ON THE STANDARD L-FUNCTION ATTACHED TO QUATERNIONIC MODULAR FORMS.

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In this paper we study the analytic properties of the standard L-function attached to vector valued quaternionic modular forms using the Rankin-Selberg method. This involves the construction of vector valued theta series, which we obtain by applying some differential operators on Jacobi-theta series studied by Krieg. Such differential operators are obtained from the Howe-Weyl duality for the pair $Sp_n(\mathbb{C}) \times GL_{2n}(\mathbb{C})$.

1. Introduction

Let B be a definite quaternion algebra over $\mathbb{Q}$ and M a B-module of rank m endowed with a skew-hermitian B-valued form and let G be the group of B-linear automorphisms of M preserving this form. Modular forms associated to such a group G are often called quaternionic. Various aspects of such modular forms have been studied before, for example Krieg in [13] has developed the classical theory of such forms, Shimura in [18] has studied the analogues of the Maass-Shimura differential operators in this setting with applications to singular forms, Karel [11] and Tsao [27] have studied the Fourier coefficients of Siegel-type Eisenstein series and Yamana the analogue of the Siegel-Weil formula in [32] and the Ikeda lift in [31].

In this paper we study the analytic properties of the standard L-function attached to a vector valued holomorphic quaternionic modular form, and more particularly the location of possible poles and their orders. Such analytic properties have been studied before by Yamana [33] building on previous work of Piatetski-Shapiro and Rallis [7]. However our approach differs by the one taken by previous works in two important aspects.

The first important difference is that the results in [7, 33] are "generic" in nature, that is, the location of possible poles is independent of the type of the corresponding automorphic representation of $G_{\mathbb{A}}$ at infinity. Our results to the contrary depend on the particular type of the quaternionic modular form. Of course the results in [7, 33] are in some sense more general since they allow to deal with a larger set of quaternionic modular forms (for example non-holomorphic) but on the other hand are not as precise as ours. Indeed as we explain right after Theorem 8.2, our results, when they apply, give a smaller set of possible poles than the results in [33]. We refer to Remark 8.3 for more on this. We also remark that we compute some gamma factors explicitly something which is well-known to be hard in general.

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The second aspect where our work differs from previous works is on the method employed. Indeed in [33] the doubling method is used whereas we use a version of the Rankin-Selberg method involving theta series, which we simply call Rankin-Selberg method (this is what Piatetski-Shapiro in [16] calls in general Shimura's method). In both methods the properties of the L-function are read off by properties of Siegel-type Eisenstein series. However the doubling method involves (the restriction of) a higher degree Siegel-type Eisenstein series, whereas the Rankin-Selberg method involves a theta series and a Siegel-type Eisenstein series of the same degree. It is true that the doubling method can cover more cases since one does not need to assume any restriction on the parity of m (the rank of the B-module M above) in contrast to the Rankin-Selberg method, which requires that the corresponding symmetric space is a tube domain and hence one needs to impose the condition that m is even. However in the cases where the Rankin-Selberg is applicable, it seems that it provides more precise results on the location of the possible poles than the doubling method due to the fact that it employs a smaller degree Eisenstein series. We note here that this is indeed the case for Siegel modular forms (see for example the discussion in [14, page 17]). Furthermore we should emphasize that another potentially important application of the Rankin-Selberg method is towards a stronger results with respect to the absolute convergence of the L-function, and hence its non-vanishing thanks to the Euler product expansion. This ought to have important applications for the theory of quaternionic modular forms and is a direction we hope to explore in the future. Again we refer to Remark (8.3) for more on this.

Before we close this introduction we mention two other interesting aspects of our work, which are perhaps of independent interest. The first is the location of poles and their orders of some normalized Siegel-type Eisenstein series in Theorem 3.8. The second is the construction of some vector valued theta series in Theorem 5.5, using differential operators, whose construction is based on the existence of "pluriharmonic" polynomials in our setting.

2. Quaternionic modular forms

In this section we introduce the notion of vector-valued quaternionic modular form both classically and adelicaly. For most of our notation here we follow the one introduced in the books [21] and [24].

We will be writing v for a finite or infinite place of $\mathbb{Q}$, the set of finite places will be denoted by $\mathbf{h}$ and the archimedean one by $\mathbf{a}$. We let $\mathbb{A}$ denote the adele ring of $\mathbb{Q}$ and we write $\mathbb{A}_{\mathbf{h}}$ for its finite part. For an adele $x \in \mathbb{A}$ we write $x = (x_v) \in \mathbb{A}$, where v runs over all finite places and the place at infinity. We moreover write $x = x_{\mathbf{h}}x_{\mathbf{a}}$ where $x_{\mathbf{a}} \in \mathbb{R}$ and $x_{\mathbf{h}} \in \mathbb{A}_{\mathbf{h}}$. We let $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and we for a $z \in \mathbb{C}$ and define $e_{\mathbf{a}}(z) = \exp(2\pi i z)$. For a finite place $v \in \mathbf{h}$ corresponding to a prime number p we will use both notations $\mathbb{Q}_p$ and $\mathbb{Q}_v$ for the corresponding local field. For a $x \in \mathbb{Q}_p$ we set $e_v(x) := e_p(x) := e_{\mathbf{a}}(-y)$ where $y \in \bigcup_{n=1}^{\infty} p^{-n}\mathbb{Z}$ such that $x - y \in \mathbb{Z}_p$. We then define the character $e_{\mathbb{A}} : \mathbb{A} \to \mathbb{T}$ by $e_{\mathbb{A}}(x) := e_{\mathbf{a}}(x_{\mathbf{a}}) \prod_{v \in \mathbf{h}} e_v(x_v)$. Furthermore for a place v of $\mathbb{Q}$ we let $|\cdot|_v := |\cdot|_p$ the p-adic norm of $\mathbb{Q}_p$ normalized as $|p|_p^{-1} = p$. We also write $|x|_{\mathbf{h}} := \prod_{v \in \mathbf{h}} |x|_v$ for $x \in \mathbb{A}_{\mathbf{h}}$ and $|x|_{\mathbb{A}} := |x|_{\mathbf{h}}|x_{\mathbf{a}}|$ for $x \in \mathbb{A}$. By a character

