Timestepping schemes for the 3d Navier–Stokes equations

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Abstract

It is well known that the solutions of the 3d Navier–Stokes equations remain bounded for all time if the initial data and the forcing are sufficiently small relative to the viscosity, and for a finite time given any bounded initial data. In this article, we consider two temporal discretisations (semi-implicit and fully implicit) of the 3d Navier–Stokes equations in a periodic domain and prove that their solutions remain bounded in H^1 subject to essentially the same respective smallness conditions (on initial data and forcing or on the time of existence) as the continuous system and to suitable timestep restrictions.

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1. Introduction

Much work has been done on the stability and convergence of various timestepping schemes for the Navier–Stokes equations in two space dimensions (2d NSE). The long time stability of Euler schemes for the 2d NSE has been treated in, e.g., [2, 6, 4, 8], and more recently extended to higher-order schemes in [9, 3]. Once the numerical solutions are shown to be bounded in suitable space, either on a

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limited interval of time or for all time, convergence can usually be established using standard techniques (cf., e.g., [5]).

In three dimensions, the existence of a solution bounded in $L^\infty(\mathbb{R}_+, H^1)$ is not known, we only know a globally bounded solution with small data and of a local bounded solution with arbitrary data; for more background on the NSE, see e.g. [1, 7]. Hence, the extension of the numerical stability results from 2d or 3d is not straightforward. We conduct it here in the two cases for which the existence of strong solution is known, namely, as we said, a globally bounded solution with small data or a locally bounded solution with arbitrary data.

In this article we consider temporal discretisations of the 3d NSE using the semi-implicit (2.1) and fully implicit (3.1) Euler schemes, and their boundedness in H^1 for interval of times corresponding to the existence of solutions. As in the earlier works cited above, we do not consider spatial discretisations, giving the advantage that our results will be free of Courant–Friedrichs–Lewy-type constraints, although some smallness of the timestep may be required.

We consider the Navier–Stokes equations in $\Omega = (0, 2\pi)^3$ with periodic boundary conditions,

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u + f, \\ &\nabla\cdot u = 0, \end{aligned} \tag{1.1}$$

plus the initial data $u(0) = u_0$. With no loss of generality, we assume that $\nabla\cdot f = 0$, and that the integrals of f and u_0 vanish over Ω. The last assumption implies that $u = u(t)$, whenever it is well-defined for $t \geq 0$, also has vanishing integral over Ω, giving us the Poincaré inequality

$$|u|_{L^2}^2 \leq c_0(\Omega)|\nabla u|_{L^2}^2. \tag{1.2}$$

For notational convenience, we redefine c_0 to give also the bound

$$|\nabla u|_{L^2}^2 \leq c_0|\Delta u|_{L^2}^2. \tag{1.3}$$

In order to facilitate comparison with the numerical solutions, in the rest of this section we briefly review the boundedness of solutions of the 3d NSE, both in L^2 and in H^1 for the two cases (small data, large time and short time for arbitrary data).

Multiplying (1.1) by u in $L^2(\Omega)$, integrating by parts and using the fact that $(u\cdot\nabla u, u) = 0$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|\nabla u|^2 = (f, u). \tag{1.4}$$

Here and henceforth, unadorned norm $|\cdot|$ and inner product $(\cdot,\cdot)$ are taken to be L^2. Bounding the rhs by the Cauchy–Schwarz inequality and using the Poincaré inequality, (1.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \frac{\nu}{c_0}|u|^2 \le \frac{1}{\nu}|f|^2_{L^\infty(H^{-1})}, \tag{1.5}$$

where $|f|_{L^\infty(H^{-1})} := \sup_{t\ge 0}|f(t)|_{H^{-1}}$. Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2 \exp\left(\frac{\nu}{c_0}t\right)\right) \le \exp\left(\frac{\nu}{c_0}t\right)\frac{1}{\nu}|f|^2_{L^\infty(H^{-1})}. \tag{1.6}$$

Integrating from 0 to t (here we change the dummy variable of integration to s), we obtain

$$|u(t)|^2 \exp\left(\frac{\nu}{c_0}t\right) \le |u(0)|^2 + \frac{c_0}{\nu^2}|f|^2_{L^\infty(H^{-1})}\left(\exp\left(\frac{\nu}{c_0}t\right) - 1\right). \tag{1.7}$$

We then find the uniform L^2 bound valid for all $t \ge 0$:

$$|u(t)|^2 \le |u(0)|^2 + (c_0/\nu^2)|f|^2_{L^\infty(H^{-1})} =: K_0(u_0, f; \nu, \Omega). \tag{1.8}$$

1.1. H^1 estimate for small data

Now multiplying (1.1) by $-\Delta u$ in $L^2(\Omega)$ and integrating by parts, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 + \nu|\Delta u|^2 = (u\cdot\nabla u, \Delta u) - (f, \Delta u). \tag{1.9}$$

Bounding the nonlinear term using the Sobolev inequality $|u|_{L^6} \le c|\nabla u|_{L^2}$, which is specific to dimension three, we find

$$\big|(u\cdot\nabla u, \Delta u)\big| \le |u|_{L^3}|\nabla u|_{L^6}|\Delta u|_{L^2} \le \frac{c_1}{2}|u|_{L^3}|\Delta u|^2_{L^2}, \tag{1.10}$$

and the forcing term in the obvious fashion

$$(f, \Delta u) \le \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|\Delta u|^2. \tag{1.11}$$

We then arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 + (3\nu/2 - c_1|u|_{L^3})|\Delta u|^2 \leq \frac{2}{\nu}|f|^2. \tag{1.12}$$

