturbation can change this. The system is immune from decoherence because no local unitary operation can change the state of the system. The stability of topological computers reflects the extreme physical, not epistemic, multiple realisability of the system.

Finally, it is worth addressing a possible concern at this point concerning our twin claims of multiple realisability and a failure of suervenience. One may think that since the state is realisable, and realisable in many different ways then there cannot be a failure of supervenience. However, recall that supervenience isn't about realisation, it is about the way changes affect a system. A FQH state is realized by having certain lower level entities correlate in certain ways, but does not supervene on the metrological sum of those entities because it is not possible to made certain changes to those lower level entities in isolation *in principle*. Consider, for example, a simple example of a single pair of maximally entangled particles such as in an EPR experiment. This state can be formed by many different *types* of particles, hence an EPR entangled state is multiple realisable, but there is still a failure of mereological supervenience of each concrete particular realising base.

3.7 Long-range Entanglement

We can now explore the microphysical dynamics that lead to FQH states being holistic and strongly emergent. As we have seen the Chern-Simons field can be thought of as tying a quantum property, quanta of vorticity, to electrons to produce composite fermions. The underlying microphysical basis for this is to be found in the electron interactions themselves (Wen 2013, Chen, Gu & Wen 2010). In the FQHE the correlation of local operators is short-ranged. However, the ground state degeneracy of the FQH state depends on the topology of space, this means the electrons in FQHE states cannot just be correlated over small distances; they must have a mechanism of achieving long-range correlations. This mechanism is Chen *et al.*'s conception of *long-range entanglement*.

Chen *et al.*'s definition of long-range entanglement (which we refer to simply as "long range entanglement" hereafter and which is described in more detail below) is the characteristic property of their conception of *topologically ordered* states of matter. Topologically order is a mathematical means of characterizing non-symmetry breaking order in those systems with an energy gap between their ground state and excited states. On the strength of all of the experimental evidence, the FQH fluid is believed to be such a topologically ordered state. It is worth noting that other topological states are thought to exist, such as gapped spin liquids (strongly interacting magnetic phases which retain rotational spin symmetry down to T=0 and have a gap in their spectrum), but at present the experimental evidence for them is less conclusive and the FQH fluid remains the only generally accepted example of such a phase of matter.

Chen, Gu & Wen (2010) define topological order by contrasting short-range and long-range entanglement. A system has short-range entanglement if it can be transformed into an unentangled state through local unitary evolution. Mathematically, we define a local unitary (LU) transformation

$$U = T\left[e^{i\int_0^1 dg \tilde{H}(g)}\right]$$

where T is a path-ordering operator, $\tilde{H}(g) = \Sigma_i \tilde{O}_i(g)$ is a sum of local Hermitian operators and g parameterizes the path between states. All short-range entangled (SRE) states can be transformed into each other via LU transformations. By contrast, long-range entangled (LRE) states cannot be disentangled via a LU transformation. It may be shown that gapped ground states achieved via Landau symmetry breaking have only short-range entanglement, while topologically ordered phases of matter have long-range entanglement.

Topological order is then underpinned by a pattern of LRE. It is this that allows the system to know the topology of the space. It is also these long-range entanglements that mean that alternative descriptions of the system, such as in terms of composite fermions (see below), cannot ignore the whole system dynamics. This entanglement underpins the radical multiple realisability of the system and its immunity to local perturbations. Any local perturbation (which is not large enough to destroy the system) is suppressed by non-local correlations between electrons.

3.8 Summary of the key physics

- The FQH state is characterised by plateaux in Hall resistivity and vanishing longitudinal resistivity around fractional filling factors. The FQH state is incompressible.
- Disorder is required to observe the plateaux, although the details of the effect are quite insensitive to the details of such disorder.
- Each state is associated with a family of anyonic quasiparticles which possess a fraction of an electron charge. Anyons are neither fermions nor bosons and obey fractional statistics.
- The fractional statistics reflect the topological constraints on the FQH state. These topological features are also seen in an EFT description of the state in terms of a topological Chern-Simons theory.
- FQH states are topologically ordered. This is quite unlike most condensed matter states which are distinguished by their symmetries.
- Particular topological orderings reflect particular patterns of long-range entanglements which knit together the system. Such long-range entanglements cannot be removed under a RG scheme, and mean that the FQH state is insensitive *in principle* to local perturbations.

4. The FQHE and E1: Particles, *quasiparticles and entity realism*

One obvious way of spelling out the emergent aspect of the FQHE is by regarding FQHE quasiparticle excitations realistically. The FQHE is not the only condensed matter system with quasiparticles; they are ubiquitous in the subject. In general, quasiparticle refers to the single particle-like excited states in interacting systems. These may be single particles (like electrons) dressed quantum mechanically (or renormalised) by interactions. Alternatively quasiparticles may be collective excitations that may be treated as single-particle like, but whose dynamics involve a change in the coordinates of all of the particles in the system. For example, a simple crystal lattice possesses particle-like vibrations known as phonons, which have this latter property.

Wallace says of quasiparticles:

> [A]re quasiparticles real? Well they can be created and annihilated; they can be detected (by scattering them off 'real' particles like neutrons); in some cases (such as so-called ballistic phonons) their time-of-flight can be measured and they play a crucial explanatory role in solid-state theories. We have no more evidence than this that real particles exist, so it seems absurd to deny the existence of quasiparticles. (Wallace 2003 p.8)

As alluded to above, the excitations that we probe experimentally in condensed matter systems are generally consistent with the quasiparticle picture. In the case of a ferromagnet, for example, the fundamental excitation of the system can be viewed in real space as the result of smearing a single flipped spin across the entire ferromagnet. This is described mathematically using a canonical coordinate transformation that turns the system of many strongly interacting spins into an equivalent system of weakly interacting quasiparticles. The resulting quasiparticle excitations interact with experimental probes similarly to single particles in free space carrying the same quantum numbers, and are known as a magnons. In the FQH case, there is very strong experimental evidence for fractional excitations with exactly the properties predicted by the theory, which also admit a quasiparticle description. Again these could be regarded in real space as being an expression of the collective motion of the electrons in the underlying system, or realistically as real, fundamental particles in their own right.

This picture seems even more compelling depending on how one views fundamental particles such as electrons. For example, Michael Levin and Xiao-Gang Wen (Levin & Wen 2005) have proposed a model where photons and electrons are emergent phenomena. The idea is that both types of particle can be modelled as different excitation modes of what is known as a string-net condensation. Just as the quasiparticles of the FQHE are a collective excitation of the sea of electrons, so the electron itself would merely be an excitation of a deeper unifying physical reality. Less exotically, it is consistent to regard 1) electrons in a metal as quasiparticle excitations of the sea of electrons that form the ground state, and also 2) electrons in the vacuum as quasiparticles excited from the vacuum and dressed by quantum fluctuations from the matter fields of the Universe.