χ we will always mean a finite Hecke character of $\mathbb{Q}$ and we will write χ_v for its local component at some place v. Moreover we let $\chi_{\mathbf{h}} = \prod_{v\in\mathbf{h}} \chi_v$, for its finite part and $\chi_{\mathbf{a}}$ for its component at the archimidean place. For an integral ideal $\mathfrak{c}$ of $\mathbb{Q}$ and $x \in \mathbb{A}^\times$ we set $\chi_{\mathfrak{c}}(x) := \prod_{v|\mathfrak{c}} \chi_v(x_v)$. The corresponding ideal character to χ will be denoted by χ^* . For an element $\alpha \in \mathbb{Q}$ we will write $\chi^*(\alpha)$ for the the value of χ^* at the fractional ideal generated by α. For a non-zero fractional ideal $\mathfrak{m}$ of $\mathbb{Q}$ generated by a positive rational number m we set $N(\mathfrak{m}) := m$, and for an $x \in \mathbb{A}_{\mathbf{h}}$ we write $x\mathbb{Z}$ for the fractional ideal of $\mathbb{Q}$ corresponding to x.

We denote by B a definite quaternion algebra over $\mathbb{Q}$. We write ρ for the main involution of B and $N_{B/\mathbb{Q}}(x) = xx^\rho \in \mathbb{Q}$ for the norm map. We fix a maximal order $\mathfrak{o}$ of B and for a place v of $\mathbb{Q}$ we set $B_v := B \otimes_{\mathbb{Q}} \mathbb{Q}_v$, and when v is a finite place we set $\mathfrak{o}_v := \mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_v$. When v is the place at infinity we also write $\mathbb{H} = B_{\mathbf{a}}$ for the Hamilton quaternions, understood with the usual notation $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The standard involution ρ of B extends to B_v, for all places v of $\mathbb{Q}$. We let $M_n(B)$ denote the $n \times n$ matrices with entries in B and we write $\mathrm{tr}\,, \det : M_n(B) \to \mathbb{Q}$ for the reduced trace and determinant. This is defined by fixing an $\mathbb{Q}$-algebra embedding $M_n(B) \hookrightarrow M_n(K)$ for some quadratic extension K of $\mathbb{Q}$ and taking the determinant or trace there. It is well known that these definitions are independent of the choice of K. In this paper when we refer to the determinant or trace of a matrix with entries in B, we will always mean the reduced determinant or trace unless it is explicitly stated otherwise. We set $\lambda(x) := \frac{1}{2}\mathrm{tr}\,(x)$ for $x \in M_n(B)$. This definitions extend to $M_n(B)_{\mathbb{A}} := M_n(\mathbb{H}) \otimes \prod'_{v\in\mathbf{h}} M_n(B_v)$, the restricted tensor product with respect to $M_n(\mathfrak{o}_v)$'s. Moreover for a matrix $\alpha \in M_n(B)$ we set $\alpha^* := {}^t\alpha^\rho$ and if α is invertible we set $\widehat{\alpha} := {}^t\alpha^{-\rho}$, that is, the inverse of α^*.

We let G denote the connected algebraic group defined over $\mathbb{Q}$ whose group of $\mathbb{Q}$-valued points is,

$$G_{\mathbb{Q}} := \{\alpha \in GL_{2n}(B) | \alpha\eta\alpha^* = \eta\},$$

where for an positive integer n we write $\eta := \eta_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in SL_{2n}(B)$, where $SL_{2n}(B)$ the invertible matrices of degree $2n$ with entries in B and (reduced) determinant equal to 1. For the group G above, and a finite place v of $\mathbb{Q}$, we write G_v for the group of the $\mathbb{Q}_v$-rational points, $G_{\mathbf{a}}$ the real points and $G_{\mathbb{A}}$ for its adelic points. We moreover write $G_{\mathbf{h}} = \prod'_{v\in\mathbf{h}} G_v$, the finite adelic points of G.

For $n = 1$ we have the isomorphism (see for example [31, Equation (1.1)]),

$$G_{\mathbb{Q}} = \{(b, \alpha) \in B^\times \times GL_2(\mathbb{Q}) \mid N_{B/\mathbb{Q}}(b) \det(\alpha) = 1\}/\{(z, z^{-1})\ z \in \mathbb{Q}^\times\},$$

and actually the case $n = 1$ will lead us to the classical theory of elliptic modular forms (see for example [31, page 1487]), which is of course well understood. In this paper it will be convenient to assume, and we will from now on, that $n \geq 2$. We note here that if we write $GL_n(B)_{\mathbb{A}}$ for the adelic points of the algebraic group $GL_n(B)$, then the assumption $n \geq 2$ implies that $GL_n(B)_{\mathbb{A}} = GL_n(B)GL_n(\mathbb{H}) \prod_{v\in\mathbf{h}} GL_n(\mathfrak{o}_v)$. Indeed the strong approximation theorem holds for $SL_n(B)_{\mathbb{A}}$ for $n \geq 2$ (see for example [31, page 2486]) and the short exact sequence,

$$1 \to SL_n(B)_{\mathbb{A}} \to GL_n(B)_{\mathbb{A}} \to \mathbb{A}^\times_{\mathbb{Q}+} \to 1,$$

where $\mathbb{A}_{\mathbb{Q}+}^{\times}$ are the adelic units x of $\mathbb{Q}$ with $x_{\mathbf{a}} > 0$. Here we note that that det : $M_n(\mathbb{H}) \to \mathbb{R}_+$ (see [29, page 195]). We moreover note that det : $GL_n(\mathfrak{o}_v) \to \mathfrak{o}_v^{\times}$ is surjective for every $v \in \mathbf{h}$. This is clear if v is a place where B splits. For a finite place where B_v is a division algebra over $\mathbb{Q}_v$ this follows from the facts that $N_{B_v/\mathbb{Q}_v}(B_v^{\times}) = \mathbb{Q}_v^{\times}$ and $\mathfrak{o}_v = \{x \in B_v \mid N_{B_v/\mathbb{Q}_v)}(x) \in \mathbb{Z}_v\}$. Using now that the narrow class number of $\mathbb{Q}$ is one we obtain the strong approximation for $GL_n(B)_{\mathbb{A}}$.

