THE REGULAR REPRESENTATIONS OF GL_N OVER FINITE LOCAL PRINCIPAL IDEAL RINGS

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Abstract. Let $\mathfrak{o}$ be the ring of integers in a non-Archimedean local field with finite residue field, $\mathfrak{p}$ its maximal ideal, and $r \geq 2$ an integer. An irreducible representation of the finite group $G_r = \mathrm{GL}_N(\mathfrak{o}/\mathfrak{p}^r)$, for an integer $N \geq 2$, is called regular if its restriction to the principal congruence kernel $K^{r-1} = 1 + \mathfrak{p}^{r-1}\mathrm{M}_N(\mathfrak{o}/\mathfrak{p}^r)$ consists of representations whose stabilisers modulo K^1 are centralisers of regular elements in $\mathrm{M}_N(\mathfrak{o}/\mathfrak{p})$.

The regular representations form the largest class of representations of G_r which is currently amenable to explicit construction. Their study, motivated by constructions of supercuspidal representations, goes back to Shintani, but the general case remained open for a long time. In this paper we give an explicit construction of all the regular representations of G_r.

1. Introduction

Let F be a non-Archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$ and finite residue field $\mathbb{F}_q$ of characteristic p. The known explicit constructions of complex supercuspidal representations of $\mathrm{GL}_N(F)$ are closely related to constructions of representations of the maximal compact subgroup $\mathrm{GL}_N(\mathfrak{o})$. These constructions go back to Shintani [19], Gérardin [7], Kutzko [13, 14], Shalika [18], Howe [10], Carayol [4], culminating in the complete construction of supercuspidal representations by Bushnell and Kutzko [2]. All of these constructions are based on induction from compact mod centre subgroups of $\mathrm{GL}_N(F)$, and as any compact subgroup is contained in a conjugate of $\mathrm{GL}_N(\mathfrak{o})$, these constructions can also be seen as giving representations of $\mathrm{GL}_N(\mathfrak{o})$. This connection goes further, because it has been shown that every supercuspidal representation determines a unique type on $\mathrm{GL}_N(\mathfrak{o})$, and two supercuspidals determine the same type if and only if they differ by twisting by an unramified character; see [1, Appendix] and [17].

While the smooth representations of $\mathrm{GL}_N(F)$ have been studied extensively, less is known about the representations of $\mathrm{GL}_N(\mathfrak{o})$. The purpose of the current paper is to give a construction of a large class of smooth complex representations of $\mathrm{GL}_N(\mathfrak{o})$ called regular representations, which we now define. For any integer $r \geq 1$ write $\mathfrak{o}_r$ for the finite local principal ideal ring $\mathfrak{o}/\mathfrak{p}^r$. We will use $\mathfrak{p}$ to denote also the maximal ideal in $\mathfrak{o}_r$. For any integer $r \geq 2$, let $G_r = \mathrm{GL}_N(\mathfrak{o}_r)$. Every smooth, or equivalently, continuous, representation of $\mathrm{GL}_N(\mathfrak{o})$ factors through some group G_r. For any integer i such that $r \geq i \geq 1$, let $G_i = \mathrm{GL}_N(\mathfrak{o}_i)$, let $\rho_{r,i} : G_r \to G_i$ be the homomorphism induced by the canonical map $\mathfrak{o}_r \to \mathfrak{o}_i$, and let $K^i = \mathrm{Ker}\,\rho_{r,i}$ be the i-th principal congruence kernel in G_r. Let $\mathfrak{g}_i = \mathrm{M}_N(\mathfrak{o}_i)$ denote the algebra of $N \times N$ matrices over $\mathfrak{o}_i$. We then have $K^i = 1 + \mathfrak{p}^i\mathfrak{g}_r$. To any irreducible representation π

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of G_r we can associate an adjoint orbit (i.e., conjugation orbit, or similarity class) in $\mathfrak{g}_1 = \mathrm{M}_N(\mathbb{F}_q) \cong K^{r-1}$, via Clifford's theorem. The representation π is called *regular* if the orbit consists of regular elements (i.e., the centraliser in $\mathrm{GL}_N(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$, of any element in the orbit has minimal dimension N). This is equivalent to the condition that the centraliser in G_1 of any element in the orbit is abelian. The main reason for focussing on regular representations is that their construction lends itself well to the methods of Clifford theory. In particular, the regular representations form the largest family of representations which has so far been constructed explicitly for all G_r.

The study of regular representations of G_r goes back to Shintani [19], and independently and later, Hill [8], who constructed all the regular representations when r is even. The general case where r is odd is much more difficult, requires new ideas, and remained incomplete until the present paper. Assume now that r is odd. In [9] Hill constructed all the cuspidal representations (i.e., the orbit has irreducible characteristic polynomial), and in [8] he gave a construction of so-called split regular representations (i.e., the orbit has all its eigenvalues in $\mathbb{F}_q$). However, it was noted by Takase [23] that the results in [8] do not give all the split regular representations, and in any case, there exist many non-split non-cuspidal regular representations. While the present work was in preparation, Krakovski, Onn and Singla [12] gave a construction of the regular representations of G_r when the residue characteristic p is not 2. In the present paper we complete the picture by giving a construction of all the regular representations of G_r. For a somewhat more detailed comparative account of the development of constructions of representations of G_r, see [22].

In the present paper we give an explicit construction of all the regular representations of G_r, without any assumption on the residue characteristic of $\mathfrak{o}$. Just like the other constructions mentioned above, our approach is based on Clifford theory and orbits. A distinguishing feature is that it is in some ways similar to the construction of types on $\mathrm{GL}_N(\mathfrak{o})$ by Bushnell and Kutzko mentioned above. The latter goes beyond regular representations but in a certain sense only deals with semisimple elements, while we need to deal directly with, for instance, regular nilpotent elements. Focussing on regular elements has several technical advantages, but allowing non-semisimple elements brings new phenomena, such as the non-triviality of the radical of the form on $J^1_{\mathrm{m}}/H^1_{\mathrm{m}}$ (see Lemmas 4.4 and 4.5). Since several of our lemmas hold also for non-regular elements (but with more difficult proofs), it would be interesting to know if the construction could be pushed further to encompass both the regular representations and the supercuspidal types on $\mathrm{GL}_N(\mathfrak{o})$ of Bushnell and Kutzko.

Although there are many irreducible representations of G_r which are not regular, the regular representations are generic in the sense that the regular elements in $\mathrm{M}_N(\overline{\mathbb{F}}_q)$ are dense. In particular, for $\mathrm{GL}_2(\mathfrak{o}_r)$ *all* the irreducible representations are either regular or pull-backs from $\mathrm{GL}_2(\mathfrak{o}_{r-1})$. Moreover, as noted by Lusztig, it is likely that the higher level Deligne-Lusztig representations $\pm R^G_T(\theta)$, for θ regular and in general position, constructed in [16] and [20] are regular in the case of $\mathrm{GL}_N(\mathfrak{o}_r)$, and are therefore subsumed in the construction of the present paper. That this is the case when r is even is proved in [5].

Organisation of the paper. In Section 2 we define parahoric subalgebras and associated filtrations of subgroups of G_r, using flags of $\mathfrak{o}_r$-modules in $\mathfrak{g}_r$. These are finite versions of the subgroups associated to lattice chains in [2, Section 1]. The

particular subalgebra $\mathfrak{A}_{\mathrm{m}}$, determined by the characteristic polynomial of a regular element in $\mathfrak{g}_r$ plays a central role in our construction. In Section 3, we describe characters of certain abelian groups defined earlier, and record well-known results about the existence of "Heisenberg lifts". In Section 4 we give the construction of regular representations of G_r. The reader who would like a quick overview of the steps of the construction, illustrated by a diagram, may look at the discussion preceding Lemma 4.1. Our main theorem summarises the consequences of the construction for the description of representations and appears Theorem 4.10. Finally, Section 5 collects a few concluding remarks.

Notation and conventions. If G is a finite group, we will write $\mathrm{Irr}(G)$ for the set of isomorphism classes of complex irreducible representations of G. For convenience, we will always consider an element $\pi \in \mathrm{Irr}(G)$ as a representation, rather than an equivalence class of representations, that is, we identify $\pi \in \mathrm{Irr}(G)$ with any representative in its isomorphism class. One can view $\mathrm{Irr}(G)$ as the set of irreducible characters of G, but we prefer to consider representations. If G is abelian, we will often refer to a one-dimensional representation of G as a character. If $H \subseteq G$ is a subgroup and π is any representation of G we write $\pi|_H$ for the restriction of π to H. If, moreover, σ is a representation of H, we will write $\mathrm{Irr}(G \mid \sigma)$ for the subset of $\mathrm{Irr}(G)$ consisting of representations which have σ as an irreducible constituent when restricted to H.

Depending on the context, we use the notation $\mathrm{M}_N(\mathfrak{o}_r)$, $\mathfrak{g}_r$ or E, respectively, to denote the algebra of $N \times N$ matrices over $\mathfrak{o}_r$. The notation E appears (only) in Section 2 where the algebra is seen as the endomorphisms of $\mathfrak{o}_r^N$, and $\mathfrak{g}_r$ appears in the rest of the paper, where it plays the role of the $\mathfrak{o}_r$-points of the Lie algebra of GL_N.

We use ϖ to denote a fixed choice of generator of $\mathfrak{p} \subset \mathfrak{o}_r$.

We will make free use of some well known results from Clifford theory (see [21, Section 2]).

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2. Parahoric subalgeberas and filtrations of subgroups

The main goal of this section is to attach to any element in $\mathrm{M}_N(\mathbb{F}_q)$ a parahoric subgroup of G_r together with a natural filtration. These filtrations are finite versions of the ones defined by lattice chains in [2, Section 1].

Let V be a free $\mathfrak{o}_r$-module of rank N, and let $\overline{V} = V \otimes_{\mathfrak{o}_r} \mathbb{F}_q \cong V/\mathfrak{p}V$ (an N-dimensional vector space over $\mathbb{F}_q$). Let $\rho_{r,1} : V \to \overline{V}$ denote the canonical map. Let $E = \mathrm{End}_{\mathfrak{o}_r}(V)$ and $\overline{E} = \mathrm{End}_k(\overline{V}) \cong E \otimes_{\mathfrak{o}_r} \mathbb{F}_q$. Let

$$V = V_0 \supset V_1 \supset \cdots \supset V_e = 0$$

be a flag of $\mathfrak{o}_r$-modules with $e \geq 1$ an integer, and such that V_i is free for $0 \leq i \leq e-1$. Let $\overline{V} = \overline{V}_0 \supset \overline{V}_1 \supset \cdots \supset \overline{V}_e = 0$ be the flag of $\mathbb{F}_q$-vector spaces obtained by setting $\overline{V}_i = V_i \otimes_{\mathfrak{o}_r} \mathbb{F}_q \cong V_i/\mathfrak{p}V_i$. We have $\mathrm{rank}_{\mathfrak{o}_r} V_i = \dim_{\mathbb{F}_q} \overline{V}_i$ for $0 \leq i \leq e-1$.

