# A LINEAR-TIME ALGORITHM FOR MAXIMUM-CARDINALITY MATCHING ON COCOMPARABILITY GRAPHS*

GEORGE B. MERTZIOS†, ANDRÉ NICHTERLEIN†‡, AND ROLF NIEDERMEIER‡

**Abstract.** Finding maximum-cardinality matchings in undirected graphs is arguably one of the most central graph problems. For general $m$-edge and $n$-vertex graphs, it is well known to be solvable in $O(m\sqrt{n})$ time. We present a linear-time algorithm to find maximum-cardinality matchings on *cocomparability* graphs, a prominent subclass of perfect graphs that strictly contains interval graphs as well as permutation graphs. Our greedy algorithm is based on the recently discovered *Lexicographic Depth First Search* (LDFS).

**1. Introduction.** The problem MATCHING (or MAXIMUM-CARDINALITY MATCHING) is, given an undirected graph, to compute a maximum-cardinality set of disjoint edges. MATCHING is arguably among the most fundamental graph-algorithmic primitives that can be computed in polynomial time. More specifically, the asymptotically fastest known algorithm for computing a maximum-cardinality matching (subsequently called maximum matching) on an $n$-vertex and $m$-edge graph runs in $O(m\sqrt{n})$ time [44]. No faster algorithm is known, even when the given graph is bipartite [24]. Improving this running time, either on general graphs or on bipartite graphs, resisted decades of research. In terms of approximation, it is known that the $O(m\sqrt{n})$ algorithm of Micali and Vazirani [44] implies a $(1 - \epsilon)$-approximation computable in $O(m\epsilon^{-1})$ time [13]. For the weighted case, Duan and Pettie [13] provided a linear-time algorithm that computes a $(1 - \epsilon)$-approximate maximum-weight matching (the constant running time factor depending on $\epsilon$ is $\epsilon^{-1} \log(\epsilon^{-1})$). In this work we take a route different from approximation and identify a large graph class, namely cocomparability graphs, on which we show that an optimal solution can be computed in linear time.

Identifying more efficiently solvable special cases for finding maximum matchings has quite some history. Yuster [56] developed an algorithm with running time $O(rn^2 \log n)$, where $r$ denotes the difference between maximum and minimum vertex degree of the input graph. Moreover, there are (quasi)*linear-time* algorithms for computing maximum matchings in several special classes of graphs, including interval graphs [30], convex bipartite graphs [51], strongly chordal graphs [10], and chordal bipartite graphs [3]. We refer the reader to Table 1 for a more thorough overview, also including results with superlinear running times. See Figure 1 for an

---

†Department of Computer Science, Durham University, DH1 3LH Durham, UK (george.mertzios@durham.ac.uk).

‡Algorithmics and Computational Complexity, Faculty IV, TU Berlin, 10587 Berlin, Germany (andre.nichterlein@tu-berlin.de, rolf.niedermeier@tu-berlin.de).

TABLE 1
*Fastest algorithms for* MATCHING *on special graph classes;* $\omega < 2.373$ *is the matrix multiplication exponent, that is, two* $n \times n$ *matrices can be multiplied in* $O(n^\omega)$ *time.*

| Graph class | Running time |
|---|---|
| general | $O(m\sqrt{n})$ [44], $O(m\sqrt{n}\log(n^2/m)/\log(n))$ [22], $O(n^\omega)$ (rand.) [46] |
| bipartite | $O(m\sqrt{n})$ [24], $O(n^\omega)$ (rand.) [46, 49], $O(m^{1.43})$ [33] |
| interval | $O(n\log n)$ (given an interval representation) [30, 45] |
| circular arcs | $O(n\log n)$ [30] |
| cointerval | $O(n\log n + m)$ [19] |
| convex | $O(n)$ [51] |
| planar | $O(n^{\omega/2})$ (rand.) [47] |
| strongly chordal | $O(n + m)$ (given the strong perfect elimination order) [10] |
| chordal bipartite | $O(n + m)$ [3] |
| regular | $O(n^2 \log n)$ [56] |
| cographs | $O(n)$ (given a cotree) [55] |
| cocomparability | $O(n + m)$ (Theorem 2.12 in section 2) |
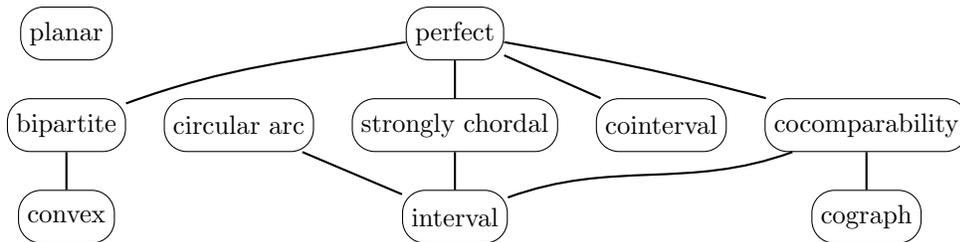


FIG. 1. *Overview of most of the graph classes mentioned in Table* 1. *An edge indicates that the class above strictly contains the class below.*

overview concerning the containment relation between the graph classes.

A graph $G$ is a *cocomparability* graph if its complement $\overline{G}$ admits a transitive orientation of its edges. These graphs (as well as their complements, i.e., comparability graphs) arise naturally in several real-world applications as they are closely related to *partially ordered sets* (also referred to as *posets*). In particular, a given cocomparability graph $G$, together with a transitive orientation of the edges of its complement $\overline{G}$, can be equivalently represented by a poset. Cocomparability graphs have been the subject of intensive theoretical research [4, 6, 7, 11, 14, 15, 26, 27, 29, 39, 43]. On the one hand, cocomparability graphs naturally generalize well-studied graph classes such as interval and permutation graphs [2, 23], *trapezoid* (or *bounded multitolerance*) graphs [25, 28, 32, 36, 38, 52], *parallelogram* (or *bounded tolerance*) graphs [20, 41, 42], *triangle* (or *PI**) graphs [5, 35], and *simple-triangle* (or *PI*) graphs [37, 53, 54]. On the other hand, cocomparability graphs form an "almost maximal" subclass of perfect graphs [2].[1] Since perfect graphs (as well as comparability graphs) properly contain bipartite graphs (for which improving the $O(m\sqrt{n})$ running time is a long-standing open question), it seems out of reach to obtain an algorithm for MATCHING with linear running time on perfect graphs. Consequently, designing a linear-time algorithm for cocomparability graphs provides a sharp boundary between $O(n+m)$-time algorithms and the known $O(m\sqrt{n})$-time algorithms for MATCHING.

---

[1]For an overview of the relation between graph classes see http://www.graphclasses.org/.

*Our contribution.* In this paper we present a linear-time algorithm for Matching on *cocomparability* graphs. It is a simple greedy algorithm, referred to as *Rightmost Matching (RMM)*, running on a specific vertex ordering. Essentially the same greedy approach was earlier considered by Dragan [12] in the context of greedy matchable graphs.[2] The vertex ordering is obtained by using (as a preprocessing step) the recently discovered *Lexicographic Depth First Search* (LDFS) algorithm [8]. Interestingly it turns out that RMM computes in a *trivial* way a maximum matching on interval graphs, when applied on the standard interval graph vertex ordering.[3] Note that the class of interval graphs is a strict subset of the class of cocomparability graphs. So far a similar phenomenon of extending an interval graph algorithm to cocomparability graphs by using an LDFS preprocessing step has also been observed for the Longest Path problem [39], the Minimum Path Cover problem [6], and the Maximum Independent Set problem [7]. Our results for the RMM algorithm, adding to the previous results [6, 7, 39], provide evidence that cocomparability graphs present an "interval graph structure" when they are considered with an LDFS preprocessing step. This insight is of independent interest and might lead to new and more efficient combinatorial algorithms.