We write $S := S_{\mathbb{Q}} := \{x \in M_n(B) \mid x^* = x\}$ for the additive group of hermitian matrices with respect to the involution $*$. Similarly we define $S_{\mathbb{A}}$, S_v and $S_{\mathbf{a}}$. For a $g \in S_{\mathbf{a}}$ we say that it is positive semi-definite (resp. definite) and write $g \geq 0$ (resp $g > 0$) if $x^* g x \geq 0$ (resp. $x^* g x > 0$) for all $0 \neq x \in \mathbb{H}^n$. We write $S_{\mathbf{a}+}$ (resp. $S_{\mathbf{a}}^{+}$) for the set of semi-positive (resp. positive) hermitian matrices in $S_{\mathbf{a}}$ and similarly S_+ (resp. S^+) for the corresponding ones in S. For a two sided fractional ideal $\mathfrak{a}$ of B we set $S(\mathfrak{a}) := S \cap M_n(\mathfrak{a})$.

We now set:

$$\mathcal{H} := \mathcal{H}_n := \left\{ z = x + iy \in S_{\mathbf{a}} \otimes_{\mathbb{R}} \mathbb{C} \;\mid\; x \in S_{\mathbf{a}}, y \in S_{\mathbf{a}}^{+} \right\}.$$

As it is explained in [13] the space $\mathcal{H}$ can be endowed with a hermitian structure, which from now on we will assume. The group $G_{\mathbf{a}}$ acts transitively on $\mathcal{H}$ by

$$\alpha z := \alpha \cdot z := (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1},\ \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G_{\mathbf{a}},\ z \in \mathcal{H},$$

where $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in M_n(\mathbb{H})$. For a $z = x + iy \in \mathcal{H}$ we set $\mathrm{Im}(z) := y$.

We now fix an embedding $\imath : M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})$ by $\imath(a + \mathbf{j}b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, where $a, b \in M_n(\mathbb{C})$. We set $J(\alpha, z) := c_\alpha z + d_\alpha$, which will be understood as an element in $GL_{2n}(\mathbb{C})$ using the identification $M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} = M_{2n}(\mathbb{C})$ induced by the map $\imath$. We note here that for an $z = x + iy \in M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ we define $z^* = x^* - iy^*$ and then $\imath(z^*) = \imath(z)^*$ where for a $M \in M_{2n}(\mathbb{C})$ we set $M^* = {}^t\overline{M}$.

Moreover we set $j_\alpha(z) := j(\alpha, z) := \det(J(\alpha, z)) \in \mathbb{C}^{\times}$. We extend the above definitions to the adelic group $G_{\mathbb{A}}$ as follows: for an element $g \in G_{\mathbb{A}}$ we set $g \cdot z := g\,z := g_{\mathbf{a}} z$, $J(g, z) := J(g_{\mathbf{a}}, z)$ and $j_g(z) := j(g_{\mathbf{a}}, z)$. Moreover for $z = x + iy \in \mathcal{H}$ with $y \in S_{\mathbf{a}}^{+}$ we set $\delta(z) := \det(y)^{1/2}$. We note here that for an element $\alpha \in G_{\mathbf{a}}$ we have that $\delta(\alpha z) = |j(\alpha, z)|^{-1} \delta(z)$ (see [13, Theorem 1.7, (3)]).

Given two fractional ideals $\mathfrak{a}$ and $\mathfrak{e}$ of $\mathbb{Q}$ such that $\mathfrak{a}\mathfrak{e} \subseteq \mathbb{Z}$, we define the subgroup of $G_{\mathbb{A}}$,

$$D[\mathfrak{a}, \mathfrak{e}] := \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_{\mathbb{A}} | a_{x_v}, d_{x_v} \in M_n(\mathfrak{o}_v), b_{x_v} \in M_n(\mathfrak{a}_v), c_{x_v} \in M_n(\mathfrak{e}_v),\, \forall v \in \mathbf{h} \right\},$$

where for a fractional ideal $\mathfrak{a}$ of $\mathbb{Q}$, we denote by $M_n(\mathfrak{a}_v)$ the matrices with entries in $\mathfrak{o} \otimes_{\mathbb{Z}} \mathfrak{a}_v$. In this notation we may as well write $M_n(\mathbb{Z}_v)$ for $M_n(\mathfrak{o}_v)$ but we prefer the latter.

We now let $\mathfrak{c}$ denote an integral ideal of $\mathbb{Q}$ and $\mathfrak{b}$ a fractional ideal. For a finite adele $q \in G_{\mathbf{h}}$ we define $\Gamma^q = \Gamma^q(\mathfrak{b}, \mathfrak{c}) := G_{\mathbb{Q}} \cap q D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] q^{-1}$, a congruence subgroup of $G_{\mathbb{Q}}$.

Given a character ψ of conductor dividing $\mathfrak{c}$ we define a character on $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ by $\psi_{\mathfrak{c}}(x) := \prod_{v|\mathfrak{c}} \psi_v(\det(d_x)_v)$, and a character ψ_q on Γ^q by $\psi_q(\gamma) = \psi_{\mathfrak{c}}(q^{-1}\gamma q)$.