Lemma 2.1. *Let notation be as above. We then have inclusions of $\mathfrak{o}_r$-modules*

$$V = L_0 \supset L_1 \supset \cdots \supset L_e \supset \cdots \supset L_{er} = 0,$$

where $L_{i+ej} = \mathfrak{p}^j \rho_{r,1}^{-1}(\overline{V}_i) = \mathfrak{p}^j(V_i + \mathfrak{p}V)$, *for* $0 \le i \le e-1$ *and* $0 \le j$.

Proof. The only thing that requires proof is that all the inclusions are strict. If $0 \le i \le e-1$ and $0 \le j \le r-1$ then multiplication by $\mathfrak{p}^j$ induces an isomorphism $L_i/L_{i+1} \cong L_{i+ej}/L_{i+ej+1}$, while the map $\rho_{r,1}$ induces an isomorphism $L_i/L_{i+1} \cong \overline{V}_i/\overline{V}_{i+1}$; in particular, L_{i+ej}/L_{i+ej+1} is non-zero. □

We put $N_i = \mathrm{rank}_{\mathfrak{o}_r} V_i$, for $i = 0, \dots, e$, so that $N = N_0 > N_1 > \cdots > N_e = 0$. Since $\mathfrak{o}_r$ is a self-injective ring (cf. [15, 3.12]), a free submodule of a free $\mathfrak{o}_r$-module is a direct summand. Hence a basis for a free submodule of V can always be extended to a basis for V, and so there exists a basis $\{x_1, \dots, x_N\}$ for V such that $\{x_1, \dots, x_{N_i}\}$ is a basis for V_i, for $i = 0, \dots, e$. Then the image $\{x_1 + \mathfrak{p}V, \dots, x_N + \mathfrak{p}V\}$ of this basis under the map $V \to \overline{V}$ is a basis for $\overline{V}$ such that $\{x_1 + \mathfrak{p}V, \dots, x_{N_i} + \mathfrak{p}V\}$ is a basis for $\overline{V}_i$, and the $\mathfrak{o}_r$-module L_{i+ej} has basis consisting of (the non-zero elements in)

$$\{\varpi^j x_1, \dots, \varpi^j x_{N_i}, \varpi^{j+1} x_{N_i+1}, \dots, \varpi^{j+1} x_N\}, \tag{2.1}$$

for $0 \le i \le e-1$ and $0 \le j$.

We define the $\mathfrak{o}_r$-algebras

$$P = \{x \in E \mid xV_i \subseteq V_i \text{ for all } 0 \le i \le e\},$$
$$\overline{P} = \{x \in \overline{E} \mid x\overline{V_i} \subseteq \overline{V_i} \text{ for all } 0 \le i \le e\}.$$

Algebras of this form are called *parabolic subalgebras* of E and $\overline{E}$, respectively. Similarly, we define the algebra

$$\mathfrak{A} = \{x \in E \mid xL_i \subseteq L_i \text{ for all } 0 \le i \le er\},$$

and an algebra of this form is called a *parahoric subalgebra* of E. The algebra P has a (two-sided) ideal I given by

$$I = \{x \in E \mid xV_i \subseteq V_{i+1} \text{ for all } 0 \le i \le e-1\},$$

and the analogous ideal $\overline{I}$ in $\overline{P}$ is defined in the obvious way. Similarly, the algebra $\mathfrak{A}$ has an ideal $\mathfrak{P}$ given by

$$\mathfrak{P} = \{x \in E \mid xL_i \subseteq L_{i+1} \text{ for all } 0 \le i \le er-1\}.$$

We have $I^e = \overline{I}^e = \mathfrak{P}^{er} = 0$, so the ideals are nilpotent. We remark that, since we have the periodicity relation $L_{i+ej} = \mathfrak{p}^j L_i$, we also have

$$\mathfrak{A} = \{x \in E \mid xL_i \subseteq L_i \text{ for all } 0 \le i \le e-1\},$$
$$\mathfrak{P} = \{x \in E \mid xL_i \subseteq L_{i+1} \text{ for all } 0 \le i \le e-1\}.$$

Using the basis $\{x_1, \dots, x_N\}$ above to identify V with $\mathfrak{o}_r^N$, whence E with $\mathrm{M}_N(\mathfrak{o}_r)$, these algebras and ideals have convenient matrix pictures. For example, writing the matrix of $a \in E$ with respect to this basis as (a_{jk}), we see that $a \in P$ if and only if $a_{jk} = 0$ whenever there is an integer $0 \le i \le e-1$ such that $j > N_i \ge k$. Thus P is the algebra of block upper-triangular matrices, with block sizes $N_1' := N_{e-1} - N_e, \dots, N_e' := N_0 - N_1$, while I is its ideal of strictly block upper-triangular matrices. Similarly, the fact that (2.1) gives a basis for L_i implies that $\mathfrak{A}$ is the algebra of block matrices which are block upper-triangular modulo $\mathfrak{p}$, with the same block sizes, while $\mathfrak{P}$ is its ideal of matrices which are strictly block upper-triangular

modulo $\mathfrak{p}$. Writing $\mathrm{M}_{m\times n}(R)$ for the set of $m \times n$ matrices over a ring R (not necessarily with identity), we have

$$\mathfrak{A} = \begin{pmatrix} \mathrm{M}_{N_1'}(\mathfrak{o}_r) & \mathrm{M}_{N_1'\times N_2'}(\mathfrak{o}_r) & \cdots & \mathrm{M}_{N_1'\times N_e'}(\mathfrak{o}_r) \\ \mathrm{M}_{N_2'\times N_1'}(\mathfrak{p}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{M}_{N_{e-1}'\times N_e'}(\mathfrak{o}_r) \\ \mathrm{M}_{N_e'\times N_1'}(\mathfrak{p}) & \cdots & \mathrm{M}_{N_e'\times N_{e-1}'}(\mathfrak{p}) & \mathrm{M}_{N_e'}(\mathfrak{o}_r) \end{pmatrix},$$

$$\mathfrak{P} = \begin{pmatrix} \mathrm{M}_{N_1'}(\mathfrak{p}) & \mathrm{M}_{N_1'\times N_2'}(\mathfrak{o}_r) & \cdots & \mathrm{M}_{N_1'\times N_e'}(\mathfrak{o}_r) \\ \mathrm{M}_{N_2'\times N_1'}(\mathfrak{p}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{M}_{N_{e-1}'\times N_e'}(\mathfrak{o}_r) \\ \mathrm{M}_{N_e'\times N_1'}(\mathfrak{p}) & \cdots & \mathrm{M}_{N_e'\times N_{e-1}'}(\mathfrak{p}) & \mathrm{M}_{N_e'}(\mathfrak{p}) \end{pmatrix}.$$

Since, with respect to the basis $\{x_1+\mathfrak{p}V, \ldots, x_N+\mathfrak{p}V\}$ of $\overline{V}$, the parabolic algebra $\overline{P}$ also consists of block upper-triangular matrices with the same block-sizes, the map $\rho_{r,1} : E \to \overline{E}$ induces surjections $P \to \overline{P}$, $I \to \overline{I}$, and we get:

Lemma 2.2.
(i) $\mathfrak{A} = \rho_{r,1}^{-1}(\overline{P}) = P + \mathfrak{p}E$,
(ii) $\mathfrak{P} = \rho_{r,1}^{-1}(\overline{I}) = I + \mathfrak{p}E$.

We remark that Lemma 2.2 implies that $\mathfrak{A}/\mathfrak{P} \cong \overline{P}/\overline{I}$, which is semisimple, so that $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$.

Although we always have a surjection $\rho_{r,1} : I^m \to \overline{I}^m$, in general $\mathfrak{P}^m$ is not equal to $I^m + \mathfrak{p}E$, for $m \geq 2$. However, we can use the matrix description of $\mathfrak{P}$ above to obtain a similar description of $\mathfrak{P}^m$, by multiplying elementary matrices with respect to the basis $\{x_1, \ldots, x_N\}$. Indeed, a straightforward induction shows that, for $m \geq 0$, the ideal $\mathfrak{P}^m$ consists of block matrices whose (i,j)-block has entries in $\mathfrak{p}^{\lceil (m+i-j)/e \rceil}$, where $\lceil y \rceil$ denotes the least integer greater than or equal to y.

Lemma 2.3. *For $m \geq 0$ and $0 \leq k \leq e(r-1)+1-m$, we have:*
(i) $\mathfrak{P}^m L_i = L_{i+m}$, *for any* $i \geq 0$,
(ii) $\mathfrak{P}^m = \{x \in E \mid xL_i \subseteq L_{i+m}$ *for all* $k \leq i \leq k+e-1\}$,
(iii) $\mathfrak{P}^m = \{x \in E \mid x\mathfrak{P}^k \subseteq \mathfrak{P}^{k+m}\}$.

Proof. Given the description of $\mathfrak{P}^m$ above, it is straightforward to check that the image of the basis (2.1) of L_i under elementary matrices in $\mathfrak{P}^m$ contains the basis (2.1) of L_{i+m}, and *(i)* follows. Similarly, it is straightforward to check that the matrix description of

$$\{x \in E \mid xL_i \subseteq L_{i+m} \text{ for all } 0 \leq i \leq e-1\}$$

is the same as that for $\mathfrak{P}^m$ above. Now *(ii)* follows since $xL_i \subseteq L_{i+m}$ if and only if $xL_{i+e} \subseteq L_{i+e+m}$, whenever $0 \leq i \leq e(r-1)-m$. Finally, for *(iii)*, suppose $x \in E$ is such that $x\mathfrak{P}^k \subseteq \mathfrak{P}^{k+m}$ so that $x\mathfrak{P}^k L_i \subseteq L_{i+k+m}$, for $i = 0, \ldots, e-1$. But then *(i)* implies $xL_{i+k} \subseteq L_{i+k+m}$, for $i = 0, \ldots, e-1$, and *(ii)* implies $x \in \mathfrak{P}^m$. □

As an immediate corollary, we get:

Corollary 2.4.

(i) $\mathfrak{p}\mathfrak{A} = \mathfrak{A}\mathfrak{p} = \mathfrak{P}^e$,
(ii) $\mathfrak{P}^m = \mathfrak{P}^{m+1}$ *if and only if* $m \geq er$.

Note also that we have $\mathfrak{P}^{e(r-1)} = \mathfrak{p}^{r-1}P$ and $\mathfrak{P}^{e(r-1)+1} = \mathfrak{p}^{r-1}I$. Let $\mathrm{tr} : E \to \mathfrak{o}_r$ denote the trace map.

Lemma 2.5. *Let $x \in E$, and let m be an integer such that $0 \leq m \leq e(r-1)+1$. Then $\mathrm{tr}(\mathfrak{P}^m x) = \{0\}$ if and only if $x \in \mathfrak{P}^{e(r-1)+1-m}$.*

Proof. Note that one implication is clear, since $\mathfrak{P}^{e(r-1)+1} \subseteq I$ so $\mathrm{tr}(\mathfrak{P}^{e(r-1)+1}) = 0$.

