

Endogenous Growth and Wave-Like Business Fluctuations*

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Abstract

This paper argues that observed long lags in innovation implementation rationalize Schumpeter's statement that "wave-like fluctuations in business ... are the form economic development takes in the era of capitalism." Adding implementation delays to an otherwise standard endogenous growth model with expanding product variety, the equilibrium path admits a Hopf bifurcation where consumption, R&D and output permanently fluctuate. This mechanism is quantitatively consistent with the observed medium-term movements of US aggregate output. In this framework, an optimal allocation may be restored at equilibrium by the mean of a procyclical subsidy, needed to generate additional consumption smoothing. Finally, a procyclical R&D subsidy rate designed to half consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% (compensation equivalent) increase in welfare.

JEL Classification O3, E32

Keywords Endogenous growth; endogenous fluctuations; innovation cycles; time delays; medium-term cycles; Hopf bifurcation

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1 Introduction

The conjecture that in the modern era business fluctuations and economic growth are two faces of the same coin comes back to Schumpeter [71], who pointed out that “wave-like fluctuations in business ... are the form economic development takes in the era of capitalism.”¹ Starting from this premise, Schumpeter raised the key question of “why is it that economic development does not proceed evenly ..., but as it were jerkily; why does it display those characteristics ups and downs?” When searching for an answer, he drew attention to the critical fact that innovations “appear en masse at intervals”, “discontinuously in groups or swarms,” which “signifies a very substantial increase in purchasing power all over the business sphere.”

Following the seminal work by Aghion and Howitt [3], Grossman and Helpman [49] and Romer [67], important developments have been undertaken in the last twenty years addressed to improve our understanding on the main channels through which innovations promote development and growth. Endogenous growth theory is in a fundamental sense Schumpeterian, since it stresses the critical role played by innovations in the observed growth of total factor productivity. However, little has been written since then on the relation between innovation and business fluctuations.

A natural candidate for the study of Schumpeterian wave-like business fluctuations is the observed long delay elapsed between the realization of R&D activities and the implementation and adoption of the associated innovations.² Schumpeter [71]’s description of the periodicity of business fluctuations is, in this sense, very appealing: “the boom ends and the depression begins after the passage of the time which must elapse before the products of the new enterprise can appear on the market.” The argument in this paper is very close to Schumpeter’s description: waves of innovations arrive en masse, moving the economy to a boom; the associated increase in productivity raises purchasing power all over the business sphere, inducing research activities to flourish; but, the new products will take a while to develop; when the new wave of innovations is eventually implemented, the new products enter the market producing a second boom, which will generate a third, then a fourth and so on and so for.

It is important to notice that Schumpeterian wave-like business fluctuations as described in the previous paragraph substantially differ from the type of fluctuations studied in modern business cycle theory. Inspired on Kydland and Prescott [58], it has focused on the study of high frequency movements, those between 4 and 40 quarters. Schumpeter, indeed, was more interested in medium (Juglar) and low (Kondratieff) frequency movements lasting around 10

¹Consistently with Schumpeter we have used the term wave-like *business* fluctuations. This term is, however, completely unrelated with the concept of wave-like fluctuations used in the partial differential equation literature.

²Comin and Hobijn [33] study the pattern of technology diffusion around the globe and find that countries on average adopt technologies 47 years after their invention. Comin et al [34] find that, when compared to the US, lags in the use of technology are measured in decades for most countries. Adams [2] estimates that academic knowledge is a mayor contributor to productivity growth, but its effects lag roughly 20 years. Mansfield [60] estimates the mean adoption delay of twelve mayor 20th-century innovations in 8 years. Jovanovic and Lach [53] estimate at 8.1% the annual diffusion rate of new products.

and 50 years, respectively. A description of economic fluctuations more in accordance with the Schumpeterian's view was recently suggested by Comin and Gertler [31]. They estimate the medium-term movements of US per capita GDP growth by analyzing frequencies between 40 and 200 quarters, and find that it permanently undulates with a periodicity of around 11 years and an amplitude of around 8 percentage points from peak to valley. This paper focuses on Juglar cycles or, equivalently, on medium terms movements.

In this paper, wave-like business fluctuations are modeled in a simple way by adding an implementation delay to an otherwise textbook endogenous growth model with expanding product variety. It shows that the equilibrium path admits a Hopf bifurcation where consumption, research and output permanently fluctuate. The main mechanism relating growth to wave-like business fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. The mechanics is the following. Let the economy initially react to a permanent positive shock by some concentration of research activities, which makes new ideas to appear en masse. This is the standard reaction of a dynamic general equilibrium model when the initial stock of (intangible) capital is relatively low. However, the economic effects of this wave of research activity will be delayed in time. When a swarm of new businesses will become eventually operative, the associated increase in productivity will inject additional resources to the economy –“a substantial increase in purchasing power” in Schumpeter's words. Consumption smoothing makes the rest by allocating the additional resources to create a second wave of innovation activities. This process will repeat again and again as time passes. In a simple quantitative exercise, where parameters are set to match some key aggregate features of the US economy, we show that the model is able to replicate medium-term movements of similar periodicity and amplitude to those observed by Comin and Gertler. In this sense, the suggested mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations resulting in a procyclical R&D policy. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. This results is due to the fact that consumption fluctuates less in the optimal allocation, implying that an optimal subsidy has to generate additional R&D investments during booms than during recessions. Interestingly, the policy does not affect output in the short run, since for a long while production is determined by past R&D investments; new investments will eventually become productive after a long delay. Finally, it quantitatively finds that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare.

The model in this paper belongs to the literature on dynamic general equilibrium with time delays, including vintage capital, time-to-build and demographic theories. Firstly, fluctuations in the vintage capital literature are the result of machine replacement, as described in Benhabib and Rustichini [22], Boucekkine et al [27] and Caballero and Hammour [29].³ Following the

³See also Boucekkine and de la Croix [24].

lumpy investment literature, initiated by Doms and Dunne [40], Cooper et al [35] find robust evidence on the existence of machine replacement, but little support for the contribution of machine replacement to the understanding of observed business fluctuations. Second, since the seminal paper by Kydland and Prescott [58], investment lags have been shown to make the business cycle highly persistent. Asea and Zak [9] and, more recently, Bambi [10] go further and prove that time-to-build may generate endogenous fluctuations. However, time-to-build delays are short relative to Juglar cycles, since they last some few quarters only. Finally, Boucekine et al [25] find that in a demographic model with realistic survival probabilities and a vintage human capital structure, the adjustment of the economy may generate output movements of the order of a Kondratieff cycle. For all these reasons, implementation delays are more appealing for understanding Schumpeterian business fluctuations than vintage (human and physical) capital or time-to-build arguments.

There is an extensive literature on endogenous competitive equilibrium cycles, along the seminal contributions of Benhabib and Nishimura [19][20], where the raising of these persistent cycles arrive through a Hopf bifurcation in a multi-sector growth model. In our paper, endogenous cycles still emerge through a Hopf bifurcation, but their main source is the presence of implementation delays instead of multiple sectors. To clearly state this point, we adopt a particular version of the expanding product variety model formally equivalent to a one-sector model. These delays change the equations describing the competitive equilibrium to a system of mixed functional differential equations. To this extent our contribution shares some similarities with the one of D’Albis et al. [8]; in fact the raising of competitive equilibrium cycles, in their model, depends on the presence of delay differential equations pinned down by a learning-by-doing process with memory effect. On the other hand, a key difference between our paper and D’Albis et al. [8] is that in an endogenous growth model, the standard assumptions used to prove the emergence of Hopf bifurcation do not hold and so, to prove that cyclical behavior arises, one has to perform a non trivial dimensional reduction⁴ of the original system of DDE’s (see Section 5) which we have not found in the current literature.

This paper is also related to Matsuyama [61] and Francois and Lloyd-Ellis [45], among the few exceptions connecting endogenous growth with cycles. Firstly, Matsuyama [61] shows that, under some conditions regarding the saving rate, endogenous cycles arrive in a discrete time Rivera-Batiz and Romer [66] endogenous growth model, where monopoly rents last only one period and implementing an innovation entails fixed costs. Along the cycle, the economy moves periodically from a Neoclassical regime to an AK regime. Research activities come en masse as in Schumpeter’s theory, but they are countercyclical. In our theory, indeed, the economy is always in an AK regime and innovation activities are procyclical. Second, Francois and Lloyd-Ellis [45] link growth and cycles combining animal spirits, such as in Schleifer [70], to a Schumpeterian endogenous growth model. In their framework, a cyclical equilibrium exists because firms are

⁴The term “dimensional reduction” is used for different purposes in physics and in data analysis; here we simply mean that the dynamic behavior of the original system of differential equations can be unveiled by studying a system having smaller dimension.

interested in delaying implementation to the boom in order to maximize the expected length of incumbency. In our model, cycles are also related to implementation delays. However, differently from Francois and Lloyd-Ellis [45], business fluctuations are not the consequence of animal spirits but result from a Hopf bifurcation, and they materialize as medium-term movements instead of happening at high frequencies.

The idea that delayed gains in productivity may generate persistence has been deeply studied in the recent literature on “news shocks” –see Beaudry and Portier [15].⁵ However, the main source of fluctuations in this literature remains exogenous. In our theory, indeed, current research activities and the associated future innovations may be seen as perfectly forecasted, endogenous news shocks. Endogenous news are at the basis of the cyclical behavior of our economy, since more resources are allocated to produce current news when past news realize.

Implementation delays in this paper are indeed very different from the delay elapsing between the arrival of a general purpose technology (GPT) and its implementation. In fact, GPT refers to major technology breakthroughs, whose implementation requires costly and very long restructuring. According to David [37], the implementation of a new GPT may generally take several decades: for example, the electric dynamos took three decades to attain a fifty percent diffusion level in the U.S.. Then the consequences of a discovery of a GPT may well reproduce the low (Kondratieff) frequency movements in the data but not the medium ones which are the objective of our analysis.

Finally, this paper shares with Comin and Gertler [31] the view that lags of technology adoption do generate medium-term movements in models of endogenous productivity growth. In Comin and Gertler’s words, medium-term movements “reflect a persistent response of economic activity to the high-frequency fluctuations normally associated with the cycle.” They are the endogenous reaction to a sequence of exogenous shocks. In our theory, indeed, medium-term movements are endogenous and self-sustained, and they could emerge independently of the existence of any high frequency exogenous productivity process.

The paper is organized as follows. Sections 2 and 3 describe the economy and define the decentralized equilibrium, respectively. The balanced growth path and its properties are studied in Section 4. The next section focuses on the transitional dynamics of the economy. In particular, it proves the existence of a persistent cycle. In Section 6 the robustness of the results are shown for a generic innovation externality. Section 7 quantitatively studies the empirical relevance of the persistent cycle. Section 8 analyzes the optimal allocation and suggests a countercyclical R&D subsidy as a Pareto improving policy. A counterfactual exercise is performed showing that a 10% R&D countercyclical subsidy halving consumption fluctuations generates first order welfare gains. Finally, Section 9 concludes.

⁵More recently, Comin et al [32] stress the importance of endogenous adoption in the amplification of these shocks.

2 The Economy

The economy is populated by a continuum of infinitely lived, identical households of unit measure, holding a constant flow labor endowment of one unit. There is a sole final good, used for consumption purposes only. Household *preferences* are represented by:

$$U = \int_0^\infty \log(c_t) e^{-\rho t} dt, \quad (1)$$

where c_t is per capita consumption and $\rho > 0$ represents the subjective discount rate.

In line with the literature on expanding product variety, the final consumption good is produced by the mean of a CES technology defined on a continuum of intermediary inputs in the support $[0, n]$. Differently from the existing literature, we assume that adopting new technologies requires a time delay $d > 0$, meaning that varieties discovered at time t become operative at time $t + d$. It can be interpreted as an *implementation delay* which elapses from the discovery of a new variety to its economic implementation. As usual, the extent of product variety n is assumed to represent also the aggregate state of knowledge. Knowledge is assumed to positively affect the productivity of the consumption sector as an externality, i.e., n has a positive effect on the production of the consumption good. Knowledge produced at time t is assumed to become public information at the time $t + d$, when for the first time the innovation is produced and sold. Under the previous assumptions, the consumption good technology is

$$c_t = n_{t-d}^{\nu+1-\frac{1}{\alpha}} \left(\int_0^{n_{t-d}} x_t(j)^\alpha dj \right)^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1 \quad (2)$$

where n_{t-d} represents the extent of operative varieties at time t , and $x_t(j)$ is the amount of the intermediary input j used at time t in the production of c_t . This consumption good technology implies a constant (and equal) elasticity of substitution between every pair of varieties, $\theta = \frac{1}{1-\alpha} > 1$.

The parameter ν is the elasticity of the externality, but also the return to specialization as explained extensively in Ethier [43] and Benassy [18]; it will emerge from the balanced growth path analysis in Section 6 that the growth rate of the main aggregate variables, namely c_t and n_t , is the same only when $\nu = 1$. For this reason, we deal with this case in the first instance, and only later we will check the robustness of our results when $\nu \neq 1$. It is also worth noting that in the case $\nu = 1$ we may distinguish between the markup charged by the monopolistic firms producing $x(j)$ from the degree of returns to specialization.

The assumption that the externality operates only through the measure of operative varieties n_{t-d} is consistent with the *love of variety* argument as suggested by Dixit and Stiglitz [39].

Technology in the intermediary sector is assumed to be symmetric across varieties

$$x_t(j) = l_t(j), \quad (3)$$

where $l_t(j)$ is labor allocated to the production of variety j . Total labor L allocated to the

production of the intermediary sector is given by

$$\int_0^{n_{t-d}} x_t(j) dj = L_t. \quad (4)$$

Notice that $L_t \in [0, 1]$, since the total labor endowment is one.

An efficient allocation of labor to the production of the consumption good, spreading through the intermediary sector, results from maximizing (2) subject to (4). It is easy to see that an efficient allocation is symmetric, meaning $x_t(j) = x_t$ for all j , which implies

$$c_t = n_{t-d}L_t \quad \text{and} \quad n_{t-d}x_t = L_t. \quad (5)$$

As stated above, labor allocated to the production of the consumption good benefits from a knowledge externality, n , which comes linearly in the reduced form of the consumption goods technology (5). In the following sections, we show that optimal and equilibrium allocations are both efficient in the sense defined above –see Koeninger and Licandro [55].

Finally, *R&D* activities are also assumed to be linear on labor and addressed to the creation of new intermediary inputs. The innovation technology creating these new varieties is assumed to be:

$$\dot{n}_t = An_{t-d}(1 - L_t), \quad (6)$$

where $1 - L_t$ is labor assigned to R&D production, its marginal productivity depending on parameter A , $A > 0$. It is also assumed that the R&D sector benefits from a positive externality depending linearly on the extent of operative varieties.

Note that consumption and R&D technologies, (5) and (6) respectively, collapse to

$$\dot{n}_t = A(n_{t-d} - c_t). \quad (7)$$

The AK structure of the model, see Rebelo [64], can be easily seen if the extent of product variety n_{t-d} is interpreted as (intangible) capital. In the following, we will refer to (7) as the feasibility constraint.⁶

The fact that equations (5) and (6) collapse into the aggregate technology (7) means that the model in this paper belongs to the long tradition of one-sector growth models. Even if consumption and R&D are different goods, since technologies (5) and (6) are symmetric, the model behaves as if there were only one good. In other words, one unit of the consumption good may be transformed into an additional new variety at the constant rate A . Consequently, the assumption $\nu = 1$ is sufficient for the expanding product variety model to become a one-sector model.⁷ Our analysis focuses on this case to differentiate from the existing literature on

⁶Equivalently, it can be assumed that labor is only used to the production of goods, and output is assigned to both consumption and R&D, with L representing the consumption to output ratio and A the rate at which the consumption good is transformed into innovations.

⁷In the general case of $\nu > 0$, the variable change $z = n^\nu$ may be introduced. It does imply that (5) becomes $c_t = z_t L_t$. Under the *alternative* R&D technology $\dot{z}_t = Az_t(1 - L_t)$, the aggregate technology (7) collapses into $\dot{z}_t = A(z_t - c_t)$. This clearly shows that the critical assumption for the economy to be one-sector is that technologies producing the different final goods are symmetric.

permanent cycles in multi-sector growth models (e.g. Benhabib and Nishimura [19][20]) while Section 6 shows how to extend some of the main results to the multi-sector case, $\nu \neq 1$.

3 Decentralized Equilibrium

The economy is decentralized in the standard way. The market for the final consumption good is supposed to be perfectly competitive, so that individuals and firms take the consumption price, normalized to unity, as given. Innovations are protected by an infinitely lived patent and the market for intermediary inputs is monopolistically competitive. The R&D sector is perfectly competitive, implying that research firms make zero profits. Finally, the labor market is also assumed to be perfectly competitive.

Final Good Firms – A representative firm produces the consumption good by the mean of technology (2). It takes intermediary prices as given and maximizes profits by choosing $x_t(j)$ for $j \in [0, n_{t-d}]$:

$$\max c_t - \int_0^{n_{t-d}} p_t(j)x_t(j)dj \quad (8)$$

subject to the consumption good technology (2), while $p(j)$ indicates the relative price of the intermediate good j . From the first order condition associated to this problem the inverse demand function

$$p_t(j) = n_{t-d}^{2\alpha-1} \left(\frac{c_t}{x_t(j)} \right)^{1-\alpha} \quad (9)$$

can be easily derived.

Intermediary Good Firms – Firms producing intermediary goods operate under monopolistic competition. They maximize profits subject to the inverse demand function (9) and the technology constraint (3), which collapses to

$$\pi_t = \max_{p_t} p_t^{\frac{1}{\alpha-1}} (p_t - w_t).$$

The optimal price rule is

$$p_t = \frac{1}{\alpha} w_t, \quad (10)$$

where the real wage w is equal to the marginal cost of production (technology is linear in labor), and $\frac{1}{\alpha}$ represents the markup over marginal costs, which depends inversely on the elasticity of substitution across varieties.