We let ρ denote a rational representation $\rho : GL_{2n}(\mathbb{C}) \to GL(V)$, where V is a finite dimensional complex vector space. Then for a function $f : \mathcal{H} \to V$ we define

$$(f|_\rho \alpha)(z) := \rho(J(\alpha, z))^{-1} f(\alpha \cdot z),\ \alpha \in G_{\mathbb{Q}},\ z \in \mathcal{H}_n.$$

When we are taking $\rho = \det^k$ for some $k \in \mathbb{Z}_+$ then we write $f|_k$ instead of $f|_{\det^k}$. We now define,

Definition 2.1. A function $f : \mathcal{H} \to V$ is called a quaternionic modular form for the congruence subgroup Γ^q of weight ρ and nebentype ψ_q if:

(1) f is holomorphic,
(2) $f|_\rho \gamma = \psi_q(\gamma) f$ for all $\gamma \in \Gamma^q$,

Remark 2.2. We make the following remarks:

(1) Since we are taking $n > 1$ we do not need to have any condition at the cusps thanks to the Koecher's principle (see [13, Lemma 1.5]).
(2) In the case where $\rho = \det^k$ for some $k \in \mathbb{Z}$, our convention with respect to the weight is different to the one used in [13]. Namely our weight k here corresponds to the weight $2k$ there. This is only a notational difference since we are taking $j(\alpha, z)^k = \det(J(\alpha, z))^k$ for $\alpha \in G_{\mathbf{a}}$ and $z \in \mathcal{H}$ where in [13, page 78] one takes $\det(J(\alpha, z))^{\frac{2k}{2}}$, as the scalar factor of automorphy.
(3) We remark here that one could give a more general definition of a quaternionic modular form, but here we restrict ourselves to the case where the corresponding symmetric space is a tube domain.

We denote the above complex vector space by $\mathcal{M}_\rho(\Gamma^q, \psi_q)$. For an integer $k \in \mathbb{Z}$, and a rational representation (ρ, V) of $GL_{2n}(\mathbb{C})$ we set $\rho_k := \rho \otimes \det^k$. For any $\gamma \in G_{\mathbb{Q}}$ we have a Fourier expansion of the form (see for example [13, Theorem 1.2]),

$$(f|_\rho \gamma)(z) = \sum_{\tau \in \mathfrak{S}_\gamma} c_f(\tau, \gamma) e_{\mathbf{a}}(\lambda(\tau z)),$$

where $\mathfrak{S}_\gamma$ a lattice in S_+. Here we have extended $\lambda : M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ as $\lambda(A + Bi) := \lambda(A) + i\lambda(B)$. We call f a cusp form if $c_f(\tau, \gamma) = 0$ for any $\gamma \in G$ and τ with $\det(\tau) = 0$. The space of cusp forms we will be denoted by $\mathcal{S}_\rho(\Gamma^q, \psi_q)$.

Given a representation (ρ, V) of $GL_{2n}(\mathbb{C})$ we can define a Hermitian inner product $\prec \cdot, \cdot \succ$ on V with the property $\prec \rho(g)v, w \succ_V = \prec v, \rho(g^*)w \succ$ for $g \in GL_{2n}(\mathbb{C})$ and $v, w \in V$ (see [24, page 96]). We take the inner product linear in the first argument and anti-linear in the second, i.e. $\prec av, bw \succ = a \prec v, w \succ \bar{b}$ for $a, b \in \mathbb{C}$ and $v, w \in V$.

For $f, g : \mathcal{H}_n \to V$ such that $f \in \mathcal{S}_\rho(\Gamma^q, \psi_q)$ and $g|_\rho \gamma = \psi_q(\gamma) g$ for all $\gamma \in \Gamma^q$, (not necessarily holomorphic), we define

$$\langle f, g \rangle := \mathrm{vol}(\Phi)^{-1} \int_\Phi \prec \rho(y) f(z), g(z) \succ d\nu(z) =$$

$$\mathrm{vol}(\Phi)^{-1} \int_\Phi \prec \rho(\sqrt{y}) f(z), \rho(\sqrt{y}) g(z) \succ d\nu(z),$$

where $\Phi := \Gamma^q \setminus \mathcal{H}_n$, and $d\nu(z) = det(y)^{-(2n-1)} dxdy$ is the volume element in $\mathcal{H}_n$ which is Γ^q invariant (see [13, Theorem 1.10]). We note that the expression $\prec \rho(y)f(z), g(z) \succ$ is Γ^q-invariant thanks to [13, Theorem 1.7 (3)]. Here $\sqrt{y}$ (or sometimes we may write $y^{1/2}$) is a positive definite hermitian matrix such that $\sqrt{y}^2 = y$. Indeed since y is positive definite there exist an invertible matrix $u \in GL_n(\mathbb{H})$ with $uu^* = I_n$ such that $y = udu^*$ for some diagonal matrix with strictly positive diagonal elements. Then we simply let $\sqrt{y} = u\sqrt{d}u^*$, with the obvious meaning of $\sqrt{d}$. We also remind the reader of our convention from above: $\rho(y) = \rho(J(1_{2n}, y)) = \rho(\imath(y))$.

We now turn to the adelic quaternionic modular forms. If we write D for a group of the form $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$, and ψ a character then we define,

Definition 2.3. A function $\mathbf{f} : G_{\mathbb{A}} \to V$ is called an adelic quaternionic modular form if

(1) $\mathbf{f}(\alpha x w) = \psi_{\mathfrak{c}}(w)\rho(J(w, \mathbf{i}))\mathbf{f}(x)$ for $\alpha \in G$, $w \in D$ with $w(\mathbf{i}) = \mathbf{i}$,
(2) For every $p \in G_{\mathbf{h}}$ there exists $f_p \in \mathcal{M}_\rho(\Gamma^p, \psi_p)$, where $\Gamma^p = G \cap pDp^{-1}$ such that $\mathbf{f}(py) = (f_p|_\rho y)(\mathbf{i})$ for every $y \in G_{\mathbf{a}}$.

Here we write $\mathbf{i} := i1_n \in \mathcal{H}$. We denote this space by $\mathcal{M}_\rho(D, \psi)$. We call $\mathbf{f}$ a cusp form if f_p is a cusp form for all $p \in G_{\mathbf{h}}$, and the space of cusp forms will be denoted by $\mathcal{S}_\rho(D, \psi)$. The following is an extension of Proposition 20.2 in [24] (see also [3] where the case of vector valued Siegel modular forms is considered) to the case of vector valued quaternionic modular forms.