We first prove the opposite implication for $m = 0$, so we assume that $x \in E$ is such that $\mathrm{tr}(\mathfrak{A}x) = \{0\}$. It is easy to show (e.g. using elementary matrix considerations) that the trace form $E \times E \to \mathfrak{o}_r$ given by $(\alpha, \beta) \mapsto \mathrm{tr}(\alpha\beta)$ is non-degenerate. Similarly, it is also easy to show that for $\gamma \in E$, the condition $\mathrm{tr}(P\gamma) = \{0\}$ implies that $\gamma \in I$. Hence, since $\mathfrak{p}E \subset \mathfrak{A}$, the condition $\mathrm{tr}(\mathfrak{A}x) = \{0\}$ implies that $x \in \mathfrak{p}^{r-1}E$. Furthermore, since $P \subset \mathfrak{A}$, the condition $\mathrm{tr}(\mathfrak{A}x) = \{0\}$ implies that $x \in I \cap \mathfrak{p}^{r-1}E = \mathfrak{p}^{r-1}I = \mathfrak{P}^{e(r-1)+1}$, as required.

Now suppose $x \in E$ is such that $\mathrm{tr}(\mathfrak{P}^m x) = \{0\}$. Then $\mathrm{tr}(\mathfrak{A}(\mathfrak{P}^m x)) = \{0\}$ so the case $m = 0$ implies that $\mathfrak{P}^m x \subseteq \mathfrak{P}^{e(r-1)+1}$. Now Lemma 2.3, *(iii)* implies that $x \in \mathfrak{P}^{e(r-1)+1-m}$, as required. □

Define the groups

$$U = U^0 = \mathfrak{A}^\times, \quad U^m = 1 + \mathfrak{P}^m, \text{ for } m \geq 1.$$

The group U is called a *parahoric subgroup* of $E^\times$. We have a filtration

$$U \supset U^1 \supset \cdots \supset U^{er-1} \supset U^{er} = \{1\},$$

where the inclusions are strict thanks to Lemma 2.4, *(ii)*. We also define $U^i = \{1\}$ for $i > er$.

Since $\mathfrak{P}$ is a (two-sided) ideal in $\mathfrak{A}$, each group U^m is normal in U. Moreover, if $1 + x \in U^m$, and $1 + y \in U^n$, then

$$(1+x)(1+y) \equiv 1 + x + y \equiv (1+y)(1+x) \pmod{\mathfrak{P}^{m+n}},$$

so we have the commutator relation

$$[U^m, U^n] \subseteq U^{m+n}.$$

Thus in particular, the group U^m is abelian whenever $m \geq er/2$, that is, when $m \geq \lceil \frac{er}{2} \rceil$.

For every $m \geq 1$ we have an isomorphism

$$U^m/U^{m+1} \xrightarrow{\sim} \mathfrak{P}^m/\mathfrak{P}^{m+1}, \qquad (1+x)U^{m+1} \longmapsto x + \mathfrak{P}^{m+1},$$

and since $\mathfrak{p}\mathfrak{P}^m \subseteq \mathfrak{P}^{m+e} \subseteq \mathfrak{P}^{m+1}$, we have an action of $\mathfrak{o}_r/\mathfrak{p} \cong \mathbb{F}_q$ on $\mathfrak{P}^m/\mathfrak{P}^{m+1}$, for each $m \geq 0$. This makes $\mathfrak{P}^m/\mathfrak{P}^{m+1}$ a finite dimensional vector space over the finite field $\mathbb{F}_q$, where the action of $\mathbb{F}_q$ is compatible with the group structure. Hence $\mathfrak{P}^m/\mathfrak{P}^{m+1}$ is an elementary abelian group.

By choosing a basis, we may identify V with $\mathfrak{o}_r^N$, $\overline{V}$ with $\mathbb{F}_q^N$, E with $\mathrm{M}_N(\mathfrak{o}_r)$, $\overline{E}$ with $\mathrm{M}_N(\mathbb{F}_q)$, and $E^\times$ with G_r. These identifications will remain in force throughout the rest of this paper. From now on, let $\Omega_r \subset \mathfrak{g}_r$ be an orbit under the adjoint

(conjugation) action of G_r. Write Ω_1 for the image of Ω_r in $\mathfrak{g}_1$. We will associate a certain parahoric subalgebra to Ω_1, which will be denoted by $\mathfrak{A}_{\mathrm{m}}$. Let

$$\prod_{i=1}^{h} f_i(x)^{m_i} \in \mathbb{F}_q[x]$$

be the characteristic polynomial of Ω_1 (i.e., the characteristic polynomial of any element in Ω_1), where the $f_i(x)$ are distinct and irreducible of degree d_i, for $i = 1, \dots, h$. This determines a partition of N:

$$\lambda = (d_1^{m_1}, \dots, d_h^{m_h}) = (\underbrace{d_1, d_1, \dots, d_1}_{m_1 \text{ times}}, \dots, \underbrace{d_h, d_h, \dots, d_h}_{m_h \text{ times}}).$$

We define $\mathfrak{A}_{\mathrm{m}} \subseteq \mathfrak{g}_r$ to be the preimage of the standard parabolic subalgebra of $\mathfrak{g}_1$ corresponding to λ (i.e., the block upper-triangular subalgebra whose block sizes are given by λ, in the order given above). Moreover, we let $\mathfrak{A}_{\mathrm{M}} = \mathfrak{g}_r = \mathrm{M}_N(\mathfrak{o}_r)$ be the full matrix algebra. Let $\mathfrak{P}_{\mathrm{m}}$ and $\mathfrak{P}_{\mathrm{M}}$ be the corresponding Jacobson radicals of $\mathfrak{A}_{\mathrm{m}}$ and $\mathfrak{A}_{\mathrm{M}}$, respectively. Then $\mathfrak{P}_{\mathrm{m}}$ is the pre-image under $\rho_{r,1} : \mathfrak{g}_r \to \mathfrak{g}_1$ of the strict block-upper subalgebra of $\mathfrak{A}_{\mathrm{m}}$, and similarly $\mathfrak{P}_{\mathrm{M}}$ is the pre-image of 0, that is, $\mathfrak{P}_{\mathrm{M}} = \mathfrak{p}\mathfrak{g}_r$. For $* \in \{\mathrm{m}, \mathrm{M}\}$ we have the corresponding groups

$$U_* = U_*^0 = \mathfrak{A}_*^\times, \quad U_*^i = 1 + \mathfrak{P}_*^i, \qquad \text{for } i \geq 1,$$

and the filtrations

$$U_* \supset U_*^1 \supset \cdots \supset U_*^{e_* r} = \{1\},$$

where $e_* = e(\mathfrak{A}_*)$ is the number of blocks of the algebra $\mathfrak{A}_*$ mod $\mathfrak{p}$. Note that $U_{\mathrm{M}}^i = K^i$ and $e_{\mathrm{M}} = 1$, while $e_{\mathrm{m}} = m_1 + \cdots + m_h$.

By definition, we have $\mathfrak{A}_{\mathrm{M}} \supseteq \mathfrak{A}_{\mathrm{m}}$, and the label m here stands for "minimal", while M stands for "maximal". From the definitions, we have

$$U_{\mathrm{m}}/U_{\mathrm{m}}^1 \cong \prod_{i=1}^{h} \mathrm{GL}_{d_i}(\mathbb{F}_q))^{m_i}, \qquad \mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}} \cong \prod_{i=1}^{h} \mathrm{M}_{d_i}(\mathbb{F}_q)^{m_i},$$
$$U_{\mathrm{M}}/U_{\mathrm{M}}^1 = G_r/K^1 \cong G_1, \qquad \mathfrak{A}_{\mathrm{M}}/\mathfrak{P}_{\mathrm{M}} \cong \mathfrak{g}_1.$$

Note that if Ω_1 has irreducible characteristic polynomial, then $\mathfrak{A}_{\mathrm{m}} = \mathfrak{A}_{\mathrm{M}} = \mathfrak{g}_r$.

Given an element $\beta \in \mathfrak{g}_r$ we will denote its image in $\mathfrak{g}_1$ by $\bar{\beta}$. Similarly, if $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$, we will let β_{m} denote the image of β in $\mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}}$. Note that by, for example, the rational Jordan normal form, $\Omega_r \cap \mathfrak{A}_{\mathrm{m}}$ is non-empty. Up to G_r-conjugation we can also arrange the diagonal irreducible blocks of $\bar{\beta}$, and hence of β_{m} in any order. In particular, we can find a $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$ such that

$$\beta_{\mathrm{m}} = \underbrace{\beta_{\mathrm{m}}^1 \oplus \cdots \oplus \beta_{\mathrm{m}}^1}_{m_1 \text{ times}} \oplus \cdots \oplus \underbrace{\beta_{\mathrm{m}}^h \oplus \cdots \oplus \beta_{\mathrm{m}}^h}_{m_h \text{ times}}, \tag{2.2}$$

where each $\beta_{\mathrm{m}}^i \in \mathrm{M}_{d_i}(\mathbb{F}_q)$ has irreducible characteristic polynomial $f_i(x)$.

There are several equivalent characterisations of regular elements in $\mathfrak{g}_1$. One of the simplest is that an element in $\mathfrak{g}_1$ is regular if its centraliser in $\mathfrak{g}_1$ has dimension N (as $\mathbb{F}_q$-vector space). One can also define regular elements in $\mathfrak{g}_i$ for $r \geq i > 1$, as those elements whose centraliser in $\mathfrak{g}_i$ has $\mathfrak{o}_i$-rank N. A result of Hill [8, Theorem 3.6] implies that an element in $\mathfrak{g}_i$ is regular if and only if its image in $\mathfrak{g}_1$ is regular.

Lemma 2.6. *Let $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$. If β is regular, then $C_{G_r}(\beta) \subseteq \mathfrak{A}_{\mathrm{m}}^\times$.*

Proof. If β is regular, we have $C_{G_r}(\beta) = \mathfrak{o}_r[\beta]^\times$, so $\beta \in \mathfrak{A}_\mathrm{m}$ implies that $C_{G_r}(\beta) \subset \mathfrak{A}_\mathrm{m}$, since $\mathfrak{A}_\mathrm{m}$ is an algebra. □

Lemma 2.7. *We have*

$$|C_{\mathfrak{A}_\mathrm{m}/\mathfrak{P}_\mathrm{m}}(\beta_\mathrm{m})| = q^N = |C_{\mathfrak{g}_1}(\bar{\beta})|.$$

Proof. The isomorphism $\mathfrak{A}_\mathrm{m}/\mathfrak{P}_\mathrm{m} \cong \prod_{i=1}^h \mathrm{M}_{d_i}(\mathbb{F}_q)^{m_i}$, induces an isomorphism

$$C_{\mathfrak{A}_\mathrm{m}/\mathfrak{P}_\mathrm{m}}(\beta_\mathrm{m}) \cong \prod_{i=1}^h C_{\mathrm{M}_{d_i}(\mathbb{F}_q)}(\beta_\mathrm{m}^i)^{m_i},$$

so $|C_{\mathfrak{A}_\mathrm{m}/\mathfrak{P}_\mathrm{m}}(\beta_\mathrm{m})| = \prod_{i=1}^h q^{d_i m_i} = q^N$. The second equality follows by definition of regularity of $\bar{\beta}$. □