Assuming that

$$|u_0|_{L^3} \leq \frac{\nu}{4c_1}, \tag{1.13}$$

we find that on some interval of time $(0, T)$

$$|u|_{L^3} \leq \nu/(2c_1). \tag{1.14}$$

We then find that

$$3\nu/2 - c_1|u|_{L^3} \geq \nu. \tag{1.15}$$

Using the Poincaré inequality, (1.12) becomes on this interval

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 + \frac{\nu}{c_0}|\nabla u|^2 \leq \frac{2}{\nu}|f|^2_{L^\infty(L^2)}, \tag{1.16}$$

where $|f|_{L^\infty(L^2)} := \sup_{t\geq 0}|f(t)|_{L^2}$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla u|^2 \exp\left(\frac{\nu}{c_0}t\right)\right) \leq \exp\left(\frac{\nu}{c_0}t\right)\frac{2}{\nu}|f|^2_{L^\infty(L^2)}. \tag{1.17}$$

Integrating from 0 to t (here we change the dummy variable of integration to s), we deduce that

$$|\nabla u(t)|^2 \exp\left(\frac{\nu}{c_0}t\right) \leq |\nabla u(0)|^2 + \frac{2c_0}{\nu^2}|f|^2_{L^\infty(L^2)}\left(\exp\left(\frac{\nu}{c_0}t\right) - 1\right). \tag{1.18}$$

We obtain

$$|\nabla u(t)|^2 \leq |\nabla u(0)|^2 + (2c_0/\nu^2)|f|^2_{L^\infty(L^2)} =: K_1(u_0, f; \nu, \Omega). \tag{1.19}$$

Using the Sobolev inequality $|u|^2_{L^3} \leq c\,|u|^2_{H^{1/2}} \leq c\,|u|\,|\nabla u|$, we find that (1.14) and then (1.19) hold for all time provided.

$$K_0K_1 = \left(|u_0|^2 + c_0\,|f|^2_{L^\infty(H^{-1})}\right)\left(|\nabla u_0|^2 + 2c_0\,|f|^2_{L^\infty(L^2)}/\nu^2\right) \leq c_2(\Omega)\,\nu^4, \tag{1.20}$$

where $c_2 = c_2(\Omega) = c/c_1^4$. It therefore follows that whenever this holds, the 3d NSE has a global solution bounded by (1.8) and (1.19).

1.2. H^1 estimate for short times

Let us now recall the uniform H^1 estimate for short time and arbitrary data. Multiplying (1.1) by $-\Delta u$ in $L^2(\Omega)$ and integrating by parts, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 + \nu|\Delta u|^2 = (u\cdot\nabla u, \Delta u) - (f, \Delta u). \tag{1.21}$$

Bounding as before the nonlinear term using the Sobolev and interpolation inequalities, we find

$$\begin{aligned} |(u\cdot\nabla u, \Delta u)| \le |u|_{L^6}|\nabla u|_{L^3}|\Delta u|_{L^2} \quad &\le c\,|\nabla u|\,|\nabla u|_{H^{1/2}}|\Delta u| \\ \le c\,|\nabla u|^{3/2}|\Delta u|^{3/2} \quad &\le \frac{c_4}{2\nu^3}|\nabla u|^6 + \frac{\nu}{2}|\Delta u|^2, \end{aligned} \tag{1.22}$$

and using the Cauchy–Schwarz inequality for the last term, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 \le \frac{c_4}{\nu^3}|\nabla u|^6 + \frac{1}{\nu}|f|^2_{L^\infty(L^2)}. \tag{1.23}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2 + F) \le \frac{c_4}{\nu^3}(|\nabla u|^2 + F)^3, \tag{1.24}$$

where $F := \big(\nu^2|f|^2_{L^\infty(L^2)}/c_4\big)^{1/3}$. Writing $z(t) := |\nabla u(t)|^2 + F$ and integrating, we find

$$\frac{z(t)^2}{z(0)^2} \le \frac{1}{1 - 2tc_4 z(0)^2/\nu^3}, \tag{1.25}$$

as long as $t < \nu^3/(2c_4 z(0)^2)$. It is clear from this that our solution will remain bounded, say, $z(t)^2 \le 2z(0)^2$, for $0 \le t \le \nu^3/(4c_4 z(0)^2)$.

In view of results in Sections 1.1 and 1.2, we aim to study the H^1 stability of two different time discretizaion schemes in the following sections; namely, the semi-implicit and the fully implicit Euler scheme.

2. Semi-implicit scheme

Given a fixed $k = \Delta t > 0$, we discretise (1.1) in time using the semi-implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + u^{n-1}\cdot\nabla u^n + \nabla p = \nu\Delta u^n + f^n, \tag{2.1}$$

where u^n and f^n are approximations such that

$$u^n = u(n\Delta t), \quad f^n = \frac{1}{k}\int_{(n-1)k}^{nk} f(t)dt. \tag{2.2}$$

For the 2d NSE, this scheme was proved in [4] to be globally stable in H^1. For the 3d NSE, its stability mirrors that (which is known) of the continuous system, subject to relatively mild timestep restrictions.

