Kalai and Muller's Possibility Theorem: A Simplified Integer Programming Version *

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Abstract

We provide a respecification of an integer programming characterization of Arrovian social welfare functions introduced by Sethuraman et al. (2003). By exploiting this respecification, we give a new and simpler proof of Theorem 2 in Kalai and Muller (1977).

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1 Introduction

In a pathbreaking paper, Sethuraman et al. (2003) proposed a characterization of Arrovian social welfare functions in terms of integer programs (for a further development of this research program, see also Sethuraman et al. (2006), Vohra (2011), and Busetto et al. (2015), among others). As remarked by these authors, integer programming is a powerful analytical tool, which makes it possible to derive in a systematic and simple way many of the already known theorems on Arrovian social welfare functions and to prove new results. In particular, it permits one to reconsider the fundamental issues concerning the characterization of domains admitting nondictatorial Arrovian social welfare functions.

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Kalai and Muller (1977) provided the first complete characterization of these domains. In their Theorem 1, they showed that there exists a n-person nondictatorial Arrovian social welfare function for a given domain if and only if there exists a 2-person nondictatorial Arrovian social welfare function for the same domain. In their Theorem 2, Kalai and Muller (1977) provided a characterization of the domains admitting nondictatorial Arrovian social welfare functions "without ties," i.e., which do not allow for indifference between distinct alternatives in their range.

Taking inspiration from Kalai and Muller (1977), Sethuraman et al. (2003) built up an integer program which incorporates a reformulation of the condition of decisiveness implication. By using this integer program, they provided a simplified version of Theorem 1 in Kalai and Muller (1977).

We propose an amended version of the integer program they used to reformulate Theorem 1 in Kalai and Muller (1977), as we show that it contains logically redundant constraints. Then, by exploiting our integer program, we give a new and simpler proof of Theorem 2 in Kalai and Muller (1977). We do this in two steps. We first propose a simpler definition of decomposability which eliminates the logical redundancies contained in the condition of decisiveness implication proposed by Kalai and Muller (1977). This enables us to prove that decomposable domains admit nondictatorial solutions to our simplified integer program. Then, Theorem 2 in Kalai and Muller (1977) can be straightforwardly obtained as a corollary of this theorem.

Busetto et al. (2015) proceeded along the way opened by Kalai and Muller (1977) and Sethuraman et al. (2003). Using integer programming, they provided a characterization of the domains admitting Arrovian social welfare functions "with ties," i.e., which allow for indifference between distinct alternatives in their range. In this paper, when referring to Arrovian social welfare functions, we consider implicitly Arrovian social welfare functions "without ties."

2 Notation and definitions

Let E be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of E, denoted by n. Elements of E are called agents.

Let $\mathcal{E}$ be the collection of all subsets of E. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

Let Σ be the set of all the complete, transitive, and antisymmetric binary relations on $\mathcal{A}$, called preference orderings.

Let Ω denote a subset of Σ such that $|\Omega| \geq 2$. An element of Ω is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if x is ranked above y under $\mathbf{p}$.

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let TR denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let NTR denote the set of nontrivial pairs.

Let Ω^n denote the n-fold Cartesian product of Ω. An element of Ω^n is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on Ω is a function $f : \Omega^n \to \Sigma$.

A SWF on Ω, f, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_i y$, for all $i \in E$, implies $xf(\mathbf{P})y$.

A SWF on Ω, f, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P'} \in \Omega^n$, $x\mathbf{p}_i y$ if and only if $x\mathbf{p}'_i y$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P'})y$, and, $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P'})x$.

An Arrovian Social Welfare Function (ASWF) on Ω is a SWF on Ω, f, which satisfies PO and IIA.

An ASWF on Ω, f, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_j y$ implies $xf(\mathbf{P})y$.

An ASWF on Ω, f, is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, 1\}$.

An Integer Program (IP) on Ω consists of a set of linear constraints, related to the preference orderings in Ω, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let d denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on Ω.

A solution d is dictatorial if there exists $j \in E$ such that $d_S(x, y) = 1$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$.

A solution d is nondictatorial if it is not dictatorial.

An ASWF on Ω, f, and a solution to an IP on the same Ω, d, are said to correspond if, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $xf(\mathbf{P})y$ if and only if $d_S(x, y) = 1$, $yf(\mathbf{P})x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

3 Arrovian social welfare functions and integer programming

The first formulation of an IP on Ω was proposed by Sethuraman et al. (2003). Their IP, which we will call IP1, consists of the following set of constraints:

$$d_E(x, y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

for all triples of alternatives x, y, z and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy the following conditions (hereafter referred to as Conditions $(*)$):

$$\begin{aligned}
&A \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}z\mathbf{p}y,\\
&B \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}x\mathbf{p}z,\\
&C \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}y\mathbf{p}x,\\
&U \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}y\mathbf{p}z,\\
&V \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}x\mathbf{p}y,\\
&W \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}z\mathbf{p}x.
\end{aligned}$$

By introducing integer programming, Sethuraman et al. (2003) were able to provide a new representation of ASWFs with respect to the axiomatic one previously used in the Arrow's tradition. In particular they showed, in their Theorem 1, that there exists a one-to-one correspondence between the set of the solutions to IP1 on Ω and the set of the ASWFs on the same Ω.