*Preliminaries.* We use standard notation from graph theory. In particular, all paths we consider are simple paths. A *matching* in a graph is a set of pairwise disjoint edges. Let $G = (V, E)$ be an undirected graph, and let $M \subseteq E$ be a matching in $G$. A vertex $v \in V$ is called *matched* with respect to $M$ if there is an edge in $M$ containing $v$; otherwise $v$ is called *free* with respect to $M$. If the matching $M$ is clear from the context, then we omit "with respect to $M$." An *alternating path* with respect to $M$ is a path in $G$ such that every second edge of the path belongs to $M$. An *augmenting path* is an alternating path whose endpoints are free. It is well known that a matching $M$ is maximum if and only if there is no augmenting path for it [31].

A graph $G = (V, E)$ is an *interval* graph if we can assign to each vertex of $G$ a closed interval on the real line such that two vertices are adjacent in $G$ if and only if the corresponding two intervals intersect. A *comparability* graph is a graph whose edges can be transitively oriented; that is, if $u \to v$ (the edge $\{u, v\}$ is oriented towards $v$) and $v \to w$, then $u \to w$. A *cocomparability* graph $G$ is a graph whose complement $\overline{G}$ is a comparability graph. The class of interval graphs is strictly included in the class of cocomparability graphs [2]. Intuitively, we can transitively orient the "nonedges" of an interval graph, using the following ordering of nonintersecting intervals from left to right: Consider three intervals $I_a, I_b, I_c$ in an interval representation of an interval graph. If $I_a$ lies completely to the left of $I_b$, and $I_b$ lies completely to the left of $I_c$, then also $I_a$ lies completely to the left of $I_c$.

**2. A linear-time algorithm for cocomparability graphs.** To begin with, we present in subsection 2.1 a simple greedy linear-time algorithm (called RMM) for computing a maximum matching $M$ on interval graphs. Subsequently we provide in subsection 2.2 all necessary background on vertex orderings for cocomparability graphs and on the LDFS, which is needed for our algorithm on cocomparability graphs. Finally, as our central result, we prove in subsection 2.3 that the algorithm RMM actually works also for cocomparability graphs.

---

[2]Refer to Remark 2 in subsection 2.3 for a discussion about the subtle but important differences from our approach.

[3]This is the vertex ordering that results from sorting the intervals according to their left endpoints. The RMM algorithm for interval graphs was discovered by Moitra and Johnson [45].

**2.1. The greedy algorithm for interval graphs.** Given an interval graph $G$ with $n$ vertices and $m$ edges, we first compute in $O(n+m)$ time an interval representation of $G$ and, at the same time, we also sort the intervals according to their left endpoint [48]. The algorithm works as follows (cf. [30, 45]):

1. Initialize $M = \emptyset$ and label all vertices as "unvisited."
2. Pick the unvisited vertex (interval) $x$ which has the rightmost left endpoint among all currently unvisited vertices in $G$. Then, label $x$ as "visited."
3. If $x$ has at least one unvisited neighbor in $G$, then pick the unvisited neighbor $y$ of $x$ which has the rightmost left endpoint among all unvisited neighbors of $x$. Then label $y$ as "visited" and add the edge $\{x,y\}$ to $M$.
4. If there is still an unvisited vertex in $G$, then go to step 2.
5. Return $M$.

We call the above algorithm *Rightmost Matching* (RMM). It can be executed in $O(n+m)$ time; with a simple exchange argument we can show that the matching $M$ returned by RMM is indeed maximum in $G$. This algorithm implicitly uses the following vertex ordering that characterizes interval graphs. It corresponds to sorting the intervals according to their left endpoints and can be computed in $O(n+m)$ time from $G$ [48].

LEMMA 2.1 (see [48]). *$G = (V, E)$ is an interval graph if and only if there exists a vertex ordering $\sigma$ of $G$ (called an I-ordering) such that, for all $x <_\sigma y <_\sigma z$, if $\{x, z\} \in E$, then also $\{x, y\} \in E$.*

**2.2. Cocomparability graphs and vertex orderings.** Before we proceed with our algorithm RMM and its analysis on cocomparability graphs (see subsection 2.3), we now state vertex ordering characterizations of cocomparability graphs and of any vertex ordering that can result from an LDFS search on an arbitrary graph. The following vertex ordering characterizes cocomparability graphs [27].

DEFINITION 2.2 (see [27]). *Let $G = (V, E)$ be a graph. An ordering $\pi$ of the vertices $V$ is an umbrella-free ordering (or a CO-ordering) if for all $x <_\pi y <_\pi z$ it holds that if $\{x, z\} \in E$, then $\{x, y\} \in E$ or $\{y, z\} \in E$ (or both).*

LEMMA 2.3 (see [27]). *A graph $G = (V, E)$ is a cocomparability graph if and only if there exists an umbrella-free ordering $\pi$ of $V$.*

Umbrella-free orderings directly generalize I-orderings for interval graphs (see Lemma 2.1). It is worth noting here that, although there exists a linear-time algorithm to *compute* an umbrella-free ordering $\pi$ of a given cocomparability graph [34], the fastest known algorithm to *verify* that a given vertex ordering is indeed umbrella-free needs the same time as boolean matrix multiplication (Spinrad [50] discusses this issue). As an example, we illustrate in Figure 2 the cocomparability graph $\overline{C_6}$, i.e., the complement of the cycle on six vertices. In this graph, it is straightforward to check by Definition 2.2 that the vertex ordering $\pi = (b, d, c, f, e, a)$ is indeed an umbrella-free ordering.

In the following we present the notion of an LDFS ordering $\sigma$ (see Definition 2.5) due to Corneil and Krueger [8]. This notion is based on *good triples* and *bad triples*, which are defined next.

DEFINITION 2.4 (see [8]). *Let $G = (V, E)$ be a graph and $\sigma$ be an arbitrary ordering of $V$. Let $a, b, c \in V$ be three vertices such that $a <_\sigma b <_\sigma c$, $\{a, c\} \in E$, and $\{a, b\} \notin E$. If there exists a vertex $d$ such that $a <_\sigma d <_\sigma b$, $\{d, b\} \in E$, and $\{d, c\} \notin E$, then $(a, b, c)$ is a good triple; otherwise it is a bad triple.*

DEFINITION 2.5 (see [8]). *Let $G = (V, E)$ be a graph. An ordering $\sigma$ of $V$ is an*
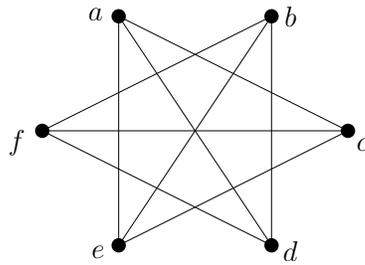
FIG. 2. *The cocomparability graph $G = \overline{C_6}$, i.e., the complement of a cycle on six vertices. The vertex ordering $\pi = (b, d, c, f, e, a)$ is an umbrella-free ordering for $G$.*
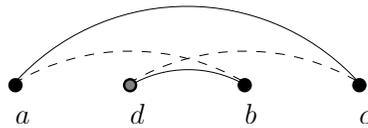


FIG. 3. *A good triple $(a, b, c)$ and its vertex $d$ as in Definition 2.4, in the vertex ordering $\sigma = (a, d, b, c)$. The edges $\{a, c\}$ and $\{d, b\}$ are indicated with solid lines and the nonedges $\{a, b\}$ and $\{d, c\}$ with dashed lines. Note that $\{a, d\}$ and $\{b, c\}$ can be edges or nonedges.*

LDFS ordering *if $\sigma$ has no bad triple.*

An example of a good triple $(a, b, c)$ and the corresponding fourth vertex $d$ is depicted in Figure 3. Now we present the generic LDFS algorithm (Algorithm 1) due to Corneil and Krueger [8]. LDFS runs on an arbitrary connected graph $G$, starting at a distinguished vertex $u$. It is a variation of the well-known Depth First Search (DFS) algorithm; the main difference is that LDFS assigns labels to the vertices and uses the lexicographic order over these labels as a tie-breaking rule. Briefly, it proceeds as follows. Initially, the label $\varepsilon$ is assigned to every vertex. Then, iteratively, an unvisited vertex $v$ with a lexicographically maximum label is chosen and removed from the graph. If $v$ is chosen as the $i$th vertex, then the label of each of its unvisited neighbors is being updated by *prepending* the digit $i$ to it. Note that the digits in the label of any vertex are always in decreasing order. Hence all neighbors of the last chosen vertex have a lexicographically greater label than all its nonneighbors, and thus all vertices are visited in a depth first search order.