R&D – Successful researchers receive a patent of infinite life. At equilibrium, the patent value v_t is equal to the present value of the associated flow of monopolistic profits. Therefore, we can write (e.g. Acemoglu [1])

$$r_t = \frac{\pi_t}{v_t} + \frac{\dot{v}_t}{v_t}. \quad (11)$$

The value of the patent has at least to cover the innovation cost,

$$v_t \geq \frac{w_t}{An_{t-d}} = \frac{w_t(1-L_t)}{\dot{n}_t}. \quad (12)$$

Households – The household’s intertemporal maximization problem is

$$\max \int_0^\infty \log(c_t) e^{-\rho t} dt \quad (13)$$

subject to the instantaneous budget constraint

$$\dot{a}_t = r_t a_{t-d} + w_t - c_t \quad (14)$$

and the initial condition $a_t = \bar{a}_t$, for $t \in [-d, 0]$, where \bar{a}_t is an exogenously given continuous positive function defined on the t domain, formally $\bar{a}_t \in C([-d, 0]; \mathbb{R}_+)$.⁸ Moreover, a_{t-d} represents the value at t of patents produced up to time $t-d$, which refer to variety already implemented in the economy; the patents are also assumed to be owned by households. Non consumed income is then saved in the form of new patents. The households problem is an optimal control problem with delay. The positivity constraints $c_t \geq 0$ and $a_t \geq 0$ are implicitly assumed. It is possible, using the concavity of utility functional and the linearity of the state equation (in line with Freni et al. [46] among others) to prove existence and uniqueness of the optimal solution for such a problem.

Following existing theory (see e.g. Kolmanovskii and Myshkis [56] for the finite horizon case and Agram et al. [5] Theorem 3.1, for the infinite horizon case), a given state-control pair (a_t, c_t) is optimal if there exists an absolutely continuous costate function μ_t such that

$$\frac{e^{-\rho t}}{c_t} = \mu_t \quad (15)$$

$$-\dot{\mu}_t = r_{t+d} \mu_{t+d}, \quad (16)$$

$$\lim_{t \rightarrow \infty} a_t \mu_t = 0, \quad \text{or, equivalently,} \quad \lim_{t \rightarrow \infty} a_t c_t^{-1} e^{-\rho t} = 0. \quad (17)$$

The optimality conditions (15)-(16) collapse into the following Euler-type equation

$$\frac{\dot{c}_t}{c_t} = \underbrace{r_{t+d}}_{R\&D \text{ returns}} \cdot \underbrace{\frac{c_t}{c_{t+d}} e^{-\rho d}}_{\text{discount factor}} - \rho. \quad (18)$$

The representative household faces then the following trade-off, consuming at time t or buying new patents which will become operative at time $t+d$.

Decentralized Equilibrium – The decentralized equilibrium is symmetric, meaning that (5) holds, and equation (9) becomes

$$p_t = n_{t-d}. \quad (19)$$

Recall that the consumption good is the numeraire, which implies that p_t is the price of the intermediary input relative to the price of consumption. An expansion in product variety improves productivity in the consumption sector, inducing an increase in the relative price of the intermediary input as reflected by (19). From (10) and (19), the wage rate at equilibrium is

$$w_t = \alpha n_{t-d}. \quad (20)$$

⁸By $C([-d, 0]; \mathbb{R}_+)$ we mean the space of continuous function from $[-d, 0]$ to $\mathbb{R}_+ := [0, +\infty)$.

Market power makes the equilibrium real wage equal to a fraction α , the inverse of the markup, of labor marginal productivity – aggregate technology is in (5). Consequently, the real wage maps the behavior of current technology.

From (5), (10) and (19), intermediary profits can be written as

$$\pi_t = (1 - \alpha) \frac{c_t}{n_{t-d}} > 0. \quad (21)$$

Profits are proportional to total sales per firm, the proportionally factor being directly related to the markup rate.

Combining the R&D technology (6), the price rule (10), equations (12), (19), and the free entry condition, we find that the patent value is constant when there is positive research:

$$v_t = \frac{\alpha}{A}, \quad \text{with} \quad \dot{n}_t \geq 0 \quad (22)$$

Moreover, R&D returns are equal to

$$r_{t+d} = \frac{1 - \alpha}{\alpha} \frac{c_{t+d}}{n_t} A. \quad (23)$$

Combining equation (18) with (23) leads to the equilibrium Euler equation

$$\frac{\dot{c}_t}{c_t} = \underbrace{\frac{1 - \alpha}{\alpha} \frac{c_{t+d}}{n_t} A}_{\text{R\&D return}} \underbrace{e^{-\rho d} \left(\frac{c_t}{c_{t+d}} \right)}_{\text{discount factor}} - \rho = \frac{1 - \alpha}{\alpha} A e^{-\rho d} \frac{c_t}{n_t} - \rho. \quad (24)$$

The private return to R&D arrives after a period of length d . For this reason, it has to be discounted using the appropriate ratio of marginal utilities. Moreover, the private return to R&D is different from the social return, which is equal to A . Under log utility, the term in c_{t+d} cancels and the Euler-type equation does not depend on it, on the consumption over number of varieties ratio $\frac{c_t}{n_t}$.

On the other hand the other equilibrium equation can be found from the instantaneous budget constraint taking into account that the assets' market clearing condition is

$$a_t = v n_t \quad \Rightarrow \quad \dot{a}_t = v \dot{n}_t \quad (25)$$

since v_t is the constant found in equation (22). We then have the following definition.

Definition 1 DECENTRALIZED EQUILIBRIUM. *A decentralized equilibrium is a path (c_t, n_t) , for $t \geq 0$, verifying the feasibility condition*

$$\dot{n}_t = A(n_{t-d} - c_t), \quad t \geq 0, \quad (26)$$

the Euler-type equation

$$\frac{\dot{c}_t}{c_t} = \frac{1 - \alpha}{\alpha} A e^{-\rho d} \frac{c_t}{n_t} - \rho, \quad t \geq 0, \quad (27)$$

the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_{++})$, the transversality condition

$$\lim_{t \rightarrow \infty} n_t c_t^{-1} e^{-\rho t} = 0, \quad (28)$$

and the irreversibility constraint $\dot{n}_t \geq 0$ for $t \geq 0$.

Remark 1 *The system of the two equations (26)-(27) is composed by one linear DDE, i.e. (26), and one nonlinear ODE, i.e. (27). For this reason, given any couple of initial conditions (\bar{n}_t, \bar{c}_0) with $\bar{n}_t \in C([-d, 0]; \mathbb{R}_{++})$, and $\bar{c}_0 > 0$ such system admits a unique local solution⁹ (applying e.g. [38], Theorem 3.1, p.209 and Proposition 6.1, p.233).*

It is worth noting that in our setting the Euler equation is an ODE and so the past values of consumption do not play any role in determining the solutions.¹⁰ In particular, at time $t = 0$, which is the date at which the first consumption-investment decision is taken, only $c(0)$ enters the Euler equation. Therefore the history of $c(t)$ for t in $[-d, 0)$ does not affect the solution. Consequently, a discontinuity at $t = 0$ in the variable $c(t)$, even when it exists, does not require any special treatment.

Interestingly enough, this is not always the case as pointed out by d’Albis and Augeraud-Veron [6] in an overlapping generation framework where the Euler equation in terms of aggregate consumption (p.464, eq. (15) of [6]) contains also terms referring to the consumption of past generations. For this kind of problems a discontinuity between $\bar{c}_t \in [-d, 0)$ and \bar{c}_0 may emerge and has to be accounted for. Therefore more sophisticated existence and uniqueness results, such as those in d’Albis et al. [7] should be considered.

4 Balanced Growth Paths

A balanced growth path (BGP) is defined as a solution of the system (26)-(27) such that, for a suitable $g \in \mathbb{R}_+$,

$$c_t = c_0 e^{gt}; \quad n_t = n_0 e^{gt}$$

for any $t \geq 0$. At a balanced growth path, the transversality condition (28) is automatically satisfied since $\rho > 0$, while the positivity of g guarantees that the irreversibility constraint is satisfied. So, a BGP is an equilibrium according to Definition 1.

From equation (24), for all $t \geq 0$, the consumption to knowledge ratio is constant

$$\frac{c_t}{n_t} = \frac{\alpha(g + \rho)e^{\rho d}}{(1 - \alpha)A}, \quad (29)$$

while the growth rate g is the unique real solution of the transcendental equation

$$Ae^{-gd} - g = \frac{\alpha(g + \rho)e^{\rho d}}{1 - \alpha}, \quad (30)$$

which is found by substituting (29) into (26). Let denote it by g_e . Such solution is positive under the following parametric condition:

$$A \geq \frac{\alpha \rho e^{\rho d}}{1 - \alpha} =: A_{\min}^e, \quad (31)$$

⁹The solution in general is not global: it is defined up to a maximal time t_1 , is continuous in $[-d, t_1)$ and differentiable in $(0, t_1)$. However, in the cases we are interested in (namely when periodic solutions emerge around a BGP), solutions are global since they stay away from the singular point $n = 0$.

¹⁰When the externality parameter ν is different from one the Euler equation becomes an advanced differential equation (see Section 6 and Appendix B).

which we will assume to hold from now on. Note that the strict inequality in (31) implies strict positivity of g_e . Moreover a straightforward application of the implicit function theorem on (30) shows that $\frac{\partial g_e}{\partial A} > 0$, and $\frac{\partial g_e}{\partial \alpha} < 0$, implying that both more productive economies and economies with larger markups grow faster along a BGP. It can be also easily verified that $\frac{\partial g_e}{\partial d} < 0$, meaning that longer innovation delays implies a lower (asymptotic) growth rate.

We summarize the discussion in the following proposition.

Proposition 1 *The solution of the system (26)-(27) is a BGP with positive growth if and only if the conditions below are satisfied:*

- i) inequality (31) holds;*
- ii) the growth rate is given by the unique positive solution g_e of (30);*
- iii) the initial condition \bar{n}_t has the form $\bar{n}_t = n_0 e^{g_e t}$ with $n_0 > 0$ and $t \in [-d, 0]$;*
- iv) given n_0 , c_0 is the solution of (29) for $t = 0$.*

For any given $n_0 > 0$ a BGP exists and is unique.

Therefore positive endogenous growth depends, as usual, on the presence of constant returns to scale in the production function of new varieties and on a productive enough *R&D* sector. In the following we describe an alternative equilibrium without *R&D* and then no economic growth. If the *R&D* sector is inactive because the up-front cost are too high, $v_t < \frac{\alpha}{A}$, then all the labor force is employed in the intermediary sector, $L_t = 1$, and the number of varieties remains constant over time, $\dot{n} = 0$. From equation (5) we have that $c_t = n_{t-d}$ and then the intermediary firms' profit becomes $\pi_t = 1 - \alpha$. Assuming an initial constant history of $a_0(t)$, the Euler equation pins down a *R&D* returns equal to $r_{t+d} = \rho e^{\rho d}$; then the rate of return is constant and greater than the discount factor, namely $r > \rho$. The solution of the no-arbitrage condition (11) is

$$v_t = \left(v_0 - \frac{\pi}{r} \right) e^{rt} - \frac{\pi}{r}.$$

Since $r > \rho$ the transversality condition $\lim_{t \rightarrow \infty} v_t \frac{n_t}{c_t} e^{-\rho t} = 0$ holds if $v_t = \frac{\pi}{r} = \frac{1-\alpha}{\rho e^{\rho d}}$. Combining this with the fact that the up-front cost are too high, we find the following condition

$$A < \frac{\alpha \rho}{1 - \alpha} e^{\rho d}.$$

In this context all the variables remain constant over time and therefore the economy is at a steady state. The last inequality is exactly the opposite of inequality (31) necessary for having positive growth.

5 Transitional Dynamics

The next key step is to analyze the dynamic behavior of the decentralized equilibrium around a BGP, i.e., when the initial condition \bar{n}_t , with $t \in [-d, 0]$, is sufficiently close but different

from those in Proposition 1, point *iii*). In particular, the objective is to study analytically the occurrence of periodic orbits through a Hopf bifurcation.

The technique to prove existence of a Hopf bifurcation is well known in the case of finite dimensional ODEs and has been generalized to infinite dimensional systems (including DDEs') starting with Crandall and Rabinowitz [36]. For the case of DDEs' systems, the main theoretical results are in Diekmann et al. [38] (Chapter X, in particular Theorem 2.7 p. 291 and Theorem 3.9, p.298) and Hassard et al.[52] (Chapter 4, Section 2).

Existing results (such as Theorem 2.7, p.291 of Diekmann et al. [38]) could be applied to prove the occurrence of a Hopf bifurcation in the nonlinear system under study, if it were not the case that the characteristic equation (35) (see Proposition 2, below) has always a zero real root. This (spurious) zero root emerges from the endogenous growth nature of the model, so there is an entire line (i.e. a one dimensional vector subspace) of equilibrium points that yields the existence of the zero eigenvalue. When a zero eigenvalue arises the bifurcation picture is, usually, much more complex and, to the best of our knowledge, even the specialized literature did not consider cases, such as the one here, where the dynamical system is 1-homogeneous and infinite dimensional.

Indeed, even restricting the attention to finite dimensional ODEs systems, there are very few papers in the endogenous growth literature which have studied the raising of periodic orbits through a Hopf bifurcation. As far as we know, the main contributions are Benhabib and Perli [21], Ben-Gad [17], and Greiner and Semmler [48]. All of them prove existence of a Hopf bifurcation in two steps: firstly, they reduce the dimension of the original (finite-dimensional) system by a change of variables; the new variables are ratios of the original ones and are constant on the BGP; secondly they write the Jacobian of the reduced system to find conditions on the parameters under which a pair of purely imaginary eigenvalues emerges.¹¹

In this paper, we deal with the problem of the spurious zero root and we prove the existence of a Hopf bifurcation through the following procedure which represents a new contribution to the existing, previously mentioned, literature. Firstly, we consider the detrended and linearized version of the system (26)-(27) around a steady state (equations (32)-(33) below).¹² Secondly, we look at the associated characteristic equation and we prove that, under suitable conditions on parameters, the spectrum of the roots of such equation displays a couple of simple purely imaginary roots which "crosses transversally" the imaginary axis. In particular, we choose d as our bifurcation parameter and the purely imaginary roots emerge moving d . Thirdly, we use the fact that the dynamics of our nonlinear (infinite-dimensional) system (32)-(33), represented by the function f in equation (96) below, is a homogeneous function of degree one to perform a suitable dimensional reduction (explained in the proof of Theorem 1 in Appendix E) which

¹¹In these contributions, this check is not always done analytically but, sometimes, only numerically. Also, the "transversal crossing condition" and the stability of the periodic orbits are not checked, and the dynamics of the original variables is derived only numerically. Finally the inequalities constraints are checked only numerically.

¹²This is based on the fact that, under some conditions, the local dynamic properties of the nonlinear system are strictly related to the behavior of the linearized system.

enables us to “throw away” the zero eigenvalue. Then, we apply standard Hopf bifurcation theory and prove that a Hopf bifurcation generically occurs and that the ratios $\frac{c_t}{n_t}$ and $\frac{n_{t-d}}{n_t}$ are periodic functions.¹³ Finally, in line with the previously mentioned literature, we show numerically the existence of persistent cycles in the original variables, c_t , and n_t , and that the inequality constraints (in our case $\dot{n}_t \geq 0$) are indeed respected.¹⁴ This quantitative analysis is done in Section 7 where we show that our model indeed replicates the U.S. medium-term cycles.

A full analytical assessment of the role of the transversality condition is also performed. It is, indeed, worth noting that the Hopf bifurcation will give rise to admissible equilibrium trajectories only if the transversality condition (28) is satisfied: this turns out to be true for every initial condition \bar{n}_t close to a BGP if the complex roots “crossing” the imaginary axis are the ones with biggest real part (except for one which has to be “killed” by the proper choice of the initial consumption c_0).¹⁵

5.1 The Linearized System

We now start writing the detrended and linearized version of the system (26)-(27) around a steady state. Define $\tilde{x}_t = x_t e^{-g_e t}$, $x_t = \{c_t, n_t\}$, with \tilde{c}_t, \tilde{n}_t representing detrended consumption and detrended knowledge capital, respectively. Equations (26)-(27) then become

$$\dot{\tilde{n}}_t = A(\tilde{n}_{t-d} e^{-g_e d} - \tilde{c}_t) - g_e \tilde{n}_t \quad (32)$$

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1-\alpha}{\alpha} A e^{-\rho d} \frac{\tilde{c}_t}{\tilde{n}_t} - (\rho + g_e). \quad (33)$$

By linearizing the Euler-type equation (33) around a steady state and using (29), we get

$$\dot{\tilde{c}}_t = (g_e + \rho)\tilde{c}_t - \frac{(g_e + \rho)^2 \alpha e^{\rho d}}{A(1-\alpha)} \tilde{n}_t. \quad (34)$$

Existence and uniqueness of a continuous solution for the linearized system of delay differential equations (32)-(34) is guaranteed, for example, by Theorem 6.2 page 171 in Bellman and Cooke [16]. Moreover, the characteristic equation associated to such linearized system is (see e.g. Kolmanovskii and Nosov [57], p.50, or Hale and Lunel [51], p.198)¹⁶

$$h(\lambda) := \lambda^2 - \rho\lambda - \lambda A e^{-(g_e + \lambda)d} + A(g_e + \rho)e^{-(g_e + \lambda)d} - A(g_e + \rho)e^{-g_e d} = 0. \quad (35)$$

¹³Such bifurcation will be called “generic”, meaning that it arises for “almost all” the values of the parameters such that purely imaginary roots arise and is obtained moving only one parameter in a neighborhood of the bifurcation locus, keeping the others fixed. The exact meaning of the word “generic” involves topological considerations which are described e.g. in the book of Ruelle [69], Section 8.7, p.44, Section 9.2 p.58 and Section 13, p.74.

¹⁴Therefore the irreversibility constraint is checked to hold ex post and only numerically. A full analytical investigation of the role of the irreversibility constraint is left for future research.

¹⁵To get this result we have to prove the following: i) under which conditions the transversality condition is satisfied in the linearized system: this fact is non trivial and is done in Propositions 7 and 8 of Appendix D; ii) how this result can be transferred to the nonlinear system: this is done in Proposition 9 (the last one of Appendix D) and used in the proof of Theorem 1.

¹⁶To get such form of the characteristic equation we use equation (30).

The set of roots of such characteristic equation is also called the *spectrum* of the linearized system (32)-(34). As explained above the spectrum plays a central role in our analysis and for this reason we devote the next subsection to investigate its properties.

5.2 The Roots of the Characteristic Equation

We start with the following result.