Despite the ubiquity of the quasiparticle interpretation, there are reasons to be cautious. Quasiparticle realism could be undermined in the FQH case by a topological feature that causes an ambiguity in representing quasiparticles, which is not present in non-topological phases of matter (Nayak et al 2008). For example, any anyonic system must have multiple types of anyons. So for a state that has anyons of statistics η, that state also has anyons of statistics of 4η and so on. We can always approximate any two anyons by a single particle whose quantum numbers are obtained by combining quantum numbers (including topological quantum numbers). This formation of a different type of anyon by bringing together two anyons is called fusion. A classic example of this is when we bring a statistic η = y particle together with a statistic η = −y particle to form a boson with statistics η = 0, dubbed the 'trivial' particle.

The characterization of the FQH state in terms of novel particles is even more complicated since another description of the FQH state is possible in terms of a different sort of particle: the composite fermion. The composite fermion model of the FQHE was proposed by Jain (1989) and extended by Halperin, Lee and Read (1993). This description here is not one of excited states, as in the case described above, but rather one of building the ground state from a new sort of particle. The idea is that instead of describing the FQHE system in terms of single electrons, we describe it in terms of electrons that have quanta of vorticity attached to them. This compound entity, an electron plus quanta of vorticity is known as a composite fermion. (Sometimes the composite fermion is described as an electron that has captured an even number of quanta of magnetic flux, but this is only an analogy, the magnetic field is uniform everywhere.) Composite fermions carry integer charge, since the only element of charge is the electron and also have regular statistics the same as any other fermion. The advantage of this description is that it allows the FQHE states to be mapped onto the IQHE states with the FQHE being interpreted as an IQHE for composite fermions. The composite fermion model has enjoyed success in providing a straightforward description of the physics of many of the FQH states and is consistent with the more general Chern-Simons field-theoretical description of the physics. It is possible that composite fermions are merely a useful conceptual stepping stone to allow a justification of Chern-Simons physics, but it may be that future experiment provides more direct evidence for their existence.

The composite fermion picture might lead one to think that the FQHE is not so different from the IQHE, since it may be straightforwardly reduced to the IQHE. However the composite fermion picture is deceptive: composite fermions are not merely individual electrons interacting with the magnetic field (Stern 2008). For example if we attempt to remove a composite fermion by adiabatically turning on a flux tube then we actually alter the charge of the system by a factor $e/(2p+1)$. Even though the composite fermion only has charge e, by annihilating it we remove a fractional charge. The reason for this discrepancy is that composite fermions are topological entities that are defined relative to the whole system. So if we remove a composite fermion in one composite fermion Landau level we also affect the composite fermions in all the other composite fermion Landau levels. Even a single composite fermion is actually a collective entity of the whole system. This is because the quantised vortex term is derived from the quantum mechanical interaction of all the electrons. A FQHE state may appear simple when translated into an IQHE state in terms of composite fermions but this is illusory. A composite fermion is actually a highly complicated entity summarising the complex strongly correlated long-range correlations of the electrons making up the system.

A lesson to be drawn from this is that even if we describe the FQH system in microphysical terms that seem to remove interactions (i.e. in terms of non-interacting composite particles), this is an illusion. Any way we describe the system, either in terms of anyons, composite fermions, Chern-Simons field terms there is a holistic character to FQH states that is not present for IQHE states.

In summary, we believe whilst one can claim the FQH state is E1 emergent this is not our position since there is no uniquely privileged set of quasiparticles with which to describe the system. Moreover, if the FQHE were only emergent in the sense of E1 then this would not mark it out particularly from other more familiar systems. If one believes in quasiparticle realism for reasons along Wallace's lines then any crystal with phonons is emergent. This may be so, but characterising the FQH state in terms of E1 emergence along would miss much interesting physics and philosophy.

5. The FQHE and E2: intensional properties

The genesis of philosophical interest in emergence in condensed matter physics is Philip Anderson's 1972 paper *More is Different*. The views expressed by Anderson, and endorsed by other scientists, (such as Laughlin & Pines (2000), Coleman (2003), Mayer (1988), Wheeler & Ford (1998)) are sometimes dubbed "New Emergentism".

New emergentism (NE) is primarily concerned with the way in which complex systems display novel properties. For example Anderson says:

> *The behaviour of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity, entirely new properties appear, and the understanding of the new behaviours requires research which I think is as fundamental in its nature as any other. (Anderson, 1972, 393)*

The sense of novelty expressed by NE is not the novelty of new values of a property. If we place an apple into a fruit bowl the mass of the bowl plus apple is greater than the items separately. This new value of mass is an uninteresting form of novelty; instead we want some sense of qualitatively novel features of a complex system emerging (Reuger 2000). Yet within NE this novelty is not in competition with the laws and theories of microphysics. For NE, complex entities are not only composed by microphysical entities they also obey the laws of microphysics and have their properties fixed (in some sense) by the microphysical properties.

NE has twin commitments then: to qualitative novelty of systemic properties and to microphysical determination of those properties. These commitments have led some to contest that NE hypothesis mixes epistemic and ontological issues (Chalmers 2006, O'Connor & Wong, 2005)

or others to claim it is incoherent (Howard 2007). According to these critics the novel properties discussed by NE are merely practically impossible to derive from microphysical properties. However, Mainwood (2006) has suggested that the NE position is not simply confused, but instead follows in the traditions of early 20th century British Emergentists such as Broad (1925). Broad's notion of emergence was also based on the idea that systemic properties of complex systems are different from the properties of the parts of those systems (and the properties of different systems made from those parts). Entities A, B, C in relation R(A,B,C) have properties that only apply to R and are not derivable in principle from a general summation rule.

As such, both BE and NE can be classified as theses concerning emergence in the sense of E2 as defined in section 2. In the specific case of the FQHE this means we are concerned with novel properties the system of electrons in a FQH state display; *where the novelty is judged relative to free electrons or electrons in another composite, such as the IQH state.* Of course, famously, Broad's mechanism for novelty was specific laws of composition and it is widely thought that empirically there is little evidence for such laws. Therefore the NE position must be explicated in terms of a different mechanism.

There have been several different attempts to spell out the sense in which E2 as proposed by the NE picks out a metaphysically robust sense of emergence. We will briefly consider three different (but interrelated) articulations here.