We note a few facts that will be useful later on. First, for any a and $b \in L^2$,

$$2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2. \tag{2.3}$$

Next, for $b > 0$ and given positive real sequences (x_n) and (r_n) satisfying

$$(1+b)\,x_n \le x_{n-1} + r_{n-1}, \tag{2.4}$$

we have

$$x_n \le (1+b)^{-n}x_0 + \frac{1+b}{b}\max_j r_j. \tag{2.5}$$

The L^2 bound works out essentially as in the continuous case: multiplying (2.1) by $2ku^n$, using (2.3) and noting that $(u^{n-1}\cdot\nabla u^n, u^n) = 0$, we find

$$|u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k|\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n). \tag{2.6}$$

Bounding the forcing term using Cauchy–Schwarz, we obtain

$$\begin{aligned} |u^n|^2 + 2\nu k|\nabla u^n|^2 &\le |u^{n-1}|^2 + 2k(f^n, u^n) \\ &\le |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu + k\nu|\nabla u^n|^2. \end{aligned} \tag{2.7}$$

Using the Poincaré inequality, we deduce that

$$(1+\nu k/c_0)|u^n|^2 \le |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu. \tag{2.8}$$

Integrating this using (2.5), we find for all $n \in \{1,2,\cdots\}$,

$$\begin{aligned} |u^n|^2 &\le |u^0|^2 + \frac{c_0+\nu k}{\nu^2}|f|_{L^\infty(H^{-1})}^2 \\ &= K_0(u_0, f; \nu, \Omega) + (k/\nu)|f|_{L^\infty(H^{-1})}^2, \end{aligned} \tag{2.9}$$

where K_0, and K_1 below, are as in the continuous case. We note that this bound (the rhs of (2.9)) tends to K_0 as $k \to 0$.

2.1. H^1 estimate for small data

Theorem 1. *For small data, let the initial data $u_0 \in H^1$, the forcing f and the timestep k satisfy*

$$(K_0 + k|f|^2_{L^\infty(H^{-1})}/\nu)(K_1 + 2k|f|^2_{L^\infty(L^2)}/\nu) \le c_2(\Omega)\nu^4, \tag{2.10}$$

where $K_0(u_0, f)$ and $K_1(u_0, f)$ are as in the continuous case, (1.8) *and* (1.19). *Then u^n is bounded in H^1 as follows,*

$$|\nabla u^n|^2 \le K_1 + (2k/\nu)|f|^2_{L^\infty(L^2)} \qquad \text{for all } n \ge 0. \tag{2.11}$$

Proof. We now turn to stability in H^1 for small solutions. Multiplying (2.1) by $-2k\Delta u^n$ and using (2.3), we find

$$\begin{aligned} &|\nabla u^n|^2 + |\nabla(u^n - u^{n-1})| + 2\nu k|\Delta u^n|^2 \\ &\qquad = |\nabla u^{n-1}|^2 - 2k(f^n, \Delta u^n) + 2k(u^{n-1} \cdot \nabla u^n, \Delta u^n). \end{aligned} \tag{2.12}$$

Bounding the nonlinear term using the Sobolev inequality,

$$\left|(u^{n-1} \cdot \nabla u^n, \Delta u^n)\right| \le |u^{n-1}|_{L^3} |\nabla u^n|_{L^6} |\Delta u^n|_{L^2} \le c_1 |u^{n-1}|_{L^3} |\Delta u^n|^2_{L^2}, \tag{2.13}$$

and using the Cauchy–Schwarz inequality for the forcing, (2.12) implies

$$|\nabla u^n|^2 + (3\nu/2 - c_1|u^{n-1}|_{L^3})k|\Delta u^n|^2 \le |\nabla u^{n-1}|^2 + (2k/\nu)|f^n|^2. \tag{2.14}$$

As in (1.14) of Section 1.1, if we now assume that

$$|u^{n-1}|_{L^3} \le \nu/(2c_1), \tag{2.15}$$

we deduce from (2.14) that

$$(1 + \nu k/c_0)|\nabla u^n|^2 \le |\nabla u^{n-1}|^2 + 2k|f^n|^2/\nu. \tag{2.16}$$

As long as (2.15) holds, we can integrate this using (2.5) to get the bound

$$\begin{aligned} |\nabla u^n|^2 &\le |\nabla u^0|^2 + \frac{2(c_0 + \nu k)}{\nu^2}|f|^2_{L^\infty(L^2)} \\ &\le K_1(u_0, f; \nu, \Omega) + (2k/\nu)|f|^2_{L^\infty(L^2)}, \end{aligned} \tag{2.17}$$

which proves (2.11). As in the continuous case, we now use the Sobolev inequality and interpolation inequalities to bound

$$|u^{n-1}|_{L^3}^2 \le c\,|u^{n-1}|_{H^{1/2}}^2 \le c\,|u^{n-1}|\,|\nabla u^{n-1}|. \tag{2.18}$$

The timestep restriction (2.10) then becomes a sufficient condition for (2.15). More explicitly, since (2.10) holds at $n = 0$, (2.9) and (2.11) imply that it will hold at $n = 1$ and, by induction, for all $n \in \{2, \cdots\}$, i.e. the solution of the scheme (2.1) is bounded uniformly in (discrete) time subject to (2.10). Comparing to (1.20), we note that this condition also depends on the timestep k in addition to u_0 and f. This timestep restriction is however relatively mild compared to that for the fully implicit scheme in §3 below. □

2.2. H^1 estimate for short times

Theorem 2. *For short times, assuming the timestep restriction* (2.26)*, we have*

$$|\nabla u^n|^2 \le 2\,|\nabla u^0|^2 + F, \quad \text{where } F := \big(\nu^2 |f|_{L^\infty(L^2)}^2 / c_4\big)^{1/3}, \tag{2.19}$$

for all n such that $nk = t_n \le \nu^3 / \big(8c_4(|\nabla u^0|^2 + F)^2\big)$.