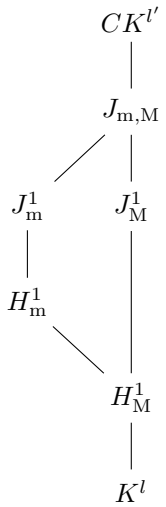
Set $l = \lceil \frac{r}{2} \rceil$, $l' = \lfloor \frac{r}{2} \rfloor$, so that $l + l' = r$. The relations $\mathfrak{A}_\mathrm{M} \supseteq \mathfrak{A}_\mathrm{m} \supseteq \mathfrak{P}_\mathrm{m} \supseteq \mathfrak{P}_\mathrm{M}$ imply that for every $i \geq 1$, $\mathfrak{P}_\mathrm{M}^i = \mathfrak{p}^i\mathfrak{g}_r$ is a two-sided ideal in $\mathfrak{A}_\mathrm{m}$. For $\beta \in \Omega_r \cap \mathfrak{A}$ and $* \in \{\mathrm{m}, \mathrm{M}\}$, we can therefore define the following groups

$$\begin{aligned} C &= C_{G_r}(\beta), \\ J_*^1 &= (C \cap U_*^1)U_*^{e_* l'}, \\ H_*^1 &= (C \cap U_*^1)U_*^{e_* l'+1}. \end{aligned}$$

Note that $J_\mathrm{M}^1 = (C \cap K^1)K^{l'}$ and $H_\mathrm{M}^1 = (C \cap K^1)K^l$, and that both of these groups are normalised by $CK^{l'}$, since C normalises both K^1 and $K^{l'}$, and $[K^{l'}, K^1] \subseteq K^l \subseteq K^{l'}$. Moreover, we define the group

$$J_{\mathrm{m},\mathrm{M}} = (C \cap U_\mathrm{m}^1)K^{l'}.$$

We have the following diagram of subgroups, where the vertical and slanted lines denote inclusions (we have only indicated the inclusions which are relevant to us and repeat the definitions of the groups, for the reader's convenience).



$$J_{\mathrm{m},\mathrm{M}} = (C \cap U_\mathrm{m}^1)K^{l'},$$

$$\begin{aligned} J_\mathrm{m}^1 &= (C \cap U_\mathrm{m}^1)U_\mathrm{m}^{e_\mathrm{m} l'}, \\ H_\mathrm{m}^1 &= (C \cap U_\mathrm{m}^1)U_\mathrm{m}^{e_\mathrm{m} l'+1}, \\ J_\mathrm{M}^1 &= (C \cap K^1)K^{l'}, \\ H_\mathrm{M}^1 &= (C \cap K^1)K^l. \end{aligned}$$

We explain the non-trivial inclusions in the above diagram. We have $\mathfrak{P}_{\mathrm{m}} \supseteq \mathfrak{P}_{\mathrm{M}}$, and so $U^1_{\mathrm{m}} \supseteq K^1$. By Corollary 2.4, we get

$$U_{\mathrm{m}}^{e_{\mathrm{m}}l'+1} = 1 + \mathfrak{p}^{l'}\mathfrak{P}_{\mathrm{m}} \supseteq 1 + \mathfrak{p}^{l'}\mathfrak{P}_{\mathrm{M}} = K^l;$$

thus $H^1_{\mathrm{m}} \supseteq H^1_{\mathrm{M}}$. Moreover,

$$U_{\mathrm{m}}^{e_{\mathrm{m}}l'} = 1 + \mathfrak{p}^{l'}\mathfrak{A}_{\mathrm{m}} \subseteq 1 + \mathfrak{p}^{l'}\mathfrak{A}_{\mathrm{M}} = K^{l'},$$

so $J_{\mathrm{m,M}}$ contains both J^1_{m} and J^1_{M} as subgroups.

The following lemma will be a crucial step in the construction of representations, and is the main reason why we work with the algebra $\mathfrak{A}_{\mathrm{m}}$ and its associated subgroups.

Lemma 2.8. *Suppose that Ω_r consists of regular elements. Then there exists a $\beta \in \Omega_r$ such that $J_{\mathrm{m,M}}$ is a Sylow p-subgroup of $CK^{l'}$.*

Proof. By the rational Jordan normal form, there is a $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$. Moreover, any permutation of the diagonal blocks can be achieved by G_r-conjugation, so there is a $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$ such that $\beta_{\mathrm{m}} \in \mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}}$ satisfies (2.2). Assume now that β is chosen in this way. We have

$$[CK^{l'} : J_{\mathrm{m,M}}] = \frac{|CK^{l'}|/|K^{l'}|}{|J_{\mathrm{m,M}}|/|K^{l'}|} = \frac{|C/(C \cap K^{l'})|}{|(C \cap U^1_{\mathrm{m}})/(C \cap K^{l'})|} = \frac{|C|}{|C \cap U^1_{\mathrm{m}}|}.$$

Thus we need to show that $C \cap U^1_{\mathrm{m}}$ is a Sylow p-subgroup in C. Since β is regular, C is abelian, and by Lemma 2.6 we have $C \subseteq U_{\mathrm{m}} = \mathfrak{A}_{\mathrm{m}}^{\times}$, so we have

$$\frac{C}{C \cap U^1_{\mathrm{m}}} = \frac{C \cap U_{\mathrm{m}}}{C \cap U^1_{\mathrm{m}}}.$$

Then the isomorphism $U_{\mathrm{m}}/U^1_{\mathrm{m}} \cong \prod_{i=1}^h \mathrm{GL}_{d_i}(\mathbb{F}_q)^{m_i}$ induces an isomorphism

$$\frac{C \cap U_{\mathrm{m}}}{C \cap U^1_{\mathrm{m}}} \cong \prod_{i=1}^h C_{\mathrm{GL}_{d_i}(\mathbb{F}_q)}(\beta^i_{\mathrm{m}})^{m_i}.$$

Each β^i_{m} has irreducible characteristic polynomial over $\mathbb{F}_q$, so $\mathbb{F}_q[\beta^i_{\mathrm{m}}]/\mathbb{F}_q$ is a field extension of degree d_i. Since $C_{\mathrm{GL}_{d_i}(\mathbb{F}_q)}(\beta^i_{\mathrm{m}}) = \mathbb{F}_q[\beta^i_{\mathrm{m}}]^{\times}$, we conclude that p does not divide the order of $C_{\mathrm{GL}_{d_i}(\mathbb{F}_q)}(\beta^i_{\mathrm{m}})$. Therefore, p does not divide the order of $\frac{C}{C \cap U^1_{\mathrm{m}}}$, so $J_{\mathrm{m,M}}$ is a Sylow p-subgroup of $CK^{l'}$. □

We remark that the above lemma holds without the hypothesis that Ω_r consists of regular elements, but the proof is slightly easier in the case of a regular orbit. Note also that when β is as in the above lemma, $J_{\mathrm{m,M}}$ is in fact normal in $CK^{l'}$, since $C = C \cap U_{\mathrm{M}}$ normalises $J_{\mathrm{m,M}}$ when C is abelian. Thus $J_{\mathrm{m,M}}$ is the unique Sylow p-subgroup of $CK^{l'}$. We will not need this fact.

3. Characters and Heisenberg lifts

As in the introduction, F denotes the fraction field of $\mathfrak{o}$. Fix an additive character $\psi : F \to \mathbb{C}^{\times}$ which is trivial on $\mathfrak{o}$ but not on $\mathfrak{p}^{-1}$ (i.e., ψ has conductor $\mathfrak{o}$). For each $r \geq 1$ we can view ψ as a character of the group $F/\mathfrak{p}^r$ whose kernel contains $\mathfrak{o}_r$. We will use ψ and the trace form $(x, y) \mapsto \mathrm{tr}(xy)$ on $\mathfrak{g}_r$ to set up a duality between

the groups $\mathrm{Irr}(K^i)$ and $\mathfrak{g}_{r-i}$, for $i \geq r/2$. For $\beta \in \mathrm{M}_N(\mathfrak{o}_r)$, define a homomorphism $\psi_\beta : K^i \to \mathbb{C}^\times$ by

$$\psi_\beta(1+x) = \psi(\varpi^{-r} \operatorname{tr}(\hat{\beta}\hat{x})), \tag{3.1}$$

where $x \in \mathfrak{p}^i\mathfrak{g}_r$, and $\hat{\beta}, \hat{x} \in \mathrm{M}_N(\mathfrak{o})$ denote arbitrary lifts of β and x, respectively. The value $\psi(\varpi^{-r} \operatorname{tr}(\hat{\beta}\hat{x}))$ is independent of the choice of lifts, since ψ is trivial on $\mathfrak{o}$. For this reason, we will abuse notation slightly from now on and write $\psi(\varpi^{-r} \operatorname{tr}(\beta x))$ instead of $\psi(\varpi^{-r} \operatorname{tr}(\hat{\beta}\hat{x}))$. The map $\beta \mapsto \psi_\beta$ is a homomorphism whose kernel is $\mathfrak{p}^{r-i}\mathfrak{g}_r$, thanks to the non-degeneracy of the trace form. Hence it induces an isomorphism

$$\mathfrak{g}_r/\mathfrak{p}^{r-i}\mathfrak{g}_r \xrightarrow{\sim} \mathrm{Irr}(K^i),$$

where we will usually identify $\mathfrak{g}_r/\mathfrak{p}^{r-i}\mathfrak{g}_r$ with $\mathfrak{g}_{r-i}$. For $g \in G_r$ we have

$$\psi_{g\beta g^{-1}}(1+x) = \psi(\varpi^{-r} \operatorname{tr}(g\beta g^{-1}x)) = \psi(\varpi^{-r} \operatorname{tr}(\beta g^{-1}xg)) = \psi_\beta(1+g^{-1}xg).$$

Let $\mathfrak{A}, \mathfrak{P}$, and $U^m, m \geq 0$ be the objects associated to an arbitrary flag of length e, as in Section 2. Let n and m be two integers such that $e(r-1)+1 \geq n > m \geq n/2 > 0$. Then U^m/U^n is abelian, and we have an isomorphism

$$\mathfrak{P}^m/\mathfrak{P}^n \xrightarrow{\sim} U^m/U^n, \qquad x + \mathfrak{P}^n \longmapsto (1+x)U^n.$$

Each $a \in \mathfrak{g}_r$ defines a character $\mathfrak{g}_r \to \mathbb{C}^\times$ via $x \mapsto \psi(\varpi^{-r} \operatorname{tr}(ax))$, and this defines an isomorphism $\mathfrak{g}_r \to \mathrm{Irr}(\mathfrak{g}_r)$. For any subgroup S of $\mathfrak{g}_r$, define

$$S^\perp = \{x \in \mathfrak{g}_r \mid \psi(\varpi^{-r} \operatorname{tr}(xS)) = 1\}.$$

Using the isomorphism $\mathfrak{g}_r \to \mathrm{Irr}(\mathfrak{g}_r)$, we can identify $S^\perp$ with the group of characters of $\mathfrak{g}_r$ which are trivial on S.