5.1. *"Good" coordinates, SSB and EFTs*

Mark Wilson (1993) follows others such as David Lewis (1999) in concluding that without differentiation properties in and of themselves are trivial. Too many properties is as bad as no properties at all for delineating the structures of the world. What is required is a way to privilege a set of properties. For example, Lewis privileges the intrinsic properties of point like objects and the spatio-temporal relations between them. Lewis suggests constructing a 'natural' sparse language with reference to only these fundamental properties. Any other property can be ranked in unnaturalness by considering how complex and long its definition is in the sparse language.

Mark Wilson's critique of Lewis is that such a way of viewing properties pays almost no attention to what are deemed the physically significant properties by physics. The sets of properties which are privileged by physics (either nomologically, predictively or causally) will often not be Lewis' natural properties. That this is the case is one of the motivations of Anderson's critique of privileging fundamental physics.[3] In Wilson's terminology the privileged set of properties are dubbed *intensional*. Importantly, for Wilson intensional properties can vary from system to system because they are defined as the properties that determine the physical evolution of a system. For example, a rigid body has its evolutionary dynamics set by its mass, velocity, angular momentum and moment of inertia.

In general in physics we can define a system as exploring a state space. Sometimes this state space can be as simple as a phase space, in which the system is described in terms of its position and velocity coordinates. But other state spaces are possible to construct, in which a system is described as exploring an abstract space defined by a particular set of coordinates. The nature of those coordinates is defined by the so-called degrees of freedom of the system (DOF). If a system is subject to a particular constraint then this can reduce the dimensionality of the state space that the system can explore. For Mark Wilson the "natural" or "good" set of coordinates for any system is those for which the state space description is optimally simple. Since these good coordinates pick out the physically significant variables of a system they also pick out the intensional properties of that system. Since the nature of the constraints that reduce the state space will vary between condensed matter systems so too will the intensional properties.

Mainwood suggests that the distinction between the novelty suggested by Broad and by NE can be understood in terms of intensional properties. For Broad specific laws of composition meant that even a Laplacian demon couldn't derive the laws applying to a specific system. Whereas for NE the laws are general and could be derived in principle (even if not in practice). However, the intensionally significant properties of composite systems could never be derived from the intensionally significant properties of the parts.

The question is then: what determines a set of good coordinates? In general it seems to be constraints of some sort. Specifically, Wilson suggests that order in the form of spontaneous symmetry breaking (SSB) is a mechanism by which the state space is reduced and good coordinates are frozen into a system. Since the broken symmetries are particular to each system so too are the good coordinates. In the case of SSB, the coordinates of interest might presumably include the so-called *order parameter* of the system. This is a field whose vacuum expectation value takes on a non-zero value only when symmetry is broken; whose variation in space determines Anderson's emergent rigidity and whose excitations constitute the emergent quasiparticles of the broken symmetry state. Another possibility is to identify the 'good coordinates' with the *normal coordinates* found via the diagonalization of the interacting Hamiltonian of the broken symmetry state to identify quasiparticle excitations, as described in section 4. The constraint that leads to either of these sets of coordinates is that a particular symmetry is broken and a symmetry element is therefore missing in the description of the system.

SSB is advocated as a mechanism for emergence in condensed matter by other authors as well. For example, Morrison (2012) suggests that SSB in an electromagnetic gauge theory is responsible for determining the defining properties of superconductors (such as flux quantization). However, it seems clear that the senses of 'determination' employed by Wilson, Mainwood and Morrison are not all equivalent. Morrison's notion of determination is much stronger; SSB is a top-down process, which makes the system behave in a certain way. Whereas the sense of determination used by Mainwood and Wilson implies no downward causation/determination in a strong sense. Rather the determination is of privileging properties, not of determining events.

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[3] It seems clear that problems with Lewis' account do not just rely on considerations from condensed mater systems. It is hard to reconcile the natural set of properties with those properties deemed foundational by current fundamental physics, since fundamental physics (e.g. QFT) arguably has no spatio-temporally separate point like objects. Of course metaphysics is free to ignore physics for its own reasons, but one should be clear that such metaphysics is not empirically supported in any way.

A distinct, but related approach to E2 emergence is suggested by Bain (Bain 2012, also see Jessica Wilson 2010 for another DOF approach). Bain considers the role effective field theories (EFTs) play in condensed matter physics and other areas. The determination of EFTs often involve the use of renormalization group methods to establish the low-energy properties of a theory. The renormalization group method involves imposing an energy cutoff to the Lagrangian describing the theory, which allows DOF above the energy cutoff to be integrated (or averaged) out (Lancaster & Blundell 2014). We may then define a new set of Euler-Lagrange equations with modified coupling constants. This is the background to the CS theory of the FQHE, where we use an EFT. For Bain such a procedure is indicative of a type of ontological emergence. We have a new set of governing Euler-Lagrange equations, so new dynamics in the new theory.

Bain's interpretation of the elimination of DOF (or the imposition of good coordinates) is distinct from Wilson & Mainwood's. For Bain the metaphysical significance of EFTs is not in allowing distinctions between systemic properties to be made. Instead, the significance is that an EFT is a new set of *governing* laws that determine both properties and events. In that respect Bain's sense of E2 is much stronger than Wilson or Mainwood's and is more akin to Morrison's account. The difference between Morrison and Bain is that Bain places the strength of determination in the *new laws* that each EFT produces, not in any one specific mechanism such as SSB. Moreover, Morrison casts doubt on whether EFTs give a sense of why or how a particular phenomenon is emergent, and suggests instead that they give provide a way of dealing with a specific phenomenon that cannot be incorporated into a fundamental theory. (Morrison, private communication).

5.2 *Is the FQHE E2 emergent?*

Given the ways of articulating E2 discussed in the previous section, is the FQHE emergent in this sense?

We are intuitively sympathetic to the approaches of Mark Wilson, Mainwood, Morrison and Bain. These accounts identify interesting features of emergence as it applies to condensed matter physics, and the FQHE shares many features in common with systems categorized as E2 emergent. For example, Bain may be correct that the FQHE is described by an EFT. It is also true that there are a set of constraints operating on the FQH state that mean it is possible to reduce the state space and define a privileged set of good coordinates to describe the system. In this sense at least then the FQHE is emergent is the sense of E2. But we believe such a quick identification is potentially misleading.

It is clearly not possible that the mechanism of E2 emergence in the FQHE is SSB. By definition all topologically ordered states of matter share the same symmetry. However, the New Emergentists have never claimed SSB is the only possible mechanism by which emergence can occur. Hence, one could argue that topological constraints are just another type of constraint that can pick out intensional properties. Another possibility is that the constraint that the FQH system is incompressible at certain filling fractions leads to the CS gauge field being identified as a set of 'good coordinates'. However, such a conclusion risks overlooking important physical differences in the way different constraints operate. It is our contention (which we shall argue for in section 6) that the nature of the topological ordering constraint is qualitatively quite unlike SSB in philosophically important respects.