Proof. For short-time H^1 stability, we bound the nonlinear term in (2.12) as in (1.22),

$$\begin{aligned} |(u^{n-1}\cdot\nabla u^n, \Delta u^n)| &\le |u^{n-1}|_{L^6} |\nabla u^n|_{L^3} |\Delta u^n|_{L^2} \\ &\le c|\nabla u^{n-1}||\nabla u^n|_{H^{1/2}}|\Delta u^n| \\ &\le c|\nabla u^{n-1}||\nabla u^n|^{1/2}|\Delta u^n|^{3/2} \\ &\le \frac{c_4}{2\nu^3}|\nabla u^{n-1}|^4|\nabla u^n|^2 + \frac{\nu}{2}\,|\Delta u^n|^2. \end{aligned} \tag{2.20}$$

Then, (2.12) becomes

$$\begin{aligned} &|\nabla u^n|^2 + 2\nu k|\Delta u^n|^2 \\ &\le |\nabla u^{n-1}|^2 - 2k(f^n, \Delta u^n) + k\nu|\Delta u^n|^2 + \frac{kc_4}{\nu^3}|\nabla u^{n-1}|^4|\nabla u^n|^2, \end{aligned} \tag{2.21}$$

this implies

$$|\nabla u^n|^2 \le |\nabla u^{n-1}|^2 + \frac{c_4 k}{\nu^3}|\nabla u^{n-1}|^4|\nabla u^n|^2 + \frac{k}{\nu}|f|_{L^\infty(L^2)}^2. \tag{2.22}$$

We can rewrite (2.22) as

$$\Big(1-\frac{c_4k}{\nu^3}|\nabla u^{n-1}|^4\Big)|\nabla u^n|^2 \\ \le \Big(1-\frac{c_4k}{\nu^3}|\nabla u^{n-1}|^4\Big)|\nabla u^{n-1}|^2+\frac{c_4k}{\nu^3}|\nabla u^{n-1}|^6+\frac{k}{\nu}|f|^2_{L^\infty(L^2)}. \tag{2.23}$$

Since we are interested in short times, we assume that $|\nabla u^{n-1}|^2 \le 2|\nabla u^0|^2$ for all relevant n and demand that k satisfy

$$k \le \frac{\nu^3}{8c_4|\nabla u^0|^4}. \tag{2.24}$$

This implies that the brackets in (2.23) are $\ge \frac{1}{2}$, that is

$$\frac{1}{2} \le 1-\frac{4c_4k}{\nu^3}|\nabla u^0|^4 \le 1-\frac{c_4k}{\nu^3}|\nabla u^{n-1}|^4; \tag{2.25}$$

For later use we add the extra F, which make a stronger condition than (2.24)

$$k \le \frac{\nu^3}{2c_4(2|\nabla u^0|^2+F)^2}. \tag{2.26}$$

With this assumption, (2.23) implies

$$\frac{|\nabla u^n|^2-|\nabla u^{n-1}|^2}{k} \le \frac{2c_4}{\nu^3}|\nabla u^{n-1}|^6+\frac{2}{\nu}|f|^2_{L^\infty(L^2)}. \tag{2.27}$$

Unlike its continuous-time analogue (1.23), this difference inequality implies $|\nabla u^n| < \infty$ for all n, although for sufficiently large time nk, it (i.e. the bound) grows without bound as $k \to 0$. This is a well-known pitfall in discretising differential equations in time. To obtain a finite-time bound on $|\nabla u^n|$, we proceed in analogy with (1.24) and define

$$z_n := |\nabla u^n|^2 + F. \tag{2.28}$$

We then get from (2.27)

$$\frac{z_n - z_{n-1}}{k} \le \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}). \tag{2.29}$$

Observe that $g(\zeta) > g(\hat{\zeta})$ whenever $\zeta > \hat{\zeta}$, that is, $g \ge 0$ is a strictly monotone increasing function.

Now let ζ_n be the positive solution of the difference *equation*,

$$\frac{\zeta_n - \zeta_{n-1}}{k} = g(\zeta_{n-1}), \tag{2.30}$$

and observe that $\zeta_n \geq 0$ if $\zeta_{n-1} \geq 0$. Denoting $t_n := nk$, we claim that

$$\zeta_n \leq \zeta(t_n) \tag{2.31}$$

where $\zeta(\cdot)$ is the solution of the *differential* equation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = g(\zeta) \qquad \text{with } \zeta(t_{n-1}) = \zeta_{n-1}. \tag{2.32}$$

To show this, we first note that $\zeta(t)$ is non-decreasing since $g \geq 0$. Then

$$\zeta(t_n) - \zeta(t_{n-1}) = \int_{t_{n-1}}^{t_n} g(\zeta(t))\,\mathrm{d}t \geq \int_{t_{n-1}}^{t_n} g(\zeta(t_{n-1}))\,\mathrm{d}t = kg(\zeta_{n-1}). \tag{2.33}$$

Thanks to (2.30), we obtain that

$$\zeta(t_n) - \zeta(t_{n-1}) = \zeta_n - \zeta_{n-1}, \tag{2.34}$$

and this prove our claim. By induction, taking $\zeta(0) = \zeta_0 > 0$ instead of the initial data in (2.30), we then have $\zeta_n \leq \zeta(t_n)$ for all $n \in \{1, 2, \cdots\}$. Comparing with the continuous case (1.24)–(1.25), we conclude that $\zeta_n \leq \zeta(t_n) \leq 2\zeta(0) = 2\zeta_0$ for $nk = t_n \leq \nu^3/(8c_4\zeta_0^2)$.