Proposition 2 *Assume that (31) holds. Then the characteristic equation (35) has countably many solutions described as follows:*

- (i) *two real roots: $\lambda_0 > g_e + \rho > 0$ and $\lambda_1 = 0$; such roots are simple;*
- (ii) *at most a countable set of complex roots of the form $\mu_r \pm i\eta_r$ ($r \geq 2$) where $\mu_r \in \mathbb{R}$, $\eta_r > 0$ for every $r \geq 2$. Such roots must have bounded real part and are then ordered taking $\mu_2 \geq \mu_3 \geq \mu_4 \geq \dots$. If, for some r_0 we have $\mu_{r_0} = 0$ (i.e. there exist purely imaginary roots) then for every other $r \neq r_0$ we have $\mu_r \neq 0$. Moreover, the purely imaginary roots $\pm i\eta_{r_0}$ are simple.*

Proof. See Appendix C. ■

Note that, in the above statement, we prove that a zero root always exists. As previously explained, this fact is an intrinsic property of our model and standard results in the literature on the occurrence of a Hopf bifurcation cannot be directly applied.

Now we proceed to study the complex roots of the spectrum under assumption (31). As already said above what is important for our purposes is to know, depending on the parameters of the problem, when a couple of complex roots has zero real part. In particular, we want to see when this happens for the conjugate complex roots with biggest real part.

The parameters of the problem are ρ , α , A and d with the restrictions $\rho > 0$, $\alpha \in (0, 1)$, $d > 0$ and (31) holds. Call E the subset of \mathbb{R}_{++}^4 where such restrictions are satisfied. Then, following, for example, Kolmanovskii and Nosov [57], p.55, we define the D-Subdivision D_j as the set of the points $(\rho, \alpha, A, d) \in E$, such that the characteristic equation (35) has j and only j roots (counted with their multiplicity) with strictly positive real part. We have the following result concerning the region of our interest: D_1 and D_3 .

Proposition 3 *The subdivisions D_1 and D_3 are nonempty regions in E with nonempty interior.*

Proof. See Appendix C. ■

Let us choose d as the bifurcation parameter. Proposition 4 shows that, starting from a “generic” point $(\rho_H, \alpha_H, A_H, d_H) \in E$ which lies on the boundary between D_1 and D_3 , namely on $\partial D_1 \cap \partial D_3$ with ∂ indicating the boundary of the sets, and moving the parameter d in a neighborhood of d_H , the couple of complex roots with the biggest real part crosses transversally the imaginary axis, i.e. the real part has nonzero derivative with respect to d in d_H .¹⁷

¹⁷It is worth noting that ∂D_1 intersects also with ∂D_0 when, for example, $A = A_{min}$; for this reason we have specified the boundary between D_1 and D_3 with $\partial D_1 \cap \partial D_3$.

Proposition 4 Consider a point $(\rho_H, \alpha_H, A_H, d_H) \in \partial D_1 \cap \partial D_3 \subset E$. Let d be in a sufficiently small neighborhood I of d_H and write $\mu_r(d)$, $\eta_r(d)$ to denote the real and imaginary part of the complex roots as functions of $d \in I$ when $\rho = \rho_H$, $\alpha = \alpha_H$ and $A = A_H$. Then $\mu_2(d)$ is $C^1(I)$, $\mu_2(d_H) = 0$ and, generically,¹⁸ we have

$$\mu_2'(d_H) \neq 0.$$

Moreover we have $\mu_2'(d_H) > 0$ if the parameters $(\rho_H, \alpha_H, A_H, d_H)$ satisfy the inequalities (dropping the subscript H for simplicity of writing)

$$2\eta + Ae^{-g_e d} [(1 + d(g_e + \rho)) \sin(\eta d) + d\eta \cos(\eta d)] > 0 \quad (36)$$

$$Ae^{-g_e d} [a - (a + \eta^2) \cos(\eta d) + \eta b \sin(\eta d)] > 0 \quad (37)$$

where $\eta := \eta_2$ is the imaginary part of the purely imaginary root, g_e is the unique real solution of (30) and where

$$a = (d(g_e + \rho) - 1) \frac{\partial g_e}{\partial d} + g_e(g_e + \rho), \quad b = d \frac{\partial g_e}{\partial d} - \rho.$$

Proof. See Appendix C. ■

The two inequalities in the Proposition are satisfied if the point $(\rho_H, \alpha_H, A_H, d_H)$ is taken in a neighborhood of the specification given in our numerical exercise and precisely set as in (45).

5.3 Hopf Bifurcation

Since prevailing theory cannot be directly applied to the system (32)-(33), existence of a Hopf bifurcation has to be proved. Theorem 1 below uses the results in the previous section to show it for the bifurcation parameter d .

Theorem 1 Consider a point $(\rho_H, \alpha_H, A_H, d_H) \in \partial D_1 \cap \partial D_3 \subset E$.

(i) A Hopf bifurcation generically occurs in $d = d_H$ for the projection of system (32)-(33) in a subspace of codimension 1, i.e. for such a projected system, there exists a family of periodic orbits $p_d(t)$ for d in a right or left neighborhood of d_H . The period T of these orbits tends to $\frac{2\pi}{\eta_2}$ as $d \rightarrow d_H$.

(ii) On such orbits the ratios

$$\frac{\tilde{c}_t}{\tilde{n}_t}, \quad \frac{\tilde{n}_{t+s}}{\tilde{c}_t}, \quad \text{and} \quad \frac{\tilde{n}_{t+s}}{\tilde{n}_t}, \quad \forall s \in [-d, 0]$$

are all periodic functions.

(iii) The family of orbits $p_d(t)$ satisfies the transversality condition (28).

¹⁸Here in the sense that this happens for every point $(\rho_H, \alpha_H, A_H, d_H) \in \partial D_1 \cap \partial D_3$ except for a set with empty interior in the topology of $\partial D_1 \cap \partial D_3$.

Proof. See Appendix E. ■

Since we cannot prove directly the occurrence of a Hopf bifurcation using the existing literature (see e.g. the already quoted Theorem 2.7, p.291 in Diekmann et al. [38]), we developed a method to “throw away” the zero root of the characteristic equation. We do this by studying a suitably projected system (which lives in a space of codimension 1) where the above theorem can be applied; we indeed prove that periodic orbits $p_d(t)$ do emerge in this reduced system. This procedure works heavily using the 1-homogeneity of the dynamic system, f (see (96)).

Moreover, the proof that the periodic orbits satisfy the transversality condition (28) is done by showing first that this holds for the linear system and then for the nonlinear system as well by using the theory developed in Diekmann et al. [38] (Chapters VII-VIII-IX). See Appendix D for details.¹⁹

The above result shows that permanent cycles may arise in an endogenous growth model with implementation delays through a generalization of the Hopf bifurcation theory. In Section 7, we also study the quantitative relevance of this finding, and we show that periodic orbits arise for realistic parameter values, and that, under this parametrization, the variable c_t and n_t persistently oscillate around the BGP. It is in this sense that our results are in line with Schumpeter’s statement that “wave-like fluctuations in business are the form economic development takes in the era of capitalism.”

5.4 Wave-Like Fluctuations

This section provides an intuitive explanation on the dynamical behavior of the model, in particular why in some cases the equilibrium paths converge by damping oscillations and in others permanent fluctuations may arise.²⁰ Let us use (30) to rewrite the equilibrium system (33)-(32) as

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{Ae^{-g_e d} - g_e}{A} \right) \frac{(\rho + g_e)A}{Ae^{-g_e d} - g_e} \quad (38)$$

$$\frac{\dot{\tilde{n}}_t}{\tilde{n}_t} = - \left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{Ae^{-g_e d} \tilde{n}_{t-d}/\tilde{n}_t - g_e}{A} \right) A. \quad (39)$$

From (30), $Ae^{-g_e d} - g_e > 0$, implying that the sign of $\dot{\tilde{c}}_t/\tilde{c}_t$ (resp. $\dot{\tilde{n}}_t/\tilde{n}_t$) depends positively (negatively) on the right hand side parentheses. Notice that both parentheses differ only on the $\tilde{n}_{t-d}/\tilde{n}_t$ term, reflecting the fact that current changes in technology take a delay d to be adopted.

¹⁹Our analysis has focused on the existence of a Hopf bifurcation and therefore of periodic orbits. However our results also imply local determinacy of the steady state as long as $(\rho, \alpha, A, d) \in D_1$. Bambi and Licandro [12] investigate the possibility of local indeterminacy in a time-to-build setting.

²⁰Damping oscillations happens when the economy is too far from the frontier between D_1 and D_3 in the sense previously explained; on the other hand, permanent fluctuations refer to the periodic orbits which emerge through a Hopf bifurcation when the economy is sufficiently close to the frontier between D_1 and D_3 .

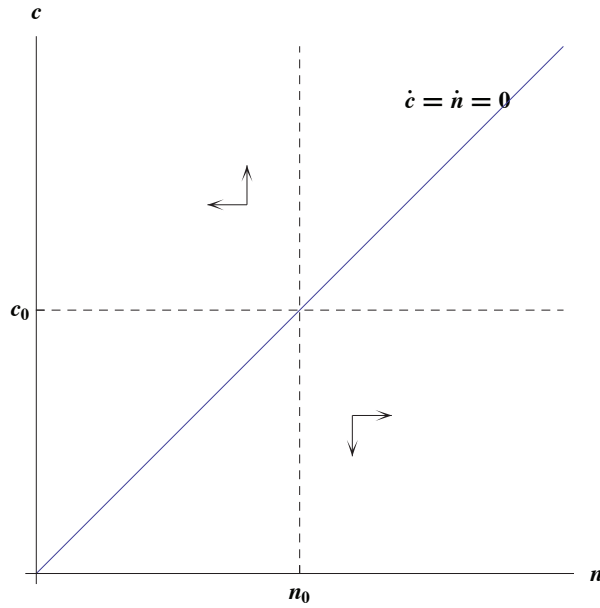


Figure 1: Phase diagram $d = 0$

We begin the analysis looking at the endogenous growth economy without implementation delays, i.e. the case $d = 0$. In this case, the right hand side parentheses on (38)-(39) become

$$\left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{A - g_e}{A} \right),$$

the same in both equations. Figure 1 represents the behavior of the economy for any feasible (A, ρ, α) , with g^e given by (30) under $d = 0$. The loci $\dot{\tilde{c}}_t = 0$ and $\dot{\tilde{n}}_t = 0$ are identical and the system diverges when (n_t, c_t) is not in these loci. As well-know in endogenous growth theory, for a given n_0 , the initial consumption has to be c_0 making the economy jump to steady state at the initial time $t = 0$.

When d is strictly positive, the system (38)-(39) changes its nature and a phase diagram cannot be used to study global dynamics – see the Appendix D for a formal analysis of the linearized system. However, the phase diagram in Figure 2 will help us understanding the oscillatory behavior of the economy. Notice that, $\tilde{n}_{t-d}/\tilde{n}_t$ in (39) is usually different from unity. It means that depending on the state of the cycle, $\dot{\tilde{n}}_t = 0$ may be above or below $\dot{\tilde{c}}_t = 0$. When it is below, a third region shows up in Figure 2, in which the system moves south-west. Notice that for a given n_{t-d} , the fact that n_t is reducing, tends to move the $\dot{\tilde{n}}_t = 0$ locus up. In the opposite case, the system moves north-east and the $\dot{\tilde{n}}_t = 0$ locus tends to move down. An equilibrium path, for given initial conditions, will tend to move then cyclically.

Existence of permanent cycles crucially depends on the implementation delay. For a relatively small d , the ratio $\tilde{n}_{t-d}/\tilde{n}_t$ tends to be close to unity, implying that oscillations dump and the

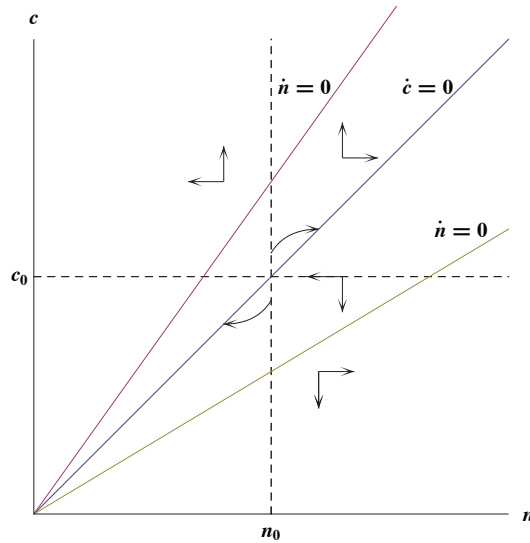


Figure 2: The behavior of the system under $d > 0$

economy converges to its steady state. As far as d increases, converging to d_H , fluctuations persist for a longer time and they tend to be permanent. In a permanent fluctuation equilibrium, the economy moves around its steady state.

Of course, the choice of the implementation delays has to respect the conditions found in the previous theorems. If this is not the case, then the economy faces explosive fluctuations and the irreversibility constraint will be violated in a finite time.

Initial conditions determine the amplitude of the cycle. In the extreme case where the detrended initial conditions are constant, the term $\tilde{n}_{t-d}/\tilde{n}_t$ is one irrespective of the value of parameters and the economy will behave as in Figure 1, meaning that it will jump to steady state at the initial time. In the general case of initial conditions close to but different from those, putting the economy on a BGP, the economy presents a cycle in the variable $\frac{c_t}{n_t}$ which correspond, as it is shown in the quantitative exercise, to a persistent cycle of $(\tilde{n}_t, \tilde{c}_t)$ whose amplitude depends on the amplitude of the initial conditions.

6 Robustness of our results for externality parameter $\nu \neq 1$

In this section, we extend our previous results to the case of $\nu \neq 1$, but close to one. Notice that it corresponds to a two-sector economy, where c and n are produced by the mean of different technologies. The first step to accomplish this task consists in rewriting equations (26) and (27) and the transversality condition (28) for ν in $[0, +\infty)$. The procedure to do it is sketched in

Appendix B. The resulting equations are

$$\dot{n}_t = An_{t-d} \left(1 - \frac{c_t}{n_{t-d}^\nu} \right), \quad (40)$$

$$\frac{\dot{c}_t}{c_t} = \left[\frac{1-\alpha}{\alpha} A \frac{c_t}{n_t^\nu} + (\nu-1) \frac{\dot{n}_t}{n_t} \frac{c_t}{c_{t+d}} \right] e^{-\rho d} - \rho, \quad (41)$$

with the transversality condition

$$\lim_{t \rightarrow +\infty} n_{t-d}^{\nu-1} n_t c_t^{-1} e^{-\rho t} = 0. \quad (42)$$

Clearly, in this more general case, the definition of decentralized equilibrium (Definition 1 above) is the same after having substituted equations (26), (27) and (28) with (40), (41) and (42), respectively.

The first relevant difference with the case $\nu = 1$ is that a balanced growth path may exist but now consumption and the number of varieties do not grow at the same rate anymore.²¹ In particular a balanced growth path (BGP) is now defined as a solution of the system (40) and (41) such that, for a suitable $g \in \mathbb{R}_+$

$$c_t = c_0 e^{\nu g t} \quad n_t = n_0 e^{g t} \quad \forall t \geq 0.$$

As for the case $\nu = 1$ any BGP is an equilibrium (according to Definition 1 suitably modified as explained just above) since the positivity of g guarantees that the irreversibility constraint is satisfied and the transversality condition (42) is automatically true by performing a simple substitution.

From the Euler equation (41)

$$\frac{c_t}{n_t^\nu} = \frac{\alpha}{(1-\alpha)A} \left[(\nu g + \rho) e^{\rho d} - (\nu-1) g e^{-g \nu d} \right]. \quad (43)$$

Evaluating equation (40) at a BGP and substituting it into the last equation, allow us to find the following equation for a BGP growth rate

$$-g + A e^{-g d} - \frac{(\nu g + \rho) \alpha}{1-\alpha} e^{[g(\nu-1)+\rho]d} + \frac{\alpha(\nu-1)}{1-\alpha} g e^{-g d} = 0. \quad (44)$$

We have the following result concerning the solutions of (44).

Lemma 1 *Assume that (31) is satisfied. Then for any $\nu \geq 0$ equation (44) has exactly one positive root $g_{e,\nu} \geq 0$. This root is zero if and only if (31) holds with equality.*

Proof. See Appendix F. ■

The above discussion and the result of Lemma 1 allows us to restate Proposition 1 as follows.

²¹On the other hand, consumption, output, and investment (i.e. $\dot{a}_t = \dot{v}_t n_t + v_t \dot{n}_t$) continue to grow at the same pace. For this reason we will continue to use the term balanced growth path.

Proposition 5 *The solution of the system (40)-(41) is a BGP if and only if the conditions (i)–(iv) of Proposition 1 holds with (30) and (29) replaced respectively by (44) and (43). Moreover if (31) holds then for any given $n_0 > 0$ a BGP exists and is unique.*

Proof. It follows immediately from Lemma 1 and the definition of a BGP. ■

Moreover a straightforward application of the implicit function theorem on (44) shows that $\frac{\partial g_e}{\partial \nu} < 0$ i.e. the positive growth rate of the economy decreases if the externality increases; this result is not surprising because a higher externality implies higher return on investment and then higher future consumption. Therefore in front of a high externality the agents have an incentive to invest less with an obvious negative effect on the growth rate of the economy.

We now look at the robustness of our main result on the existence of periodic solutions (Theorem 1) when $\nu \neq 1$.²²

Theorem 2 *There exists a neighborhood J of $\nu = 1$ such that, for every $\nu \in J$, one can define two nonempty regions $D_{1,\nu}, D_{3,\nu} \subset E$. Given a point $(\rho_H, \alpha_H, A_H, d_H) \in \partial D_{1,\nu} \cap \partial D_{3,\nu}$ we have that:*

(i) *A Hopf bifurcation generically occurs in $d = d_H$ for the projection of the detrended version of system (40)-(41) in a subspace of codimension 1, i.e. there exists, for such such projected system, a family of periodic orbits $p_d^\nu(t)$ (of period T which tends to $\frac{2\pi}{\eta_{2,\nu}}$ as $d \rightarrow d_H$) arising for d in a right or left neighborhood of d_H .*

(ii) *On such orbits the ratios*

$$\frac{\tilde{c}_t}{\tilde{n}_t^\nu}, \quad \frac{\tilde{c}_{t+s}}{\tilde{c}_t}, \quad \text{and} \quad \frac{\tilde{n}_{t+s}}{\tilde{n}_t}, \quad \forall s \in [-d, 0]$$

are all periodic functions.