A similar criticism, of overlooking important particulars, can be made of Bain's characterization of the FQHE as just one example of many systems which can be described by an EFT. In Bain's scheme there is nothing special about the FQHE. Any system for which an EFT is used is ontologically emergent. However, this overlooks the fact that there can be heterogeneous physical reasons for why an EFT can be used at all. For example, the reasons why an EFT can be used in FQH systems are quite different from the reasons why an EFT can be used for other systems. These two processes (the description by an EFT and the microphysical dynamics that allow that) should not be confused (Kitaev & Preskill 2006).

Although using an EFT is a general strategy, applicable to many diverse systems, it can have many different physical meanings for each of those systems. For instance, one can make General Relativity non-relativistic by approximating it by an EFT which reproduces the post-Newtonian expansion (essentially Newtonian physics with small correction terms). In both cases (FQHE and non-relativistic GR) we remove predictively irrelevant DOF for our chosen level of precision. But the physical meaning behind that elimination of DOF (precisely why those DOF are predictively irrelevant) is different. To anticipate the next section, it is the particular type of long-range entanglements present in FQH systems that mean they can be described by an EFT; specifically it is the topological entanglement entropy that allows a particular EFT to be used.

This leads to our final concern: *the metaphysical significance of E2 is still highly contentious*. For example, it is far from obvious that the picking gout of good coordinates is anything more than an epistemic consideration. For any complex system there will be parametrizations that aid our understanding and ability to explain and predict. But understanding, explanation and predictive power are not automatically metaphysically important considerations. For example, if one has a purely epistemic notion of explanation then explanatory power has no metaphysical significance (Hughes 1993). We can see by comparing Mark Wilson, Mainwood, Morrison, and Bain, that although they all claim metaphysical significance, they differ as to its strength and source.

Take Bain's account for instance. Bain offers no account as to why a Lagrangian should not be taken as merely descriptive of a system. If the Lagrangian is merely a description then the elimination of DOF and the resultant new Euler-Lagrange equations are just new descriptions that have thrown away predictively/explanatorily irrelevant details. But throwing away explanatorily irrelevant details is not enough for ontological emergence. There is no obvious reason why Bains ontological emergence is not simply a relation between theories only, rather than a description of the relation between physical systems. EFTs might be pragmatically useful for representing the physical systems but there is no automatic implication of physical emergence from a failure of theoretical reduction.[4]

Similarly, one criticism of Mark Wilson's approach is that the nature of constraint is vague. After all, any system is subject to multiple constraints. These constraints can be general (e.g. conservation of energy, momentum) or system specific. But it is only a subset of all constraints applicable to a given system that pick out a set of good-coordinates. Furthermore there will be no general rule for which constraints matter

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[4] There may be simply a Humean/non-Humean distinction here as to whether laws in general, and the Euler-Lagrange equations in particular determine what a system will do or describe what a system will do.

from system to system. The notion of good co-ordinates is relativized to a particular aspect of a particular system. This can have an epistemic interpretation: it is with respect to describing/predicting a particular behaviour or property that a set of constraints/good-coordinates is relevant. In other words, our interests pick out the sub-set of constraints that lead to a particular set of good-coordinates.

Alternatively we can have an ontological interpretation: objective properties of a system are determined by particular operative constraints acting on it. The link between those properties and constraints implies a privileged language to describe the system in terms of (the good co-ordinates). This second position could be argued for by contending that the reduction of the state space is a global optimum of simplicity rather than a local optimum of simplicity. Of course establishing that there could never be any other set of co-ordinates that are better would be an extremely difficult task.

Our contention is not that E2 is not metaphysically important, or that the FQHE is not E2 emergent. Rather it is that the philosophical implications of the FQHE are much richer than can be captured by using only the notion of E2. For example, in the next section we shall argue that at the heart of the FQH state is a metaphysically robust intrinsic holism. E2, in any formulation, fails to identify what is unique about topological orderings. It is the failure of mereological supervenience which is the best evidence that there is a very real sense in which the properties that determine the evolution of the FQH state are metaphysically irreducibly whole system properties. Notions of holism play no role in E2, hence describing the FQHE as only E2 emergent misses something out.

6. *The FQHE & E3: failure of mereological supervenience*

We suggest that the basis for the emergent properties of the FQHE could be located in the long-range entanglements that characterise topological states of matter. Quantum entanglements have been used as evidence for strong emergence elsewhere. Silberstein and McGeever (1999) argue that the maximally entangled state of two spatially separated EPR correlated particles is a paradigm case of strong emergence. Silberstein and McGreever follow Humpreys (1997), Healy (1994), and Teller (1986) in suggesting that the appropriate ontology of entangled states is to regard the particle pairs as forming a spatially extended whole. It is then not so much that one particle influences the other, rather that with respect to certain properties there are no separate particles at all. Any measurement records the properties of this new whole entity.

This type of emergence is what was referred to as E3 in section 2. We are concerned with properties of a whole that are novel with respect to the properties of the parts when they are in that same whole. In such cases we have emergence due to a failure of mereological supervenience of properties. One particular account of emergence as a failure of mereological supervenience is Humphreys' fusion account.

6.1 *Fusion*

Let us imagine we have two different levels, denoted i and $i+1$. (The two levels could be well separated with respect to some relevant scale, such as length, time or energy.) $\Lambda_\alpha^i\big[\alpha^i(t_1)\big]$ denotes an i-th level entity (α) instantiating an i-th level property Λ_α as an event at time t_1. Let $\otimes$ be the fusion operator, so that the fusion of these two property instantiation events is given by:

$$\Lambda_\alpha^i\big[\alpha^i(t_1)\big] \otimes \Lambda_\beta^i\big[\beta^i(t_1)\big]$$

This fusion leads to an $i+1$ level event:

$$\big(\Lambda_\alpha^i \otimes \Lambda_\beta^i\big)\big[\alpha^i(t_2) + \beta^i(t_2)\big]$$

Which is equivalent to:

$$\Lambda_\sigma^{i+1}\big[\{\alpha^i + \beta^i\}(t_2)\big]$$

So the fusion of properties at the *i*-th level produces a new property at the *i*+1 th level.

The essential feature of a fused property instance is that it is a property instance of the pair of entities in a certain relation only, and the individual 'basal' properties instances are lost. We cannot understand event instances of fused properties as simply additions of event instances of lower level properties. Note that this type of emergence is diachronic: it occurs as a process in time. One way of understanding the basal loss is by taking a causal theory of properties such as Shoemaker's (1980). This is suggested by Wong (2006). In this case, since properties are individuated by the causal powers they bestow and the lower level causal powers are fused so are the properties. This seems to capture Humphrey's intent when he says that basal properties are causally used up in forming the fused state.