We generalise the definition of ψ_β to allow β to lie in an appropriate power of $\mathfrak{P}$. For any $\beta \in \mathfrak{P}^{e(r-1)+1-n}$ define a character $\psi_\beta : U^m \to \mathbb{C}^\times$ by

$$\psi_\beta(1+x) = \psi(\varpi^{-r} \operatorname{tr}(\beta x)).$$

Lemma 3.1. *Let $e(r-1)+1 \geq n > m \geq n/2 > 0$. Then*

(i) For any integer i such that $0 \leq i \leq e(r-1)+1$, we have

$$(\mathfrak{P}^i)^\perp = \mathfrak{P}^{e(r-1)+1-i}.$$

(ii) The map $\beta \mapsto \psi_\beta$ induces an isomorphism

$$\mathfrak{P}^{e(r-1)+1-n}/\mathfrak{P}^{e(r-1)+1-m} \xrightarrow{\sim} \mathrm{Irr}(U^m/U^n).$$

Proof. Let $\rho_r : \mathfrak{o} \to \mathfrak{o}_r$ be the canonical map. For any $x \in \mathfrak{g}_r$ the set $\varpi^{-r}\rho_r^{-1}(\operatorname{tr}(x\mathfrak{P}^i))$ is a fractional ideal of $\mathfrak{o}$, so by our choice of ψ we have

$$\psi(\varpi^{-r} \operatorname{tr}(x\mathfrak{P}^i)) := \psi(\varpi^{-r}\rho_r^{-1}(\operatorname{tr}(x\mathfrak{P}^i)) = 1$$

if and only if $\varpi^{-r}\rho_r^{-1}(\operatorname{tr}(x\mathfrak{P}^i) \subseteq \mathfrak{o}$. Thus $x \in (\mathfrak{P}^i)^\perp$ if and only if $\operatorname{tr}(x\mathfrak{P}^i) = 0$ in $\mathfrak{o}_r$, so *(i)* follows from Lemma 2.5. Moreover, *(i)* together with the isomorphism $\mathfrak{P}^m/\mathfrak{P}^n \cong U^m/U^n$ implies *(ii)*. □

Let G be a finite group and N a normal subgroup, such that G/N is an elementary abelian p-group. Then the group G/N has a structure of $\mathbb{F}_p$-vector space. Let χ be a one-dimensional representation of N which is stabilised by G. Define an alternating bilinear form

$$h_\chi : G/N \times G/N \longrightarrow \mathbb{C}^\times, \qquad h_\chi(xN, yN) = \chi([x,y]) = \chi(xyx^{-1}y^{-1}).$$

By bilinearity we simply mean that $h_\chi(xN, yzN) = h_\chi(xN, yN)h_\chi(xN, zN)$ for all $x, y, z \in G$, and similarly for linearity in the first variable. This follows from the commutator relation $[x, yz] = [x, y]^y[x, z]$ and its analogue for the first variable. Note that linearity with respect to scalar multiplication follows from this since for $\bar{n} \in \mathbb{F}_p$ we have $\bar{n}(xN) = x^n N$, for any lift $n \in \mathbb{Z}$ of $\bar{n}$.). An easy computation shows that h_χ is well-defined, thanks to the stability of χ under G. Define the subspace

$$\overline{R}_\chi = \{xN \in G/N \mid h_\chi(xN, yN) = 1 \text{ for all } y \in G\}.$$

This is the *radical* of the form h_χ, and we say that h_χ is *non-degenerate* if $\overline{R}_\chi = 0$. We will make use of the following result (cf. [3, 8.3.3]):

Lemma 3.2. *Assume that the form h_χ is non-degenerate. Then there exists a unique $\eta_\chi \in \mathrm{Irr}(G \mid \chi)$, and $\dim \eta_\chi = [G : N]^{1/2}$.*

Note that if χ and χ' are two representations of N such that $\eta_\chi = \eta_{\chi'}$, then by Clifford's theorem, the restriction $\eta_\chi|_N = \eta_{\chi'}|_N$ is a multiple of χ and of χ', so we must have $\chi = \chi'$.

We will encounter situations where the form h_χ is not non-degenerate. In these cases we will apply the following generalisation, which is a corollary of the above lemma:

Corollary 3.3. *Let G, N and $\overline{R}_\chi$ be as above, and let R_χ be the inverse image of $\overline{R}_\chi$ under the map $G \to G/N$. Then χ has an extension to R_χ, and for any extension $\tilde{\chi}$ of χ, there exists a unique $\eta_{\tilde{\chi}} \in \mathrm{Irr}(G \mid \tilde{\chi})$. Moreover,*

$$\dim \eta_{\tilde{\chi}} = [G : R_\chi]^{1/2}.$$

Proof. By definition, $\chi([R_\chi, R_\chi]) = 1$, so $R_\chi / \operatorname{Ker} \chi$ is abelian and thus χ extends to R_χ. Let $\tilde{\chi}$ be an extension and $x \in G$. Then, for any $r \in R_\chi$, we have $[x, r] \in N$, so

$$\tilde{\chi}([x, r]) = \chi([x, r]) = 1.$$

Hence $\tilde{\chi}(xrx^{-1}) = \tilde{\chi}(r)$, that is, x stabilises $\tilde{\chi}$. Moreover, R_χ is normal in G since for any $x, y \in G$, $r \in R_\chi$, we have

$$\begin{aligned}\chi([xrx^{-1}, y]) &= \tilde{\chi}([xrx^{-1}, y]) = \tilde{\chi}(xrx^{-1}yxr^{-1}x^{-1}y^{-1}) \\ &= \tilde{\chi}(x^{-1}y^{-1}xrx^{-1}yxr^{-1}) = \tilde{\chi}(x^{-1}y^{-1}xrx^{-1}yx)\tilde{\chi}(r^{-1}) \\ &= \tilde{\chi}(r)\tilde{\chi}(r^{-1}) = 1.\end{aligned}$$

We thus have a well-defined form $h_{\tilde{\chi}}$ on G/R_χ, which is non-degenerate. The remaining statements now follow immediately from Lemma 3.2. □

In the situation of the above corollary, we call $\eta_{\tilde{\chi}}$ a *Heisenberg lift* of χ.

We will apply the above results to the group J^1_* and its normal subgroup H^1_*, for $* \in \{\mathrm{m}, \mathrm{M}\}$. We have an isomorphism

$$J^1_*/H^1_* \cong \frac{U_*^{e_* l'}}{(C \cap U_*^{e_* l'})U_*^{e_* l' + 1}},$$

and this is a quotient of $U_*^{e_* l'}/U_*^{e_* l' + 1} \cong \overline{\mathfrak{A}}_*/\overline{\mathfrak{P}}_*$ (where $\overline{\mathfrak{A}}_*$ is the image of $\mathfrak{A}_*$ in $\mathfrak{g}_1$, as in Section 2), with the latter isomorphism being induced by the map $1 + \pi^{l'} x \mapsto \bar{x} + \mathfrak{P}_*$. Since $\overline{\mathfrak{A}}_*/\overline{\mathfrak{P}}_*$ is a product of additive groups of matrix rings over $\mathbb{F}_q$, it is an elementary abelian p-group (where as before $p = \operatorname{char} \mathbb{F}_q$), and

in fact has a structure of $\mathbb{F}_q$-vector space. Thus J^1_*/H^1_*, being a quotient of an elementary abelian group, is itself elementary abelian.

4. Construction of regular representations

For any $x \in \mathfrak{g}_r$, let x_i denote the image of x in $\mathfrak{g}_i$, for $r \geq i \geq 1$. We also write $\bar{x}$ for x_1. Recall from the paragraph preceding Lemma 2.6 that a regular element in $\mathfrak{g}_i$ can be defined by the property of having centraliser in $\mathfrak{g}_i$ of $\mathfrak{o}_i$-rank N. A result of Hill [8, Corollary 3.7] implies that if $\beta_i \in \mathfrak{g}_i$ is regular, then $C_{G_i}(\beta_i) = \mathfrak{o}_i[\beta_i]^\times$. It follows that if β is regular, then $C = C_{G_r}(\beta)$ is abelian and the homomorphisms

$$\rho_{r,i} : C \to C_{G_i}(\beta_i)$$

are surjective for every $r \geq i \geq 1$.

Suppose that π is an irreducible representation of G_r. By Clifford's theorem, the restriction of π to the abelian group K^l defines an orbit of characters ψ_β, and hence by the results of Section 3, an orbit of elements in $\mathfrak{g}_r/\mathfrak{p}^{r-l}\mathfrak{g}_r \cong \mathfrak{g}_{l'}$ under the conjugacy action of G_r (i.e., the adjoint action). We call π *regular* if this orbit in $\mathfrak{g}_{l'}$ consists of regular elements.

Fix an orbit $\Omega_{l'} \subset \mathfrak{g}_{l'}$ consisting of regular elements. We will construct all the irreducible representations of G_r with orbit $\Omega_{l'}$. When r is even, the construction is well known and amounts to taking any $\beta + \mathfrak{p}^{l'}\mathfrak{g}_r \in \Omega_{l'}$, extending ψ_β to $CK^{l'}$ and inducing to G_r. To show that ψ_β extends to $CK^{l'}$ is straightforward in this case; see for example [8, Theorem 4.1].

From now on, assume that $r \geq 3$ is odd, so that $l' = l - 1$. We highlight the hypotheses that will remain in force throughout this section:

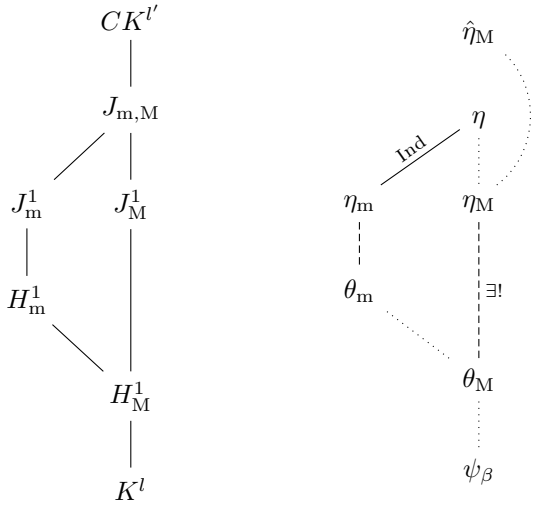
(i) $r \geq 3$ is odd,
(ii) $\beta \in \mathfrak{g}_r$ is regular,
(iii) $\beta \in \mathfrak{A}_\mathrm{m}$ and the image $\beta_\mathrm{m} \in \mathfrak{A}_\mathrm{m}/\mathfrak{P}_\mathrm{m} \cong \prod_{i=1}^h \mathrm{M}_{d_i}(\mathbb{F}_q)^{m_i}$ is

$$\beta_\mathrm{m} = \underbrace{\beta^1_\mathrm{m} \oplus \cdots \oplus \beta^1_\mathrm{m}}_{m_1 \text{ times}} \oplus \cdots \oplus \underbrace{\beta^h_\mathrm{m} \oplus \cdots \oplus \beta^h_\mathrm{m}}_{m_h \text{ times}},$$

where $\beta^i_\mathrm{m} \in \mathrm{M}_{d_i}(\mathbb{F}_q)$ have irreducible characteristic polynomial.