(iii) *The family of orbits $p_d^\nu(t)$ satisfies the transversality condition (28).*

Proof. See Appendix F. ■

It is worth mentioning that the structure of the method to prove Theorem 2 is very similar to the one for the case $\nu = 1$. Let us write the detrended and linearized version of the system (40)-(41), study the associated characteristic equation, and then apply the same arguments used to prove Theorem 1. The big differences with the case $\nu = 1$ are two: first the dynamics of the system is not any more a homogeneous function; second the Euler equation becomes an advanced differential equation and so the characteristic equation admits infinitely many complex solutions with positive real parts. In the Appendix F we briefly explain how to get rid of such difficulties.

²²Since a complete proof of Theorem 2 requires a lot of space (several very hard technical difficulties emerge in this case), Appendix F only gives a sketch of the proof.

7 Quantitative Analysis and Medium-Term Movements

In this section, we undertake a quantitative exercise to show that the conditions required for our economy to be on a permanent cycle equilibrium are quantitatively sensible. For this purpose, we set the model parameters to

$$d = 8.2, \quad \rho = 0.03, \quad \alpha = 0.9, \quad \nu = 1 \quad \text{and} \quad A = 0.786, \quad (45)$$

which allows us to replicate some key features of the US economy. The adopted value of d is consistent with Mansfield's estimations, and $\alpha = .9$ is in line with estimated markups in Basu and Fernald [14], implying a markup rate of 11%. By setting $\rho = .03$, A was chosen for the growth rate g_e to be equal to 2.4% as in Comin and Gertler [31]. Crucially, the model not only matches the US long run growth but the economy is in D-Subdivision D_1 , very close to the Hopf bifurcation.

We use the software DDE-BIFTOOL developed by Engelborghs and Roose [42] to compute the subset of the rightmost roots of the characteristic equations (35). The spectrum of roots is represented in Figure 3. As said above, the detrended system has a spurious zero root and a strictly positive real root, the latter being ruled out by the transversality condition as proved in Theorem 1. Under this parametrization, the spectrum shows two conjugate complex roots very close to the imaginary axis, all the other conjugate roots having strictly negative real part. According to our parametrization, we have that $\eta_2 = 0.57$ and the numerical value of the two inequalities (36) and (36) of Proposition 4, is respectively 0.1135 and 0.0264.

A slight increase of the innovation delay d and these two roots cross the imaginary axis; when this happens a periodic orbit of the projected system, and so of $\frac{c_t}{n_t}$, emerges through a Hopf bifurcation, consistently with Theorem 1. To quantitatively simulate the dynamics in the original variables, c_t and n_t , we need to specify the initial condition.

To set the initial conditions, we assume that during the years 1948 to 1959 the US economy faced medium term movements similar to those estimated by Comin and Gertler for the same period. We interpret it as the US adjusting to the new economic environment emerging after World War II. Initial (detrended) conditions are represented by the trigonometric function $\bar{n}_t e^{-g_e t} = 1 + a \cos(bt/\pi)$, where parameter a is set to 0.0375 and parameter b to 20/11 for the amplitude be close to 8%; the simulated path is a persistent cycle of periodicity 11 years; this value is close to period T in point (i) of Theorem 1; using the information on the spectrum of roots, we have indeed found a $T = 11.21$.²³

To compute the numerical solution, we use the strategy proposed by Collard et al [30], which combines the method of steps suggested by Bellman and Cooke [16] with a shooting algorithm –see Judd [54]. We apply this strategy to the nonlinear system (33)-(32). The solution for detrended output, measured as $A\tilde{n}_{t-d}$ and normalized to turn around zero, is represented in

²³The particular choice $n_0 = 1$ comes without any loss of generality, since the profile of the solution does not depend on the level of the state variable, as usual in endogenous growth models, but on the profile of the initial conditions.

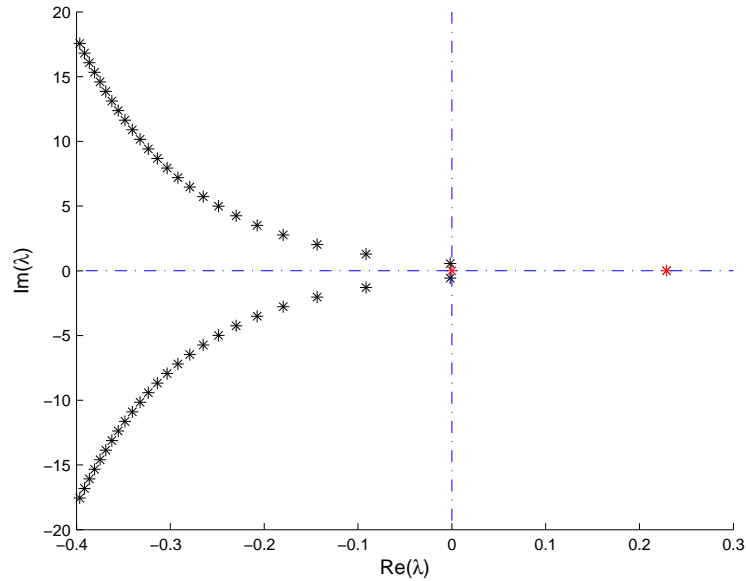


Figure 3: Spectrum of Roots.

Figure 4. The decentralized equilibrium converges to a regular Juglar cycle with periodicity close to 11 years and an amplitude of around 8 percentage points.²⁴ The periodicity of the cycle depends on the model's parameters, in particular on the implementation delay d , but the amplitude of the cycle crucially depends on the amplitude of the initial conditions. Given that initial conditions are periodic with a periodicity close to the permanent cycle period, the economy converges to it very fast. The first recession and boom reflect the behavior of initial conditions, and maps on the following regular recessions around 1973, 1984, 1995 and 2006.

As can be observed in Figure 4, the approximately 11-years period of the solution is larger than the 8.2-years implementation delay. In facts, it is easy to see that the implementation delay has to be close to three fourth of the cycle period. Figure 5 represents the stationary solution for a period just larger than a cycle. Let for example the economy be at the boom at time t , with c_t at its maximum level. Consequently, \dot{c}/\tilde{c} has to be zero at time t . From the Euler-type equation, to \dot{c}_t be zero, c_t/n_t has to be at its stationary value. As can be observed in Figure 5, equilibrium output crosses zero around $t + d$, meaning that c_{t+d}/n_t is around its stationary value. Consequently, d has to be close to $3/4$ of the the 11-years cycle period.

²⁴The inequality constraint, $\dot{n}_t \geq 0$ has been checked numerically to hold both at the initial conditions and on the simulated equilibrium path. It is also worth noting that the numerical exercise is consistent with the presence of a *stable* periodic orbit, i.e. the Hopf bifurcation being supercritical.

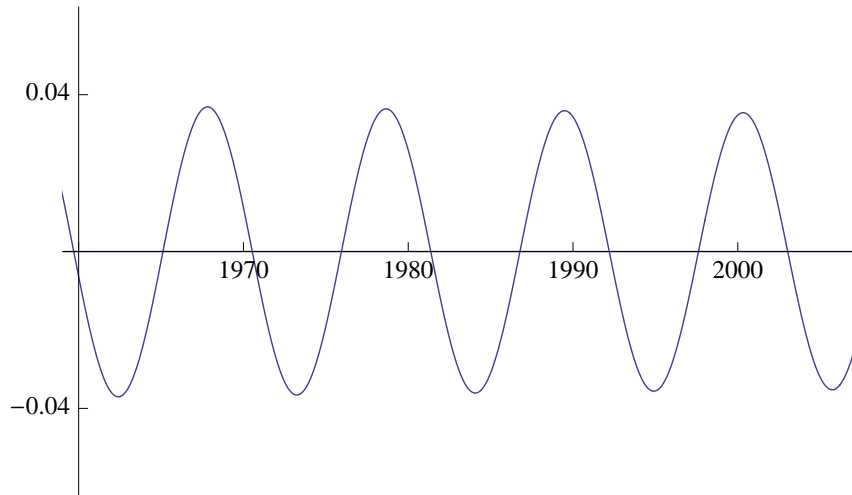


Figure 4: Simulated path for US output normalized around zero.

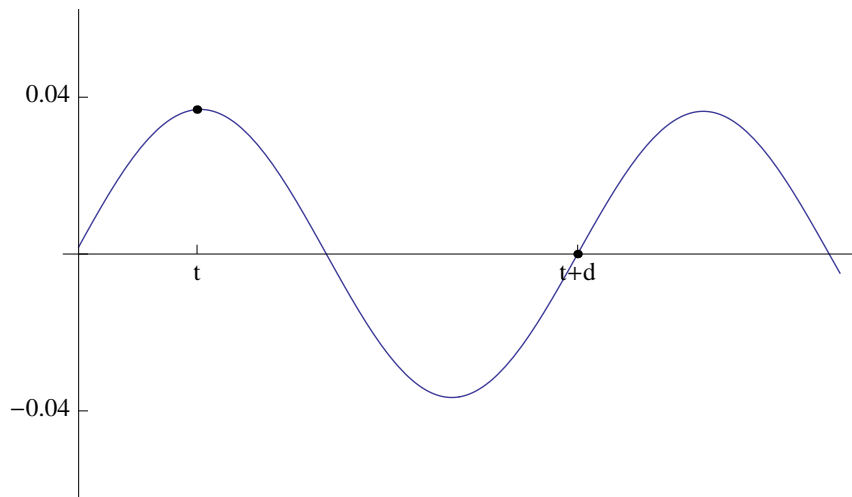


Figure 5: Periodicity of the cycle and the implementation delay.

8 Optimal Allocations and R&D Subsidies

An optimal allocation solves the following social planner problem

$$\max \int_0^\infty \log(c_t) e^{-\rho t} dt$$

subject to the feasibility constraint

$$\dot{n}_t = A(n_{t-d} - c_t), \quad (7)$$

the irreversibility constraint $\dot{n} \geq 0$ and the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, with \bar{n}_t the same as in the decentralized equilibrium. Notice that for $d = 0$ the variable change $\hat{c} = Ac$ renders this problem formally identical to the AK model in Rebelo [64].²⁵

Following Kolmanovskii and Myshkis [56] and operating as in the decentralized economy, optimality requires the Euler-type condition

$$\frac{\dot{c}_t}{c_t} = A e^{-\rho d} \frac{c_t}{c_{t+d}} - \rho, \quad (46)$$

and the transversality condition

$$\lim_{t \rightarrow \infty} n_t c_t^{-1} e^{-\rho t} = 0, \quad (47)$$

The social planner faces a trade-off between consuming at time t or saving and consuming at $t + d$. For this reason, in (46) the R&D productivity, A , is weighted by the ratio of marginal utilities of consuming at $t + d$ and t , which multiplied by $e^{-\rho d}$ represents the discount factor on a period of length d . It is useful to observe that, as in the AK model, the Euler-type mixed functional differential equation (46) does not depend on the state variable n . Consequently, since the social return to R&D is constant, the planner may allocate consumption over time without caring about the path of knowledge n . So the optimal consumption path is in its balanced growth path from time zero.²⁶ However, since initial conditions affect production from zero to time d , R&D has to adjust to fulfill the feasibility condition. This mechanism will repeat again and again making the optimal allocations to fluctuate, converging by damping oscillations under reasonable restrictions on parameters. This will be precisely shown in Proposition 6 below.

An optimal allocation is then a path (c_t, n_t) , for $t \geq 0$, verifying the mixed functional differential equations system (7) and (46), the transversality condition (47), the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, and the irreversibility constraint $\dot{n} \geq 0$. At a balanced growth path, from (46), consumption grows at the constant rate g solving

$$g + \rho = A e^{-(g+\rho)d}. \quad (48)$$

²⁵We implicitly assume that the solution is interior, meaning that the inequality constraint holds. Bambi et al [11] in a similar framework explicitly states the needed parameters' restriction.

²⁶To be more precise the path n only influence the choice of the initial consumption c_0 : this affect the size of the optimal consumption path but not its exponential form.

It is not difficult to show that the transcendental equation (48) has always a unique real solution which is strictly positive if and only if the following condition holds

$$A > \rho e^{\rho d} \equiv A_{\min}^*. \quad (49)$$

Note that, when $d = 0$, this condition collapses to the standard assumption in the AK model that $A > \rho$.

To study the transitional dynamics we need to look at the complex roots of the characteristic equation (which is a translation of the above (48))

$$h_0(z) := z - Ae^{-zd} = 0 \quad (50)$$

This analysis is done in Bambi et al. [11], Proposition 1. In particular, we use the fact that such equation has only one real root $z_0 = g + \rho$ and infinitely many simple complex roots whose real part is always negative if and only if $Ad < \frac{3\pi}{2}$. The next proposition presents the main properties of the transitional dynamics.

Proposition 6 *Assume that $A > A_{\min}^*$, then the optimal equilibrium paths for n_t and c_t are*

$$n_t^* = a_L e^{gt} + \sum_{j=1}^{+\infty} a_j e^{z_j t} \quad (51)$$

$$c_t^* = c_0 e^{gt} \quad (52)$$

where g is the unique real solution of (48), a_L and $\{a_j\}_{j=1}^{+\infty}$ are the residues associated to the complex roots $\{z_j\}_{j=1}^{+\infty}$ of the characteristic equation (50),

$$a_L = A \sum_{j=0}^{+\infty} \frac{c_0^*}{(z_j - g)h'(z_j)}, \quad a_j = \frac{\bar{n}_0 + z_j \int_{-d}^0 \bar{n}_s e^{-z_j s} ds}{h'(z_j)} - \frac{Ac_0^*}{(z_j - g)h'(z_j)} \quad (53)$$

and the initial value of consumption, c_0 , equals to

$$c_0^* = \frac{\rho}{A} \left(\bar{n}_{-d} + \int_{-d}^0 \dot{\bar{n}}_s e^{(g+\rho)s} ds \right). \quad (54)$$

Moreover if we assume $Ad < \frac{3\pi}{2}$ or $Ad > \frac{3\pi}{2}$ and $g > \max\{Re(z_j)\}_{j=1}^{\infty}$, then $n_t - a_L e^{gt}$ converges to 0 by damping oscillations.

Proof. See Appendix F. ■

Under log utility, consumption equals the return on wealth, the latter being represented by the term within brackets at the right hand side of (54) divided by the relative productivity A —see (7). Notice that initial wealth is the sum at time zero of the value of operative varieties n_{-d} plus the value of produced but still non operative varieties, i.e., those produced between $-d$ and zero. The factor $e^{(g+\rho)s}$, multiplying the mass of varieties $\dot{\bar{n}}_s$ created at time s , $s \in [-d, 0]$, discounts the varieties' value for the period still remaining until those varieties will become operative. The set of initial conditions which make the irreversibility constraint hold is characterized in Bambi et al [11].

8.1 Comparing Centralized and Decentralized Allocations

Optimal and equilibrium allocations differ in at least two dimensions. First, consumption is perfectly smoothed in the optimal allocation, but fluctuates at equilibrium. Second, the growth rates are different at the balanced growth path. We develop these two arguments below, before suggesting an optimal R&D policy.

The fact that consumption does not fluctuate at the optimal allocation comes from the same analytical argument used in Bambi [10] and Bambi et al. [11] while a deep discussion on the consumption smoothing mechanism can be found in Bambi et al. [11]. It is also worth noting that in the social planner case the system of MFDE describing the economy is a block recursive or triangular system where the advance and the delay parts are split. Therefore the social planner may and will decide a smooth path of consumption because the risk adverse agents always prefer a smooth consumption path to a path which alternates periods of high consumption to periods with low consumption.

Therefore consumption does not fluctuate at the optimal allocation. This is not the case at equilibrium, since the private return to R&D depends on future profits, which are a negative function of the future market share; this implies that the Euler equation depends also on the number of varieties and therefore the system describing the economy is no more block recursive. On the balanced growth path we have that

$$g_e \left(e^{-\rho d} + \frac{\alpha}{A(1-\alpha)} \right) + \frac{\alpha \rho}{A(1-\alpha)} = A e^{-(g_e+\rho)d}, \quad (30)$$

$$g + \rho = A e^{-(g+\rho)d} \quad (48)$$

where g_e and g represent the equilibrium and optimal growth rates, respectively. As shown in Appendix, there exist cutoff level for α ,

$$\underline{\alpha} \equiv \frac{g + \rho - g e^{-\rho d}}{2(g + \rho) - g e^{-\rho d}}, \quad (55)$$

$\underline{\alpha} \in (0, 1/2)$, such that the equilibrium growth rate, g_e , is equal to the optimal growth rate, g , iff $\alpha = \underline{\alpha}$. Equilibrium growth is smaller than optimal growth, i.e. $g_e < g$, iff $\underline{\alpha} < \alpha < 1$. Otherwise, it is larger. Remind that equilibrium profits are declining in α , which represents the inverse of the markup, and optimal growth does not depend on it.

This result is consistent with Benassy [18], who shows for $d = 0$ that the equilibrium growth rate is smaller than the optimal rate if and only if the knowledge externality, ν in equation (2), is small enough or, equivalently, the elasticity of substitution α is large enough. Since in our framework ν is assumed to be unity, let argue in terms of the elasticity of substitution for a given knowledge externality. For $d = 0$, $\underline{\alpha} = \left(1 + \frac{A}{\rho}\right)^{-1}$, meaning that there is a range of parameters for which the optimal growth rate is smaller than the equilibrium growth rate at the balanced growth path. Increasing α makes goods more substitutable, reducing markups, the return to R&D and the growth rate. Consequently, there is a degree of substitutability beyond which the optimal growth rate is larger than the equilibrium rate.

Since returns to private R&D are different from public returns, optimality may be restored by the mean of a time dependent subsidy/tax scheme imposed on current R&D investments or, equivalently, on the return to R&D. By comparing the equilibrium (24) and the optimal (46) Euler-type conditions, it is easy to see that private and public returns equalize when the subsidy rate is

$$1 + s_t = \frac{\alpha}{1 - \alpha} \frac{n_t}{c_{t+d}} = \underbrace{\frac{\alpha}{1 - \alpha} \frac{n}{c}}_{\text{constant}} \underbrace{\frac{n_t/c_{t+d}}{n/c}}_{\text{procyclical}},$$

where the stationary ratio n/c is defined in (29).