Fusion is a physical and not a logical relation. Also note that the entities themselves, i.e. α and β, strictly speaking retain their separate identities. It is only with respect to their fused basal property instances that they can no longer be individuated. The higher level property, which is the product of fusion, does not supervene on the lower level properties since the individual basal properties are no longer well defined. Because the fused basal properties no longer exist there is no challenge from Kim's (2007) causal exclusion argument. (As there is no supervenience there is no possible overdetermination of events. Contrast this with E2 as discussed in the previous section. In E2, being able to define macroscopic sets of intensional properties alone does not avoid causal exclusion.)

E3 as articulated as a form of property instance fusion accounts for the striking features of the FQHE. It is the long range order in the FQH state that characterizes the topological states of matter in general. The long range order of these states reflects the fact that the system is criss-crossed by long range entanglements (see below). So if fusion (or holism in general) is a good way of understanding standard entanglement then it is an appropriate framework for explicating the FQHE. Holism is present in any empirically adequate way of representing the FQH state. For example, the FQH quasiparticles are collective excitations only definable within the system, that obey fractional statistics that depend on the topological properties of the whole system. Even though quasiparticles are localised to some extent they cannot be treated in isolation. The same is true of the Chern-Simons field approach. In this representation there is a gauge field which pervades all space and is emergent from the microphysical details. This gauge field ties quanta of vorticity to electrons making them composite fermions. But composite fermions are topological entities that depend on the whole system to determine their properties. Finally at the level of electrons we know that FQHE

states are those for which there are non-local entanglements. In all three representations the FQHE state is one in which some of the individual properties of free electrons are determined by the strong global correlations which coordinate the FQHE states, hence the FQHE states are best thought of as collective properties.

Characterising emergence in terms of E3 in general, and as a case of fusion in particular, is a controversial claim. We will now proceed to try and address some of the general criticisms levelled at E3/fusion by considering how these criticism would apply to such an interpretation of the FQH state.

6.2 Criticisms of fusion

Criticism1: triviality

One of the often cited criticism of E3 emergence applied to entangled states is that this form of emergence is trivial. It is realised so widely that it cannot help us pick out an interesting distinction between resultant and emergent systems. So, for instance Mainwood states:

> [A] failure of mereological supervenience is both well-defined and empirically realised in many systems. In fact it is realised too frequently. Any everyday macroscopic system exhibits entanglement, both internally, and with its environment. On this criterion, all large-scale systemic properties are emergent (with the possible exception of a few prepared in astonishingly delicate experimental situations). (Mainwood, 2006 p.53)

Even if one does not hold entanglement as completely trivial as Mainwood, one may wonder if the FQHE is interestingly different in a philosophical sense from other condensed matter systems. For example, Howard claims that both superfluidity and superconductivity are emergent because of entanglement of microphysical components, so what would be special about the FQHE?

We believe that both of these criticisms are misplaced. We are not claiming that the FQHE is emergent in an interesting sense merely because there is *some* entanglement. Rather, it is the *specific type of entanglement* that makes the E3 emergence of the FQHE non-trivial.

Recall from section 3 that short-range entanglements are equivalent to unitary operations. That is in some key respects a system possessing only short-range entanglements can be represented as system with no entanglements at all. Short-range entanglements can be converted into direct product states and thereby removed in a renormalisation scheme. So the everyday entanglements described by Mainwood, or the type of entanglements described by Howard in other condensed matter systems do not create the same type of order, i.e. topological, as the long-range entanglements of the FQHE.

It is the topological order due to the long-range entanglements that knit the FQH state into a unified whole with respect to certain properties. This is profoundly distinct from other condensed matter phases or everyday system/environment entanglements. To see this easily, consider the difference between a topological computer and a quantum computer. *It is only a topological computer possessing long-range entanglements that is insensitive to local perturbations.* This insensitivity is because the system responds as a whole in a manner quite unlike non-topologically ordered states.

Each FQH state includes short-range and long-range entanglements, but it is the topological properties of the long-range entanglements that define universal equivalence classes and determine the properties of the FQHE. Fixed points in the renormalisation group flow correspond to particular topological arrangements of the long-range entangled order. The system cannot be deformed from one of these topological equivalence classes to another without passing through a quantum phase boundary. This phase boundary is indicative of entering a qualitatively different property regime.

Criticism 2: inadequacy

Another criticism levelled by Mainwood is that, even if non-trivial, E3 emergence does not capture commonly identified features of emergence. To quote Mainwood again:

> [M]aking out emergence as a failure of mereological supervenience makes no attempt to capture the emergentist intuitions of novelty or unexpectedness between micro- and macro-physical properties. The systemic properties can be entirely predictable and derivable from those of the component parts (as indeed they are, in the examples of entangled electrons). (Mainwood 2006, p.53)

We agree with this criticism in general. Indeed we endorse the view that E2 (Mainwood's preferred framework) is a fruitful way to understand many cases of emergence in condensed matter physics, whereas E3 isn't. However, in the specific case of topologically ordered states it is not correct to say that systemic properties are entirely predictable from component parts. In the FQH state there are a vast number of electrons which become holistically tied together by long-range entanglements. This holism means that it is impossible to start (even in principle) by tracking all the correlations involved by building up one pair of electrons at a time. Since, by definition, the holism means that long range correlations are not simply aggregations of short range correlations.

As a quick aside: one might worry that our response to the triviality concern means that the examples Humphreys himself has described do not count as fusion emergent. This is not the case. Where Humphreys has a maximally entangled pair (an EPR pair of particles say) we would have fusion emergence; there is no way one could 'remove' the entanglement from an adequate description of an EPR pair of particles. In most examples of macroscopic matter, entanglements can be removed by local unitary operations and we still have an adequate description of the dynamics of that matter. However, in the FQH state this same procedure leaves the long-range entanglements. Only a topological state of matter has this latter feature. So entanglement can be the basis for fusion emergentism but only in certain circumstances, e.g. maximally entangled pairs and topological states of matter.

Criticism 3: dual basal roles

Wong (2006) has criticized Humphrey's fusion account, suggesting that it faces severe problems. Wong's criticisms stem from the basal loss feature of fusion. The difficulty is the *problem of structural properties*. For Wong a structural property is a non-emergent aggregate property. Wong points out that basal property instances which are lost on fusion may also play other roles in sustaining non-emergent structural properties. The basal properties that are lost may play multiple roles in a system, some emergent, and some merely structural. If these property instances are really lost though, how can they play their structural roles? According to Wong, there are two potential responses the fusion emergentist could make.