Proposition 2.4. *Let $\mathbf{f} \in \mathcal{M}_\rho(D, \psi)$, then for all $q \in GL_n(B)_{\mathbb{A}}$ and $s \in S_{\mathbb{A}}$ we have,*

$$\mathbf{f}\left(\begin{pmatrix} q & s\widehat{q} \\ 0 & \widehat{q} \end{pmatrix}\right) = \rho(q_{\mathbf{a}}^*) \sum_{\tau \in S_+} c_{\mathbf{f}}(\tau, q) e_{\mathbf{a}}(\lambda(iq^*\tau q)) e_{\mathbb{A}}(\lambda(\tau s)),$$

where $c_{\mathbf{f}}(\tau, q) \in V$. They satisfy the following properties:

(1) $c_{\mathbf{f}}(\tau, q) \neq 0$ only if $e_{\mathbf{h}}(\lambda(q^*\tau q s)) = 1$ for any $s \in S_{\mathbf{h}}(\mathfrak{b}^{-1})$,
(2) $c_{\mathbf{f}}(\tau, q) = c_{\mathbf{f}}(\tau, q_{\mathbf{h}})$,
(3) $c_{\mathbf{f}}(b^*\tau b, q) = \rho(b^*) c_{\mathbf{f}}(\tau, bq)$ for any $b \in GL_n(B)$,
(4) $\psi_{\mathbf{h}}(\det(e)) c_{\mathbf{f}}(\tau, qe) = c_{\mathbf{f}}(\tau, q)$ for any $e \in \prod_{v \in \mathbf{h}} GL_n(\mathfrak{o}_v)$.

Proof. Let $x = \begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}$ be as above, and put $p = x_{\mathbf{h}}$. The functions $f_p \in \mathcal{M}_{\rho_k}(\Gamma^p, \psi_p)$ given in definition 2.3 have Fourier expansions

$$f_p(z) = \sum_{\tau \in S_+} c_f(\tau, q, s) e_{\mathbf{a}}(\lambda(\tau z))$$

with $c_f(\tau, q, s) \in V$. Since $x_{\mathbf{a}}\mathbf{i} = q_{\mathbf{a}} q_{\mathbf{a}}^* i + s_{\mathbf{a}}$, by the definition 2.3 we have that

$$\mathbf{f}(x) = (f_p|_\rho x)(\mathbf{i}) = \rho(q_{\mathbf{a}}^*) \sum_{\tau \in S_+} c_f(\tau, q, s) e_{\mathbf{a}}(\lambda(iq^*\tau q)) e_{\mathbf{a}}(\lambda(\tau s)).$$

We let

$$c_{\mathbf{f}}(\tau, q, s) := e_{\mathbf{h}}(-\lambda(\tau s)) c_f(\tau, q, s), \tag{1}$$

and hence
$$\mathbf{f}(x) = \rho(q_{\mathbf{a}}^*) \sum_{\tau \in S_+} c_{\mathbf{f}}(\tau, q, s) e_{\mathbf{a}}(\lambda(iq^*\tau q)) e_{\mathbb{A}}(\lambda(\tau s)).$$
As it is explained in [24, page 168], the fact that $\mathbf{f}(\alpha x w) = \mathbf{f}(x)$ for any
$$\alpha = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \in G \text{ and } w = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \in G_{\mathbf{h}} \cap D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]),$$
allows us to establish that $c_{\mathbf{f}}(\tau, q, s)$ does not depend on s. Denoting this simply by $c_{\mathbf{f}}(\tau, q)$ gives the Fourier expansion. The proof of the properties for the coefficients can be shown as in [24, page 168]; we only explain (3) here.

We consider $\mathrm{diag}[b\ \hat{b}] = \begin{pmatrix} b & 0 \\ 0 & \hat{b} \end{pmatrix} \in G$ for $b \in GL_n(B)$. Then we have $\mathbf{f}(\mathrm{diag}[b\ \hat{b}]x) = \mathbf{f}(x)$ for all $x \in G_{\mathbb{A}}$. In particular for any $q \in GL_n(B)_{\mathbb{A}}$ and $s \in S_{\mathbb{A}}$ we have,
$$\mathbf{f}\left(\begin{pmatrix} q & s\widehat{q} \\ 0 & \widehat{q} \end{pmatrix}\right) = \mathbf{f}\left(\mathrm{diag}[b\ \hat{b}] \begin{pmatrix} q & s\widehat{q} \\ 0 & \widehat{q} \end{pmatrix}\right) = \mathbf{f}\left(\begin{pmatrix} bq & bsb^*\widehat{bq} \\ 0 & \widehat{bq} \end{pmatrix}\right).$$
By the Fourier expansion,
$$\rho(q_{\mathbf{a}}^*) \sum_{\tau \in S_+} c_{\mathbf{f}}(\tau, q) e_{\mathbf{a}}(\lambda(iq^*\tau q)) e_{\mathbb{A}}(\lambda(\tau s)) =$$
$$\rho(q_{\mathbf{a}}^* b^*) \sum_{\tau \in S_+} c_{\mathbf{f}}(\tau, bq) e_{\mathbf{a}}(\lambda(iq^* b^* \tau bq)) e_{\mathbb{A}}(\lambda(b^*\tau bs)).$$
Equating Fourier coefficients we obtain $c_{\mathbf{f}}(b^*\tau b, q) = \rho(b^*) c_{\mathbf{f}}(\tau, bq)$, which is property (3). $\square$

Note: We will sometimes use the notation $c(\tau, q; \mathbf{f})$ for the coefficient $c_{\mathbf{f}}(\tau, q)$.