An optimal policy has two components. First, as in the expanding product variety model, it has to equalize the (average) private return to the (average) social return. The magnitude of it corresponds to the *constant* term in the equation above, which depends negatively on both the markup, $1/\alpha$, and the average market share of intermediary firms, c/n . Second, it has to compensate for the *countercyclical* fluctuations in the private return. The social return to R&D is constant and equal to A , but the private return fluctuates countercyclically, being small than the mean during expansions and large during contractions –due to consumption smoothing, market shares are small during booms. To render the equilibrium allocation optimal, the subsidy has to move procyclically to balance fluctuations in the private return.²⁷

8.2 Welfare Gains

This section suggests a R&D policy designed to partially remedy the distortions underlined in the previous section, with the purpose of undertaking some counterfactual exercise around the equilibrium computed in Section 7 and evaluate the corresponding welfare gains. The model is then extended to study a time varying R&D subsidy addressed to increase the average return to R&D and reduce the volatility of consumption. Let assume the R&D policy follows

$$1 + s_t = (1 + s) \left(\frac{c_t}{n_t} \right)^{\sigma-1},$$

where s is a constant rate and $\sigma < 1$ represents the additional smoothing introduced by the R&D policy. The equilibrium Euler-type equation (24) becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1 - \alpha}{\alpha} (1 + s) A e^{-\rho d} \left(\frac{c_t}{n_t} \right)^{\sigma} - \rho.$$

Notice that an equilibrium without R&D policy requires $s = 0$ and $\sigma = 1$.

In order to make welfare comparisons, we compute a consumption equivalent measure defined as the constant rate at which consumption in the decentralized equilibrium should increase all over the equilibrium path to make equilibrium welfare equal to the corresponding welfare of the equilibrium path with subsidies. Since utility is logarithmic, our welfare measure collapses to

$$\omega = e^{\rho(W_{R\&D} - W_e)} - 1,$$

²⁷The policy described in this section shares some similarities with the one introduced by Reichlin [65]. In fact, in both cases the policy is welfare enhancing and at the same time implies a smooth consumption path.

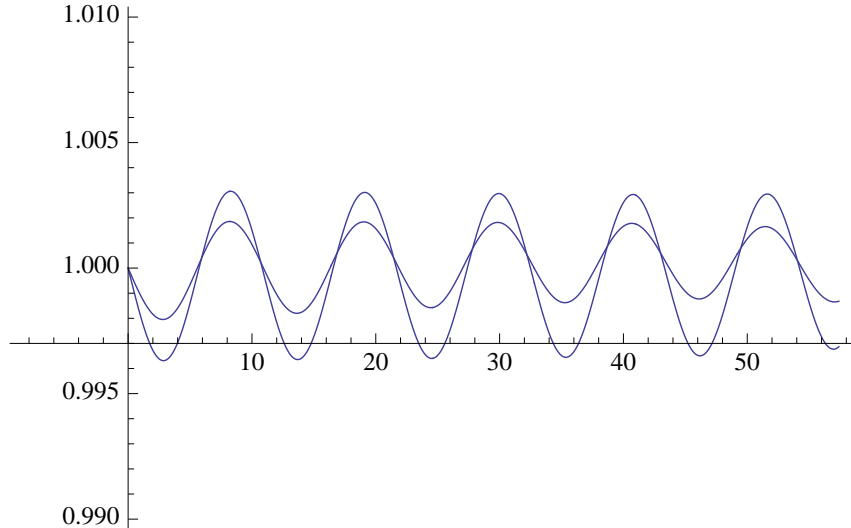


Figure 6: Consumption paths with and without subsidy.

where $W_{R\&D}$ and W_e measure welfare, as defined by the utility function (1), evaluated at equilibrium with and without subsidies, respectively.

When the R&D policy pays a 10% average subsidy, $s = .10$, and the subsidy rate moves in order to smooth consumption, with a smoothing parameter $\sigma = 1/2$, the growth rate increases from 2.4% to 3.4%. In Figure 6, detrended consumption paths, relative to initial consumption, are represented for the economies with and without subsidies. The smoother corresponds to the economy with procyclical subsidies. As can be observed, the subsidy halves consumption fluctuations. There are welfare gains of 9.6% as measured by ω . The order of magnitude is consistent with the findings in Barlevy [13]. If the 10% subsidy were constant, the growth rate would be 2.8% and the welfare gains 3.3%. Consequently, a 6.3% welfare gain may be attributed to consumption smoothing alone, however consumption smoothing affects welfare mainly through the raise in the growth rate.

9 Conclusions

This paper studies the relation between Schumpeterian wave-like business fluctuations and economic development in an endogenous growth framework with implementation delays. The paper shows that the equilibrium path admits a Hopf bifurcation and that consumption and the number of varieties persistently fluctuate around a balanced growth path. The main mechanism relating growth to wave-like business fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. A simple quantitative exercise shows that such an endogenous mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. Finally, it quantitatively find that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare.

Appendix

A More on Equilibrium

For completeness, we now present the theorem giving the optimality conditions for the *unconstrained* households problem who maximize (13) subject to the instantaneous budget constraint (14), the control constraint $c_t \geq 0$ ($t \geq 0$) and the initial condition $n_t = \bar{n}_t$, for $t \in [-d, 0]$, where \bar{n}_t is a known continuous positive function defined on the t domain. For brevity we will call this problem (UHP).

Theorem 3 *Assume that the function r_t is bounded and that the function w_t is such that*

$$w_t \leq k_1 e^{k_2 t}, \quad \forall t \geq 0,$$

for suitable constants $k_1, k_2 > 0$. Then an admissible state control path (n_t, c_t) for the problem (UHP) is optimal if there exists an absolutely continuous costate function μ_t such that

$$\frac{e^{-\rho t}}{c_t} = \mu_t \tag{56}$$

$$\dot{\mu}_t = -r_{t+d} \mu_{t+d}, \tag{57}$$

and, for every other admissible path (\hat{a}_t, \hat{c}_t) , the transversality condition

$$\overline{\lim}_{t \rightarrow \infty} (\hat{a}_t - a_t) \mu_t \geq 0, \tag{58}$$

holds. The conditions (56) and (57) above are also necessary.

Proof. It is a special case of Theorem 3.1 of [5]. ■

A straightforward consequence of such result (that, we recall, holds only for the unconstrained problem) is that, if we find a solution (a_t, c_t, μ_t) to the system of equations (14)-(56)-(57) satisfying the new transversality condition

$$\lim_{t \rightarrow \infty} a_t \mu_t = 0, \quad \text{or, equivalently,} \quad \lim_{t \rightarrow \infty} a_t c_t^{-1} e^{-\rho t} = 0, \tag{59}$$

then such solution is optimal. If, moreover, the state constraint $a_t \geq 0$ is satisfied for every $t \geq 0$, then the such solution is optimal for the constrained problem, too.

B Derivation of the key equations when $\nu \neq 1$

To show how the equations (40) and (41) arise we quickly run over the procedure used to write (26) and (27) showing what changes when $\nu \neq 1$. First of all equation (5) becomes:

$$c_t = n_{t-d}^{1+\nu} x_t \quad \text{and} \quad c_t = n_{t-d}^\nu L_t. \tag{60}$$

Then equation (26) in our paper becomes

$$\dot{n}_t = An_{t-d}(1 - n_{t-d}^{-\nu}c_t) \quad (61)$$

A bit longer is to find the new Euler equation. We start rewriting equation (9) which now is equal to

$$p_t(j) = n_{t-d}^{\alpha(\nu+1)-1} \left(\frac{c_t}{x_t(j)} \right)^{1-\alpha}$$

Substituting (60) into this last expression give us the price p_t at the symmetric equilibrium

$$p_t = n_{t-d}^\nu \quad (62)$$

while the wage rate is now determined by the following equation

$$w_t = \alpha n_{t-d}^\nu \quad (63)$$

and then equation (22) becomes

$$v_t = \frac{\alpha}{A} n_{t-d}^{\nu-1} \quad \Rightarrow \quad \frac{\dot{v}_t}{v_t} = (\nu - 1) \frac{\dot{n}_{t-d}}{n_{t-d}} \quad (64)$$

Then the profit of the intermediary firm is equal to:

$$\pi_t = p_t x_t - w_t \frac{L_t}{n_{t-d}} = (1 - \alpha) \frac{c_t}{n_{t-d}}$$

while combining this with (64) leads to the returns

$$r_t = \frac{1 - \alpha}{\alpha} A \frac{c_t}{n_{t-d}^\nu} + (\nu - 1) \frac{\dot{n}_{t-d}}{n_{t-d}}$$

and therefore the new Euler equation at equilibrium will be

$$\frac{\dot{c}_t}{c_t} = \left[\frac{1 - \alpha}{\alpha} A \frac{c_{t+d}}{n_t^\nu} + (\nu - 1) \frac{\dot{n}_t}{n_t} \right] \frac{c_t}{c_{t+d}} e^{-\rho d} - \rho \quad (65)$$

Therefore the two key equations (26) and (27) becomes respectively (61) and (65) for a generic externality.

Now we see how the transversality condition (28) should be modified. We start from the initial transversality condition (17). Now, when $\nu \neq 1$, we still have as in (25) that $a_t = v_t n_t$ but with v_t which is not constant and is indeed equal to

$$v_t = \frac{\alpha}{A} n_{t-d}^{\nu-1}$$

So, substituting into (25) we get

$$a_t = \frac{\alpha}{A} n_{t-d}^{\nu-1} n_t$$

which implies that (17) becomes

$$\lim_{t \rightarrow +\infty} \frac{\alpha}{A} n_{t-d}^{\nu-1} n_t c_t^{-1} e^{-\rho t} = 0$$

which is equivalent to (42) since $\frac{\alpha}{A} > 0$.

C Proof of the properties of the solutions of the characteristic equation

Proof of Proposition 2.

Proof of (i). To prove this statement we study the function $h(\lambda)$ for $\lambda \in \mathbb{R}$. It is easy to check that

$$h(0) = 0, \quad \lim_{\lambda \rightarrow +\infty} h(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow -\infty} h(\lambda) = +\infty.$$

Moreover

$$h'(\lambda) = 2\lambda - \rho + Ae^{-(g_e + \lambda)d} [-1 + d(\lambda - g_e - \rho)]$$

with

$$h'(0) = -\rho + Ae^{-g_e d} [-1 - d(g_e + \rho)] < 0, \quad \lim_{\lambda \rightarrow +\infty} h'(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow -\infty} h'(\lambda) = -\infty.$$

and

$$h''(\lambda) = 2 + Ade^{-(g_e + \lambda)d} [2 - d(\lambda - g_e - \rho)].$$

with

$$h''(0) = 2 + Ade^{-g_e d} [2 + d(g_e + \rho)] > 0, \quad \lim_{\lambda \rightarrow +\infty} h''(\lambda) = 2, \quad \lim_{\lambda \rightarrow -\infty} h''(\lambda) = +\infty.$$

By simple computations it is easy to prove that the function $h''(\lambda)$ has a minimum point at $\bar{\lambda} = g_e + \rho + \frac{3}{d}$ and that the value of the minimum is $2 - Ade^{-(2g_e + \rho)d-3}$. We have now two cases.

- If $2 - Ade^{-(2g_e + \rho)d-3} \geq 0$ then the minimum value of $h''(\lambda)$ is positive so $h''(\lambda) \geq 0$ for every $\lambda \in \mathbb{R}$. This implies (since h'' is zero in at most one point) that h' is strictly increasing and there exists a unique point $\hat{\lambda} > 0$ such that $h'(\hat{\lambda}) = 0$. The claim follows from the fact that $h(0) = 0$ and $\lim_{\lambda \rightarrow +\infty} h(\lambda) = +\infty$.
- If $2 - Ade^{-(2g_e + \rho)d-3} < 0$ then the minimum value of $h''(\lambda)$ is negative so there exists an interval $(\bar{\lambda}_1, \bar{\lambda}_2)$ (with $0 < \bar{\lambda}_1 < \bar{\lambda} < \bar{\lambda}_2 < +\infty$) such that $h''(\lambda) < 0$ iff $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$. Since $h''(g_e + \rho + \frac{2}{d}) = 2 > 0$ then $\bar{\lambda}_1 > g_e + \rho + \frac{2}{d}$. Now this means that $h'(\lambda)$ is strictly increasing on $(-\infty, \bar{\lambda}_1)$ and on $(\bar{\lambda}_2, +\infty)$ and strictly decreasing on $(\bar{\lambda}_1, \bar{\lambda}_2)$. Since $h'(0) < 0$ and $h'(g_e + \rho + \frac{2}{d}) = 2g_e + \rho + Ae^{-(2g_e + \rho)d-2} > 0$ then h' has a unique zero in the interval $(0, g_e + \rho + \frac{2}{d})$. Moreover from the expression of h' it easily follows that, for every $\lambda > g_e + \rho + \frac{2}{d}$ it must be $h'(\lambda) > 0$. So, as before, there exists a unique point $\hat{\lambda} > 0$ such that $h'(\hat{\lambda}) = 0$. The claim follows again from the fact that $h(0) = 0$ and $\lim_{\lambda \rightarrow +\infty} h(\lambda) = +\infty$.

Finally, the fact that $\lambda_0 > g_e + \rho$ follows since, by simple computations, we have

$$h(g_e + \rho) = (g_e + \rho)[g_e - Ae^{-g_e d}] = -\frac{\alpha}{1 - \alpha}(g_e + \rho)e^{\rho d} < 0,$$

where in the last equality we used the fact that g_e is the unique positive solution of the equation (30).

Proof of (ii). It follows from part (i) that all the other roots of (35) are not real. By the standard spectral theory for delay equation (see e.g. [38], Theorem 4.4 p.29 and Theorem 4.18, p.120) it follows that the roots are at most countable and that they must live in a left half plane. Since the coefficients of the characteristic equation are real it is clear that given any complex root $\mu_r + i\eta_r$ also its conjugate $\mu_r - i\eta_r$ is a root.

To find the complex roots we need to solve the following system when the real and imaginary part of $h(\mu + i\eta)$ are equal to zero.

$$\begin{cases} \mu^2 - \eta^2 - \rho\mu - Ae^{-g_e d}(g_e + \rho) + Ae^{-(g_e + \mu)d} [(g_e + \rho - \mu) \cos(\eta d) - \eta \sin(\eta d)] = 0 \\ 2\mu\eta - \rho\eta - Ae^{-(g_e + \mu)d} [(g_e + \rho - \mu) \sin(\eta d) + \eta \cos(\eta d)] = 0. \end{cases} \quad (66)$$

Assume now that for some r_0 , $\mu_{r_0} = 0$ and $\eta_{r_0} > 0$. then we have that η_{r_0} satisfy the following system, where we omit, for simplicity, the subscript r_0 ,

$$\begin{cases} \eta^2 + Ae^{-g_e d}(g_e + \rho) = Ae^{-g_e d} [(g_e + \rho) \cos(\eta d) - \eta \sin(\eta d)] \\ \rho\eta = -Ae^{-g_e d} [(g_e + \rho) \sin(\eta d) + \eta \cos(\eta d)] = 0. \end{cases} \quad (67)$$

Squaring each equation of the above system and summing them up²⁸ we get

$$\left[\eta^2 + Ae^{-g_e d}(g_e + \rho) \right]^2 + \rho^2 \eta^2 = A^2 e^{-2g_e d} \left[(g_e + \rho)^2 + \eta^2 \right]$$

which simplifies to

$$\eta^2 \left[\eta^2 + \rho^2 + 2Ae^{-g_e d}(g_e + \rho) - A^2 e^{-2g_e d} \right] = 0. \quad (68)$$

Clearly such equations can have only one positive solution, so there cannot be two couples of purely imaginary roots.

Finally assume that the purely imaginary roots are not simple. Then the number η_{r_0} must solve the equations $h(i\eta_{r_0}) = h'(i\eta_{r_0}) = 0$. We show that this is impossible dropping, again for simplicity, the subscript r_0 . Indeed putting $h'(i\eta) = 0$ we get

$$\begin{cases} \rho = Ae^{-g_e d} [-(1 + d(g_e + \rho)) \cos(\eta d) + d\eta \sin(\eta d)] \\ -2\eta = Ae^{-g_e d} [(1 + d(g_e + \rho)) \sin(\eta d) + d\eta \cos(\eta d)] = 0. \end{cases} \quad (69)$$

From (69) above and (67) we easily get then that it must be $\sin(\eta d) < 0$ and $\cos(\eta d) < 0$. This implies, in particular, that

$$\eta d > \pi \quad (70)$$

²⁸We thank an anonymous referee for suggesting us to use this method to simplify the proof.

Now, as we have done for the system (67) we square each equation of (69) and sum them, obtaining

$$\rho^2 + 4\eta^2 = A^2 e^{-2g_e d} [(1 + d(g_e + \rho))^2 + d^2 \eta^2] \quad (71)$$

Now from (68) we get that it must be

$$\eta^2 = A^2 e^{-2g_e d} - 2Ae^{-g_e d} (g_e + \rho) - \rho^2 \quad (72)$$

so, substituting in (71) (in the left hand side only) we get

$$\rho^2 + 4 \left[A^2 e^{-2g_e d} - 2Ae^{-g_e d} (g_e + \rho) - \rho^2 \right] = A^2 e^{-2g_e d} [(1 + d(g_e + \rho))^2 + d^2 \eta^2].$$

Now, summing and bringing the term $4A^2 e^{-2g_e d}$ from the left to the right hand side we get

$$-3\rho^2 - 8Ae^{-g_e d} (g_e + \rho) = A^2 e^{-2g_e d} [-4 + (1 + d(g_e + \rho))^2 + d^2 \eta^2].$$

Since the left hand side is negative we must have

$$(1 + d(g_e + \rho))^2 + d^2 \eta^2 < 4$$

which contradicts (70). ■

Proof of Proposition 3. We use classical results on the roots of analytic functions. To apply them we need first to reduce our problem to avoid the presence of the root $\lambda = 0$. To do so we consider the function $h_1 : \mathbb{C} \rightarrow \mathbb{C}$ defined as

$$h_1(\lambda) := \frac{h(\lambda)}{\lambda} = \lambda - \rho - A(g_e + \rho)e^{-(g_e + \lambda)d} + A(g_e + \rho)e^{-g_e d} \cdot \frac{e^{-\lambda d} - 1}{\lambda}, \quad \lambda \neq 0,$$

while, for $\lambda = 0$

$$h_1(\lambda) := -\rho - A(g_e + \rho)e^{-g_e d} - dA(g_e + \rho)e^{-g_e d} < 0.$$

Clearly the roots of h_1 are exactly the nonzero roots of h so D_1 and D_3 can be defined in term of the roots of h_1 . Since h_1 is analytic on \mathbb{C} we can apply to it the theorems concerning the zeros of analytic functions. By [38][Theorem 4.4, p.29] we know that the real part of all the roots of h_1 is bounded from above and that in each vertical strip the root of h_1 are a finite number. Then, by continuous dependence theorems (see e.g. [63] discussion before Theorem 1, p.97) we know that the roots of h_1 in a given vertical strip depend equicontinuously on the parameters ρ, α, A, d .