Firstly: the missing basal property instances could be replaced by other 'backup' property instances that take over and replace them in the structure. However, there is little empirical evidence for this, and it seems unlikely that there would *always* be backups no matter the precise circumstances of fusion. Secondly: one could contend that there are multiple copies of the basal property instances and only some of them fuse and are lost, leaving the rest of the unfused identical basal property instances to play a structural role. Wong regards this inherently more feasible than backup properties, but suggests that this resolution means the fusion program is undermined. The reason is that if not all of the basal property instances are lost then Wong thinks that fusion cannot overcome the problem of causal overdetermination (which was one of Humphreys' main motivations). If some of the basal features remain then they will retain their causal powers and those powers will provide an alternative causal chain to produce systemic effects. There will then be two causal stories, one involving the fused basal property instances and one involving the non-fused basal property instances.[5]

We are in sympathy with Wong's criticisms. But instead of abandoning fusion we suggest a modification to it, inspired by Teller's (1986) account of relational properties, can overcome these difficulties.

6.3 *Intensional relational properties*

Teller's analysis of the special nature of quantum relations begins with a definition of what he calls *local physicalism*: all non-relational properties of an individual, i, supervene on i's physical non-relational properties; and any relations holding among individual relata i, j, k supervene on the non-relational properties of the relata i, j, k. (Teller, 1986 p. 73). In short, we have local physicalism if relational properties supervene on non-relational properties. For example, to say that x is longer than y supervenes on the non-relational properties of x's length and y's length.

Teller's notion of relational holism is based on the idea that there can be circumstances in which entities (which can otherwise be identified as individuals) have inherent relations. Such inherent relations do not supervene on the non-relational properties of the individually identified entities. Teller contends that in quantum mechanics we still have individuals but they have inherent relations. Teller runs through a series of arguments to suggest that most (if not all) classical physics deals with non-relational properties, but that entangled systems in quantum mechanics are inherently relational.

> [C]onsider two particles with the non-classical property of spin, and we construct a certain simple superposition of two non-relational spin states of a and of b. In this superposition quantum mechanics does not assign a definite spin to either particle in any direction…even if one postulates specific spins for particles in the superimposed state, the property described by the superposition does not supervene on the postulated individual properties…the superposition characterises an independently identifiable property with distinctive experimental implications for the a-b system as a whole. (Teller 1986 p.79/80)

For Teller quantum mechanics abounds with inherently relational properties; relations that do not supervene on individual non-relational properties. Inherent relations are all pervasive; every interaction produces them.

Given that quantum mechanics underlies all condensed matter, one may wonder why we argue for the special status of the FQH state? After all, if Teller's analysis is correct, and quantum systems are all inherently relational, why aren't the macroscopic aggregate systems formed by those quantum systems also inherently relational? The answer is that in the FQH state it is the inherently relational properties that are important for physically determining what the system does. In many other condensed matter systems we may have entanglements, but they are short range, and therefore removable.

We suggest that Teller is correct and his account can be combined with fusion emergentism. Rather than the fusion relation using up basal property instances, it instead *turns non-relational properties into inherently relational properties*. Thinking of fusion in this way has the advantage that the basal property instances are still there, it is just that they are inherently relational. Since there is no basal loss there is no problem with those property instances playing other structural roles. But there is also no threat from overdetermination. This is because we can identify the higher level property instances with the now inherently relational basal properties. However, this does not imply a reduction to merely the causal powers of the basal properties. Since, when fused, they are inherently relational, one should conceive of the fused system as the bearer of the novel causal power, not any aggregation of basal property bearing elements.

In the FQH case we may provide an example via the properties of quasiparticles. The consideration of quasiparticle properties does not commit us to a claim about the reality of quasiparticles. Rather, since the quasiparticles constitute the collective motion of all of the constituent electrons in the system, quasi-particle properties are a way of indirectly considering the way the properties of those electrons are interacting. The quasi-particles of a particular system may be just a representational tool, but that representation is still tied to the underlying physical features of the system in question, and can therefore reveal features of it. Specifically, recall that the quasi-particles of the FQH state bear a fractional charge. Now the charge of an isolated electron is as a good a candidate for non-relational property as any. But when a system of many electrons

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[5] Wong suggests that basal loss is unmotivated in any case. Since Wong conceives of emergence in exclusively causal terms, he contends that any emergent property must have genuinely novel causal powers, which generate novel effects the basal causal powers cannot. Different effects mean that there is no competition and no overdetermination.

condenses into a FQH state, the fractional charge on a quasiparticle excitation of that state now arises only by virtue of the correlations (or entanglements) existing across the whole system. More dramatically, we can consider adding an extra electron (carrying its single charge) to the FQH system. If we try to keep track of the extra charge, it will appear to decay into several fractional charges. Of course the electrons do still carry charge, but it is no longer a charge intrinsic to them as individual particles. The electron gives up its charge property to the entanglements. Thus, it is not that the basal properties of electrons (charge, mass etc.) disappear in the transition to a FQH state, it is that (some at least) of those basal properties are now inherently relational for the FQH state. [6]

Any non-relational basal property of an isolated electron that in a FQH state becomes entwined in the web of long-range entanglements is a property that is made inherently relational. One way this relationality can be seen is in the system's robustness to unitary operations/local perturbations. This reflects the fact that the individual elements that make up the FQH state cannot respond as individuals to interactions. The FQH state responds as a whole or not at all (recall the unique robustness of a topological computer in comparison with any other type of quantum computer).   Furthermore, as well as the transformation of basal properties to become relational, the web of correlations of the electrons in a FQH state allow new properties to manifest which can only be associated with a collective state and hence are inherently relational. For example, the properties of the Chern-Simons gauge field cannot be considered intrinsic properties associated with a particular component of the FQH system. These properties are not straightforwardly assignable as non-relational basal properties of the electrons outside of the FQH state though. So, the FQH state has inherently relational features derived from the conversion of basal features of the components of the FQH state and from the production of new features definable only for the collective state. Note that not all basal properties of the electrons need to become inherently relational, but the properties that are responsible for determining the FQH state are those that are fused.