Sethuraman et al. (2003) also built up a second IP on Ω, for many respects related to the work of Kalai and Muller (1977) on nondictatorial ASWFs.

Kalai and Muller (1977) introduced the following condition of decomposability to characterize the domains of antisymmetric preference orderings admitting nondictatorial ASWFs.

Ω is said to be decomposable (henceforth, KM decomposable) if there exists a set R, with $TR \subsetneqq R \subsetneqq \mathcal{A}^2$, satisfying the following conditions.

Condition I. For every two pairs $(x,y),(x,z) \in NTR$, *if there exist* $\mathbf{p},\mathbf{q} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$, *then* $(x,y) \in R$ *implies that* $(x,z) \in R$.

Condition II. For every two pairs $(x,y),(x,z) \in NTR$, *if there exist* $\mathbf{p},\mathbf{q} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$, *then* $(z,x) \in R$ *implies that* $(y,x) \in R$.

Condition III. For every two pairs $(x,y),(x,z) \in NTR$, *if there exists* $\mathbf{p} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$, *then* $(x,y) \in R$ *and* $(y,z) \in R$ *imply that* $(x,z) \in R$.

Condition IV. For every two pairs $(x,y),(x,z) \in NTR$, *if there exists* $\mathbf{p} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$, *then* $(z,x) \in R$ *implies that* $(y,x) \in R$ *or* $(z,y) \in R$.

In the second IP introduced by Sethuraman et al. (2003), which we will call IP1$'$, constraint (3) is replaced by the following set of constraints:

$$d_S(x,y) \le d_S(x,z), \tag{4}$$

$$d_S(z,x) \le d_S(y,x), \tag{5}$$

for all triples x,y,z such that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x,y) + d_S(y,z) \le 1 + d_S(x,z), \tag{6}$$

$$d_S(z,y) + d_S(y,x) \ge d_S(z,x), \tag{7}$$

for all triples x,y,z such that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and for all $S \in \mathcal{E}$.

Constraints (4) and (5) translate, in terms of variables $d_S(x,y)$, Conditions I and II of Kalai and Muller (1977). In their Claim 1, Sethuraman et al. (2003) showed that these constraints are special cases of (3). Constraints (6) and (7) translate Conditions III and IV of Kalai and Muller (1977). In their Claim 2, Sethuraman et al. (2003) showed that also these constraints are special cases of (3). Their analysis established that any solution d to IP1 on Ω is a solution to IP1$'$ on the same domain and that IP1 and IP1$'$ are equivalent in the case where $n = 2$.

We now prove that the set of constraints (4)-(7) exhibits problems of logical dependence. The following proposition shows that one of the constraints (4) and (5) is redundant.

Proposition 1. *d satisfies (1), (2), and (4) if and only if it satisfies (1), (2), and (5).*

Proof. Suppose that d satisfies (1), (2), and (4). Consider a triple x, y, z. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(z, x) > d_S(y, x),$$

for some $S \in \mathcal{E}$. Then, $d_S(z, x) = 1$, $d_S(y, x) = 0$. But then, $d_{S^c}(x, z) = 0$, $d_{S^c}(x, y) = 1$. This implies that

$$d_{S^c}(x, y) > d_{S^c}(x, z),$$

contradicting (4). Therefore, d satisfies (1), (2), and (5). Suppose that d satisfies (1), (2), and (5). Consider a triple x, y, z. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(x, y) > d_S(x, z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x, y) = 1$, $d_S(x, z) = 0$. But then, $d_{S^c}(y, x) = 0$, $d_{S^c}(z, x) = 1$. This implies that

$$d_{S^c}(z, x) > d_{S^c}(y, x),$$

contradicting (5). Therefore, d satisfies (1), (2), and (4). ■

The following proposition shows that one of the constraints (6) and (7) is redundant.