Before presenting the details of the construction, we give an informal overview. Schematically, the construction is illustrated by the following diagrams (dotted lines are extensions, dashed are Heisenberg lifts, and the solid one between η_m and

η is an induction):



The diagram of representations on the right should be read from bottom to top. We have seen in Lemma 2.8 that there exists a $\beta \in \Omega_r$ such that $J_{\mathrm{m,M}}$ is a Sylow p-subgroup of $CK^{l'}$, and fix one such β. We then show that the character ψ_β of K^l has an extension θ_{M} to H^1_{M} and that θ_{M} extends further to θ_{m} on H^1_{m}. Next, we use Corollary (3.3) to show that there exists a unique representation $\eta_{\mathrm{M}} \in \mathrm{Irr}(J^1_{\mathrm{M}} \mid \theta_{\mathrm{M}})$, as well as a (non-unique) representation $\eta_{\mathrm{m}} \in \mathrm{Irr}(J^1_{\mathrm{m}} \mid \theta_{\mathrm{m}})$. Moreover, we compute the dimensions of η_{M} and η_{m}. Then we show that $\eta := \mathrm{Ind}_{J^1_{\mathrm{m}}}^{J_{\mathrm{m,M}}} \eta_{\mathrm{m}}$ has the same dimension as η_{M}, from which it follows that η is an extension of η_{M}. We can then apply a general lemma to conclude that η_{M} has an extension $\hat{\eta}_{\mathrm{M}}$ to $CK^{l'}$. Finally, we show that by choosing all possible extensions θ_{M} of ψ_β and all possible extensions $\hat{\eta}_{\mathrm{M}}$ of η_{M}, we have constructed all the representations in $\mathrm{Irr}(CK^{l'} \mid \psi_\beta)$, without redundancy. Since $CK^{l'} = \mathrm{Stab}_{G_r}(\psi_\beta)$, a standard result from Clifford theory then yields all the irreducible representations of G_r with orbit Ω_r by induction from $CK^{l'}$.

We now give the details and proofs of the construction. Some of the steps can be carried out for the groups arising from the algebras $\mathfrak{A}_{\mathrm{m}}$ and $\mathfrak{A}_{\mathrm{M}}$ simultaneously. For this purpose, we will let $\mathfrak{A}$ denote either $\mathfrak{A}_{\mathrm{m}}$ or $\mathfrak{A}_{\mathrm{M}}$, and let $\mathfrak{P}$ be the radical in $\mathfrak{A}$, with $e = e(\mathfrak{A})$. The associated subgroups will be denoted by U^i, H^1, J^1.

Lemma 4.1. *The character ψ_β has an extension θ_{M} to H^1_{M}. Moreover, θ_{M} has an extension θ_{m} to H^1_{m}.*

Proof. By Lemma 3.1 *(ii)*, if we take

$$m = el' + 1, \qquad n = 2m - 1 = e(r-1) + 1,$$

then the coset $\beta + \mathfrak{P}^{el'}$ defines a character on $U^{el'+1}$, trivial on $U^{e(r-1)+1}$ by the same formula as the one defining ψ_β. Since $\mathfrak{P}^{el'} = \mathfrak{p}^{l'}\mathfrak{A}$, we have a map

$$\mathfrak{A}/\mathfrak{P}^{el'} \longrightarrow \mathfrak{g}_r/\mathfrak{p}^{l'}\mathfrak{g}_r,$$

which sends $\beta + \mathfrak{P}^{el'}$ to $\beta + \mathfrak{p}^{l'}\mathfrak{g}_r$ (note that this is neither surjective nor injective). Thus the different choices of lift of the latter coset give the different choices of

extension of ψ_β to $U^{el'+1}$. Our element $\beta \in \mathfrak{A}$ therefore gives rise to an extension (which we still denote by ψ_β) of ψ_β to $U^{el'+1}$, defined by

$$\psi_\beta(1+x) = \psi(\varpi^{-r} \operatorname{tr}(\beta x)), \quad \text{for } x \in \mathfrak{P}^{el'+1}.$$

We now show the existence of the extensions θ_{M} and θ_{m}. If $c \in C \cap U^1$ and $x \in \mathfrak{P}^{el'+1}$, then

$$\begin{aligned}[c, 1+x] &\in c(1+x)c^{-1}(1-x+\mathfrak{P}^{e(r-1)+2}) \\ &= 1 + cxc^{-1} - x + \mathfrak{P}^{e(r-1)+2}.\end{aligned}$$

Since we have

$$U^{e(r-1)+1} \subseteq \operatorname{Ker} \psi_\beta, \tag{4.1}$$

we obtain

$$\psi_\beta([c, 1+x]) = \psi(\varpi^{-r} \operatorname{tr}(\beta(cxc^{-1} - x))) = \psi(\varpi^{-r} \operatorname{tr}(c\beta x c^{-1} - \beta x)) = 1,$$

where we have used that c commutes with β.

Thus $C \cap U^1$ stabilises the character ψ_β on $U^{el'+1}$, and since $C \cap U^1$ is abelian, this implies that ψ_β extends to $H^1 = (C \cap U^1)U^{el'+1}$. □

Remark 4.2. The extension θ_{M} can be written as a character $\theta_0\psi_\beta$, where $\theta_0 \in \operatorname{Irr}(C \cap K^1)$ is a character which agrees with ψ_β on $C \cap K^l$ and

$$\theta_0\psi_\beta(zk) := \theta_0(z)\psi_\beta(k),$$

for $z \in C \cap K^1$ and $k \in K^1$. We will use this later in the proof of Lemma 4.9. One can write θ_{m} similarly, but we will not need that.

We fix arbitrary extensions θ_{M} and θ_{m} as in the above lemma. For $* \in \{\mathrm{m}, \mathrm{M}\}$, we will now construct the irreducible representations η_* of J^1_* containing θ_*. In particular, we will show that there exists a unique representation η_{M} of J^1_{M} containing θ_{M}. We will treat both cases simultaneously, denoting either θ_{m} or θ_{M} by θ. We need to verify the hypotheses of Corollary 3.3. To this end, first note that θ is stabilised by J^1. Indeed, it is enough to show that $U^{el'}$ stabilises θ. For $x \in \mathfrak{P}^{el'}$, $c \in (C \cap U^1)$ and $y \in \mathfrak{P}^{el'+1}$, we have

$$\begin{aligned}[1+x, c(1+y)] &\in (1+x)c(1+y)(1-x+x^2+\mathfrak{P}^{e(r-1)+1})(1-y+\mathfrak{P}^{e(r-1)+1})c^{-1} \\ &\subseteq (c+xc-cx+cy)(1-y)c^{-1} + \mathfrak{P}^{e(r-1)+1} \\ &\subseteq 1 + x - cxc^{-1} + \mathfrak{P}^{e(r-1)+1}.\end{aligned}$$

Hence, since ψ_β is trivial on $U^{e(r-1)+1}$ (see (4.1)) and c commutes with β, we have

$$\theta([1+x, c(1+y)]) = \psi_\beta([1+x, c(1+y)]) = \psi(\varpi^{-r} \operatorname{tr}(c\beta x c^{-1} - \beta x)) = 1,$$

that is, θ is stabilised by the element $c(1+y)$, hence by all of J^1. We saw at the end of Section 3 that J^1/H^1 is an elementary abelian p-group. Define the alternating bilinear form

$$h_\beta : J^1/H^1 \times J^1/H^1 \longrightarrow \mathbb{C}^\times, \qquad h_\beta(xH^1, yH^1) = \theta([x, y]) = \psi_\beta([x, y]).$$

Let $\overline{R}_\beta$ be the radical of the form h_β, and let R_β denote the preimage of $\overline{R}_\beta$ under the map $J^1 \to J^1/H^1$. If we need to specify which parabolic subalgebra $\mathfrak{A}_*$ we are working with, we will write $\overline{R}_{\beta,*}$, $R_{\beta,*}$, for $* \in \{\mathrm{m}, \mathrm{M}\}$. For our purposes, we need

to determine the dimension of the unique representation $\eta \in \mathrm{Irr}(J^1 \mid \psi_\beta)$, which by Corollary 3.3 equals $[J^1 : R_\beta]^{1/2} = [J^1/H^1 : \overline{R}_\beta]^{1/2}$.

In order to determine the radical of the form h_θ, we need the following result:

Lemma 4.3. *Let $x, y \in J^1$ and write $x = z_1(1+s)$ and $y = z_2(1+t)$, where $z_1, z_2 \in C \cap U^1$ and $s, t \in \mathfrak{P}^{el'}$. Then*

$$\theta([x,y]) = \psi_\beta(1 + (st - ts)).$$

Proof. Note that for any $s_1, s_2 \in \mathfrak{P}^{el'}$ we have $z_1 s_1 s_2 \in s_1 s_2 + \mathfrak{P}^{e(r-1)+1}$ and

$$z_1 s_1 z_2 s_2 \in z_1(s_1 + \mathfrak{P}^{el'+1})s_2 \subseteq z_1 s_1 s_2 + \mathfrak{P}^{e(r-1)+1} \subseteq s_1 s_2 + \mathfrak{P}^{e(r-1)+1}.$$

Thus

$$\begin{aligned}
[x,y] &\in z_1(1+s)z_2(1+t)(1-s+s^2+\mathfrak{P}^{e(r-1)+1})z_1^{-1}(1-t+t^2+\mathfrak{P}^{e(r-1)+1})z_2^{-1} \\
&\subseteq (z_1 z_2 + z_1 s z_2)(1+t)(1-s+s^2)(z_1^{-1}z_2^{-1} - z_1^{-1}tz_2^{-1} + z_1^{-1}t^2 z_2^{-1}) + \mathfrak{P}^{e(r-1)+1} \\
&\subseteq (1 - z_1 z_2 s z_1^{-1} z_2^{-1} + z_1 z_2 t z_1^{-1} z_2^{-1} - ts + z_1 s z_1^{-1} + st \\
&\quad - z_2 t z_2^{-1} + st - t^2 - st + t^2) + \mathfrak{P}^{e(r-1)+1} \\
&\subseteq 1 - z_1 z_2 s z_1^{-1} z_2^{-1} + z_1 z_2 t z_1^{-1} z_2^{-1} + z_1 s z_1^{-1} - z_2 t z_2^{-1} + st - ts + \mathfrak{P}^{e(r-1)+1}.
\end{aligned}$$

Using the facts that z_1 and z_2 commute with β and that $U^{e(r-1)+1} \subseteq \mathrm{Ker}\,\theta$, we get

$$\begin{aligned}
\theta([x,y]) &= \psi_\beta(1 + (-z_1 z_2 s z_1^{-1} z_2^{-1} + z_1 z_2 t z_1^{-1} z_2^{-1} + z_1 s z_1^{-1} - z_2 t z_2^{-1} + st - ts)) \\
&= \psi_\beta(1 + (st - ts)).
\end{aligned}$$

□

Note that the above lemma implies that the value of the form h_β on the elements $x = z_1(1+s)$ and $y = z_2(1+t)$ does not depend on $z_1, z_2 \in C \cap U^1$. Consider the map

$$\rho : U^{el'} \longrightarrow U^{el'}/U^{el'+1} \xrightarrow{\sim} \mathfrak{A}/\mathfrak{P},$$

where the isomorphism is given by $(1 + \varpi^{l'}x)U^{el'+1} \mapsto x + \mathfrak{P}$. Let $\beta + \mathfrak{P}$ be the image of β in $\mathfrak{A}/\mathfrak{P}$ under this map.