We conclude this section with a final remark. Note that we have an isomorphisms $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \to D[\mathbb{Z}, \mathfrak{c}]$ by $x \mapsto bxb^{-1}$ where $b := \mathrm{diag}[\beta 1_n, 1_n]$ for some $\beta \in \mathbb{Q}$ such that $\mathfrak{b} = (\beta)$. In particular we have an isomorphisms between $\mathcal{M}_\rho(D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]) \to \mathcal{M}_\rho(D[\mathbb{Z}, \mathfrak{c}])$, given by $\mathbf{f}(x) \mapsto \mathbf{h}(x) := \mathbf{f}(xb)$. Since $\beta \in \mathbb{Q}$ we have that this induces an isomorphism between $M_\rho(\Gamma(\mathfrak{b}, \mathfrak{c})) \to M_\rho(\Gamma(\mathbb{Z}, \mathfrak{c}))$ by $f(z) \mapsto h(z) := f(\beta z)$, where we write $\Gamma(\mathfrak{b}, \mathfrak{c})$ for $\Gamma^1(\mathfrak{b}, \mathfrak{c})$. In particular we do not loose much if we set $\mathfrak{b} = \mathbb{Z}$ below. In this case we set $\Gamma_0(\mathfrak{c}) := \Gamma(\mathbb{Z}, \mathfrak{c})$.

3. Siegel type Eisenstein Series

In this section we define Siegel type Eisenstein series and study their analytic properties. The main theorem of this section is Theorem 3.8. In particular we will determine, after multiplying with some normalising factors, the location of possible poles, as well as their orders, of this Eisenstein series. Our approach is similar to the one taken by Shimura in the symplectic [19] and unitary case [21] (see also Remark (3.9) below).

We start by defining some subgroups of G. We define the Siegel parabolic subgroup by
$$P := P_{\mathbb{Q}} := \left\{ \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in G_{\mathbb{Q}} \mid c_\gamma = 0 \right\},$$

where $a_\gamma, b_\gamma, c_\gamma, d_\gamma \in M_n(B)$ and similarly we define P_v for a place v of $\mathbb{Q}$ or $P_{\mathbb{A}}$. It can be shown, see for example [13, Chapter II, section 1], that $P = LR$ where

$$L = \{\gamma \in P \mid d_\gamma = \widehat{a_\gamma} \in GL_n(B),\ \ b_\gamma = 0\} \cong GL_n(B),$$

where we recall that for a matrix $\alpha \in GL_n(B)$ we define $\widehat{\alpha} = (\alpha^*)^{-1}$. The group R can be identified with the group of hermitian matrices by the map $\tau : S \to R$, $\tau(\sigma) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}$.

We write $\mathfrak{r}$ for the product of all prime ideals of $\mathbb{Q}$, where the division algebra B ramifies, that is if p is a prime such that $M_n(B_p) \ncong M_{2n}(\mathbb{Q}_p)$ then $p|\mathfrak{r}$. Let $\ell \in \mathbb{Z}$ and take an integral ideal $\mathfrak{c}$ in $\mathbb{Q}$, such that $\mathfrak{r}$ divides $\mathfrak{c}$. In particular $\mathfrak{c} \neq \mathbb{Z}$. Moreover we consider a Dirichlet character χ of $\mathbb{Q}$ whose conductor divides $\mathfrak{c}$. For a fixed fractional ideal $\mathfrak{b}$ we write $D(\mathfrak{c})$ for $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, and simply D for $D[\mathfrak{b}^{-1}, \mathfrak{b}]$. Moreover $D_0(\mathfrak{c})$ for the subgroup of $D(\mathfrak{c})$ with $x \in D_0(\mathfrak{c})$ if $x(\mathbf{i}) = \mathbf{i}$. Similarly we define D_0.

We now define the Siegel type Eisenstein series of weight ℓ associated to the character χ in a similar manner as in the symplectic and unitary case in [24, page 131]. In particular we set,

$$\mathbf{E}(x,s) := \mathbf{E}(x,s;\chi,D(\mathfrak{c})) := \sum_{\gamma \in P\backslash G} \phi(\gamma x, s), \quad \mathrm{Re}(s) >> 0,$$

where $(x,s) \in G_{\mathbb{A}} \times \mathbb{C}$, and the function $\phi(x,s) : G_{\mathbb{A}} \times \mathbb{C} \to \mathbb{C}$ is supported on $P_{\mathbb{A}} D_0(\mathfrak{c}) \subset G_{\mathbb{A}}$ in the x variable, and is defined as follows: for $x \in P_{\mathbb{A}} D_0(\mathfrak{c})$ we write $x = pw$ with $p \in P_{\mathbb{A}}$ and $w \in D_0(\mathfrak{c})$ and we define $\phi(x,s) := \prod_v \phi_v(x,s)$ where,

$$\phi_v(x,s) := \begin{cases} \chi_v(\det(d_{p_v}))^{-1} |\det(d_{p_v})|_v^{-s}, \ \text{ if } v \in \mathbf{h}, \text{ and } v \not| \mathfrak{c}, \\ \chi_v(\det(d_{p_v}))^{-1} \chi_v(\det(d_{w_v}))^{-1} |\det(d_{p_v})|_v^{-s}, \ \text{ if } v \in \mathbf{h} \text{ and } v \mid \mathfrak{c}, \\ j_x(\mathbf{i})^{-\ell} |j_x(\mathbf{i})|^{\ell - s}, \text{if } v = \mathbf{a}. \end{cases}$$

One can check that $\phi(x,s)$ is well-defined, i.e. independent of the choice of p and w above. Furthermore we note that $|j_x(\mathbf{i})| = |\det(d_{p_\mathbf{a}})|$. Indeed we have that $j_x(\mathbf{i}) = j(p,\mathbf{i}) j(w,\mathbf{i})$. But $|j(w,\mathbf{i})| = 1$ for $w \in D_0$ since $\imath(c_w \mathbf{i} + d_w) \in U(2n)$ (see [31, page 2486]), where $\imath : M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})$ was defined in the previous section. Hence $|j_x(\mathbf{i})| = |\det(d_{p_\mathbf{a}})|$. A simple substitution shows that for $\alpha \in G$ and $w \in D_0(\mathfrak{c})$ we have,

$$\mathbf{E}(\alpha x w, s) = \chi_{\mathfrak{c}}(\det(d_w))^{-1} j_w(\mathbf{i})^{\ell} \mathbf{E}(x,s),$$

and hence the Eisenstein series has the required modular properties.