---

**Algorithm 1.** LDFS$(G, u)$ [8].

**Input:** A connected graph $G = (V, E)$ with $n$ vertices and a vertex $u \in V$.
**Output:** An LDFS ordering $\sigma_u$ of the vertices of $G$.

1: Assign the label $\varepsilon$ to all vertices and mark all vertices as unnumbered
2: label$(u) \leftarrow \{0\}$
3: **for** $i = 1$ to $n$ **do**
4:    Pick an unnumbered vertex $v$ with the lexicographically largest label
5:    $\sigma_u(i) \leftarrow v$                          {assign to $v$ the number $i$; $v$ is now numbered}
6:    **for** each unnumbered vertex $w \in N(v)$ **do**
7:       prepend $i$ to label$(w)$

8: **return**  the ordering $\sigma_u = (\sigma_u(1), \sigma_u(2), \ldots, \sigma_u(n))$

---

The execution of the LDFS algorithm is illustrated with the running example of Figure 2. In this example, suppose that the LDFS algorithm starts at vertex $a$. Suppose that LDFS chooses vertex $c$ next. Now, ordinary DFS could choose either $e$ or $f$ next, but LDFS has to choose $e$, since $e$ has a greater label than $f$ ($e$ is a neighbor of the previously visited vertex $a$). The next visited vertex has to be $b$, since it is the only unvisited neighbor of $e$. The vertex following $b$ in the LDFS ordering $\sigma_a$ must be $f$ rather than $d$, since $f$ has a greater label than $d$ ($f$ is a neighbor of vertex $c$ which has been visited more recently than $d$'s neighbor $a$). Finally, LDFS visits the last vertex $d$, completing the LDFS ordering as $\sigma_a = (a, c, e, b, f, d)$.

It is important here to connect the vertex ordering $\sigma_u$ that is returned by the LDFS algorithm (i.e., Algorithm 1) with the notion of an LDFS ordering, as defined in Definition 2.5. The next theorem due to Corneil and Krueger [8] shows that a vertex ordering $\sigma$ of an arbitrary graph $G$ can be returned by an application of the LDFS algorithm to $G$ (starting at some vertex $u$ of $G$) if and only if $\sigma$ is an LDFS ordering.

THEOREM 2.6 (see [8]).   *For an arbitrary graph $G = (V, E)$, an ordering $\sigma$ of $V$ can be returned by an application of Algorithm 1 to $G$ if and only if $\sigma$ is an LDFS ordering.*

In the generic LDFS, there can be some choices to be made at Line 4 of Algorithm 1. More specifically, at some iteration there may be two or more vertices that have the same label; in this case the algorithm must break ties and choose one of these vertices. Generic LDFS (i.e., Algorithm 1) allows an arbitrary choice here. We present in the following a special type of LDFS algorithm, called LDFS$^+$ (see Algorithm 2 below), which chooses a specific vertex in such a case of equal labels, as follows. Along with the graph $G = (V, E)$, an ordering $\pi$ of $V$ is also given as input. The algorithm LDFS$^+$ operates exactly as a generic LDFS that starts at the *rightmost* vertex of $V$ in the ordering $\pi$, with the only difference being that, in the case where at some iteration at least two unvisited vertices have the same label, LDFS$^+$ chooses the *rightmost* vertex among them in the input ordering $\pi$. The resulting ordering is then denoted $\sigma = \text{LDFS}^+(G, \pi)$.

---

**Algorithm 2.** LDFS$^+$ $(G, \pi)$.

**Input:** A connected graph $G = (V, E)$ with $n$ vertices and an ordering $\pi$ of $V$.
**Output:** An LDFS ordering $\sigma$ of the vertices of $G$.

1: Assign the label $\varepsilon$ to all vertices and mark all vertices as unnumbered
2: **for** $i = 1$ to $n$ **do**
3:    Pick the rightmost vertex $v$ in $\pi$ among the unnumbered vertices with the lexicographically largest label
4:    $\sigma(i) \leftarrow v$                    {assign to $v$ the number $i$; $v$ is now numbered}
5:    **for** each unnumbered vertex $w \in N(v)$ **do**
6:       prepend $i$ to label($w$)

7: **return**  the ordering $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$

---

Consider our running example of Figure 2. In this graph $G = \overline{C_6}$, suppose that LDFS$^+$ is given as input the umbrella-free ordering $\pi = (b, d, c, f, e, a)$. Then the ordering $\sigma = \text{LDFS}^+(G, \pi)$ is computed using Algorithm 2 as follows. The first visited vertex is $a$, since $a$ is the rightmost vertex in the ordering $\pi$. Now, LDFS (see Algorithm 1) could choose any of the neighbors $c, d, e$ of $a$ next, but LDFS$^+$ (see

Algorithm 2) has to choose $e$, since $e$ is the rightmost among these vertices in the ordering $\pi$. In this example there exists no further tie among vertices with the same label. Thus, proceeding similarly to our example vertex ordering for LDFS above, it follows that the resulting ordering $\sigma = \mathrm{LDFS}^+(G, \pi)$ is $\sigma = (a, e, c, f, b, d)$.

For the purposes of our algorithm RMM for computing a maximum matching on cocomparability graphs in subsection 2.3, we will consider an arbitrary umbrella-free vertex ordering $\pi$ of the input cocomparability graph $G$, and we will then compute the LDFS ordering $\sigma = \mathrm{LDFS}^+(G, \pi)$, by applying Algorithm 2 (i.e., $\mathrm{LDFS}^+$) to $\pi$. Our RMM algorithm (see Algorithm 3) will then take this LDFS ordering $\sigma$ as input, together with the graph $G$. It is important to note here that, starting from an umbrella-free ordering $\pi$, the LDFS vertex ordering $\sigma = \mathrm{LDFS}^+(G, \pi)$ remains umbrella-free [6]. That is, $\sigma$ satisfies both the conditions of Definition 2.2 and Definition 2.5, and thus $\sigma$ is *simultaneously* an LDFS ordering and an umbrella-free ordering. For this reason we refer to $\sigma$ as an *LDFS umbrella-free* vertex ordering of the input cocomparability graph $G$. Finally, note that, given an umbrella-free ordering $\pi$ of a cocomparability graph $G$ with $n$ vertices and $m$ edges, the ordering $\mathrm{LDFS}^+(G, \pi)$ can be computed in $O(n + m)$ time [26].

**2.3. The algorithm for cocomparability graphs.** Once we have computed in $O(n + m)$ time the LDFS umbrella-free ordering $\sigma = \mathrm{LDFS}^+(G, \pi)$, we apply our simple linear-time algorithm *Rightmost Matching* (RMM) (see Algorithm 3) to compute a new vertex ordering $\widehat{\sigma}$ and a maximum matching $M$ of $G$. RMM is a simple greedy algorithm which operates as follows. At every step it visits the rightmost unvisited vertex $x$ in $\sigma$ and it labels $x$ as visited. Then, if $x$ does not have any unvisited neighbor, then RMM proceeds at the next step by visiting the rightmost currently unvisited vertex in $\sigma$; note that this vertex is now different from $x$, as $x$ has already been labeled as visited. Otherwise, if $x$ has at least one unvisited neighbor, then RMM visits after $x$ its rightmost unvisited neighbor $y$ in the ordering $\sigma$, and it also adds the edge $\{x, y\}$ to the computed matching $M$.

---

**Algorithm 3.** $\mathrm{RMM}(G, \sigma)$.

**Input:** A cocomparability graph $G$ with an LDFS umbrella-free ordering $\sigma$ of $G$.
**Output:** A vertex ordering $\widehat{\sigma}$ of $G$ and a maximum matching of $G$.