Proposition 2. *d satisfies (1), (2), and (6) if and only if it satisfies (1), (2), and (7).*

Proof. Suppose that d satisfies (1), (2), and (6). Consider a triple x, y, z. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(z, y) + d_S(y, x) < d_S(z, x),$$

for some $S \in \mathcal{E}$. Thus, $d_S(z, y) = 0$, $d_S(y, x) = 0$, and $d_S(z, x) = 1$. But then, $d_{S^c}(y, z) = 1$, $d_{S^c}(x, y) = 1$, and $d_{S^c}(x, z) = 0$. This implies that

$$d_{S^c}(x, y) + d_{S^c}(y, z) > 1 + d_{S^c}(x, z),$$

contradicting (6). Therefore, d satisfies (1), (2), and (7). Suppose that d satisfies (1), (2), and (7). Consider a triple x, y, z. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(x, y) + d_S(y, z) > 1 + d_S(x, z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x,y) = 1$, $d_S(y,z) = 1$, and $d_S(x,z) = 0$. But then, $d_{S^c}(y,x) = 0$, $d_{S^c}(z,y) = 0$, and $d_{S^c}(z,x) = 1$. This implies that

$$d_{S^c}(z,y) + d_{S^c}(y,x) < d_{S^c}(z,x),$$

contradicting (7). Therefore, d satisfies (1), (2), and (6). ■

Busetto et al. (2015) considered a "ternary" version of IP1, i.e., generalized IP1 to the case where $d_S(x,y) = \frac{1}{2}$, for some $(x,y) \in NTR$ and for some $S \in \mathcal{E}$. They also introduced a second "ternary" IP on Ω, which incorporates, like IP1$'$ proposed by Sethuraman et al. (2003), a reformulation of Conditions I-IV of Kalai and Muller (1977). In constructing it, they eliminated the redundancies inherent in IP1$'$, we have exhibited in Propositions 1 and 2. The "binary" version of that program, which we will call IP2, consists of constraints (1), (2), (4), and the following reformulation of constraint (6).

$$d_S(x,y) + d_S(y,z) \leq 1 + d_S(x,z), \tag{6 bis}$$

for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, and for all $S \in \mathcal{E}$.

We will consider now the relationship between IP1 and IP2. The following result is a straightforward consequence of Claims 1 and 2 proved by Sethuraman at. al. (2003), mentioned above.

Proposition 3. *If d is a solution to IP1 on Ω, then it is a solution to IP2 on the same Ω.*

The following result shows that the converse of Proposition 3 holds, and IP1 and IP2 coincide, when $n = 2$. An analogous equivalence result between IP1 and IP1$'$ for $n = 2$ was provided, without an explicit proof, by Sethuraman et al. (2003). We provide here a "binary" version of the proof of Proposition 2 in Busetto et al. (2015) which states the equivalence result for their "ternary" version of the program.

Proposition 4. *Let $n = 2$. If d is a solution to IP2 on Ω, then it is a solution to IP1 on the same Ω.*

Proof. Let $n = 2$. Let d be a solution to IP2 on Ω. Consider a triple x, y, z and disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$. Suppose that

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > 2.$$

Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) > 1 + d_{\{2\}}(y, x),$$

contradicting (6 bis). The cases where $B \neq \emptyset$, $V \neq \emptyset$, and $C \neq \emptyset$, $U \neq \emptyset$ lead, *mutatis mutandis*, to the same contradiction. Consider the case where $U \neq \emptyset$ and $V \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}x\mathbf{q}y$. Suppose that $U = \{1\}$ and $V = \{2\}$. Then,

$$d_{\{2\}}(z, x) > d_{\{2\}}(z, y),$$

contradicting (4). The cases where $V \neq \emptyset$, $W \neq \emptyset$, and $U \neq \emptyset$, $W \neq \emptyset$, lead, *mutatis mutandis*, to the same contradiction. Therefore, d satisfies (3). Hence, d is a solution to IP1 on Ω. ■

4 A simplified version of Kalai and Muller's possibility theorem

Kalai and Muller (1977) were the first who provided a complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs. They did this by means of two theorems. In their Theorem 1, they showed that, for a given domain Ω, there exists a nondictatorial ASWF for $n = 2$ if and only if, for the same Ω, there exists a nondictatorial ASWF for $n > 2$. In their Theorem 2, they showed that there exists a nondictatorial ASWF on Ω for $n \geq 2$ if and only if Ω satisfies the conditions of KM decomposability introduced in Section 3.

Sethuraman et al. (2003) opened the way to an analysis of the problem of dictatorship in terms of integer programming. More precisely, they showed, in their Theorem 8, a result establishing a one-to-one correspondence between the nondictatorial solutions of IP1 for $n = 2$ and its nondictatorial solutions for $n > 2$.

We go forward along the line opened by Sethuraman et al. (2003), providing a characterization of domains admitting nondictatorial solutions to IP1. This result is the heart of a new and simpler proof of Theorem 2 in Kalai and Muller (1977) for nondictatorial ASWFs in terms of integer programming.