Now we prove that D_1 and D_3 are non empty.

As a first step, we take the parameters in the wider region

$$E_1 := (0, +\infty) \times [0, 1) \times [0, +\infty) \times (0, +\infty) \supset E$$

and consider the regions D_1 and D_3 in E_1 . For simplicity of writing we set from now on $\hat{\alpha} = \frac{\alpha}{1-\alpha}$; when $\alpha \in [0, 1)$ we have $\hat{\alpha} \in [0, +\infty)$ and we have a one-to-one strictly increasing correspondence between the values of α and the values of $\hat{\alpha}$.

If we take $A = A_{min}^e$ then $g_e = 0$ and so the right hand side of the characteristic equation (35) becomes

$$h(\lambda) = \lambda^2 - \rho\lambda - \lambda e^{-\lambda d} \hat{\alpha} \rho e^{\rho d} + e^{-\lambda d} \hat{\alpha} \rho^2 e^{\rho d} - \hat{\alpha} \rho^2 e^{\rho d} \quad (73)$$

so

$$h_1(\lambda) = \lambda - \rho - e^{-\lambda d} \hat{\alpha} \rho e^{\rho d} + \hat{\alpha} \rho^2 e^{\rho d} \cdot \frac{e^{-\lambda d} - 1}{\lambda} \quad (74)$$

It is clear that, if we set $\hat{\alpha} = 0$ (i.e. $\alpha = 0$), then also $A_{min}^e = 0$ and the above (74) reduces to

$$h_1(\lambda) = \lambda - \rho.$$

Since this function has only the root ρ , it follows that, for any $\rho > 0$ and $d > 0$ the point $(\rho, 0, 0, d)$ belongs to D_1 . Now we can apply Theorem 2.1 and Corollary 2.3 of [68]. So, if we take a parameters' region $B_1 \subseteq E_1$, with B_1 closed bounded and connected, such that, for every point in B_1 , h_1 has no zeros on the imaginary axis, then if one point of B_1 belongs to D_1 we must have $B_1 \subseteq D_1$.

From formula (67) we get that, if there exists a couple of purely imaginary roots $\pm i\eta$ (with $\eta > 0$) it must be $\sin(\eta d) < 0$ so $\eta d > \pi$. Moreover from (72) we get that if such couple exists it must be

$$\eta^2 = A^2 e^{-2g_e d} - \rho^2 - 2A e^{-g_e d} (g_e + \rho)$$

which implies

$$A^2 e^{-2g_e d} > \rho^2 + 2A e^{-g_e d} (g_e + \rho) + \frac{\pi^2}{d^2}. \quad (75)$$

This condition, if we take $A = A_{min}^e$, becomes

$$\hat{\alpha}^2 \rho^2 e^{2\rho d} > 2\hat{\alpha} \rho^2 e^{\rho d} + \rho^2 + \frac{\pi^2}{d^2} \iff \rho^2 (\hat{\alpha}^2 e^{2\rho d} - 2\hat{\alpha} e^{\rho d} - 1) > \frac{\pi^2}{d^2}$$

which is clearly not satisfied when $\hat{\alpha} \in [0, e^{-\rho d}(1 + \sqrt{2})]$. Since the terms in (75) are continuous functions of the parameters ρ, α, A, d then we can use Theorem 2.1 of [68] to conclude that, given any $\rho_0 > 0, d_0 > 0$, for (ρ, α, A, d) in a neighborhood of $(\rho_0, 0, 0, d_0)$ we remain in D_1 . The set of such points intersected with E is a subset of E with nonempty interior.

Now we prove that D_3 is nonempty. To do this we fix $A = A_{min}^e$ and $d > 0$ and we find when purely imaginary roots appear. The system (67) becomes

$$\begin{cases} \eta^2 + \hat{\alpha} \rho^2 e^{\rho d} = \hat{\alpha} \rho e^{\rho d} [\rho \cos(\eta d) - \eta \sin(\eta d)] \\ \rho \eta = -\hat{\alpha} \rho e^{\rho d} [\rho \sin(\eta d) + \eta \cos(\eta d)] = 0. \end{cases} \quad (76)$$

Now we find $\hat{\alpha}$ and ρ above as functions of η . From the second equation we get

$$\hat{\alpha} = \frac{-\eta}{e^{\rho d} [\rho \sin(\eta d) + \eta \cos(\eta d)]} \quad (77)$$

when $\eta \cos(\eta d) + \rho \sin(\eta d) < 0$; then substituting (77) into the first equation leads, with easy computations, to

$$\rho^2 = \frac{\eta^2 \cos(\eta d)}{1 - \cos(\eta d)}, \quad i.e. \quad \rho = \sqrt{\frac{\eta^2 \cos(\eta d)}{1 - \cos(\eta d)}}. \quad (78)$$

This implies that $\cos(\eta d) > 0$. Since we already know that $\sin(\eta d) < 0$, then it must be

$$\eta d \in \left(\left(2k - \frac{1}{2}\right) \pi, 2k\pi \right), \quad k = 1, 2, \dots$$

Now substituting (78) into (77) we see by simple computations that for such values of ηd the right hand side of (77) is strictly positive so also $\hat{\alpha} > 0$. This means that, for every fixed $k = 1, 2, \dots$, for η running over $\left(\left(2k - \frac{1}{2}\right) \frac{\pi}{d}, 2k\frac{\pi}{d}\right)$, the equations (77) and (78) define a curve in the positive quadrant of the plane $(\rho, \hat{\alpha})$. For values of ρ and $\hat{\alpha}$ on such curve (once we fix $d > 0$ and set $A = A_{min}^e$) we have occurrence of pure imaginary roots (e.g for $k = 1$ and $\eta = \frac{7}{4}\frac{\pi}{d}$). It is not difficult to see, by straightforward computations, that on such curves, when $\eta \rightarrow \left(2k - \frac{1}{2}\right) \frac{\pi}{d}$ then $\rho \rightarrow 0^+$ and $\hat{\alpha} \rightarrow +\infty$ while, when $\eta \rightarrow 2k\frac{\pi}{d}$ then $\rho \rightarrow +\infty$ and $\hat{\alpha} \rightarrow 0^+$. So such curves are in E ; moreover they do not intersect for different values of k and they increase²⁹ with k . So, if we take a point $(\rho_1, \hat{\alpha}_1)$ belonging to the region between the curve for $k = 1$ and the curve for $k = 3$, we have that the point $(\rho_1, \alpha_1, A_{min}^e, d)$ (where α_1 is the value corresponding to $\hat{\alpha}_1$) belong to D_3 and is in E . This shows that D_3 is not empty. Using the same argument as for D_1 we also see that D_3 has nonempty interior. ■

Proof of Proposition 4. First of all by definition of D_1 and D_3 for $(\rho, \alpha, A, d) = (\rho_H, \alpha_H, A_H, d_H)$ it must be $\mu_2 = 0$, $\eta_2 > 0$ and $\mu_r < 0$ for $r \geq 3$. We apply the implicit function theorem (IFT) to the characteristic equation (35) taking now h as function of (λ, d) . First of all we observe that, since every purely imaginary root must be simple, we have $\frac{\partial h}{\partial \lambda}(i\eta_2(d_H), d_H) \neq 0$ so IFT says that, for d in a sufficiently small neighborhood I of d_H , μ_2 and η_2 must be C^1 functions of d . Moreover it must be

$$\mu_2'(d_H) = - \left(\operatorname{Re} \frac{\partial h}{\partial \lambda}(i\eta_2(d_H), d_H) \right) = - \frac{(\operatorname{Re} \frac{\partial h}{\partial d})(\operatorname{Re} \frac{\partial h}{\partial \lambda}) + (\operatorname{Im} \frac{\partial h}{\partial d})(\operatorname{Im} \frac{\partial h}{\partial \lambda})}{\left| \frac{\partial h}{\partial \lambda} \right|^2}(i\eta_2(d_H), d_H)$$

For our purposes it is enough to compute the numerator of such fraction. By (69) we have, at any purely imaginary root $\lambda = i\eta$,

$$\operatorname{Re} \frac{\partial h}{\partial \lambda} = -\rho + Ae^{-g_e d} [-(1 + d(g_e + \rho)) \cos(\eta d) + d\eta \sin(\eta d)] \quad (79)$$

and

$$\operatorname{Im} \frac{\partial h}{\partial \lambda} = 2\eta + Ae^{-g_e d} [(1 + d(g_e + \rho)) \sin(\eta d) + d\eta \cos(\eta d)]. \quad (80)$$

²⁹In the sense that for fixed ρ the values of $\hat{\alpha}$ corresponding to increasing k , increase, too. The same if we exchange the role of ρ and $\hat{\alpha}$.

Moreover, by direct computation we have

$$\frac{\partial h}{\partial d}(\lambda) = Ae^{-g_e d} \left[a + e^{-\lambda d} (\lambda^2 + b\lambda - a) \right],$$

where

$$a = (d(g_e + \rho) - 1) \frac{\partial g_e}{\partial d} + g_e(g_e + \rho), \quad b = d \frac{\partial g_e}{\partial d} - \rho$$

where, by applying IFT to (30),

$$\frac{\partial g_e}{\partial d} = -\frac{g_e A e^{-g_e d} + \rho \hat{\alpha} e^{\rho d} (g_e + \rho)}{d A e^{-g_e d} + 1 + \hat{\alpha} e^{\rho d}} < 0.$$

So

$$\operatorname{Re} \frac{\partial h}{\partial d} = A e^{-g_e d} \left[a - (a + \eta^2) \cos(\eta d) + \eta b \sin(\eta d) \right] \quad (81)$$

and

$$\operatorname{Im} \frac{\partial h}{\partial d} = A e^{-g_e d} \left[(a + \eta^2) \sin(\eta d) + \eta b \cos(\eta d) \right]. \quad (82)$$

Using the expressions (79), (80), (81) and (82) above we can then compute the numerator of $\mu'_2(d_H)$ and see when it can become zero. For our purposes it is enough to show that, on $\partial D_1 \cap \partial D_3$ we have $\mu'_2(d_H) \neq 0$ except at most for a set with empty interior. First of all we characterize the points of $\partial D_1 \cap \partial D_3$ with suitable equations. This region is a manifold of dimension 3 in \mathbb{R}^4 . Then we show that, on this manifold, the set of points where $\mu'_2(d_H) = 0$ can be at most a set with empty interior in $\partial D_1 \cap \partial D_3$.

By (67) we have, for every purely imaginary root $\lambda = i\eta$,

$$\cos(\eta d) = e^{g_e d} \cdot \frac{g_e \eta^2 + A e^{-g_e d} (g_e + \rho)^2}{A(\eta^2 + (g_e + \rho)^2)}, \quad \sin(\eta d) = -\eta e^{g_e d} \cdot \frac{\eta^2 + \rho(g_e + \rho) + A e^{-g_e d} (g_e + \rho)}{A(\eta^2 + (g_e + \rho)^2)} \quad (83)$$

As already noted in the proof of Proposition 3 the above implies that, at every pure imaginary root, it must be $\sin(\eta d) < 0$ and $\cos(\eta d) > 0$ so $\eta d \in ((2k - \frac{1}{2})\pi, 2k\pi)$, for some $k = 1, 2, \dots$. Moreover, by adapting the argument of the proof of Proposition 3, it is not hard to see that it must be $\eta_2(d_H) \in (\frac{3\pi}{2}, 2\pi)$. By (72) we also know that, again for every purely imaginary root $\lambda = i\eta$,

$$\eta^2 = A^2 e^{-2g_e d} - \rho^2 - 2A e^{-g_e d} (g_e + \rho),$$

which rewrites, using (30), as

$$\eta^2 = (g_e + \rho)^2 \left(\hat{\alpha}^2 e^{2\rho d} - 2\hat{\alpha} e^{\rho d} - 1 \right),$$

from which we see that we must have $\hat{\alpha}^2 e^{2\rho d} - 2\hat{\alpha} e^{\rho d} - 1 > 0$. Calling for simplicity $c^2 := \hat{\alpha}^2 e^{2\rho d} - 2\hat{\alpha} e^{\rho d} - 1$ we can rewrite (83) as

$$\cos(\eta d) = e^{g_e d} \cdot \frac{c^2 g_e + A e^{-g_e d}}{A(1 + c^2)}, \quad \sin(\eta d) = -c e^{g_e d} \cdot \frac{\rho + c^2 (g_e + \rho) + A e^{-g_e d}}{A(1 + c^2)}. \quad (84)$$

or, using (30), as

$$\cos(\eta d) = \frac{e^{g_e d}}{A} \cdot \left[g_e + \frac{g_e + \rho}{\hat{\alpha} e^{\rho d} - 2} \right], \quad \sin(\eta d) = -c \frac{e^{g_e d}}{A} (g_e + \rho) \cdot \left[1 + \frac{1}{\hat{\alpha} e^{\rho d} - 2} \right]. \quad (85)$$

So the points in $\partial D_1 \cap \partial D_3$ are characterized by the relations

$$c > 0, \quad d(g_e + \rho)c \in \left(\frac{3\pi}{2}, 2\pi \right)$$

and one of the following two³⁰:

$$\begin{aligned} \cos(d(g_e + \rho)c) &= e^{g_e d} \cdot \frac{c^2 g_e + A e^{-g_e d}}{A(1 + c^2)} \in (0, 1), \\ \sin(d(g_e + \rho)c) &= -c e^{g_e d} \cdot \frac{\rho + c^2(g_e + \rho) + A e^{-g_e d}}{A(1 + c^2)} \in (-1, 0). \end{aligned} \quad (86)$$

Now we can compute $-[(\operatorname{Re} \frac{\partial h}{\partial d})(\operatorname{Re} \frac{\partial h}{\partial \lambda}) + (\operatorname{Im} \frac{\partial h}{\partial d})(\operatorname{Im} \frac{\partial h}{\partial \lambda})]$ substituting there the relations (84) and $\eta = (g_e + \rho)c$. The expression we find is an analytic function in the four parameters (ρ, α, A, d) which is not identically zero on $\partial D_1 \cap \partial D_3$. Thanks to the properties of analytic functions (see e.g. [4], Lecture 26) such function can be zero at most on the set with empty interior in $\partial D_1 \cap \partial D_3$.

The last statement of the proposition follows by observing that we always have

$$\operatorname{Re} \frac{\partial h}{\partial \lambda}(i\eta_2(d_H), d_H) < 0, \quad \operatorname{Im} \frac{\partial h}{\partial d}(i\eta_2(d_H), d_H) < 0$$

so a straightforward condition for the positivity of $\mu'(d_H)$ is to ask that

$$\operatorname{Re} \frac{\partial h}{\partial d}(i\eta_2(d_H), d_H) > 0, \quad \operatorname{Im} \frac{\partial h}{\partial \lambda}(i\eta_2(d_H), d_H) > 0$$

Substituting the numerical values of (45) and using continuity we get the claim. ■

D The linearized system and the transversality conditions

Proposition 7 *The series expansion of the Laplace transform solution of the system (32)-(34), given the initial conditions $\tilde{n}_t = \bar{n}_t e^{-g_e t}$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, and $\tilde{c}_0 = c_0$, is*

$$\tilde{n}_t = \sum_{r=0}^{+\infty} p_r(t) e^{\lambda_r t} \quad (87)$$

$$\tilde{c}_t = \frac{1}{A} \sum_{r=0}^{+\infty} \left(A e^{-(g_e + \lambda_r)d} p_r(t-d) - (g_e + \lambda_r) p_r(t) - p_r'(t) \right) e^{\lambda_r t} \quad (88)$$

³⁰The other will follow by the fundamental trigonometric identity $\cos^2 \beta + \sin^2 \beta = 1$ for all $\beta \in \mathbb{R}$.

where $\{\lambda_r\}_{r=0}^{+\infty}$ are the roots of the characteristic equation (35) and $\{p_r(t)\}_{r=0}^{+\infty}$ are polynomials of degree $k-1$ where k is the multiplicity of λ_r . When $k=1$ we have

$$p_r = \frac{\phi(\lambda_r)}{h'(\lambda_r)} \quad (89)$$

where

$$\phi(\lambda) = -Ac_0 + (\lambda - \rho - g_e) \left[\bar{n}_0 + Ae^{-(g_e+\lambda)d} \int_{-d}^0 \bar{n}_t e^{-(g_e+\lambda)t} dt \right].$$

Proof of Proposition 7. The fact that the system (32)-(34) admits a solution with a series expansion follows e.g. from Corollary 6.4 (p.168) of [38]. Here we explicitly compute the coefficients of such solutions by using the Laplace transform. We first differentiate equation (32) and then substitute there $\dot{\tilde{c}}_t$ from (34) and \tilde{c}_t from (32). We then find the following second order delay differential equation for \tilde{n}_t

$$\ddot{\tilde{n}}_t - \rho \dot{\tilde{n}}_t - Ae^{-g_e d} \dot{\tilde{n}}_{t-d} - \left(g_e(g_e + \rho) + \frac{\alpha(g_e + \rho)^2 e^{\rho d}}{1 - \alpha} \right) \tilde{n}_t + A(g_e + \rho) e^{-g_e d} \tilde{n}_{t-d} = 0, \quad (90)$$

where the initial data, in terms of the initial data of the system (32)-(34), are $\tilde{n}_t = \bar{n}_t e^{-g_e t}$ for $t \in [-d, 0]$ and $\dot{\tilde{n}}_0 = A(\bar{n}_{-d} - c_0) - g_e \bar{n}_0$. Recalling that the Laplace transformation of a function f (with subexponential growth at infinity) is defined as $L(f)(\lambda) = \int_0^\infty f(t) e^{-\lambda t} dt$, observing that the solution of (90) satisfies such subexponential growth (see e.g. Theorem 5.4, p.34 of [38]), we have, for λ sufficiently big,

$$\begin{aligned} L(\dot{\tilde{n}}_t)(\lambda) &= -\tilde{n}_0 + \lambda L(\tilde{n}_t)(\lambda) \\ L(\ddot{\tilde{n}}_t)(\lambda) &= -\dot{\tilde{n}}_0 + \lambda L(\dot{\tilde{n}}_t)(\lambda) = -\dot{\tilde{n}}_0 - \lambda \tilde{n}_0 + \lambda^2 L(\tilde{n}_t)(\lambda) \\ L(\tilde{n}_{t-d})(\lambda) &= e^{-\lambda d} \left[\int_{-d}^0 \tilde{n}_t e^{-\lambda t} dt + L(\tilde{n}_t)(\lambda) \right] \\ L(\dot{\tilde{n}}_{t-d})(\lambda) &= -\tilde{n}_{-d} + \lambda L(\tilde{n}_{t-d})(\lambda) = -\tilde{n}_{-d} + \lambda e^{-\lambda d} \left[\int_{-d}^0 \tilde{n}_t e^{-\lambda t} dt + L(\tilde{n}_t)(\lambda) \right] \end{aligned}$$