1: Label all vertices "unvisited"; $i \leftarrow 0$; $M \leftarrow \emptyset$
2: **while** there are unvisited vertices **do**
3:    Pick the rightmost unvisited vertex $x$ in $\sigma$ and label $x$ as "visited"
4:    $i \leftarrow i + 1$; $\widehat{\sigma}(i) \leftarrow x$           {add vertex $x$ to the ordering $\widehat{\sigma}$}
5:    **if** $x$ has at least one unvisited neighbor **then**
6:      Pick the rightmost unvisited neighbor $y$ of $x$ and label $y$ as "visited"
7:      $i \leftarrow i + 1$; $\widehat{\sigma}(i) \leftarrow y$         {add vertex $y$ to the ordering $\widehat{\sigma}$}
8:      $M \leftarrow M \cup \{\{x, y\}\}$                    {match $x$ and $y$}
9: **return**   the ordering $\widehat{\sigma}$ and the matching $M$

---

*Remark* 1. Since any I-ordering of an interval graph is also an LDFS umbrella-free ordering (see Lemma 2.1 and Definitions 2.2, 2.4, and 2.5), note that Algorithm 3 also works with an interval graph $G$ and an I-ordering $\sigma$ of $G$ as input. In this case, $\mathrm{RMM}(G, \sigma)$ is actually exactly the same RMM algorithm as we sketched in subsection 2.1 for interval graphs.

*Remark* 2. Essentially the same greedy approach as our RMM algorithm was already considered by Dragan [12].[4] More specifically, he characterized those graphs $G$ which admit a vertex ordering $\tau$ such that the greedy algorithm computes a maximum matching on *every induced subgraph* $F$ of $G$ when applied to the induced subordering of $\tau$ on the vertices of $F$. These graphs $G$ having the above property are called *greedy matchable* graphs [12]. We prove that cocomparability graphs admit a vertex ordering $\sigma$ (namely an LDFS umbrella-free ordering) such that the greedy algorithm computes a maximum matching in the input graph $G$ itself (and *not* in every induced subgraph of $G$). That is, Dragan [12] studied a problem that is very different from computing a maximum matching in a given graph.

Dragan [12] proved that greedy matchable graphs form a subclass of weakly triangulated graphs; a graph is weakly triangulated if it contains neither (as an induced subgraph) a chordless cycle of length at least five nor the complement of such a chordless cycle. On the contrary, cocomparability graphs are not a subclass of weakly triangulated graphs since, for every $k \geq 3$, the complement $\overline{C_{2k}}$ of a chordless cycle with length $2k$ is a cocomparability graph. Indeed, the complement of a $\overline{C_{2k}}$ (i.e., the chordless cycle $C_{2k}$) can be transitively oriented. Therefore, our results do not follow from the paper of Dragan [12].

More specifically, one of the main results of Dragan (see Theorem 1 in [12]) is that a graph $G$ is greedy matchable if and only if $G$ admits an *admissible vertex ordering* (as defined in Definition 3 of [12]). Admissible orderings are characterized by the nine forbidden suborderings as shown in Figure 1 of Dragan [12]. However, three of these forbidden suborderings (namely the 2nd, the 5th, and the 9th) are in fact LDFS umbrella-free orderings (see Definitions 2.2 and 2.5 of our paper). To see this, observe that each of these three orderings (i) is umbrella-free (see our Definition 2.2) and (ii) does not contain any triple $a, b, c$ of vertices such that $a <_\sigma b <_\sigma c$, $\{a, c\} \in E$, and $\{a, b\} \notin E$ (see our Definitions 2.4 and 2.5).

To illustrate this with an example, consider the graph $G = \overline{C_6}$ of our Figure 2 and recall from subsection 2.2 that $\sigma = (a, e, c, f, b, d)$ is an LDFS umbrella-free vertex ordering of $G$. Note that this ordering $\sigma$ contains the orderings $(a, e, c, d)$ and $(a, e, c, b)$ as induced suborderings. Furthermore, note that these suborderings correspond to the 2nd and the 5th forbidden suborderings of Figure 1 in Dragan's paper [12], respectively. Hence, $\sigma$ is an example of an LDFS umbrella-free ordering which is not an admissible ordering; this is an alternative explanation of why our results do not follow from Dragan's paper [12].

In the remainder of this section, we show that the matching $M$ returned by RMM$(G, \sigma)$ is indeed a maximum matching of $G$. The proof is by contradiction and uses an appropriate *potential function* $f$ that is defined over all matchings of $G$:

DEFINITION 2.7 (potential function). *Let $G = (V, E)$ be a cocomparability graph and $\sigma$ be an LDFS umbrella-free ordering of $V = \{v_1, \ldots, v_n\}$ with $v_1 <_\sigma \cdots <_\sigma v_n$. Let $M$ be a matching of $G$. Then the potential function is $f(M) := \sum_{i=1}^n g_M(v_i)$, where for each $v_i \in V$*

$$g_M(v_i) := \begin{cases} 0 & \text{if } \{v_i, v_j\} \in M \text{ and } i < j, \\ (i - j) \cdot (n + 1)^i & \text{if } \{v_i, v_j\} \in M \text{ and } j < i, \\ i \cdot (n + 1)^i & \text{if } v_i \text{ is not matched within } M. \end{cases}$$

---

[4]The only difference is that Dragan's algorithm visits the vertices from left to right and always matches a vertex with its leftmost unvisited neighbor.

Note by Definition 2.7 that, for the empty matching, we have $f(\emptyset) = \sum_{i=1}^{n} i \cdot (n+1)^i$. Then, as we add an edge $\{v_i, v_j\}$ to the current matching $M$, where $j < i$ and $v_i$ and $v_j$ are unmatched, we have that

$$f(M \cup \{\{v_i, v_j\}\}) = f(M) - i(n+1)^i - j(n+1)^j + (i-j)(n+1)^i$$
$$= f(M) - j((n+1)^j + (n+1)^i) < f(M).$$

Thus, adding edges to a matching decreases the potential function value. The exponential dependency on the vertex-index in $g_M$ ensures that matching vertices with higher index has a larger impact than matching vertices with lower index. Furthermore, aiming at a small potential function value also means that the endpoints of the matched edges have only a small index difference. We formalize this intuition in the next observation.

OBSERVATION 2.8. *Let $G = (V, E)$ be a graph and $\sigma$ be an arbitrary ordering of $V = \{v_1, \ldots, v_n\}$ with $v_1 <_\sigma \cdots <_\sigma v_n$. Let $M$ and $M'$ be two different matchings of $G$ such that at $v_i$ is the rightmost difference between $M$ and $M'$, that is, each vertex $v_\ell$, $\ell > i$, is either free in both $M$ and $M'$ or matched with the same $v_{\ell'}$ in both $M$ and $M'$. Suppose that*
   - *$\{v_j, v_i\} \in M' \setminus M$, $j < i$, and*
   - *$v_i$ is in $M$ either free or matched to some $v_{j'}$, $j' < j$.*
*Then, $f(M') < f(M)$.*

*Proof.* We have

$$f(M') - f(M) = \sum_{k=1}^{n} g_{M'}(v_k) - g_M(v_k) = \sum_{i=k}^{i} g_{M'}(v_k) - g_M(v_k)$$

as by assumption $v_i$ is the rightmost vertex where $M$ and $M'$ differ. Then,

$$f(M') - f(M) = g_{M'}(v_i) - g_M(v_i) + \sum_{k=1}^{i-1} g_{M'}(v_k) - g_M(v_k)$$

$$< (i-j)(n+1)^i - (i-x)(n+1)^i + \sum_{k=1}^{i-1} g_{M'}(v_k),$$

where $x = 0$ if $v_i$ is free in $M$ or $x = j'$ if $v_i$ is matched to $v_{j'}$ in $M$. In both cases we have $j > x$ and thus

$$f(M') - f(M) < -(n+1)^i + \sum_{k=1}^{i-1} g_{M'}(v_k) < -(n+1)^i + \sum_{k=1}^{i-1} n(n+1)^k$$

$$= -(n+1)^i + n\frac{(n+1)^i - 1}{n} - 1$$

$$= -(n+1)^i + (n+1)^i - 2 < 0. \qquad \square$$

With the above observation the connection between the RMM algorithm and the potential function $f$ is easy to see.