 It is in this sense that the FQH state cannot be reduced to its parts, and must be regarded as a unified whole. Moreover, in the case of the FQHE it is the inherently relational properties that determine the physical state of the system. One way of thinking about this is that the physical features of the FQH state that pick out the good-coordinates are inherently relational. (Even if one thinks that the notion of good-coordinates in general is merely an epistemic category, the fact that *any* plausible representation of the FQH state/any set of good-coordinates must include, explicitly or implicitly, these relational aspects suggests that an inherent relationalism of some sort or another is crucial to the ontology of the FQH state.) Using the notion of good-coordinates then, in Wilson's terminology these inherently relational physical features are the intensional properties. We therefore suggest a modification to Teller's terminology to make this connection clear. A fused system (such as the FQH state) is characterised by *intensional relational properties.*

Another way of thinking about this is that what we have under this 'relational fusion' is the creation of a new supervenience base which is not the mereological sum of the individual parts and their intrinsic properties in relation to one another. Instead, in the relationally fused state the supervenience base is the parts and their now inherently relational properties. There is no causal exclusion possible since this new supervenience base, although microphysical, is not reductive. We cannot say the causal powers of the FQH state are merely a summation of causal powers of the electrons; rather the causal powers of the FQH state are the collective causal powers of the now *inherently relational* electrons fused together, but this does not necessitate basal loss. We do not have to contend, contra Humphreys, that there are no properties of the electrons, merely that the properties of the electrons are not their own alone anymore.

7. *Conclusion*

We can summarise some key features of the FQH state as follows:

• The FQH fluid is incompressible and hosts fractional quasiparticle excitations. Its origin is electron-electron interactions in a 2D plane subject to a magnetic field. Impurities are necessary in order to realize the experimental observables of the state, although the state is realised independent of the precise nature of that disorder.

• Interactions between electrons dominate the system[7]; no perturbation analysis is possible.

• Many aspects of the FQH system can be described by a Chern-Simons EFT (specifically a topological quantum field theory (TQFT)) whose excitations are fractional quasiparticles. The system can also be described in terms of an IQHE of composite fermions.

• Both of these reflect non-local constraints: topological features constrain the system. The quasiparticles are anyons following fractional statistics defined by topological winding equivalence classes, while the composite fermions are also topological objects that have non-local properties.

• These facts reflect a deeper set of facts about the electron interactions. The FQH system is dominated by long-range entanglements that criss-cross the system. All the electrons are entangled with each other, this means the FQH system responds as a whole. It has capacities to respond to its environment which are only features of the whole system taken together. This is why there are no local perturbations possible which can change the state of the system, since non-local entanglements suppress them.

We contend that there are multiple senses in which FQHE might be classed as emergent.[8] There is a privileged subset of the macrophysical parameter state space due to topological constraints which picks out a set of "good" co-ordinates to define intensional properties. However,

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[6] Of course in light of QFT it is an open question whether electrons do have any genuinely individual separate properties. Space prevents an exploration of this here. But if one's view is that electrons do not have any non-relational properties to begin with then our argument concerning the FQH state can be reformulated such that the FQH state has a different set of inherently relational properties physically determining it than the relational properties determining other systems' behaviour.

[7] It is worth noting that the fact that there is interaction energy does not tease out the holism of the FQH state, since there are so many other systems that are dominated/significantly affected by interaction energy but which are not E3 emergent. The relevant distinction is between a relational state and an inherently relational state. The former supervenes on non-relational states; the latter does not. Interactions, in and of themselves, do not imply an ontological form of emergence, since the collective might simply be the mereological sum of all the interactions (which may well be impossible to compute). The FQH state is not like that because of the long-range entanglements, hence it is these that are the crux of where the failure of mereological supervenience is found.

[8] We note that the Hall effects have also recently been discussed by Lederer (2015). Lederer uses dialectical materialism to critically evaluate a variety of realist/anti-realist positions extant in the literature and how one might apply to them to the Hall effects. The unconventionality of Lederer's position makes difficult to compare to other literature on emergence, but ultimately he stresses a realist stance. The form of realism is close in some ways (though not all) to Hacking's in spirit, in that it

this type of emergence is possessed by many other systems, and the metaphysical significance of this is a matter of intense debate. Whether this E2 emergentism warrants metaphysical or ontological significance depends on wider attitudes to issues such as realism and property privileging.

However, the FQHE also displays the much stronger sense of emergence captured by E3. We suggest that FQH systems behave as collective holistic entities, just as EPR-particle pairs do. Certain individual properties of the electrons are lost to a collective set of properties of the FQH state. Electrons are still present in the system but, with respect to salient properties for the FQHE, they cannot be treated separately due to long-range entanglements. The range of the entanglements is important. Unlike other more familiar condensed matter systems, such as metals or magnets, electrons in the FQHE are entangled right across all length scales appropriate to the system. This holism can be expressed within a modified version of Humphrey's fusion framework. In the modification basal properties are not lost but instead become inherently relational under fusion.

Topological phases are an exciting area of condensed matter research and the list of potential systems categorised as topological is continually being expanded. For example, spin liquids, chiral p-wave superconductors and ultra-cold atom lattices, are types of system that probably support topological phases. Currently it is difficult to identify topological phases since they are hard to detect directly by experiment. Indeed, in the future the best way of detecting topological phases (Wen 2004, Fradkin 2013) may well be to create quasiparticles and braid them to test for topological computing abilities (Nayak et al 2008). Finally, we note that although this paper focuses on the FQHE it is potentially the case that other topological phases of matter, which are similarly emergent, are widespread in nature.

References

Anderson, P. W. (1972). More is different. *Science*, *177* (4047), 393-396.

Anderson P.W. (1997). *Basic Notions of Condensed Matter Physics*. Boulder: Westview.

Bain, J. (2012) Emergence in effective field theories. *European Journal for the Philosophy of Science* 1-17.

Batterman, R., (2002). *The Devil in the Details. Asymptotic Reasoning in Explanation, Reduction, and Emergence*, Oxford: Oxford University Press. 27

Bedau, M. A. (2008). Is weak emergence just in the mind*? Minds and Machines*,18(4), 443-459.

Bell, J. S. (1986). EPR correlations and EPW distributions. *Annals of the New York Academy of Sciences*, 480(1), 263-266.

Butterfield, J. N. & Isham, C. J. (1999) On the emergence of time in quantum gravity. In J. N. Butterfield, editor, *The Arguments of Time*. Oxford University Press.

Broad, C. D. (1925). *Mind and its Place in Nature*. London: Routledge & Kegan Paul,.

Chalmers, D. J. (2006). Strong and weak emergence. *The reemergence of emergence*, 244-256.

Chen X, Gu Z.-C., and Wen X.-G. (2010). Local unitary transformation, long-range quantum entanglement, wave function renormalization,

and topological order. *Phys. Rev. B* 82, 155138.

Coleman, P. (2003). Many body physics: Unfinished revolution. In *Annales Henri Poincare* (Vol. 4, No. 2, pp. 559-580). Birkhäuser-Verlag.

Fradkin, E. (2013). *Field Theories of condensed matter physics*. (Second edition). Cambridge: CUP..

Gibb, S. (2012). Nonreductive Physicalism and the problem of strong closure. *American philosophical quarterly*, *49*(1), 29-41.

Hacking, I. (1988). The participant irrealist at large in the laboratory. *British Journal for the Philosophy of Science*, 277-294.