Taking $\zeta_0 = z_0$, clearly $z_n \leq \zeta_n$ for all $n \geq 0$. We therefore have

$$\begin{aligned} &z_n = |\nabla u^n|^2 + F \leq \zeta_n \leq 2\zeta_0 = 2\,|\nabla u^0|^2 + 2F \\ &\Longrightarrow\ |\nabla u^n|^2 \leq 2\,|\nabla u^0|^2 + F, \end{aligned} \tag{2.35}$$

for all n such that $nk = t_n \leq \nu^3/\big(8c_4(|\nabla u^0|^2 + F)^2\big)$, which is half as long as the bound in the continuous case. □

3. Fully implicit scheme

We now consider the fully implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + u^n\cdot\nabla u^n + \nabla p = \nu\Delta u^n + f^n, \tag{3.1}$$

with $\nabla\cdot u^n = 0$ for all n and $u^0 = u_0$. In two space dimensions, uniform boundedness in H^1 for this scheme was proved in [8].

The L^2 bound obtains as before: multiplying $(3.1)_1$ by $2ku^n$ and using (2.3),

$$|u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k|\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n). \tag{3.2}$$

Bounding the forcing term in the obvious manner and using Poincaré, this implies

$$(1+\nu k/c_0)|u^n|^2 \le |u^{n-1}|^2 + k|f^n|^2_{H^{-1}}/\nu. \tag{3.3}$$

Integrating this using (2.5), we find for all $n \in \{1,2,\cdots\}$,

$$\begin{aligned} |u^n|^2 &\le |u^0|^2 + \frac{c_0+\nu k}{\nu^2}|f|^2_{L^\infty(H^{-1})} \\ &= K_0(u_0,f;\nu,\Omega) + (k/\nu)|f|^2_{L^\infty(H^{-1})}. \end{aligned} \tag{3.4}$$

As before, this bound tends to K_0 as $k \to 0$. For later use, we define

$$\tilde{K}_0(u_0,f;\nu,\Omega) := |u_0|^2 + \frac{2c_0}{\nu^2}|f|^2_{L^\infty(H^{-1})}. \tag{3.5}$$

The central ingredient for our main results of the H^1 stability is the following local-in-time estimate:

Lemma 1. *We assume the L^2 uniform bound* (3.4) *and that $u^{n-1} \in H^1$. Assuming further the timestep restrictions*

$$K^{(n-1)} \le \frac{1}{2}\Big(\frac{\nu^3}{3c_4k}\Big)^{1/2}, \tag{3.6}$$

$$\Big(1+\frac{c_5}{\nu^4}\tilde{K}_0K^{(n-1)}\Big)K^{(n-1)} + |f|^2_{L^\infty(H^{-1})}/\nu^2 \le \Big(\frac{\nu^3}{12c_4k}\Big)^{1/2}, \tag{3.7}$$

where $K^{(n-1)} := |\nabla u^{n-1}|^2 + (10c_0/\nu)|f|^2_{L^\infty(L^2)}$, then the solution u^n of (3.1) *is bounded as $|\nabla u^n|^2 \le y_1$ where y_1 is the smallest positive root of the cubic equation* (3.10).

Proof. Multiplying (3.1) by $-2k\Delta u^n$, we have

$$\begin{aligned} &|\nabla u^n|^2 + |\nabla(u^n-u^{n-1})|^2 + 2\nu k|\Delta u^n|^2 \\ &\qquad = |\nabla u^{n-1}|^2 + 2k(u^n\cdot\nabla u^n, \Delta u^n) - 2k(f^n, \Delta u^n). \end{aligned} \tag{3.8}$$

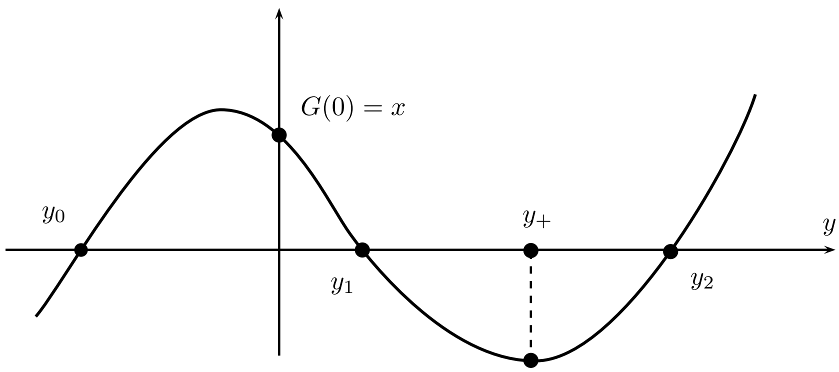

Figure 1: The graph of $G(y)$ in (3.11): y_+ is a local minimum.

Bounding the nonlinear term as we did in (1.22),

$$2k\big|(u^n \cdot \nabla u^n, \Delta u^n)\big| \leq \frac{c_4 k}{\nu^3}|\nabla u^n|^6 + \nu k|\Delta u^n|^2, \tag{3.9}$$

we find

$$\begin{aligned} &0 \leq \frac{c_4 k}{\nu^3}|\nabla u^n|^6 - |\nabla u^n|^2 - \frac{\nu k}{2}|\Delta u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f^n|^2 \\ \Rightarrow\quad &0 \leq \frac{c_4 k}{\nu^3}|\nabla u^n|^6 - \Big(1 + \frac{\nu k}{2c_0}\Big)|\nabla u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)}. \end{aligned} \tag{3.10}$$

Let $y := |\nabla u^n|^2$, $x := |\nabla u^{n-1}|^2 + 2k|f|^2_{L^\infty(L^2)}/\nu$ and

$$G(y;x) := (c_4 k/\nu^3)y^3 - (1 + \nu k/(2c_0))y + x. \tag{3.11}$$

We write $G(y)$ instead of $G(y;x)$ when there is no risk of confusion. We are of course interested in the solution set of $G(y) \geq 0$.