OBSERVATION 2.9. *The matching $M$ returned by $RMM(G, \sigma)$ minimizes the function $f(M)$.*

*Proof.* Let $M'$ be a matching such that $f(M')$ is minimum. Consider the rightmost vertex $v_n$ in the ordering $\sigma$ and let $v_i$ be the rightmost neighbor of $v_n$ in $\sigma$. Assume that $\{v_i, v_n\} \notin M'$. Then let $M''$ be the matching obtained by removing from $M'$ any edges with endpoints $v_i$ or $v_n$, and by adding to it the edge $\{v_i, v_n\}$. By Observation 2.8, we have $f(M'') < f(M')$, a contradiction. Thus $\{v_i, v_n\} \in M'$. We can now recursively apply the same argument in the induced subgraph $G[(V \setminus \{v_i\} \setminus) \{v_n\}]$, which eventually implies that $M'$ is the matching returned by RMM$(G, \sigma)$. $\square$

Before we prove our main result in Theorem 2.12, we need to prove a crucial technical lemma (Lemma 2.11). On a high level, our proof strategy is as follows: We consider a maximum matching $M$ minimizing $f$ and a matching $M'$ produced by Algorithm 3. If $M = M'$, then we are done. Otherwise, we take the "rightmost" difference between $M$ and $M'$, that is the rightmost vertex $v$ in $M$ that is not matched in the same way in $M'$ (or $v$ is free in exactly one of the two matchings). Then we show that matching $v$ in $M$ as in $M'$ leads to another maximum matching $M''$ such that $f(M'') < f(M)$. To show this, we make a case distinction where in two cases we need to exclude the special scenario described in Lemma 2.11. The existence of $M''$ would be a contradiction to our choice of $M$. This shows that $M = M'$.

In the next definition we introduce for every vertex $v$ the induced subgraph $G_\sigma(v)$ with respect to the ordering $\sigma$, which is fundamental for the statement and the proof of Lemma 2.11.

DEFINITION 2.10. *Let $G = (V, E)$ be a cocomparability graph and let $\sigma = (v_1, v_2, \ldots, v_n)$ be an LDFS umbrella-free vertex ordering of $G$. Then, for every $v_i \in V$, the graph $G_\sigma(v_i)$ is the induced subgraph of $G$ on the vertices $\{v_1, v_2, \ldots, v_i\}$.*

LEMMA 2.11. *Let $G = (V, E)$ be a cocomparability graph and $\sigma$ be an LDFS umbrella-free ordering of $V$. Let $M$ be a maximum matching of $G$ such that $f(M)$ is minimum among all maximum matchings. Then, there is no quadruple $(a, b, c, x)$ of vertices in $G$ satisfying all of the following six conditions:*
1. *$a <_\sigma b <_\sigma c \leq_\sigma x$,*
2. *$\{a, c\}, \{b, c\} \in E$ and $\{a, b\} \notin E$,*
3. *$\{a, c\} \in M$,*
4. *there is no odd-length alternating path from $a$ to $b$ within $G_\sigma(x)$,*
5. *there is no odd-length alternating path from $a$ to any free vertex $v$ within $G_\sigma(x)$, and*
6. *there is no odd-length alternating path from $b$ to any free vertex $v$ within $G_\sigma(x)$.*

*Proof.* Let $G$, $\sigma$, and $M$ be as described in the statement of the lemma (see Figure 4). The proof is by contradiction. Towards a contradiction let $(a, b, c, x)$ be a quadruple of vertices satisfying all six conditions of the lemma. Fix now vertex $x$. Among all such quadruples with fixed $x$, let $(a, b, c, x)$ be such that $a$ is leftmost in $\sigma$; that is, for any other such quadruple $(a', b', c', x)$ we have $a \leq_\sigma a'$. Since $\sigma$ is an LDFS ordering, it follows from conditions 1 and 2 and Definitions 2.4 and 2.5 that there is a vertex $d$ such that $a <_\sigma d <_\sigma b$, $\{d, b\} \in E$, and $\{d, c\} \notin E$. Since $\{d, c\} \notin E$ and $\sigma$ is umbrella-free, it follows that $\{a, d\} \in E$. Observe that $d$ is matched in $M$ as otherwise conditions 5 and 6 would be violated. Thus, there is a vertex $e \in V$ with $\{e, d\} \in M$. Now we distinguish three cases with respect to the position of $e$ in the ordering $\sigma$.

**Case 1: $c <_\sigma e$.** In this case we have that $a <_\sigma c <_\sigma e$, $\{d, e\} \in E$, and $\{d, c\} \notin E$. Thus, since $\sigma$ is umbrella-free, it follows that $\{c, e\} \in E$. However, in this case for the matching $M' = (M \setminus \{\{a, c\}, \{d, e\}\}) \cup \{\{e, c\}, \{a, d\}\}$ we invoke Observation 2.8
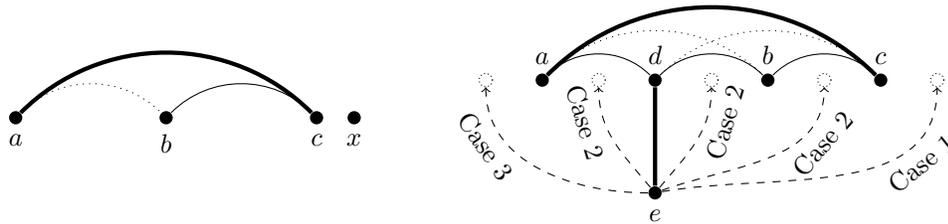
FIG. 4. *Bold lines indicate matched edges; dotted lines indicate a nonedge. Left: Situation for invoking Lemma* 2.11 *displaying conditions* 1 *to* 3 *where* $c \neq x$. *Right: The case distinction in the proof of Lemma* 2.11 *over the position of e in the order.*

with $v_i = e$ to obtain $f(M') < f(M)$, a contradiction.

**Case 2: $a <_\sigma e <_\sigma c$.** If $\{a, e\} \in E$, then there exists the length-three alternating path $(a, e, d, b)$ from $a$ to $b$ within $G_\sigma(x)$, which is a contradiction to condition 4. Thus, $\{a, e\} \notin E$. Furthermore, $\{c, e\} \in E$, since $\sigma$ is umbrella-free and $\{a, c\} \in E$. Hence, for the matching $M' = (M \setminus \{\{a, c\}, \{d, e\}\}) \cup \{\{e, c\}, \{a, d\}\}$ we invoke Observation 2.8 with $v_i = c$ to obtain $f(M') < f(M)$, a contradiction.

**Case 3: $e <_\sigma a$.** In this case it follows similarly to Case 2 that $\{a, e\} \notin E$ (proof by contradiction due to condition 4). Furthermore, observe that $\{e, d\}, \{a, d\} \in E$ and $\{e, d\} \in M$. Thus the triple $(e, a, d)$ satisfies conditions 1 to 3. Furthermore, if there exists an odd-length alternating path from $e$ to $a$ within $G_\sigma(x)$, then this alternating path can be extended through $d$ to an odd-length alternating path from $a$ to $b$ within $G_\sigma(x)$, which is a contradiction to condition 4. Hence there is no odd-length alternating path from $e$ to $a$ within $G_\sigma(x)$. Similarly, odd-length alternating paths from $e$ (resp., from $a$) to a free vertex $v$ within $G_\sigma(x)$ are excluded as well due to condition 5 (resp., due to condition 6). Thus the quadruple $(e, a, d, x)$ satisfies the six conditions of the lemma and it holds that $e <_\sigma a$, a contradiction to the choice of the initial quadruple $(a, b, c, x)$.                    □

We are now ready to prove our central result.