Remark 3.1. We note here that we imposed no condition on the sign of the character χ (i.e $\chi(-1) = \pm 1$) and the weight ℓ, which is different to the situation of the symplectic and unitary group (see for example [21, page 149], or [24, page 131]). This is due to the fact that $\det(g) \in \mathbb{R}_+$ for $g \in GL_n(\mathbb{H})$, and hence the sign of the character does not play any role. Indeed we can see that $\phi(qx,s) = \phi(x,s)$ for $q \in P$ for any character χ and $\ell \in \mathbb{Z}$. For this we write as above $x = pw$, and then we have that

$$\phi(qx,s) = \prod_v \phi_v(qpw,s) = \chi_{\mathbf{h}}(\det(d_{qp}))^{-1} \chi_{\mathfrak{c}}(\det(d_w))^{-1} j_{qx}(\mathbf{i})^{-\ell} |j_{qx}(\mathbf{i})|^{\ell} |\det(d_{qp})|_{\mathbb{A}}^{-s}.$$

But $\chi_{\mathbf{h}}(\det(d_{qp})) = \chi_{\mathbf{h}}(\det(d_q)) \chi_{\mathbf{h}}(\det(d_p)) = \chi_{\mathbf{h}}(\det(d_p))$ since $\chi_{\mathbf{a}}(\det(d_q)) = 1$ as $\det(d_q) \in \mathbb{R}_+$. Moreover $j_{qx}(\mathbf{i})^{-\ell} |j_{qx}(\mathbf{i})|^{\ell} = \left(\frac{|j_q(x(\mathbf{i}))|}{j_q(x(\mathbf{i}))}\right)^{\ell} \times j_x(\mathbf{i})^{-\ell} |j_x(\mathbf{i})|^{\ell}$. But $\frac{|j_q(x(\mathbf{i}))|}{j_q(x(\mathbf{i}))} =$

$\frac{|\det(d_q)|}{\det(d_q)} = 1$ as $\det(d_q) \in \mathbb{R}_+$. Moreover $|\det(d_{qp})|_{\mathbb{A}}^{-s} = |\det(d_p)|_{\mathbb{A}}^{-s}$ as $d_q \in GL_n(B)$. In particular $\phi(qx, s) = \phi(x, s)$ for all $q \in P$.

We now define the function $E(z, s)$ by $E(x \cdot \mathbf{i}, s) = j_x(\mathbf{i})^\ell \mathbf{E}(x, s)$, (see Definition 2.3) where $x \in G_{\mathbf{a}}$ such that $x \cdot \mathbf{i} = z \in \mathcal{H}$. Then we have

$$E(z, s) = \sum_{\alpha \in A} |\det(d_{p_\alpha})|_{\mathbf{h}}^{-s} \chi_{\mathbf{h}}(\det(d_{p_\alpha})))^{-1} \chi_{\mathfrak{c}}(\det(d_{w_\alpha}))^{-1} \delta(z)^{s-\ell}|_\ell \alpha, \tag{2}$$

where $A = P \backslash (G \cap P_{\mathbb{A}} D(\mathfrak{c}))$ and $p_\alpha \in P_{\mathbf{h}}$ and $w_\alpha \in D(\mathfrak{c})_{\mathbf{h}}$ such that $\alpha_{\mathbf{h}} = p_\alpha w_\alpha$. Indeed $\phi_{\mathbf{h}}(x, s) = \prod_{v \in \mathbf{h}} \phi_v(x, s)$ is supported on $P_{\mathbf{h}} D(\mathfrak{c})_{\mathbf{h}}$ and $(\alpha x)_{\mathbf{h}} = \alpha_{\mathbf{h}}$ since $x_{\mathbf{h}} = 1$. Furthermore $\delta(\alpha z) = |j(\alpha, z)|^{-1} \delta(z)$ and $\delta(z) = \delta(x(\mathbf{i})) = |j_x(\mathbf{i})|^{-1}$ and so $\delta(\alpha z) = |j_{\alpha x}(\mathbf{i})|^{-1}$. In particular we have that

$$\phi(\alpha x, s) = |\det(d_{p_\alpha})|_{\mathbf{h}}^{-s} \chi_{\mathbf{h}}(\det(d_{p_\alpha}))^{-1} \chi_{\mathfrak{c}}(\det(d_{w_\alpha}))^{-1} j_{\alpha x}(\mathbf{i})^{-\ell} |j_{\alpha x}(\mathbf{i})|^{\ell - s}.$$

Using that $j_{\alpha x}(\mathbf{i})^\ell = j_\alpha(z)^\ell j_x(\mathbf{i})^\ell$ we conclude equation (2). We now establish the following lemma,

Lemma 3.2. *We have,*

$$P_{\mathbb{A}} D(\mathfrak{c}) = P D(\mathfrak{c}).$$

Proof. Similarly to [21, Lemma 9.6] we may decompose $P_{\mathbb{A}} = \bigsqcup_{q \in Q} Pq(P_{\mathbb{A}} \cap D(\mathfrak{c}))$, where q can be chose of the form $q = \mathrm{diag}[r, \hat{r}]$ with $r \in GL_n(B)_{\mathbf{h}}$. As it was discussed in the previous section, after taking $\det(r)$ we establish a bijection between the set Q and $\mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}_+^\times \left(\prod_p \mathbb{Z}_p^\times\right) \mathbb{R}_+$, which is of course nothing else than the trivial group. $\square$

Let us now set $\Gamma := G \cap D(\mathfrak{c})$ and define the Eisenstein series

$$E_\ell(z, s; \chi, \Gamma) := \sum_{\gamma \in (\Gamma \cap P) \backslash \Gamma} \chi_{\mathfrak{c}}(\det(d_\gamma))^{-1} \delta(z)^{s-\ell}|_\ell \gamma.$$

By the lemma above we have that $P \backslash (G \cap P_{\mathbb{A}} D(\mathfrak{c})) = (\Gamma \cap P) \backslash \Gamma$. In particular in the notation of Equation (2) above we may pick $p_\alpha = 1$ for all $\alpha \in A$, and $w_\alpha = \gamma$, and from this we obtain that

$$E_\ell(z, s; \chi, \Gamma) = E(z, s).$$

We remark here that one can show (see for example [13, Chapter V]) that this Eisenstein series is absolutely convergent for $\mathrm{Re}(s) > 2n - 1$. In particular for $\ell > 2n - 1$ we have that $E(z, \ell)$ is a quaternionic modular form of weight ℓ with respect to the congruence subgroup Γ.