Halperin, B. I., Lee, P. A., & Read, N. (1993). Theory of the half-filled Landau level. *Physical Review B*, 47(12), 7312.

is a piecemeal local realism, based on a confluence of factors derived from both experiment (manipulability) and theory (predictive and explanatory success). With regards to emergence, Lederer's notion is much more permissive than ours in that he argues for emergent aspects of all of the Hall effects, as is illustrated in one of his few direct references to emergence:

"The transformation of quantity in quality. This has been mentioned at different times in this study, along with P. W. Anderson's paper (More is different). This is probably the easiest thing to admit in the dialectics of nature. Even though it seems trivial to many, it is at variance with the classical Aristotle dichotomy which opposes categories of quantity and quality. In the QHE study, trivial examples abound, such as the transition from a classical behaviour of the Hall resistivity at low fields to that at high fields, where low, resp. high refer to the large, resp. small number of occupied LL. Another example is the qualitative change of the QH Effects if disorder in the sample is low or high. A less trivial example, and a spectacular one at that, is the transition of the 2D electron liquid under magnetic field from the Quantum Hall topological insulator at LL filling v = p/(2ps + 1)) to a 2D metallic state at v = 1/2. A more trivial one is that the QHE has no meaning for a few electrons: it only exists for a large enough number of electrons. We have seen along this work that this ubiquitous property of matter is important in discussing the validity of the reductionist analysis of phenomena." (Lederer 2015 p. 36/37)

Although there are similarities, there are important differences between our account and Lederer's. 1) The property emergence Lederer identifies is broad in scope. Sometimes emergence is the inapplicability of a kind term at a lower level, a notion we would firmly call epistemic emergence. Other senses of emergence Lederer implies refer to large scale qualitative differences in behaviour, consistent with our definition of E2 emergence. 2) We argue that the fractional quantum Hall effect displays a different form of emergence, E3, which Lederer does not distinguish from E2.

Healey, R. (1994). Nonseparable processes and causal explanation. *Studies in History and Philosophy of Science Part A,* 25(3), 337-374.

Hendry, R. F. (2010). Ontological reduction and molecular structure. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, *41*(2), 183-191.

Howard, D. (2007). Reduction and emergence in the physical sciences: some lessons from the particle physics and condensed matter debate. In: Murphy N, Stoeger WR (eds) *Evolution and emergence: systems, organisms, persons*. Oxford University Press, Oxford, pp 141–157.

Hughes, R. I. G. (1993). Theoretical explanation. *Midwest Studies in Philosophy*, *18*(1), 132-153.

Humphreys, P. (1997). How properties emerge. *Philosophy of Science*, 1-17.

Jain, J. K. (1989). Composite-fermion approach for the fractional quantum Hall effect. *Physical review letters*, 63(2), 199.

Kim, J. (2007). *Physicalism, or something near enough*. Princeton University Press.

Kitaev, A., & Preskill, J. (2006). Topological entanglement entropy. *Phys Rev. Lett.*,6, 110404.

Lancaster T. and Blundell S.J. (2014). *Quantum field theory for the gifted amateur.* Oxford: OUP.

Laughlin, R. B. (1983). Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations. *Physical Review Letters*, 50(18), 1395-1398.

Laughlin, R. B., & Pines, D. (2000). The theory of everything. *Proceedings of the National Academy of Sciences*, *97*(1), 28-31.

Lederer, P. (2014). The Quantum Hall Effects: Philosophical Approach. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, *50*, 25-42.

Levin, M., & Wen, X. G. (2005). Colloquium: photons and electrons as emergent phenomena. *Reviews of modern physics*, 77(3), 871.

Lewis, D. K. (1999) *Papers in Metaphysics and Epistemology*. Cambridge University Press.

Mainwood, P. (2006). *Is more different? Emergent properties in physics* (Doctoral dissertation, University of Oxford).

Mayr, E. (1988) The limits of reductionism. *Nature*, 331:475.

Morrison, M., (2012). Emergent Physics and Micro-Ontology, *Philosophy of Science* 79 (1) 141-166.

Morrison, M. (1990). Theory, intervention and realism. *Synthese*, 82(1), 1-22.

Murthy G. and Shankar R. (2003). Hamiltonian theories of the fractional quantum Hall effect. *Rev. Mod. Phys*. 75, 1101.

Nayak, C., Simon, S. H., Stern, A., Freedman, M., & Sarma, S. D. (2008). Non-Abelian anyons and topological quantum computation. *Reviews of Modern Physics*, 80(3), 1083.

Newman, D. V. (1996). Emergence and strange attractors. *Philosophy of Science*, 245-261.

O'Connor, T., & Wong, H. Y. (2005). The metaphysics of emergence. *Noûs*, *39*(4), 658-678.

Reiner, R., & Pierson, R. (1995). Hacking's experimental realism: an untenable middle ground. *Philosophy of Science*, 60-69.

Rueger, A. (2000). Physical emergence, diachronic and synchronic. *Synthese*, *124*(3), 297-322.

Sachdev S. (2011). *Quantum Phase Transitions* (Second Edition). Cambridge: CUP.

Shoemaker, S. (1980). *Causality and properties* Netherlands: Springer. (pp. 109-135).

Silberstein, M., & McGeever, J. (1999). The search for ontological emergence. *The Philosophical Quarterly*, 49(195), 201-214.

Singleton J. (2001). *Band structure and electronic properties of solids*. Oxford: OUP..

Stern, A. (2008). Anyons and the quantum Hall effect a pedagogical review. *Annals of Physics*, 323(1), 204-249.

Teller, P. (1986). Relational holism and quantum mechanics. *British Journal for the Philosophy of Science*, 71-81.

Tsui, D. C., Stormer, H. L., & Gossard, A. C. (1982). Two-dimensional magnetotransport in the extreme quantum limit. *Physical Review Letters*, 48, 1559-1562.

Wallace, D. (2003). Everett and structure. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 34(1), 87-105.

Wen X.-G. (2004). *Quantum field theory of many body systems*. Oxford: OUP.

Wen X.-G. (2013). ISRN Condensed Matter Physics, 2013, 198710.

Wheeler, J. A. (2000). *Geons, black holes, and quantum foam: A life in physics*. WW Norton & Company.

Wilson, J. (2010) Non-reductive physicalism and degrees of freedom, *The British Journal for the Philosophy of Science* 61 (2), 279-311.

Wilson, M. (1993). Honorable intensions. In S. J. Wagner and R. Warner, editors*, Naturalism: A Critical Appraisal,* pages 53–94. Notre Dame University Press.

Wong, H. Y. (2006). Emergents from Fusion*. *Philosophy of Science*, *73*(3), 345-367.