Lemma 4.4. *We have*

$$R_\beta = (C \cap U^1) \cdot \rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\beta + \mathfrak{P})).$$

Proof. By definition, $x \in R_\beta$ if and only if $\theta([x,y]) = 1$ for all $y \in J^1$. Writing $x = z_1(1+s)$, $y = z_2(1+t)$ as in Lemma 4.3, we have

$$\theta([x,y]) = \psi_\beta(1 + (st - ts)) = \psi(\varpi^{-r} \operatorname{tr} t(\beta s - s\beta)),$$

so $\theta([x,y]) = 1$ for all $y \in J^1$ is equivalent to $\psi(\varpi^{-r} \operatorname{tr}(\mathfrak{P}^{el'}(\beta s - s\beta))) = 1$, that is, $\beta s - s\beta \in (\mathfrak{P}^{el'})^\perp = \mathfrak{P}^{el'+1}$ (see Lemma 3.1). The latter is equivalent to $\rho(1+s) \in C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P})$ because if we write $s = \varpi^{l'} s_0$, we have $\rho(1+s) = s_0 + \mathfrak{P}$ and $\beta s - s\beta \in \mathfrak{P}^{el'+1}$ is then equivalent to $\beta s_0 - s_0 \beta \in \mathfrak{P}$, that is, $s_0 + \mathfrak{P} \in C_{\mathfrak{A}/\mathfrak{P}}(\beta + \mathfrak{P})$. Thus we have shown that $x = z_1(1+s) \in R_\beta$ if and only if $1 + s \in \rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\beta + \mathfrak{P}))$. □

Lemma 4.5. *With notation as above, the following holds:*

(i) $[J^1 : R_\beta] = \left| \frac{\mathfrak{A}/\mathfrak{P}}{C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P})} \right|$.

(ii) *Suppose that* $\mathfrak{A} = \mathfrak{A}_{\mathrm{M}}$. *Then the form* h_β *on* $J^1_{\mathrm{M}}/H^1_{\mathrm{M}}$ *is non-degenerate. Thus, for every* θ_{M}, *there exists a unique* $\eta_{\mathrm{M}} \in \mathrm{Irr}(J^1_{\mathrm{M}} \mid \theta_{\mathrm{M}})$ *and*

$$\dim \eta_{\mathrm{M}} = q^{N(N-1)/2}.$$

(iii) *Suppose that* $\mathfrak{A} = \mathfrak{A}_{\mathrm{m}}$. *Then, for every extension* $\tilde{\theta}_{\mathrm{m}}$ *of* θ_{m} *to* R_β, *there exists a unique* $\eta_{\mathrm{m}} \in \mathrm{Irr}(J^1_{\mathrm{m}} \mid \tilde{\theta}_{\mathrm{m}})$ *and*

$$\dim \eta_{\mathrm{m}} = \prod_{i=1}^{h} q^{d_i m_i (d_i-1)/2}.$$

Proof. By Lemma 4.4, we have

$$\begin{aligned} J^1/R_\beta &\cong \frac{U^{el'}}{(C \cap U^{el'})\rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P}))} = \frac{U^{el'}}{\rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P}))} \\ &\cong \frac{U^{el'}/U^{el'+1}}{\rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P}))/U^{el'+1}} \cong \frac{\mathfrak{A}/\mathfrak{P}}{C_{\mathfrak{A}/\mathfrak{P}}(\beta+\mathfrak{P})}. \end{aligned}$$

Next, suppose that $\mathfrak{A} = \mathfrak{A}_{\mathrm{M}} = \mathfrak{g}_r$. We then have $\mathfrak{A}/\mathfrak{P} = \mathfrak{g}_1$ and $\beta + \mathfrak{P} = \bar{\beta}$. By Lemma 4.4, we need to show that $\rho^{-1}(C_{\mathfrak{g}_1}(\bar{\beta})) \subseteq H^1$, and this holds if and only if the map

$$C_{K^{l'}}(\beta) \longrightarrow C_{\mathfrak{g}_1}(\bar{\beta}) \tag{4.2}$$

induced by ρ, is surjective. To show the latter, first note that the map $C_{K^{l'}}(\beta) \to C_{\mathfrak{g}_l}(\beta_l)$, $1 + \pi^{l'} x \mapsto x_l$ is easily seen to be an isomorphism. Now, $C_{\mathfrak{g}_l}(\beta_l)$ is an $\mathfrak{o}_l$-module so the map

$$C_{\mathfrak{g}_l}(\beta_l) \longrightarrow C_{\mathfrak{g}_1}(\bar{\beta}) \cong C_{\mathfrak{g}_l}(\beta_l)/\mathfrak{p}C_{\mathfrak{g}_l}(\beta_l)$$

given by $x \mapsto \bar{x} = x + \mathfrak{P}$ is surjective. Hence the map (4.2) is surjective, so the form h_β is indeed non-degenerate on $J^1_{\mathrm{M}}/H^1_{\mathrm{M}}$. By Lemma 3.2, there exists a unique $\eta_{\mathrm{M}} \in \mathrm{Irr}(J^1_{\mathrm{M}} \mid \theta_{\mathrm{M}})$ and by *(i)* together with Lemma 2.7, its dimension is

$$\dim \eta_{\mathrm{M}} = [J^1_{\mathrm{M}} : R_{\beta,\mathrm{M}}]^{1/2} = \left|\mathfrak{g}_1/C_{\mathfrak{g}_1}(\bar{\beta})\right|^{1/2} = q^{N(N-1)/2}.$$

Finally, suppose that $\mathfrak{A} = \mathfrak{A}_{\mathrm{m}}$, and let $\tilde{\theta}_{\mathrm{m}}$ be an arbitrary extension of θ_{m} to R_β. By Corollary 3.3, there exists a unique $\eta_{\mathrm{m}} \in \mathrm{Irr}(J^1_{\mathrm{m}} \mid \tilde{\theta}_{\mathrm{m}})$ and by *(i)* together with Lemma 2.7, its dimension is

$$\begin{aligned} \dim \eta_{\mathrm{m}} = [J^1_{\mathrm{m}} : R_{\beta,\mathrm{m}}]^{1/2} &= \left|\frac{\mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}}}{C_{\mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}}}(\beta_{\mathrm{m}})}\right|^{1/2} \\ &= \left(\frac{\prod_{i=1}^{h} q^{d_i^2 m_i}}{q^N}\right)^{1/2} = \prod_{i=1}^{h} q^{(d_i^2 m_i)/2} \prod_{i=1}^{h} q^{-d_i m_i/2} \\ &= \prod_{i=1}^{h} q^{d_i m_i (d_i-1)/2}. \end{aligned}$$

□

Remark 4.6. In the proof of the second part of the above lemma we used the fact that the map $C_{U^{e_{\mathrm{M}}l'}_{\mathrm{M}}}(\beta) \longrightarrow C_{\mathfrak{A}_{\mathrm{M}}/\mathfrak{P}_{\mathrm{M}}}(\bar{\beta})$ induced by ρ is surjective. We remark that the corresponding map $C_{U^{e_{\mathrm{m}}l'}_{\mathrm{m}}}(\beta) \longrightarrow C_{\mathfrak{A}_{\mathrm{m}}/\mathfrak{P}_{\mathrm{m}}}(\beta_{\mathrm{m}})$ is *not* surjective in general. Consequently, η_{m} is not the only representation containing θ_{m}. This will not matter

for us, since we will only need the existence of an η_{m}. However, the uniqueness of η_{M} expressed in the above lemma is crucial, because without it we would not be able to deduce that the representation η defined below is an extension of η_{M}.

Note also that a quicker proof of the non-degeneracy of h_β in the second part of the lemma is to observe that $[J^1_{\mathrm{M}} : R_{\beta,\mathrm{M}}] = [J^1_{\mathrm{M}} : H^1_{\mathrm{M}}]$. In the above proof, we have emphasised the surjectivity of (4.2).

We now prove a series of lemmas whose purpose is to show the existence of an extension of η_{M} to $CK^{l'}$. Once this is achieved, the construction is easily completed.

Lemma 4.7. *Let*

$$\eta := \mathrm{Ind}_{J^1_{\mathrm{m}}}^{J_{\mathrm{m,M}}} \eta_{\mathrm{m}}.$$

Then $\dim \eta = \dim \eta_{\mathrm{m}}$ *and thus* η *is an extension of* η_{M}.

Proof. We first need to determine the dimension of the induced representation Given Lemma 4.5, we only need to compute the index $[J_{\mathrm{m,M}} : J^1_{\mathrm{m}}]$. We claim that $C \cap K^{l'} \subseteq U_{\mathrm{m}}^{e_{\mathrm{m}} l'}$. Indeed, we have the relations

$$C \cap K^{l'} = 1 + \mathfrak{p}^{l'} \rho_{r,l}^{-1}(C_{\mathfrak{g}_l}(\beta_l)) \subseteq 1 + \mathfrak{p}^{l'} \mathfrak{A}_{\mathrm{m}},$$

where the inclusion follows from our assumption that $\beta \in \mathfrak{A}_{\mathrm{m}}$ together with Lemma (2.6), guaranteeing that $\rho_{l,1}(C_{\mathfrak{g}_l}(\beta_l)) \subseteq \overline{\mathfrak{A}}_{\mathrm{m}}$, and so $\rho_{r,l}^{-1}(C_{\mathfrak{g}_l}(\beta_l)) \subseteq \rho_{r,1}^{-1}(\overline{\mathfrak{A}}_{\mathrm{m}}) = \mathfrak{A}_{\mathrm{m}}$. We now have

$$J_{\mathrm{m,M}}/J^1_{\mathrm{m}} \cong \frac{K^{l'}}{(C \cap K^{l'}) U_{\mathrm{m}}^{e_{\mathrm{m}} l'}} = \frac{K^{l'}}{U_{\mathrm{m}}^{e_{\mathrm{m}} l'}},$$

where the equality follows from the above claim. Furthermore, the map

$$\frac{K^{l'}}{U_{\mathrm{m}}^{e_{\mathrm{m}} l'}} \longrightarrow \mathfrak{g}_1/\overline{\mathfrak{A}}_{\mathrm{m}}, \qquad (1 + \pi^{l'} x) U_{\mathrm{m}}^{e_{\mathrm{m}} l'} \longmapsto \bar{x} + \overline{\mathfrak{A}}_{\mathrm{m}}$$

(recall that $\overline{\mathfrak{A}}_{\mathrm{m}}$ is the image of $\mathfrak{A}_{\mathrm{m}}$ in $\mathfrak{g}_1$) is an isomorphism, and we have

$$\left|\mathfrak{g}_1/\overline{\mathfrak{A}}_{\mathrm{m}}\right| = \frac{q^{N^2}}{q^{N^2/2} \prod_{i=1}^h q^{d_i^2 m_i/2}} = q^{N^2/2} \prod_{i=1}^h q^{-d_i^2 m_i/2}.$$

Thus, by Lemma 4.5, we have

$$\begin{aligned} \dim \eta &= \dim \eta_{\mathrm{m}} \cdot \left|\mathfrak{g}_1/\overline{\mathfrak{A}}_{\mathrm{m}}\right| = \prod_{i=1}^h q^{d_i m_i (d_i - 1)/2} q^{N^2/2} \prod_{i=1}^h q^{-d_i^2 m_i/2} \\ &= q^{N(N-1)/2} = \dim \eta_{\mathrm{M}}. \end{aligned}$$

By construction, the representation η contains θ_{M} on restriction to H^1_{M}. Hence, the representation $\eta|_{J^1_{\mathrm{M}}}$ contains θ_{M} on restriction to H^1_{M}. By Lemma 4.5 *(ii)* η_{M} is the unique representation of J^1_{M} which contains θ_{M}, so it follows that η contains η_{M} on restriction to J^1_{M}. The equality of dimensions $\dim \eta = \dim \eta_{\mathrm{M}}$ then forces $\eta|_{J^1_{\mathrm{M}}} = \eta_{\mathrm{M}}$, so that η is an extension of η_{M}. □

Lemma 4.8. *Let* G *be a finite group,* N *a normal* p*-subgroup of* G*, and* P *a Sylow* p*-subgroup of* G*. Suppose that* $\chi \in \mathrm{Irr}(N)$ *is stabilised by* G *and that* χ *has an extension to* P*. Then* χ *has an extension to* G.