Under the assumption (3.14) below on the timestep k, the graph of the cubic G is (qualitatively) as shown in Figure 1. We note in particular that $G(y) = 0$ has a negative root y_0 and two positive roots y_1 and y_2. To verify this, we note the following. First, $G(y) \to \pm\infty$ as $y \to \pm\infty$. Next, $G(y)$ has two local extrema,

$$y_\pm = \pm\Big(\frac{\nu^3}{3c_4 k}\Big[1 + \frac{\nu k}{2c_0}\Big]\Big)^{1/2}, \tag{3.12}$$

with $y_- < 0$ being a local maximum and $y_+ > 0$ a local minimum, as verified by computing $G''(y_\pm)$. Since $G(0) = x > 0$ (the problem is trivial if $x = 0$), we have $G(y_-) > 0$. Finally, computing

$$G(y_+) = -\frac{2}{3}\Big(1+\frac{\nu k}{2c_0}\Big)\Big(\frac{\nu^3}{3c_4 k}\Big[1+\frac{\nu k}{2c_0}\Big]\Big)^{1/2} + x, \tag{3.13}$$

we conclude that $G(y_+) < 0$ if (this is essentially a restriction on k)

$$|\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)} < \frac{2}{3}\Big(\frac{\nu^3}{3c_4 k}\Big)^{1/2}. \tag{3.14}$$

This implies the existence of the two positive roots y_1 and y_2 with $y_1 < y_+ < y_2$.

Now (3.10) implies that $|\nabla u^n|^2 = y$ lies in the disjoint set $[0, y_1] \cup [y_2, \infty)$. However, $y_2 > y_+ \sim k^{-1/2}$, which is absurd for small k. To prove that $y \notin [y_2, \infty)$, we multiply $(3.1)_1$ by $2k(u^n - u^{n-1})$ in L^2 to get

$$\begin{aligned} 2|u^n - u^{n-1}|^2 &+ \nu k|\nabla u^n|^2 - \nu k|\nabla u^{n-1}|^2 + \nu k|\nabla(u^n - u^{n-1})|^2 \\ &= -2k(u^n\cdot\nabla u^n, u^n - u^{n-1}) + 2k(f^n, u^n - u^{n-1}) =: I_1 + I_2. \end{aligned} \tag{3.15}$$

Bounding the rhs as

$$\begin{aligned} |I_2| &\le \frac{k}{\nu}|f^n|^2_{H^{-1}} + \nu k|\nabla(u^n - u^{n-1})|^2 \\ |I_1| &= 2k\big|(u^n\cdot\nabla u^n, u^{n-1})\big| && \le 2k|u^n|_{L^3}|\nabla u^n|_{L^2}|u^{n-1}|_{L^6} \\ &\le ck|u^n|_{H^{1/2}}|\nabla u^n|\,|\nabla u^{n-1}| && \le ck\,|u^n|^{1/2}|\nabla u^n|^{3/2}|\nabla u^{n-1}| \\ &\le \frac{\nu k}{2}|\nabla u^n|^2 + \frac{ck}{\nu^3}|u^n|^2|\nabla u^{n-1}|^4, \end{aligned}$$

and dropping the $2|u^n - u^{n-1}|^2$ on the lhs in (3.15), we arrive at

$$|\nabla u^n|^2 \le \Big(2 + \frac{2c_5}{\nu^4}|u^n|^2|\nabla u^{n-1}|^2\Big)|\nabla u^{n-1}|^2 + \frac{2}{\nu^2}|f|^2_{L^\infty(H^{-1})}. \tag{3.16}$$

If we now assume that (effectively a timestep restriction)

$$\Big(2 + \frac{2c_5}{\nu^4}|u^n|^2|\nabla u^{n-1}|^2\Big)|\nabla u^{n-1}|^2 + \frac{2}{\nu^2}|f|^2_{L^\infty(H^{-1})} \le \Big(\frac{\nu^3}{3c_4 k}\Big)^{1/2}, \tag{3.17}$$

noting that the rhs $< y_+ < y_2$, we can conclude that $|\nabla u^n|^2 < y_2$ and therefore $|\nabla u^n|^2 \in [0, y_1]$. This gives us the local H^1 integrability of the scheme (3.1): if k (is small enough that it) satisfies (3.14) and (3.17), the one-step solution of (3.1) is bounded in H^1. □

The crucial point which is not immediately obvious is that $y_1 = |\nabla u^{n-1}|^2 + \mathsf{O}(k)$ for small k. By estimating the $\mathsf{O}(k)$ more carefully, we obtain our main results.

3.1. H^1 estimate for small data

Theorem 3. *For small data, let u_0 and f be such that*

$$|\nabla u_0|^2 + \frac{2c_0}{\nu^2}|f|^2_{L^\infty(L^2)} \le \frac{\nu^2}{2\sqrt{c_0 c_4}}, \tag{3.18}$$

and let the timestep k satisfy (3.24)–(3.26) below. Then u^n is bounded as

$$|\nabla u^n|^2 \le \tilde{K}_1(u_0, f; \nu, \Omega) := |\nabla u_0|^2 + \frac{10c_0}{\nu^2}|f|^2_{L^\infty(L^2)}, \tag{3.19}$$

for all $n \in \{0, 1, \cdots\}$.