So applying the Laplace transform to the equation (90) we have

$$L(\tilde{n}_t)(\lambda) \cdot h(\lambda) = \phi(\lambda)$$

where $h(\lambda)$ is the left hand side of the characteristic equation (35) and

$$\phi(\lambda) = \dot{\tilde{n}}_0 + \tilde{n}_0(\lambda - \rho) - Ae^{-g_e d} \tilde{n}_{-d} + Ae^{-(g_e+\lambda)d} (\lambda - g_e - \rho) \int_{-d}^0 \tilde{n}_t e^{-\lambda t} dt$$

or, in terms of \bar{n}_t and c_0 ,

$$\phi(\lambda) = -Ac_0 + \bar{n}_0(\lambda - \rho - g_e) + Ae^{-(g_e+\lambda)d} (\lambda - g_e - \rho) \int_{-d}^0 \bar{n}_t e^{-(g_e+\lambda)t} dt.$$

Since \tilde{n}_t is a continuously differentiable function in $[0, +\infty)$,³¹ and therefore certainly continuous and of bounded variation on any finite interval, then we can use the inversion formula for the Laplace transformation to obtain that, for $t > 0$, (see e.g. [16], Theorem 6.3, p. 175-176)

$$\tilde{n}_t = \int_{a-i\infty}^{a+i\infty} \frac{\phi(\lambda)}{h(\lambda)} e^{\lambda t} d\lambda. \quad (91)$$

Then one can compute such complex integral by means of the Residue Theorem as in [16], Section 6.7 (in particular Theorem 6.5) obtaining

$$\tilde{n}_t = \sum_{r=0}^{\infty} p_r(t) e^{\lambda_r t} \quad (92)$$

where $\{\lambda_r\}_{r \in \mathbb{N}}$ is the sequence of the roots of the characteristic equation (35) and the $p_r(t)$ are polynomials of degree less or equal to $k(r) - 1$ where $k(r)$ is the multiplicity of λ_r . More precisely they are given by (setting for simplicity $k(r) = k$ in the formula below)

$$e^{-\lambda_r t} \cdot \lim_{\lambda \rightarrow \lambda_r} \frac{1}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} \left((\lambda - \lambda_r) \frac{\phi(\lambda) e^{\lambda t}}{h(\lambda)} \right),$$

so, when $k(r) = 1$, p_r is independent of t and is given by $p_r = \frac{\phi(\lambda_r)}{h'(\lambda_r)}$. Finally the solution of \tilde{c}_t can be derived from (87) and (32). ■

Proposition 8 *Assume that the parameters belong to the nonempty region \bar{D}_1 . Then for any exogenously given nondecreasing initial condition $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$ there exists a unique $c_0 > 0$ such that the solution path (n_t, c_t) of the system (32)-(34) satisfies the transversality condition. Such c_0 is given by*

$$c_0 = (\lambda_0 - g_e - \rho) \left[\frac{\bar{n}_0}{A} + e^{-(g_e + \lambda_0)d} \int_{-d}^0 \bar{n}_t e^{-(g_e + \lambda_0)t} dt \right] > 0. \quad (93)$$

Moreover the path (n_t, c_t) is

$$\begin{aligned} n_t &= p_1 e^{g_e t} + \sum_{r=2}^{+\infty} p_r(t) e^{(g_e + \lambda_r)t} \\ c_t &= \frac{1}{A} \left[\left(A e^{-g_e d} - g_e \right) p_1 e^{g_e t} + \sum_{r=2}^{+\infty} \left(A e^{-(g_e + \lambda_r)d} p_r(t-d) - (g_e - \lambda_r) p_r(t) - p_r'(t) \right) e^{\lambda_r t} \right] \end{aligned} \quad (94)$$

Proof. Given our assumptions, the only positive root to be ruled out in order to have convergence to the balanced growth path is λ_0 . To do that we have to specify c_0 as in (93) so that $p_0 = 0$. Uniqueness of the equilibrium path is a direct consequence of the fact that (93) is the only choice of the initial condition of consumption which rules out λ_0 . Oscillatory convergence follows from the properties of the spectrum of roots as discussed in the previous proposition.

³¹See the previously mentioned theorem of existence and uniqueness of solution in Bellman and Cooke [16].

Finally the general equilibrium path converges to the balanced growth path and for this reason it respects the transversality conditions. In fact, convergence implies that $\lim_{t \rightarrow \infty} \frac{n_t}{c_t} = \frac{(1-\alpha)A}{\alpha(g_e+\rho)e^{\rho d}}$ and then $\lim_{t \rightarrow \infty} \frac{n_t}{c_t} e^{-\rho t} = 0$. ■

Proposition 9 *Let us fix α, ρ, A, d such that they belong to the nonempty region D_1 or to a sufficiently small neighborhood of a point in $\partial D_1 \cap \partial D_3$ where, for the projected system introduced in Theorem 1, a Hopf bifurcation occurs moving parameter d . Then for any exogenously given nondecreasing initial condition $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, close enough to the BGP, there exists a $c_0 > 0$ such that the solution path (n_t, c_t) of the system (26)-(27) satisfies the transversality condition (28). Such c_0 converges to the one defined in the linearized system - see equation (93) in Proposition 8 above - as the distance $\sup_{t \in [-d, 0]} |\bar{n}_t - \bar{n}_0 e^{g_e t}|$ tends to 0.*

Proof of Proposition 9. First of all we observe that, thanks to Diekmann et al. [38], Theorem 6.8 p.240, since the characteristic equation admits a strictly positive real root λ_0 , then all equilibrium points of the detrended system (32)-(33) are unstable. Moreover thanks to Theorem 6.1 p.257 and to Theorem 5.3 p.266 the detrended system (32)-(33) admits, in a neighborhood of any equilibrium point, a stable manifold \mathcal{W}_S and a center manifold \mathcal{W}_C which contains the set of initial conditions (\bar{n}_t, c_0) which gives rise to a BGP (i.e. that satisfy Proposition 1).

Assume first that parameters are in the D_1 -subdivision. If the initial conditions (\bar{n}_t, c_0) belong to the linear stable manifold then there exists a small real number δ (which goes to zero when the distance between \bar{n}_t and the BGP goes to zero) such that $(\bar{n}_t, c_0 + \delta)$ belong to \mathcal{W}_S (see Theorem 6.1 (ii) in Diekmann et al. [38]). Now, given an initial datum \bar{n}_t , choose c_0 as in (93). If we start the linearized system (32)-(34) from such (\bar{n}_t, c_0) then we know that the solution converges to a BGP so (\bar{n}_t, c_0) belongs to the linear stable manifold. Thanks to the Theorem 6.1 (iv), p.257 in Diekmann et al. [38], we get that, if \bar{n}_t is sufficiently close to a BGP, then for suitable small δ as above, the solution of the nonlinear detrended system (32)-(33) converges to a BGP, too. This in particular implies that the transversality condition

$$\lim_{t \rightarrow \infty} \tilde{n}_t \tilde{c}_t^{-1} e^{-\rho t} = 0, \quad (95)$$

holds. As a conclusion, if we prove that the solution of the nonlinear detrended system (32)-(33) satisfies the irreversibility constraint $\dot{\tilde{n}}_t + g_e \tilde{n}_t \geq 0$ ($t \geq 0$), then this is an equilibrium associated to \bar{n}_t .

If the parameters are in a sufficiently small neighborhood of a point in $\partial D_1 \cap \partial D_3$ where, for the projected system introduced in Theorem 1, a Hopf bifurcation then all the above considerations remain true except for the fact that we have two purely imaginary elements ($\pm i\eta_2$) of the spectrum coming out when we move the parameter d crossing $\partial D_1 \cap \partial D_3$. In this case, given any \bar{n}_t , the linearized detrended system (32)-(34) starting from (\bar{n}_t, c_0) where c_0 is given by (93) has a solution whose principal components (the one coming from the eigenvalues crossing the imaginary axis) oscillates around a BGP with oscillations possibly unbounded when, moving d ,

we enter in D_3 . However, thanks to the Hopf bifurcation, even in this case, the corresponding component of the nonlinear system must keep the ratio $\frac{c_t}{n_t}$ periodic and so bounded. Concerning the component on the unstable manifold we can argue exactly as above to get that, for a suitable small δ , the solution of the nonlinear detrended system with datum $(\bar{n}_t, c_0 + \delta)$ stays out of the unstable manifold and so it remains bounded and satisfies the transversality condition (95). ■

E Proof of Theorem 1

Proof of Theorem 1. *Proof of (i).* The statement (i) should follow from Theorem 2.7 p.291 in Diekmann et al. [38] if zero were not a root of the characteristic equation. In fact, all the other assumptions of the theorem are verified thanks to Proposition 4. Therefore, we will show here that it is still possible to prove the emergence of periodic orbits by a suitable reduction procedure that we outline below.

Consider the nonlinear detrended system (32)-(33). For notational purposes, we denote by f the dynamics of such system, i.e. the function on the right hand side

$$f(n., c) = \begin{pmatrix} A(n_{-d}e^{-ge^d} - c) - g_en_0 \\ \frac{1-\alpha}{\alpha} A e^{-\rho d} \frac{c^2}{n_0} - (\rho + g_e)c \end{pmatrix} \quad (96)$$

This function is defined on the space $C^0([-d, 0]; \mathbb{R}_{++}) \times \mathbb{R}_{++}$ with values in \mathbb{R}^2 and is positively homogeneous of degree 1 in the sense that, for every $a > 0$ and for every $(n., c) \in C^0([-d, 0]; \mathbb{R}_{++}) \times \mathbb{R}_{++}$ we have $f(an., ac) = af(n., c)$. We already know (see Section 4) that, calling $\bar{\mathbf{v}}$ the element in $C^0([-d, 0]; \mathbb{R}_{++}) \times \mathbb{R}_{++}$ given by

$$\bar{\mathbf{v}} := \left(e^{ge}, \frac{\hat{\alpha}}{A} (g + \rho) e^{\rho d} \right)$$

then we have $f(\bar{\mathbf{v}}) = 0$, so also $f(a\bar{\mathbf{v}}) = 0$ for every $a > 0$. Introducing a suitable infinite dimensional formalism, as in [38], in [52] or in [23], Df can be considered as a linear operator on the space $C^0([-d, 0]; \mathbb{R}) \times \mathbb{R}$ (or on the space $\mathbb{R} \times L^2([-d, 0]; \mathbb{R}) \times \mathbb{R}$) and its eigenvalues are exactly the solutions of the characteristic equation (35). In this context the above facts implies that $Df(\bar{\mathbf{v}})$ has always a zero eigenvalue with eigenvector $\bar{\mathbf{v}}$.

Call now, for simplicity, $\mathbf{x}_0 := (\bar{n}., c_0)$ the generic initial datum of our system and $\mathbf{x}(t; \mathbf{x}_0)$ the associated solution. Call \mathcal{K} the Banach space $C^0([-d, 0]; \mathbb{R}) \times \mathbb{R}$ and \mathcal{H} the Hilbert space $\mathbb{R} \times L^2([-d, 0]; \mathbb{R}) \times \mathbb{R}$. Clearly $\mathcal{K} \subseteq \mathcal{H}$ with continuous embedding. Take a given vector $\mathbf{u} \in \mathcal{H}$, $\mathbf{u} > 0$ and consider the new variables

$$r(t) = \langle \mathbf{u}, \mathbf{x}(t) \rangle_{\mathcal{H}}, \quad \mathbf{z}(t) := \frac{\mathbf{x}(t)}{r(t)}.$$

Using the 1-homogeneity of f we have, by simple computations, that

$$r'(t) = r(t) \langle \mathbf{u}, f(\mathbf{z}(t)) \rangle_{\mathcal{H}} \quad (97)$$

$$\mathbf{z}'(t) = f(\mathbf{z}(t)) - \mathbf{z}(t) \langle \mathbf{u}, f(\mathbf{z}(t)) \rangle_{\mathcal{H}} \quad (98)$$

By the definition of $\mathbf{z}(t)$ we clearly have that $\langle \mathbf{u}, \mathbf{z}(t) \rangle_{\mathcal{H}} = 1$ for every $t \geq 0$. This means that the variable $\mathbf{z}(t)$ lives in the affine hyperplane

$$\mathcal{E} := \{\mathbf{z} \in \mathcal{H} : \langle \mathbf{u}, \mathbf{z} \rangle_{\mathcal{H}} = 1\} \subset \mathcal{H}.$$

If we choose \mathbf{u} such that $\langle \mathbf{u}, \bar{\mathbf{v}} \rangle_{\mathcal{H}} = 1$ then, calling

$$\mathcal{E}_0 := \{\mathbf{z} \in \mathcal{H} : \langle \mathbf{u}, \mathbf{z} \rangle_{\mathcal{H}} = 0\} \subset \mathcal{H}.$$

we get that any $\mathbf{z} \in \mathcal{E}$ can be written as

$$\mathbf{z} = \bar{\mathbf{v}} + \mathbf{w}, \quad \mathbf{w} \in \mathcal{E}_0.$$

Now we set $\mathbf{w}(t) = \mathbf{z}(t) - \bar{\mathbf{v}}$; by (98) the variable $\mathbf{w}(t)$ satisfies the equation

$$\mathbf{w}'(t) = f(\bar{\mathbf{v}} + \mathbf{w}(t)) - (\mathbf{w}(t) + \bar{\mathbf{v}}) \langle \mathbf{u}, f(\bar{\mathbf{v}} + \mathbf{w}(t)) \rangle_{\mathcal{H}} =: g(\mathbf{w}(t)). \quad (99)$$

By what said above, the variable $\mathbf{w}(t)$ always remains in \mathcal{E}_0 which is an hyperplane in \mathcal{H} . Now we show that, choosing appropriately \mathbf{u} , such equation admits a Hopf bifurcation in \mathcal{E}_0 . First of all the point $\mathbf{0}$ is clearly an equilibrium point of such system in \mathcal{E}_0 . Moreover³²

$$Dg(\mathbf{0}) = Df(\bar{\mathbf{v}}) - [Df(\bar{\mathbf{v}})^* \mathbf{u}] \otimes \bar{\mathbf{v}}$$

so, if we choose \mathbf{u} such that $Df(\bar{\mathbf{v}})^* \mathbf{u} = \mathbf{0}$ ³³ then we get

$$Dg(\mathbf{0}) = Df(\bar{\mathbf{v}}).$$

This means that, when we restrict the system (99) into \mathcal{E}_0 its characteristic equation is exactly $\frac{h(\lambda)}{\lambda} = 0$ where h is given by (35). So, thanks to the analysis done in Subsection 5.1, we get that a Hopf bifurcation occurs for the system (99) in \mathcal{E}_0 in a neighborhood of the equilibrium point $\mathbf{0}$. This concludes the proof of (i).

Proof of (ii).

From the definition of $\mathbf{z}(t)$, calling $z_1(t)(\cdot)$ its first component (infinite dimensional) and $z_2(t)$ its second component, we easy obtain that

$$\frac{c_t}{n_t} = \frac{z_2(t)}{z_1(t)(0)} \quad \frac{n_{t+s}}{c_t} = \frac{z_1(t)(s)}{z_2(t)} \quad \frac{n_{t+s}}{n_t} = \frac{z(t)(s)}{z_1(t)(0)}$$

so the periodicity of such ratios immediately follows from the periodicity of $\mathbf{z}(t)$.

Proof of (iii). The statement (ii) follows from Proposition 9 above. ■

³²Given $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ we call $\mathbf{x} \otimes \mathbf{y}$ the linear functional in \mathcal{H} given by:

$$\mathbf{x} \otimes \mathbf{y}(\mathbf{z}) = \langle \mathbf{x}, \mathbf{z} \rangle \mathbf{y}.$$

Moreover given a linear operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ we denote by \mathcal{A}^* the adjoint of \mathcal{A} .

³³This is always possible as $\lambda = 0$ is an eigenvalue of $Df(\bar{\mathbf{v}})^*$ and it implies that $\langle \mathbf{u}, \mathbf{v}^i \rangle_{\mathcal{H}} = 0$ for every other eigenvector of $Df(\bar{\mathbf{v}})$.

F Other proofs

Proof of Lemma 1. Fix $\nu \geq 0$. Let us define $\varphi_\nu : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\varphi_\nu(g) := -g + Ae^{-gd} - \frac{(\nu g + \rho)\alpha}{1 - \alpha} e^{[g(\nu-1) + \rho]d} + \frac{\alpha(\nu - 1)}{1 - \alpha} g e^{-gd} \quad (100)$$

Then, as for the case $\nu = 1$ we have, by simple computations,

$$\varphi_\nu(0) = A - A_{min}^e \quad \text{and} \quad \lim_{g \rightarrow +\infty} \varphi_\nu(g) = -\infty, \quad \forall \nu > 0.$$

This implies the existence of a positive root of (44) when (31) is satisfied. Concerning uniqueness we divide the proof in two cases. First take the simpler case when $\nu > 1$. Then

$$\varphi'_\nu(g) = -1 - dAe^{-gd} - \frac{\alpha}{1 - \alpha} e^{-gd} \left\{ e^{(g\nu + \rho)d} [(\nu - 1)d(g\nu + \rho) + \nu] - (\nu - 1)(1 - gd) \right\}$$

and we can easily observe that, for $g > 0$ the term in the big braces is always positive because it is clearly bigger than $\nu e^{(g\nu + \rho)d} - (\nu - 1) > 0$.