THEOREM 2.12. *For any $n$-vertex and $m$-edge cocomparability graph $G$, Algorithm* 3 *returns a maximum matching $M$ of $G$ in $O(n + m)$ time.*

*Proof.* Let $G = (V, E)$ be a cocomparability graph, and let $\sigma$ be an umbrella-free LDFS ordering of $G$. First we prove that Algorithm 3 runs in $O(n + m)$ time. To this end, we denote with $\deg(v)$ the degree of a vertex $v \in V$. During the execution of the algorithm we maintain the *unvisited vertices* in a doubly linked list $A$ (initially of size $n$), according to their position in $\sigma$. Furthermore, we maintain for each vertex $u$ its *unvisited neighbors* in a doubly linked list $N_u$ (initially of size $\deg(u)$), again according to their position in $\sigma$. Once we have computed the ordering $\sigma$, the construction of the list $A$ can be done in $O(n)$ time. The construction of all lists $N_u$, where $u \in V$, can be done in $O(n + m)$ time as follows. We initialize $N_u = \emptyset$ for every $u \in V$. Then we iterate for each vertex $u \in V$ in the list $A$ from left to right. For every such vertex $u$ we scan (in an arbitrary order) through its neighborhood $N(u)$ (note that $N_u$ is at this point still incomplete), and for each $v \in N(u)$ we append vertex $u$ in the list $N_v$.

Line 16 can be clearly executed in $O(n)$ time. The rightmost unvisited vertex $x$ in Line 18 can be found in $O(1)$ time as the rightmost vertex in the list $A$. Once $x$ is detected in Line 18, $x$ is removed from $A$ also in $O(1)$ time. Furthermore, $x$ is removed from all lists $N_u$, where $\{x, u\} \in E$, in $O(\deg(x))$ time since $x$ is always

the last element in the respective list. Moreover, Line 19 can be clearly executed in $O(1)$ time. The if-condition of Line 20 can be checked in $O(1)$ time by just checking whether the list $N_x$ is empty. Similarly to Line 18, Line 21 can be executed in $O(\deg(y))$ time. Furthermore, each of Lines 22 to 24 can be clearly executed in $O(1)$ time. Summarizing, the total running time of Algorithm 3 is $O(n + \sum_{u \in V} \deg(u)) = O(n + m)$.

For the correctness part, the proof is done by contradiction. Let $M$ be the matching returned by $\text{RMM}(G, \sigma)$. Assume towards a contradiction that $M$ is not a maximum matching. For the rest of the proof, let $M'$ denote a maximum matching that minimizes $f(M')$ among all maximum matchings of $G$. Let $x$ be the rightmost vertex in $\sigma$ on which $M$ differs from $M'$. Then $x$ is matched in at least one of the two matchings $M$ and $M'$. Now we distinguish three cases with respect to the vertex that is matched with $x$ in $M$ and $M'$.

**Case 1: $x$ is matched in $M'$ to some $y \in V$ but is free in $M$.** Then $M$ and $M'$ also differ at vertex $y$. Thus $y <_\sigma x$, since $x$ is the rightmost vertex in which $M$ and $M'$ differ. Consider the iteration $t$ of Algorithm 3 during which the algorithm visits $x$. If $y$ is free in $M$, then this leads to a contradiction; indeed, otherwise Algorithm 3 would have matched $x$ in iteration $t$ as $x$ has at least one unvisited neighbor, namely $y$. Hence, the vertex $y$ is matched in $M$ with a vertex $z$ at an earlier iteration $t' < t$. Then $M$ differs from $M'$ also at vertex $z$. If $z <_\sigma x$, then Algorithm 3 visits $x$ at an earlier iteration than $z$, which is a contradiction to the assumption on $z$. Hence $x <_\sigma z$. This is a contradiction to the assumption that $x$ is the rightmost vertex in $\sigma$ in which $M$ differs from $M'$.

**Case 2: $x$ is matched in $M$ to some vertex $y \in V$ but is free in $M'$.** If $y$ is free in $M'$, then the matching $M' \cup \{\{x, y\}\}$ is larger than $M'$, which is a contradiction to the maximality assumption on $M'$. Therefore, $y$ is matched in $M'$ to some vertex $z \in V$. Note that $M$ and $M'$ differ also on $y$ and $z$. Thus, it follows by the choice of $x$ that $y <_\sigma x$ and $z <_\sigma x$. Consider now the matching $M'' := (M' \setminus \{\{y, z\}\}) \cup \{\{x, y\}\}$, which is maximum since $|M''| = |M'|$. However, invoking Observation 2.8 with $v_i = x$ yields $f(M'') < f(M')$, which is a contradiction to the assumption on the minimality of $f(M')$.

**Case 3: $\{x, y\} \in M$ and $\{x, z\} \in M'$ with $z \neq y$.** Then $M$ and $M'$ differ also on $y$ and $z$. Thus, it follows by the choice of $x$ that $y <_\sigma x$ and $z <_\sigma x$. Consider the iteration $t$ of Algorithm 3 during which the algorithm visits $x$. Suppose that vertex $z$ is matched in $M$ with a vertex $p$ at an earlier iteration $t' < t$. Then $M$ differs from $M'$ also at vertex $p$. If $p <_\sigma x$, then Algorithm 3 visits $x$ at an earlier iteration than $p$, which is a contradiction to the assumption on $p$. If $x <_\sigma p$, then we have again a contradiction to the assumption that $x$ is the rightmost vertex in $\sigma$ in which $M$ differs from $M'$. Thus $z$ is unmatched in $M$ at the iteration $t$ of Algorithm 3 during which the algorithm visits $x$. Furthermore, $z$ is also unvisited at iteration $t$ since $z <_\sigma x$. Now, if $y <_\sigma z$, then Algorithm 3 would not match $x$ to $y$ at the execution of Line 21, which is a contradiction. Hence $z <_\sigma y$.

Suppose that $y$ is free in $M'$. Then $M'' := (M' \setminus \{\{x, z\}\}) \cup \{\{x, y\}\}$ is another maximum matching. Invoking Observation 2.8 with $v_i = x$ yields $f(M'') < f(M')$, a contradiction to the assumption on $M'$. Hence, $y$ is matched in $M'$ to some vertex $w \in V$ with $w <_\sigma x$ by the choice of $x$. If $\{w, z\} \in E$, then the matching $M'' := (M' \setminus \{\{x, z\}, \{w, y\}\}) \cup \{\{x, y\}, \{z, w\}\}$ is another maximum matching. Invoking Observation 2.8 with $v_i = x$ yields $f(M'') < f(M')$, which a contradiction to the choice of $M'$. Hence $\{z, w\} \notin E$.

Suppose that within $G_\sigma(x)$ (Definition 2.10) there exists an odd-length alternating

path $P_0$ with respect to $M'$ from $w$ to $z$. Let $E_0$ be the edges in the path $P_0$. Then, swapping in $M'$ all edges on the path $P_0$ (that is, replacing in $M'$ the edges $M' \cap E_0$ with the edges $E_0 \setminus M'$), removing $\{x, z\}$ and $\{w, y\}$ from $M'$, and adding $\{x, y\}$ yields another maximum matching $M''$. Recall that $x$ is the rightmost vertex in which $M$ and $M'$ differ. Thus, since the alternating path $P_0$ belongs to the induced subgraph $G_\sigma(x)$, it follows from Observation 2.8 with $v_i = x$ that $f(M'') < f(M')$, a contradiction to the choice of $M'$. Thus, within $G_\sigma(x)$ there exists no odd-length alternating path with respect to $M'$ from $w$ to $z$.

Similarly, suppose that within $G_\sigma(x)$ there exists an odd-length alternating path $P_1$ with respect to $M'$ from $w$ (resp., from $z$) to a free vertex $v$. Then, swapping in $M'$ all edges on the path $P_1$, removing $\{x, z\}$ and $\{w, y\}$ from $M'$, and adding $\{x, y\}$ yields another maximum matching $M''$ for which Observation 2.8 with $v_i = x$ implies $f(M'') < f(M')$, which is again a contradiction to the choice of $M'$. Thus there exists within $G_\sigma(x)$ no odd-length alternating path with respect to $M'$ from $w$ (resp., from $z$) to a free vertex $v$.