Proof. For small data, we first assume the hypotheses (local in n) of Lemma 1. We then derive a more useful explicit bound for $|\nabla u^{n-1}|^2$. We claim that with the assumption (3.18), $|\nabla u^n|^2 \le y_1$ implies

$$\Big(1 + \frac{\nu k}{4c_0}\Big)|\nabla u^n|^2 \le |\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)}, \tag{3.20}$$

where y_1 is the smallest positive root of the cubic equation (3.10). To prove this, we set $y_* := \big(|\nabla u^{n-1}|^2 + 2k|f|^2_{L^\infty(L^2)}/\nu\big)/\big(1 + \nu k/(4c_0)\big)$ and compute

$$G(y_*) = y_* \Big(1 + \frac{\nu k}{4c_0}\Big)^{-2}\Big\{-\frac{\nu k}{4c_0}\Big(1 + \frac{\nu k}{4c_0}\Big)^2 + \frac{c_4 k}{\nu^3}\, x^2\Big\}, \tag{3.21}$$

where $G(y)$ is as in (3.11) such that

$$G(y) = (c_4 k/\nu^3)y^3 - (1 + \nu k/(2c_0))y + x,$$

and $x := |\nabla u^{n-1}|^2 + 2k|f|^2_{L^\infty(L^2)}/\nu$. Now $G(y_*) \le 0$ implies that $y_* \ge y_1$, and the former is true if

$$-\frac{\nu k}{4c_0}\Big(1 + \frac{\nu k}{4c_0}\Big)^2 + \frac{c_4 k}{\nu^3}\, x^2 \le 0,$$

and this implies

$$|\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)} = x \le \frac{\nu^2}{2\sqrt{c_0 c_4}}. \tag{3.22}$$

Hence, $|\nabla u^n|^2 \le y_1 \le y_*$ implies (3.20). Then, the claim is proved.

To obtain the uniform bound, we sum (3.20) using (2.5) to find

$$|\nabla u^n|^2 \le \Big(1+\frac{\nu k}{4c_0}\Big)^{-n}|\nabla u^0|^2 + \frac{8c_0}{\nu^2}|f|^2_{L^\infty(L^2)} + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)}. \tag{3.23}$$

Assuming that

$$k \le c_0/\nu, \tag{3.24}$$

we can absorb the last term into the penultimate one to obtain (3.19). Consolidating our assumptions, the smallness condition (3.22) is now implied by (3.18), while the timestep restrictions (3.14) and (3.17) can both be satisfied by taking k sufficiently small to satisfy

$$\tilde{K}_1 \le \frac{1}{2}\Big(\frac{\nu^3}{3c_4 k}\Big)^{1/2}, \tag{3.25}$$

$$\Big(1+\frac{c_5}{\nu^4}\tilde{K}_0\tilde{K}_1\Big)\tilde{K}_1 + |f|^2_{L^\infty(H^{-1})}/\nu^2 \le \Big(\frac{\nu^3}{12c_4 k}\Big)^{1/2}. \tag{3.26}$$

This proves the theorem. □

We note that, up to the constant depending only on the domain Ω, the bound (3.19) is the same as that in the continuous case (1.19). In addition, considering $|f|^2_{H^{-1}} \le c_0^*|f|^2_{L^2}$ and using Poincaré inequality, we estimate K_0 in (1.20) such that

$$K_0 := |u_0|^2 + c_0|f|^2_{L^\infty(H^{-1})} \le c_0|\nabla u_0|^2 + c_0 c_0^*|f|^2_{L^\infty(L^2)}. \tag{3.27}$$

Assuming $k \le (c_0^*\nu)/2$, we deduce from (3.22) that

$$c_0\Big(|\nabla u_0|^2 + c_0^*|f|^2_{L^\infty(L^2)}\Big) \le \frac{\nu^2\sqrt{c_0}}{2\sqrt{c_4}}. \tag{3.28}$$

From (3.18) and (3.28), we find, up to the constant depending only on the domain Ω, the same smallness condition as in (1.20).

3.2. H^1 estimate for short times

Theorem 4. *For short times, let the timestep k satisfy* (3.40), (3.41) *and* (3.42) *below. Then there exists a $t_f^* = \nu^3/(8c_4\zeta_0^2)$, as long as $0 \le nk \le t_f^*$ we have*

$$|\nabla u^n|^2 \le 2\,|\nabla u_0|^2 + \,(\nu^2|f|^2_{L^\infty(L^2)}/c_4)^{1/3}. \tag{3.29}$$

Proof. We first assume the hypotheses (local in n) of Lemma 1. Since y_1 is the root of a cubic $G(y_1) = 0$, the bound $|\nabla u^n|^2 \le y_1$ is not very convenient, so we compute a more useful bound. Recalling that $x > 0$, we consider for some $a > 0$

$$\begin{aligned} G\big((1+ak)x; x\big) &= xk\left[\frac{c_4}{\nu^3}(1+ak)^3x^2 - \Big(\frac{\nu}{2c_0} + a\Big) - \frac{a\nu k}{2c_0}\right] \\ &< xk\left[\frac{c_4}{\nu^3}(1+ak)^3x^2 - a\right]. \end{aligned} \tag{3.30}$$