In order to obtain our characterization theorem, we need to use the reformulation of the concept of KM decomposability suitable to be applied within the analytical context of IP2. This reformulation is based on the existence of two sets, $R_1, R_2 \in \mathcal{A}^2$, instead of only one, which satisfy the two conditions introduced below.

Given a set $R \subset \mathcal{A}^2$, consider the following conditions on R.

Condition 1. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition 2. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be decomposable if and only if there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2.

With regard to this definition of a decomposable domain, let us remind the main differences with the original notion of KM decomposability already noticed by Busetto et al. (2015). Conditions 1 and 2 differ from the corresponding Conditions I and III as the former refer to triples, rather than pairs, of alternatives. Moreover, Condition 2 is reformulated in terms of a pair of preference orderings, instead of only one. This is consistent with the formulation of constraint (6 bis), which is in fact a reinterpretation of Condition 2 in terms of integer programming. Also, this new definition of a decomposable domain does not require that R_1 and R_2 contain TR, whereas KM decomposability requires that R contains TR. Finally, our new definition imposes that R_1 and R_2 satisfy only two conditions, instead of four, as required by KM decomposability. As Proposition 5 below makes it clear, this implies a redundancy of Conditions II and IV of KM decomposability, which parallels the redundancy of constraints (5) and (7) proved in Propositions 1 and 2.[1]

On the basis of the reformulation of the concept of decomposability, we now state and prove the characterization theorem.

Theorem. *There exists a nondictatorial solution to IP2 on Ω, d, for $n = 2$, if and only if Ω is decomposable.*

Proof. Let d be a nondictatorial binary solution to IP2 on Ω, for $n = 2$. Let $R_1 = \{(x, y) \in NTR : d_{\{1\}}(x, y) = 1\}$ and $R_2 = \{(x, y) \in NTR : d_{\{2\}}(x, y) = 1\}$. Then, for all $(x, y) \in NTR$, $(x, y) \in R_1$ if and only if

[1] Busetto et al. (2015), in their Proposition 3, provided a direct proof of the equivalence between the new notion of decomposability and KM decomposability.

$(y,x) \notin R_2$, as d satisfies (2). Moreover, $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$, as d is nondictatorial. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1$ and $(x,z) \notin R_1$ Then, $d_{\{1\}}(x,y) = 1$ and

$$d_{\{1\}}(x,y) > d_{\{1\}}(x,z),$$

contradicting (4). Hence, R_i, $i = 1,2$, satisfies Condition 1. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x,y),(y,z) \in R_1$, and $(x,z) \notin R_1$. Then, $d_{\{1\}}(x,y) = 1$, $d_{\{1\}}(y,z) = 1$, and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) > 1 + d_{\{1\}}(x,z),$$

contradicting (6 bis). Hence, R_i, $i = 1,2$, satisfies Condition 2. We have proved that Ω is decomposable. Conversely, suppose that Ω is decomposable. Then, there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$, such that, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$; moreover, R_i, $i = 1,2$, satisfies Conditions 1 and 2. Determine d as follows. For each $(x,y) \in NTR$, let $d_{\emptyset}(x,y) = 0$, $d_E(x,y) = 1$; moreover, let $d_{\{i\}}(x,y) = 1$ if and only if $(x,y) \in R_i$; $d_{\{i\}}(x,y) = 0$ if and only if $(x,y) \notin R_i$, for $i = 1,2$. Then, d satisfies (1) and (2) as, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x,y) > d_{\{1\}}(x,z).$$

Then, we have $(x,y) \in R_1$ and $(x,z) \notin R_1$, contradicting Condition 1. Therefore, d satisfies (4). Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) > 1 + d_{\{1\}}(x,z).$$

Then, we have $(x,y),(y,z) \in R_1$ and $(x,z) \notin R_1$, contradicting Condition 2. Therefore, d satisfies (6 bis). d is nondictatorial as $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$. Hence, d is a nondictatorial binary solution to IP2 on Ω. ■

Theorem 2 in Kalai and Muller (1977) can be obtained as a corollary of the previous result.

Corollary. *There exists a nondictatorial ASWF on Ω, f, for $n \geq 2$, if and only if Ω is decomposable.*

Proof. It is a consequence of the Theorem, Theorems 1 and 8 in Sethuraman et al. (2003), and Propositions 3 and 4. ■

From the previous corollary, we obtain a result, which, as anticipated above, establishes the equivalence between the new notion of decomposability and KM decomposability, and implies that Conditions II and IV are redundant.

Proposition 5. *Ω is KM decomposable if and only if it is decomposable.*

Proof. It is an immediate consequence of Theorem 2 in Kalai and Muller (1977) and the Corollary. ■

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