Now suppose that $w <_\sigma z$. That is, $w <_\sigma z <_\sigma y$, where $\{w, y\} \in E$ and $\{z, w\} \notin E$. Hence, $\{z, y\} \in E$ since $\sigma$ is umbrella-free. Thus, since $\{w, y\} \in M'$, it follows that the quadruple $(w, z, y, x)$ satisfies all six conditions in the statement of Lemma 2.11. This is a contradiction to Lemma 2.11, since $M'$ is assumed to be a maximum matching of $G$ such that $f(M')$ is minimum among all maximum matchings.

Finally, suppose that $z <_\sigma w$. Recall that $M$ differs from $M'$ in $w$, since $\{y, w\} \in M'$ and $\{x, y\} \in M$. Thus, since $x$ is the rightmost vertex in $\sigma$ in which $M$ differs from $M'$, it follows that $w <_\sigma x$. That is, $z <_\sigma w <_\sigma x$, where $\{x, z\} \in E$ and $\{z, w\} \notin E$. Hence, $\{w, x\} \in E$ since $\sigma$ is umbrella-free. Thus, since $\{x, z\} \in M'$, it follows that the quadruple $(z, w, x, x)$ satisfies all six conditions in the statement of Lemma 2.11. This is again a contradiction to Lemma 2.11, since $M'$ is assumed to be a maximum matching of $G$ such that $f(M')$ is minimum among all maximum matchings.

Summarizing, the matching $M$ returned by Algorithm 3 is a maximum matching. $\square$

**3. Conclusion.** We presented a thorough mathematical analysis of an efficient and easy-to-implement linear-time greedy algorithm for computing maximum matchings on cocomparability graphs. This contributes to a long list of polynomial-time algorithms for problems on cocomparability graphs. Notably, most of this previous work showed polynomial-time (typically far from linear) algorithms for problems that are NP-hard on general graphs, while we improved a problem solvable in polynomial time on general graphs to linear time on cocomparability graphs.

Apart from being of interest on its own, our result might also be useful in a more general approach towards deriving faster algorithms for computing maximum matchings in relevant special cases. The fundamental idea behind this, as described in the companion work [40], is as follows. First observe that, once a matching is given which has $k$ fewer edges than an optimal one, then using $k$ iterated augmenting path computations (each taking linear time [18]), one can improve it to a maximum matching. If for a graph $G$ we also have a vertex subset set $X$, $|X| = k$, such that $G - X$ is a cocomparability graph, then for constant $k$ we could get a linear-time algorithm for MATCHING as follows: First, delete the $k$ vertices from $G$, then apply our linear-time algorithm, and then apply (as described above) at most $k$ iterations of augmenting path computations again with respect to the original graph $G$, starting with the maximum matching for the cocomparability graph. A drawback of this approach is that we do not even know how to compute in linear time a constant-factor approximation (which would be good enough) for the mentioned vertex deletion set

of size $k$. Hence, we consider it as an interesting challenge for future work to give a linear-time (constant-factor approximation) algorithm for computing a "minimum-vertex-deletion-to-cocomparability" set. Based on the above considerations, for now we only can state the following result.

COROLLARY 3.1. MATCHING *can be solved in* $O(k \cdot (n + m))$ *time when given a size-k vertex set subset X such that deleting X from the given graph yields a cocomparability graph.*

From a more general point of view, Corollary 3.1 is a contribution to the "FPT in P" program [21], heading for more efficient polynomial-time algorithms based on problem parameterizations (also cf. [1, 9, 16, 17]).

**Acknowledgment.** We are deeply grateful to two anonymous reviewers for their very careful reading and their constructive feedback, which helped us significantly clarify and improve the presentation.

## REFERENCES

[1] M. BENTERT, T. FLUSCHNIK, A. NICHTERLEIN, AND R. NIEDERMEIER, *Parameterized aspects of triangle enumeration*, in Proceedings of the 21st International Symposium on Fundamentals of Computation Theory (FCT '17), Lecture Notes in Comput. Sci. 10472, Springer, Berlin, 2017, pp. 96–110.

[2] A. BRANDSTÄDT, V. B. LE, AND J. P. SPINRAD, *Graph Classes: A Survey*, SIAM Monogr. Discrete Math. Appl. 3, SIAM, Philadelphia, 1999, https://doi.org/10.1137/1.9780898719796.

[3] M. CHANG, *Algorithms for maximum matching and minimum fill-in on chordal bipartite graphs*, in Proceedings of the 7th International Symposium on Algorithms and Computation (ISAAC '96), Lecture Notes in Comput. Sci. 1178, Springer, Berlin, 1996, pp. 146–155.

[4] S. R. COORG AND C. P. RANGAN, *Feedback vertex set on cocomparability graphs*, Networks, 26 (1995), pp. 101–111.

[5] D. CORNEIL AND P. KAMULA, *Extensions of permutation and interval graphs*, in Proceedings of the 18th Southeastern Conference on Combinatorics, Graph Theory and Computing, 1987, pp. 267–275.

[6] D. G. CORNEIL, B. DALTON, AND M. HABIB, *LDFS-based certifying algorithm for the minimum path cover problem on cocomparability graphs*, SIAM J. Comput., 42 (2013), pp. 792–807, https://doi.org/10.1137/11083856X.

[7] D. G. CORNEIL, J. DUSART, M. HABIB, AND E. KÖHLER, *On the power of graph searching for cocomparability graphs*, SIAM J. Discrete Math., 30 (2016), pp. 569–591, https://doi.org/10.1137/15M1012396.

[8] D. G. CORNEIL AND R. M. KRUEGER, *A unified view of graph searching*, SIAM J. Discrete Math., 22 (2008), pp. 1259–1276, https://doi.org/10.1137/050623498.

[9] D. COUDERT, G. DUCOFFE, AND A. POPA, *Fully polynomial FPT algorithms for some classes of bounded clique-width graphs*, in Proceedings of the 29th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA '18), ACM, New York, SIAM, Philadelphia, 2018, pp. 2765–2784, https://doi.org/10.1137/1.9781611975031.176.

[10] E. DAHLHAUS AND M. KARPINSKI, *Matching and multidimensional matching in chordal and strongly chordal graphs*, Discrete Appl. Math., 84 (1998), pp. 79–91.

[11] J. S. DEOGUN AND G. STEINER, *Polynomial algorithms for Hamiltonian cycle in cocomparability graphs*, SIAM J. Comput., 23 (1994), pp. 520–552, https://doi.org/10.1137/S0097539791200375.

[12] F. F. DRAGAN, *On greedy matching ordering and greedy matchable graphs*, in Proceedings of the 23rd International Workshop Graph-Theoretic Concepts in Computer Science (WG 1997), Lecture Notes in Comput. Sci. 1335, Springer, Berlin, 1997, pp. 184–198.

[13] R. DUAN AND S. PETTIE, *Linear-time approximation for maximum weight matching*, J. ACM, 61 (2014), 1.

[14] J. DUSART AND M. HABIB, *A new LBFS-based algorithm for cocomparability graph recognition*, Discrete Appl. Math., 216 (2017), pp. 149–161.

[15] J. DUSART, M. HABIB, AND D. G. CORNEIL, *Maximal Cliques Structure for Cocomparability Graphs and Applications*, preprint, https://arxiv.org/abs/1611.02002, 2016.

[16] T. FLUSCHNIK, C. KOMUSIEWICZ, G. B. MERTZIOS, A. NICHTERLEIN, R. NIEDERMEIER, AND N. TALMON, *When can graph hyperbolicity be computed in linear time?*, in Proceedings of the 15th International Workshop on Algorithms and Data Structures (WADS '17), Lecture Notes in Comput. Sci. 10389, Springer, Berlin, 2017, pp. 397–408; Algorithmica, to appear, https://doi.org/10.1007/s00453-018-0522-6.

[17] F. V. FOMIN, D. LOKSHTANOV, S. SAURABH, M. PILIPCZUK, AND M. WROCHNA, *Fully polynomial-time parameterized computations for graphs and matrices of low treewidth*, ACM Trans. Algorithms, 14 (2018), 34.