More complicated is to deal with the case $\nu \in [0, 1)$. In this case, indeed, it is not difficult to see that, in some cases, $\varphi'_\nu(g)$ can be positive for some $g > 0$. So we proceed as follows:

- (i) we first set $\nu = 0$ and prove that the function φ_0 has a unique positive root g_e^0 .
- (ii) we prove that, for every $\nu > 0$ and for every $g > 0$ we have

$$\varphi_\nu(g) < \varphi_0(g)$$

- (iii) we finally prove that, on the interval $(0, g_e^0)$ the derivative $\varphi'_\nu(g)$ is negative.

The above three facts give, as a straightforward consequence, the uniqueness of the root g_e' .

We prove first (i). Setting $\nu = 0$ we have

$$\varphi_0(g) = -g + Ae^{-gd} - \frac{\alpha}{1 - \alpha} \rho e^{(-g + \rho)d} - \frac{\alpha}{1 - \alpha} g e^{-gd} =$$

so, using the definition of A_{min}^e in (31)

$$\varphi_0(g) = -g + e^{-gd} \left[A - A_{min}^e - \frac{\alpha}{1 - \alpha} g \right] \quad (101)$$

Differentiating twice we have

$$\varphi'_0(g) = -1 + e^{-gd} \left[\frac{\alpha}{1 - \alpha} (gd - 1) - (A - A_{min}^e) d \right]$$

and

$$\varphi''_0(g) = dAe^{-gd} \left[\frac{\alpha}{1 - \alpha} (2 - gd) + (A - A_{min}^e) d \right]$$

Now it is easy to see that

$$\begin{aligned}\varphi_0''(g) > 0 &\iff \frac{\alpha}{1-\alpha}(2-gd) + (A - A_{min}^e)d > 0 \\ \iff g < \frac{1}{d} \left[2 + \frac{(1-\alpha)(A - A_{min}^e)d}{\alpha} \right] &=: g_0\end{aligned}$$

So $\varphi_0'(g)$ is strictly increasing in $(0, g_0)$, strictly decreasing in $(g_0, +\infty)$ and has a maximum point at g_0 . The value of the maximum is

$$\varphi_0'(g_0) = -1 + \frac{\alpha}{1-\alpha} e^{-\left[2 + \frac{(1-\alpha)(A - A_{min}^e)d}{\alpha} \right]}$$

and it can be positive for values of α close to 1. If $\varphi_0'(g_0) \leq 0$ we have that $\varphi_0(g)$ is always strictly decreasing (since $\varphi_0'(g)$ would be 0 in at most one point) so uniqueness of g_e^0 immediately follows. Assume then $\varphi_0'(g_0) > 0$. In this case we have that there exists two values g_1 and g_2 such that $0 < g_1 < g_0 < g_2$ where $\varphi_0'(g_1) = \varphi_0'(g_2) = 0$. So φ_0 is strictly decreasing in $(0, g_1)$ and in $(g_2, +\infty)$ and is strictly increasing in (g_1, g_2) . Now we have, by simple computations,

$$\varphi_0(g_0) = -g_0 - \frac{2}{d} \cdot \frac{\alpha}{1-\alpha} e^{-g_0 d} < 0.$$

Moreover, since $g_2 > g_0$, then

$$\frac{\alpha}{1-\alpha} g_2 > \frac{\alpha}{1-\alpha} g_0 > A - A_{min}^e$$

so also $\varphi_0(g_2) < 0$. This implies that $\varphi_0(g)$ must be strictly negative on $[g_1, +\infty)$. Since it is strictly decreasing in $(0, g_1)$ we get uniqueness of g_e^0 which must also belong to $(0, g_1)$.

Now we prove (ii). Rewriting (100) we have

$$\varphi_\nu(g) = -g + e^{-gd} \left[A - A_{min}^e e^{g\nu d} + \frac{\alpha}{1-\alpha} g \left(\nu - 1 - \nu e^{(g\nu+\rho)d} \right) \right]$$

So, using (101) we have

$$\varphi_\nu(g) - \varphi_0(g) = e^{-gd} \left[A_{min}^e (1 - e^{g\nu d}) + \frac{\alpha}{1-\alpha} g \nu \left(1 - e^{(g\nu+\rho)d} \right) \right] < 0, \quad \forall g > 0.$$

Finally, to prove (iii) it is enough to differentiate $\varphi_\nu(g) - \varphi_0(g)$ getting the claim by straightforward computations. ■

Proof of Theorem 2 (Sketch). We give only the main ideas on which the proof can be built.

The key point is to observe that the solutions of the system (40)-(41) when ν is close to 1 are close to (and have the same topological behavior of) the ones when $\nu = 1$ when the data are bounded away from 0. This is due to the fact that the terms containing ν depends analytically on $\nu, n_t, n_{t-d}, c_t, c_{t-d}$ (when they are bounded away from 0) and on the parameters $(\rho, \alpha, A, d) \in E$.

The main problems in the proof are the following.

First of all, since the dynamics f is not any more an homogeneous function one has to generalize the argument used to prove (i) of Theorem 1. The idea is to define the new variable $\mathbf{z}(t)$ in a different way, as a nonlinear function (depending on ν) of the starting variable $\mathbf{x}(t)$.

Second, the characteristic equation associated to the corresponding linearized system is close to (35) when ν is close to 1 but , due to the advanced term in the Euler equation (41) complex roots with positive real part may arise. To get rid of them, one can follow an argument suggested in Boucekkine et al. [28], which uses the transversality condition (42). Then we remain with the other roots, which are continuous and infinitely differentiable functions of the parameter ν , hence all the properties stated in Proposition 4 and used Theorem 1 remain true by a simple perturbation argument, for ν sufficiently close to 1. ■

Proof of Proposition 6. The proof follows from the maximum principle approach developed by Bambi [10] and the dynamic programming approach in Bambi et al. [11], Proposition 1 and Theorem 4. ■

Proof of the results based on equation (55). Let's assume $g = g_e$. Combining (30) and (48) to solve for α gives $\underline{\alpha}$ as defined above. Notice that from (48), g does not depend on α , meaning that $\underline{\alpha}$ in (55) only depends on the other three parameters A, d, ρ . It is straightforward to observe that $\underline{\alpha}$ is always smaller than $1/2$. Finally $g_e < g$ iff $\alpha > \underline{\alpha}$, since from (30)

$$\frac{dg_e}{d\alpha} = -\frac{\frac{(g_e+\rho)e^{\rho d}}{(1-\alpha)^2}}{1 + dAe^{-g_e d} + \frac{\alpha}{1-\alpha}e^{\rho d}} < 0,$$

and g in (48) does not depend on α . ■

References

- [1] Acemoglu, Daron (2009) *Introduction to Modern Economic Growth*. Princeton University Press.
- [2] Adams, James (1990) “Fundamental stocks of knowledge and productivity growth,” *Journal of Political Economy* 98(4), 673-702.
- [3] Aghion, Philippe and Peter Howitt (1992) “A model of growth through creative destruction,” *Econometrica* 60(2), 323-351.
- [4] Agarwal, Ravi P., Kanishka Perera, and Sandra Pinelas (2011) *An introduction to complex analysis*. Springer, New York.
- [5] Agram N., S. Haadem, B. Oksendal and F. Proske (2012) *A maximum principle for infinite horizon delay equations*. <http://arxiv.org/abs/1206.6670v1>.
- [6] D’Albis Hippolyte and Emmanuelle Augeraud-Veron (2009) “Competitive growth in a life-cycle model: existence and dynamics,” *International Economic Review* 50(2), 459-484.
- [7] D’Albis Hippolyte, Emmanuelle Augeraud-Veron and Hermen Jan Hupkes (2012) “Discontinuous initial value problems for functional differential-algebraic equations of mixed type,” *Journal of Differential Equations* 253(7), 1959-2024.
- [8] D’Albis Hippolyte, Emmanuelle Augeraud-Veron and Alain Venditti (2012) “Business cycle fluctuations and learning-by-doing externalities in a one-sector model” *Journal of Mathematical Economics* 48, 295-308.
- [9] Asea, Patrick and Paul Zak (1999) “Time-to-build and cycles,” *Journal of Economic Dynamics and Control* 23(8), 1155-1175.
- [10] Bambi, Mauro (2008) “Endogenous growth and time to build: the AK case,” *Journal of Economic Dynamics and Control* 32(4), 1015-1040.
- [11] Bambi, Mauro, Giorgio Fabbri and Fausto Gozzi (2012) “Optimal policy and consumption smoothing effects in the time-to-build AK model,” *Economic Theory* 50(3), 635-669.
- [12] Bambi, Mauro and Omar Licandro (2005) “(In)determinacy and time-to-build,” EUI Working paper ECO No. 2004/17.
- [13] Barlevy, Gadi (2004) “The cost of business cycles under endogenous growth,” *American Economic Review* 94(4), 964-990.
- [14] Basu, Susanto and John G. Fernald (1997) “Returns to scale in U.S. production: Estimates and implications,” *Journal of Political Economy* 105(2), 249-283.

- [15] Beaudry, Paul and Franck Portier (2006) “Stock prices, news, and economic fluctuations,” *American Economic Review* 96(4), 1293-1307.
- [16] Bellman, Richard and Kenneth Cooke (1963) *Differential-difference equations*. New York Academic Press.
- [17] Ben-Gad, Michael (2003) “Fiscal policy and indeterminacy in models of endogenous growth,” *Journal of Economic Theory* 108, 322-344.
- [18] Benassy, Jean Pascal (1998) “Is there always too little research in endogenous growth with expanding product variety?,” *European Economic Review* 42(1), 61-69.
- [19] Benhabib, Jess and Kazuo Nishimura (1979) “The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth,” *Journal of Economic Theory* 21(3), 421-444.
- [20] Benhabib, Jess and Kazuo Nishimura (1985) “Competitive equilibrium cycles,” *Journal of Economic Theory* 35(2), 284-306.
- [21] Benhabib, Jess and Roberto Perli (1994) “Uniqueness and indeterminacy: on the dynamics of endogenous growth” *Journal of Economic Theory* 63, 113-142.
- [22] Benhabib, Jess and Aldo Rustichini (1991) “Vintage capital, investment and growth,” *Journal of Economic Theory* 55(2), 323-339.
- [23] A. Bensoussan, G. DaPrato, M. Delfour, S. Mitter (2007) *Representation and Control of infinite dimensional systems*, Birkhauser.
- [24] Boucekkine, Raouf and David de la Croix (2003) “Information technologies, embodiment and growth,” *Journal of Economic Dynamics and Control* 27(11/12), 2007-2034.
- [25] Boucekkine, Raouf, David de la Croix and Omar Licandro (2002) “Vintage human capital, demographic trends, and endogenous growth,” *Journal of Economic Theory* 104, 340-375.
- [26] Boucekkine, Raouf, Fernando del Rio and Omar Licandro (1999) “Endogenous vs exogenously driven fluctuations in vintage capital models,” *Journal of Economic Theory* 88(1), 161-187.
- [27] Boucekkine, Raouf, Marc Germain and Omar Licandro (1997) “Replacement echoes in the vintage capital growth Model,” *Journal of Economic Theory* 74(2), 333-348.
- [28] Boucekkine, Raouf, Omar Licandro, Luis Puch and Fernando del Río (2005) “Vintage capital and the dynamics of the AK model,” *Journal of Economic Theory* 120, 39-72.
- [29] Caballero, Ricardo and Mohamad Hammour (1994) “The cleansing effect of recessions,” *American Economic Review* 84(5), 1350-68.

- [30] Collard, Fabrice, Omar Licandro and Luis Puch (2008) “The short-run dynamics of optimal growth models with delays,” *Annales d’Economie et de Statistique* 90, 127-143.
- [31] Comin, Diego and Mark Gertler (2006) “Medium-term business cycles,” *American Economic Review* 96(3), 523-551.
- [32] Comin, Diego, Mark Gertler and Ana Maria Santacreu (2009) “Technology innovation and diffusion as sources of output and asset price fluctuations,” Harvard Business School WP 09-134.
- [33] Comin, Diego and Bart Hobijn (2010) “An exploration of technology diffusion,” *American Economic Review* 100(5), 2031-2059.
- [34] Comin, Diego, Bart Hobijn and Emilie Rovito (2008) “Technology usage lags,” *Journal of Economic Growth* 13(4), 237-256.
- [35] Cooper, Russell, John Haltiwanger and Laura Power (1999) “Machine replacement and the business cycle,” *American Economic Review* 89(4), 921-946.
- [36] Crandall, M.C. and P.H. Rabinowitz (1978) “The Hopf bifurcation theorem in infinite dimensions,” *Arch. Rat. Mech. Anal.* 67, 53-72.
- [37] David, Paul (1990) “The dynamo and the computer: A historical perspective on the modern productivity paradox,” *American Economic Review* 80(2), 355-360.
- [38] Diekmann, O., S.A. Van Gils, S.M. Verduyn Lunel and H.O. Walther (1995) *Delay equations*. Springer.
- [39] Dixit, Avinash and Joseph Stiglitz (1977) “Monopolistic competition and optimum product diversity,” *American Economic Review* 67(3), 297-308.
- [40] Doms, Mark and Timothy Dunne (1998) “Capital adjustment patterns in manufacturing plants,” *Review of Economic Dynamics* 1(2), 409-429.
- [41] Dontchev, Asen L. and R. Tyrrell Rockafellar (2009) *Implicit Functions and Solution Mappings: A View from Variational Analysis*. Springer.
- [42] Engelborghs, Koen and Dirk Roose (1999) “Numerical computation of stability and detection of Hopf bifurcations of steady state solutions of delay differential equations,” *Advanced Computational Mathematics* 10(3/4), 271-289.
- [43] Ethier, Wilfred J. (1982) “National and international returns to scale in modern theory of international trade,” *American Economic Review* 72(3), 389-405.
- [44] Fatás, Antonio (2000) “Do business cycles cast long shadows? Short-run persistence and economic growth,” *Journal of Economic Growth* 5(2), 147-162.

- [45] Francois, Patrick and Huy Lloyd-Ellis (2003) “Animal spirits through creative destruction,” *American Economic Review* 93(3), 530-550.
- [46] Freni, Giuseppe, Fausto Gozzi and Neri Salvadori (2006) “Existence of optimal strategies in linear multisector models,” *Economic Theory* 29(1), 25–48.
- [47] Geroski, Paul and Chris Walters (1995) “Innovative activity over the business cycle,” *Economic Journal* 105(431), 916-928.
- [48] Greiner, Alfred and Willi Semmler (1996) “Multiple steady states, indeterminacy, and cycles in a basic model of endogenous growth,” *Journal of Economics* 63(1), 79-99.
- [49] Grossman, Gene and Elhanan Helpman (1991) *Innovation and growth in the global economy*. The MIT Press.
- [50] Guckenheimer, John, and Philip Holmes (1985) *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Applied Mathematical Sciences 42, Springer.
- [51] Hale, J.K., and S.M. Verduyn Lunel (1993) *Introduction to functional differential equations*, Springer.
- [52] Hassard, B.D., N.D. Kazarinoff and Y.H. Wan (1980) *Theory and applications of Hopf bifurcation*, Cambridge University Press.
- [53] Jovanovic, Boyan and Saul Lach (1997) “Product innovation and the business cycle,” *International Economic Review* 38(1), 3-22.
- [54] Judd, Kenneth (1988) *Numerical methods in economics*. The MIT Press.
- [55] Koeninger, Winfred and Omar Licandro (2006) “On the use of substitutability as a measure of competition,” *The B.E. Journal of Macroeconomics* 6(1).
- [56] Kolmanovskii, V. and A. Myshkis (1998) *Introduction to the theory and applications of functional differential equations*. Kluwer Academic Publishers.
- [57] Kolmanovskii, V. and V.R. Nosov (1986) *Stability of functional differential equations*. Academic Press.
- [58] Kydland, Finn and Edward Prescott (1982) “Time to build and aggregate fluctuations,” *Econometrica* 50(6), 1345-70.
- [59] Kuznetsov, Yuri A. (1998) *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences 112, Springer.
- [60] Mansfield, Edwin (1989) “The diffusion of industrial robots in Japan and the United States,” *Research Policy* 18, 183-192.

- [61] Matsuyama, Kiminori (1999) “Growing through cycles,” *Econometrica* 67(2), 335-348.
- [62] Matsuyama, Kiminori (2001) “Growing through cycles in an infinitely lived agent economy,” *Journal of Economic Theory* 100(2), 220-234.
- [63] Neimark, Yu. I. (1949) “D-decomposition of the space of quasipolynomials (on the stability of linearized distributive systems)”, (Russian) *Akad. Nauk SSSR. Prikl. Mat. Meh.* 13, 349–380. English Translation in American Mathematical Society Translations, Series 2. Vol. 102: Ten papers in analysis. American Mathematical Society, Providence, R.I., 1973. pp.95-131.
- [64] Rebelo, Sergio (1991) “Long-run policy analysis and long-run growth,” *Journal of Political Economy* 99(3), 500-521.
- [65] Reichlin, Pietro (1986) “Equilibrium cycles in an overlapping generations economy with production,” *Journal of Economic Theory* 40(1), 89–102.
- [66] Rivera-Batiz, Luis A. and Paul Romer (1991) “Economic integration and endogenous growth,” *Quarterly Journal of Economics* 106(2), 532-555.
- [67] Romer, Paul (1990) “Endogenous technological change,” *Journal of Political Economy* 98(5), 71-102.
- [68] Ruan, Shigui and Junjie Wei (2003) “On the zeros of transcendental functions with applications to stability of delay differential equations with two delays”, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 10, 863-874.
- [69] Ruelle, David (1989) *Elements of Differentiable Dynamics and Bifurcation Theory*. Academic Press.
- [70] Schleifer, Andrei (1986) “Implementation cycles,” *Journal of Political Economy* 94(6), 1163-90.
- [71] Schumpeter, Joseph (1934) *The theory of economic development*. Harvard University Press.
- [72] Walde, Klaus and Ulrich Woitek (2004) “R&D expenditure in G7 countries and the implications for endogenous fluctuations and growth,” *Economics Letters* 81(1), 91-97.