Proof. By [11, (11.31)], χ will extend to G if it extends to every $H \subseteq G$ such that H/N is a Sylow subgroup of G/N. By assumption, χ has an extension, say $\tilde{\chi}$, to P, so if Q is any other Sylow p-subgroup of G, then $Q = gPg^{-1}$ for some $g \in G$, and so ${}^g\tilde{\chi}$ is an extension of χ to Q, because ${}^g\chi = \chi$. Suppose now that $P' \subseteq G$ is a subgroup such that P'/N is a Sylow p'-subgroup of G/N, for some prime $p' \neq p$. Then p does not divide the index $[P' : N]$, so by a theorem of Gallagher (see [6, Theorem 6] or [11, (8.16)]) χ extends to P'. Thus χ extends to G. □

Lemma 4.9. *We have* $CK^{l'} = \mathrm{Stab}_{CK^{l'}}(\eta_{\mathrm{M}})$.

Proof. Recall that we saw in Section (2) that H^1_{M} and J^1_{M} are normal in $CK^{l'}$. Let $g \in CK^{l'}$. Since η_{M} is the unique representation in $\mathrm{Irr}(J^1_{\mathrm{M}} \mid \theta_{\mathrm{M}})$, we have $g \in \mathrm{Stab}_{CK^{l'}}(\eta_{\mathrm{M}})$ if and only if $g \in \mathrm{Stab}_{CK^{l'}}(\theta_{\mathrm{M}})$, so we need to show that g stabilises θ_{M}. To this end, write $g = zu$ with $z \in C$, $u \in K^{l'}$, and $x = z'v$ with $z' \in C \cap K^1$, $v \in K^l$. Then

$$gxg^{-1} = z' \cdot z([z'^{-1}, u](uvu^{-1}))z^{-1},$$

where $z([z'^{-1}, u](uvu^{-1}))z^{-1} \in K^l$. Write $\theta_{\mathrm{M}} = \theta_0\psi_\beta$ as in Remark 4.2. Then

$$\begin{aligned}\theta_{\mathrm{M}}(gxg^{-1}) &= \theta_0(z')\psi_\beta(z([z'^{-1}, u](uvu^{-1}))z^{-1}) \\ &= \theta_0(z')\psi_\beta([z'^{-1}, u])\psi_\beta(v) \\ &= \theta_{\mathrm{M}}(x)\psi_\beta([z'^{-1}, u]),\end{aligned}$$

where the second equality follows since $CK^{l'}$ stabilizes ψ_β. To show that θ_{M} is stabilized by g it thus remains to show that $\psi_\beta([z'^{-1}, u]) = 1$. To this end, write $u = 1 + s$, with $s \in \mathfrak{p}^{l'}\mathfrak{g}_r$ and observe that

$$[z'^{-1}, u] = z'^{-1}(1 + s)z'(1 - s + s^2) = z'^{-1}(1 + s)z' - s,$$

where we have used that $z'^{-1}sz's = s^2$. Thus $\psi_\beta([z'^{-1}, u]) = \psi(\varpi^{-r} \operatorname{tr} \beta(z'^{-1}sz' - s)) = \psi(\varpi^{-r} \operatorname{tr}(z'^{-1}(\beta s)z' - \beta s)) = 1$, as required. □

Theorem 4.10. *Suppose that the orbit $\Omega_{l'}$ consists of regular elements and let $\beta \in \Omega_r \cap \mathfrak{A}_{\mathrm{m}}$. Then, for any extension θ_{M} of ψ_β, the representation η_{M} has an extension $\hat{\eta}_{\mathrm{M}}$ to $CK^{l'}$, where $C = C_{G_r}(\beta)$. Any representation in $\mathrm{Irr}(G_r \mid \psi_\beta)$ is of the form*

$$\pi(\theta_{\mathrm{M}}, \hat{\eta}_{\mathrm{M}}) := \mathrm{Ind}_{CK^{l'}}^{G_r} \hat{\eta}_{\mathrm{M}},$$

for some θ_{M} and $\hat{\eta}_{\mathrm{M}}$, and if another representation $\pi(\theta'_{\mathrm{M}}, \hat{\eta}'_{\mathrm{M}}) \in \mathrm{Irr}(G_r \mid \psi_\beta)$ is isomorphic to $\pi(\theta_{\mathrm{M}}, \hat{\eta}_{\mathrm{M}})$, then $\theta_{\mathrm{M}} \cong \theta'_{\mathrm{M}}$ and $\hat{\eta}_{\mathrm{M}} \cong \hat{\eta}'_{\mathrm{M}}$.

Proof. The first assertion follows from Lemma 4.8, using Lemma 2.8, Lemma 4.9 and Lemma 4.7.

Choose $\beta \in \Omega_r$. If G is a finite group with a normal subgroup N, such that G/N is abelian and $\chi \in \mathrm{Irr}(N)$ has an extension to G, then any representation in $\mathrm{Irr}(G \mid \chi)$ is an extension of χ; see [11, (6.17)]. Thus, any representation in $\mathrm{Irr}(H^1_{\mathrm{M}} \mid \psi_\beta)$ is of the form θ_{M} (i.e., an extension), any representation in $\mathrm{Irr}(J^1_{\mathrm{M}} \mid \theta_{\mathrm{M}})$ is isomorphic to η_{M} (by construction) and any representation in $\mathrm{Irr}(CK^{l'} \mid \eta_{\mathrm{M}})$ is of the form $\hat{\eta}_{\mathrm{M}}$ (i.e., an extension). Thus any representation in $\mathrm{Irr}(CK^{l'} \mid \psi_\beta)$ is of the form $\hat{\eta}_{\mathrm{M}}$. By a standard result from Clifford theory of finite groups [11, (6.11)], this means that any representation in $\mathrm{Irr}(G_r \mid \psi_\beta)$ is of the form $\pi(\theta_{\mathrm{M}}, \hat{\eta}_{\mathrm{M}})$.

Suppose that $\pi(\theta_{\mathrm{M}}, \hat{\eta}_{\mathrm{M}})$ and $\pi(\theta'_{\mathrm{M}}, \hat{\eta}'_{\mathrm{M}})$ are isomorphic. By [11, (6.11)] we must have $\hat{\eta}_{\mathrm{M}} \cong \hat{\eta}'_{\mathrm{M}}$. Thus the corresponding representations η_{M} and η'_{M} are isomorphic, and since these uniquely determine θ_{M} and θ'_{M}, respectively, we must have $\theta_{\mathrm{M}} \cong \theta'_{\mathrm{M}}$. □

Note that even though η is an extension of η_{M} to $J_{\mathrm{m,M}}$ and $\hat{\eta}_{\mathrm{M}}$ is an extension of η_{M} to $CK^{l'}$, we do not know, and do not need to know, whether $\hat{\eta}_{\mathrm{M}}$ is an extension of η.

5. Concluding remarks

The obstruction to extending a representation in $\mathrm{Irr}(K^{l'} \mid \psi_\beta)$ to $CK^{l'}$ is given by an element in the Schur multiplier $H^2(\mathbb{F}_q[\bar{\beta}]^\times, \mathbb{C}^\times)$. Takase conjectured that for $p = \mathrm{char}\, \mathbb{F}_q$ large enough, this element is trivial; see [23, Conjecture 4.6.5]. Using our main result, we deduce a strong form of Takase's conjecture, valid for any prime p (this is also proved, for $p \neq 2$, in [12]).

Corollary 5.1. *Suppose $\beta \in \mathfrak{g}_r$ is regular. Every representation in $\mathrm{Irr}(K^{l'} \mid \psi_\beta)$ extends to $CK^{l'}$, and hence Takase's conjecture* [23, Conjecture 4.6.5] *holds for $\mathbb{F}_q$ of arbitrary characteristic.*

Proof. Let $\sigma \in \mathrm{Irr}(K^{l'} \mid \psi_\beta)$. It is straightforward (cf. [8, Proposition 4.2]) that the bilinear form h_β on $K^{l'}/K^l$ defined by ψ_β has radical $(C_{G_r}(\beta) \cap K^{l'})K^l/K^l$. We have

$$|(C_{G_r}(\beta) \cap K^{l'})K^l/K^l| = |C_{\mathfrak{g}_1}(\bar{\beta})| = q^N,$$

so that by Corollary 3.3, σ has dimension $q^{N(N-1)/2}$. Let $\tilde{\sigma} \in \mathrm{Irr}(CK^{l'} \mid \sigma)$ be a constituent of $\mathrm{Ind}_{K^{l'}}^{CK^{l'}} \sigma$. By Theorem 4.10, any representation in $\mathrm{Irr}(CK^{l'} \mid \psi_\beta)$, so in particular $\tilde{\sigma}$, is an extension of some representation in $\mathrm{Irr}(J^1_{\mathrm{M}} \mid \psi_\beta)$. Let $\eta_{\mathrm{M}} \in \mathrm{Irr}(J^1_{\mathrm{M}} \mid \psi_\beta)$ be such that $\tilde{\sigma}$ is an extension of η_{M}. By Lemma 4.5 $\dim \eta_{\mathrm{M}} = q^{N(N-1)/2}$. Thus, $\tilde{\sigma}$ is an irreducible representation of dimension $q^{N(N-1)/2}$ whose restriction to $K^{l'}$ has a constituent σ of the same dimension. Thus $\tilde{\sigma}$ is an extension of σ. □

In [21] the construction of split regular representations of $\mathrm{GL}_2(\mathfrak{o}_r)$ appealed to Hill's construction [8, Theorem 4.6]. Since Takase [23] has realised that this construction does not produce all the split regular representations, one should view the construction of the current paper as superseding that of [21], while at the same time unifying the split regular case with the cuspidal.

In [12] the dimensions and multiplicities of the regular representations of $\mathrm{GL}_N(\mathfrak{o}_r)$ were determined for $p \neq 2$. This can also be done using Theorem 4.10, and our construction implies that the dimension and multiplicity formulas there remain valid for $p = 2$.

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