[18] H. N. GABOW AND R. E. TARJAN, *A linear-time algorithm for a special case of disjoint set union*, J. Comput. System Sci., 30 (1985), pp. 209–221.

[19] F. GARDI, *Efficient algorithms for disjoint matchings among intervals and related problems*, in Proceedings of the 4th International Conference on Discrete Mathematics and Theoretical Computer Science (DMTCS '03), Lecture Notes in Comput. Sci. 2731, Springer, Berlin, 2003, pp. 168–180.

[20] A. C. GIANNOPOULOU AND G. B. MERTZIOS, *New geometric representations and domination problems on tolerance and multitolerance graphs*, SIAM J. Discrete Math., 30 (2016), pp. 1685–1725, https://doi.org/10.1137/15M1039468.

[21] A. C. GIANNOPOULOU, G. B. MERTZIOS, AND R. NIEDERMEIER, *Polynomial fixed-parameter algorithms: A case study for longest path on interval graphs*, Theoret. Comput. Sci., 689 (2017), pp. 67–95.

[22] A. V. GOLDBERG AND A. V. KARZANOV, *Maximum skew-symmetric flows and matchings*, Math. Program., 100 (2004), pp. 537–568.

[23] M. C. GOLUMBIC, *Algorithmic Graph Theory and Perfect Graphs*, 2nd ed., Ann. Discrete Math. 57, North–Holland, Amsterdam, 2004.

[24] J. E. HOPCROFT AND R. M. KARP, *An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs*, SIAM J. Comput., 2 (1973), pp. 225–231, https://doi.org/10.1137/0202019.

[25] A. ILIC, *Efficient algorithm for the vertex connectivity of trapezoid graphs*, Inform. Process. Lett., 113 (2013), pp. 398–404.

[26] E. KÖHLER AND L. MOUATADID, *Linear time LexDFS on cocomparability graphs*, in Proceedings of the 14th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 14), Lecture Notes in Comput. Sci. 8503, Springer, Berlin, 2014, pp. 319–330.

[27] D. KRATSCH AND L. STEWART, *Domination on cocomparability graphs*, SIAM J. Discrete Math., 6 (1993), pp. 400–417, https://doi.org/10.1137/0406032.

[28] T. KRAWCZYK AND B. WALCZAK, *Extending partial representations of trapezoid graphs*, in Proceedings of the 43rd International Workshop on Graph-Theoretic Concepts in Computer Science (WG), Lecture Notes in Comput. Sci. 10520, Springer, Berlin, 2017, pp. 358–371.

[29] Y. D. LIANG AND M. CHANG, *Minimum feedback vertex sets in cocomparability graphs and convex bipartite graphs*, Acta Informatica, 34 (1997), pp. 337–346.

[30] Y. D. LIANG AND C. RHEE, *Finding a maximum matching in a circular-arc graph*, Inform. Process. Lett., 45 (1993), pp. 185–190.

[31] L. LOVÁSZ AND M. D. PLUMMER, *Matching Theory*, Ann. Discrete Math. 29, North–Holland, Amsterdam, 1986.

[32] T. MA AND J. P. SPINRAD, *On the 2-chain subgraph cover and related problems*, J. Algorithms, 17 (1994), pp. 251–268.

[33] A. MADRY, *Navigating central path with electrical flows: From flows to matchings, and back*, in Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science (FOCS '13), IEEE, Washington, DC, 2013, pp. 253–262.

[34] R. M. MCCONNELL AND J. SPINRAD, *Linear-time modular decomposition and efficient transitive orientation of comparability graphs*, in Proceedings of the Tenth Annual ACM–SIAM Symposium on Discrete Algorithms (SODA), ACM, New York, SIAM, Philadelphia, 1994, pp. 536–545.

[35] G. B. MERTZIOS, *The recognition of triangle graphs*, Theoret. Comput. Sci., 438 (2012), pp. 34–47.

[36] G. B. MERTZIOS, *An intersection model for multitolerance graphs: Efficient algorithms and hierarchy*, Algorithmica, 69 (2014), pp. 540–581.

[37] G. B. MERTZIOS, *The recognition of simple-triangle graphs and of linear-interval orders is polynomial*, SIAM J. Discrete Math., 29 (2015), pp. 1150–1185, https://doi.org/10.1137/140963108.

[38] G. B. MERTZIOS AND D. G. CORNEIL, *Vertex splitting and the recognition of trapezoid graphs*, Discrete Appl. Math., 159 (2011), pp. 1131–1147.

[39] G. B. MERTZIOS AND D. G. CORNEIL, *A simple polynomial algorithm for the longest path problem on cocomparability graphs*, SIAM J. Discrete Math., 26 (2012), pp. 940–963, https://doi.org/10.1137/100793529.

[40] G. B. MERTZIOS, A. NICHTERLEIN, AND R. NIEDERMEIER, *The power of linear-time data reduction for maximum matching*, in Proceedings of the 42nd International Symposium on Mathematical Foundations of Computer Science (MFCS), 2017, 46.

[41] G. B. MERTZIOS, I. SAU, AND S. ZAKS, *A new intersection model and improved algorithms for tolerance graphs*, SIAM J. Discrete Math., 23 (2009), pp. 1800–1813, https://doi.org/10.1137/09075994X.

[42] G. B. MERTZIOS, I. SAU, AND S. ZAKS, *The recognition of tolerance and bounded tolerance graphs*, SIAM J. Comput., 40 (2011), pp. 1234–1257, https://doi.org/10.1137/090780328.

[43] G. B. MERTZIOS AND S. ZAKS, *On the intersection of tolerance and cocomparability graphs*, Discrete Appl. Math., 199 (2016), pp. 46–88.

[44] S. MICALI AND V. V. VAZIRANI, *An $O(\sqrt{|V|}|E|)$ algorithm for finding maximum matching in general graphs*, in Proceedings of the 21st Annual IEEE Symposium on Foundations of Computer Science (FOCS '80), IEEE, Washington, DC, 1980, pp. 17–27.

[45] A. MOITRA AND R. C. JOHNSON, *A parallel algorithm for maximum matching on interval graphs*, in Proceedings of the International Conference on Parallel Processing (ICPP '89), Pennsylvania State University Press, University Park, PA, 1989, pp. 114–120.

[46] M. MUCHA AND P. SANKOWSKI, *Maximum matchings via Gaussian elimination*, in Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS '04), IEEE, Washington, DC, 2004, pp. 248–255.

[47] M. MUCHA AND P. SANKOWSKI, *Maximum matchings in planar graphs via Gaussian elimination*, Algorithmica, 45 (2006), pp. 3–20.

[48] S. OLARIU, *An optimal greedy heuristic to color interval graphs*, Inform. Process. Lett., 37 (1991), pp. 21–25.

[49] P. SANKOWSKI, *Maximum weight bipartite matching in matrix multiplication time*, Theoret. Comput. Sci., 410 (2009), pp. 4480–4488.

[50] J. P. SPINRAD, *Efficient Graph Representations*, Fields Inst. Monogr. 19, AMS, Providence, RI, 2003.

[51] G. STEINER AND J. S. YEOMANS, *A linear time algorithm for maximum matchings in convex bipartite graphs*, Comput. Math. Appl., 31 (1996), pp. 91–96.

[52] A. TAKAOKA, *Graph isomorphism completeness for trapezoid graphs*, IEICE Trans., 98-A (2015), pp. 1838–1840.

[53] A. TAKAOKA, *A Recognition Algorithm for Simple-Triangle Graphs*, preprint, https://arxiv.org/abs/1710.06559, 2017.

[54] A. TAKAOKA, *Recognizing simple-triangle graphs by restricted 2-chain subgraph cover*, in Proceedings of the 11th International Conference and Workshops on Algorithms and Computation (WALCOM), 2017, pp. 177–189.

[55] M. YU AND C. YANG, *An $O(n)$ time algorithm for maximum matching on cographs*, Inform. Process. Lett., 47 (1993), pp. 89–93.

[56] R. YUSTER, *Maximum matching in regular and almost regular graphs*, Algorithmica, 66 (2013), pp. 87–92.