Setting $a = 2c_4x^2/\nu^3$, this implies $G\big((1+ak)x\big) < 0$ if

$$1 + ak \le 2^{1/3} \quad \Leftrightarrow \quad (|\nabla u^{n-1}|^2 + 2k|f|^2_{L^\infty(L^2)}/\nu)^2 2c_4k/\nu^3 \le 2^{1/3} - 1. \tag{3.31}$$

Assuming this, Lemma 1 then gives us the explicit one-step estimate

$$\begin{aligned} &|\nabla u^n|^2 \le y_1 \le (1+ak)\,x \\ &\qquad = |\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)} + \frac{2c_4k}{\nu^3}\big(|\nabla u^{n-1}|^2 + (2k/\nu)|f|^2_{L^\infty(L^2)}\big)^3, \end{aligned} \tag{3.32}$$

which we can rewrite as

$$\frac{|\nabla u^n|^2 - |\nabla u^{n-1}|^2}{k} \le \frac{2c_4}{\nu^3}\left[\Big(|\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)}\Big)^3 + \frac{\nu^2}{c_4}|f|^2_{L^\infty(L^2)}\right]. \tag{3.33}$$

To obtain a finite-time bound on $|\nabla u^n|$, we proceed in analogy with (1.24) and define

$$z_n := |\nabla u^n|^2 + F \quad \text{where } F^3 = \frac{2\nu^2}{c_4}|f|^2_{L^\infty(L^2)}. \tag{3.34}$$

By expanding both sides, we have

$$\Big(|\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|^2_{L^\infty(L^2)}\Big)^3 + \frac{\nu^2}{c_4}|f|^2_{L^\infty(L^2)} \le z^3_{n-1}, \tag{3.35}$$

subject to the timestep restriction

$$k \le \frac{\nu^{5/3}}{2c_4^{1/3}|f|^{4/3}_{L^\infty(L^2)}} \quad \Rightarrow \quad \left\{ \begin{aligned} &4^{1/3}c_4^{1/3}|f|^{4/3}_{L^\infty(L^2)}\,k \le \nu^{5/3}, \\ &4^{2/3}c_4^{2/3}|f|^{8/3}_{L^\infty(L^2)}\,k^2 \le \nu^{10/3}, \\ &8c_4|f|^4_{L^\infty(L^2)}\,k^3 \le \nu^5. \end{aligned} \right. \tag{3.36}$$

The time step restriction above can be found by direct computations. Indeed, expanding (3.35), we have

$$
\begin{aligned}
&|\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 \left(\frac{2k}{\nu}|f|^2_{L^\infty(L^2)}\right) + 3|\nabla u^{n-1}|^2 \left(\frac{2k}{\nu}|f|^2_{L^\infty(L^2)}\right)^2 \\
&\qquad + \left(\frac{2k}{\nu}|f|^2_{L^\infty(L^2)}\right)^3 + \frac{\nu^2}{c_4}|f|^2_{L^\infty(L^2)} \\
&\le |\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 F + 3|\nabla u^{n-1}|^2 F^2 + F^3 \qquad (3.37)\\
&= |\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 \left(\frac{2\nu^2}{c_4}|f|^2_{L^\infty(L^2)}\right)^{1/3} \\
&\qquad + 3|\nabla u^{n-1}|^2 \left(\frac{2\nu^2}{c_4}|f|^2_{L^\infty(L^2)}\right)^{2/3} + \frac{2\nu^2}{c_4}|f|^2_{L^\infty(L^2)}.
\end{aligned}
$$

Hence, assuming the three conditions in the brace of (3.36), the inequality (3.37) holds term by term. Then (3.32) implies

$$\frac{z_n - z_{n-1}}{k} \le \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}). \qquad (3.38)$$

Arguing as we did in the semi-implicit case [cf. (2.30)–(2.33)], we conclude that $z_n \le 2z_0$ for all $n \ge 0$ such that

$$nk = t_n \le \nu^3/(8c_4\zeta_0^2) =: t_f^*. \qquad (3.39)$$

This proves the theorem subject to the timestep restrictions, which we collect here. First, (3.6) and (3.7) are implied by

$$\tilde{K} \le \frac{1}{2}\Big(\frac{\nu^3}{3c_4 k}\Big)^{1/2}, \qquad (3.40)$$

$$\Big(1 + \frac{c_5}{\nu^4}\tilde{K}_0\tilde{K}\Big)\tilde{K} + |f|^2_{L^\infty(H^{-1})}/\nu^2 \le \Big(\frac{\nu^3}{12c_4 k}\Big)^{1/2}, \qquad (3.41)$$

where unlike in Lemma 1, here

$$\tilde{K} := 2\,|\nabla u^0|^2 + 2\,(\nu^2|f|^2_{L^\infty(L^2)}/c_4)^{1/3} + (10c_0/\nu)|f|^2_{L^\infty(L^2)}.$$

Next, (3.36) is good as it stands. Finally, using (3.36) to handle the k inside the bracket, (3.31) is implied by

$$\left(2\,|\nabla u^0|^2 + \frac{(1+2^{1/3})\nu^{2/3}|f|^{2/3}_{L^\infty(L^2)}}{c_4^{1/3}}\right)^2 \le \frac{(2^{1/3}-1)\nu^3}{2c_4 k}. \qquad (3.42)$$

This proves the short-time case.

□

The time bound t_f^* is essentially that in the continuous case (smaller by a factor of $\frac{1}{2}$ which can be improved to $1-\varepsilon$ with some work and more restriction on k).

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