Even though these Eisenstein series are the ones which are relevant to our applications, we need to introduce yet another kind of Eisenstein series for which we have explicit information about their Fourier expansion. We define $\eta_{\mathbf{h}} \in G_{\mathbb{A}}$ by $\eta_v = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ for all $v \in \mathbf{h}$ and $\eta_{\mathbf{a}} = 1$. Then we define the $\mathbf{E}^*(x, s) := \mathbf{E}(x \eta_{\mathbf{h}}^{-1}, s)$, we write the Fourier expansion of $\mathbf{E}^*(x, s)$ as,

$$\mathbf{E}^*\left(\begin{pmatrix} q & \sigma \hat{q} \\ 0 & \hat{q} \end{pmatrix}, s\right) = \sum_{h \in S} c(h, q, s) \mathbf{e}_{\mathbb{A}}(\lambda(h\sigma)), \tag{3}$$

where $q \in GL_n(B)_{\mathbb{A}}$ and $\sigma \in S_{\mathbb{A}}$.

We now focus on the coefficients $c(h,q,s)$. We first introduce some notation. For a finite place $v \in \mathbf{h}$ corresponding to some prime p and an element $x \in M_n(B_v)$ we can find $c \in M_n(\mathfrak{o}_v)$ and $d \in GL_n(B_v) \cap M_n(\mathfrak{o}_v)$ such that $x = d^{-1}c$ and the matrix $[c\ d] \in M_{n,2n}(B)$ is primitive. That is, there exists $a, b \in M_n(B)$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(B)$. We then define an ideal in $\mathbb{Z}_p$, which is usualy called the denominator ideal of x, by $\nu_0(x) = \det(d)\mathbb{Z}_p$ and we set $\nu(x) := |\det(d)|_v^{-1}$. For all these we refer to [23] or [27].

We now set $T := T^n := \{x \in S | \lambda(xy) \subset \mathbb{Z},\ \forall y \in S(\mathfrak{o})\}$ where we recall $S(\mathfrak{o}) = S \cap M_n(\mathfrak{o})$. Similarly we define $T_{\mathbb{A}}$ and T_v for a place $v \in \mathbf{h}$. For an element $\zeta \in T_{\mathbb{A}}$, a character χ of conductor dividing the ideal $\mathfrak{c}$, and $s \in \mathbb{C}$ we define the Siegel series,

$$\alpha_{\mathfrak{c}}(\zeta, s, \chi) := \prod_{v \nmid \mathfrak{c}} \left(\sum_{\sigma \in S_v/\Lambda_v} e_v(-\lambda(\zeta_v \sigma))\chi_v(\nu_0(\sigma))\nu(\sigma)^{-s} \right),$$

where the product is over all finite places not dividing the ideal $\mathfrak{c}$ and for each such place v, we set $\Lambda_v := S(\mathfrak{o}_v) := S(\mathfrak{o})_v = S_v \cap M_n(\mathfrak{o}_v)$.

Moreover for $s, s' \in \mathbb{C}$ with $\mathrm{Re}(s), \mathrm{Re}(s') >> 0$, $y \in S_{\mathbf{a}}^+$ and $h \in S_{\mathbf{a}}$ we introduce the function,

$$\xi(y, h, s, s') = \int_{S_{\mathbf{a}}} e_{\mathbf{a}}(-\lambda(hx)) \det(x+iy)^{-s/2} \det(x-iy)^{-s'/2} dx.$$

The properties of this function have been studied in [17], and we will recall some of its properties a little bit later. We now can state:

Proposition 3.3. *With notation and assumption as above we have $c(h,q,s) \neq 0$ only if $(\beta q^* h q)_v \in T_v \cap M_n(\mathfrak{c}_v^{-1})$ for all $v \in \mathbf{h}$, where $\beta \in \mathbb{A}_{\mathbf{h}}^\times$ such that $\beta\mathbb{Z} = \mathfrak{b}$. In this case we have*

$$c(h,q,s) = A(q,n,\chi) \times |\det(q)|_{\mathbf{h}}^{-s} \times \alpha_{\mathfrak{c}}(\beta q_{\mathbf{h}}^* h q_{\mathbf{h}}, s, \chi) \times \xi(q_{\mathbf{a}} q_{\mathbf{a}}^*, h, s+\ell, s-\ell),$$

where $A(q,n,\chi) \in \mathbb{C}^\times$ a constant depending on q, χ and n.

Proof. This is quite standard in the theory of Eisenstein series. The cases of the symplectic and of the unitary group have been considered by Shimura in [21, Proposition 18.14]. The proof of the proposition follows quite similarly, and in the appendix of this paper we explain how one can extend the proof there to the quaternionic case. □

We now collect various facts about the Siegel series $\alpha_{\mathfrak{c}}(\zeta, s, \chi)$ that appeared above. We fix a finite place v and write p for the corresponding prime. We set $\alpha(\zeta_v, s, \chi_v) = \sum_{\sigma \in S_v/\Lambda_v} e_v(-\lambda(\zeta_v\sigma))\chi_v(\nu_0(\sigma))\nu(\sigma)^{-s}$. Since we are assuming that the ideal $\mathfrak{c}$ is divisible by all primes where B is ramified we have that $\phi_p : B_p \cong M_2(\mathbb{Q}_p)$, when p does not divide $\mathfrak{c}$. We may actually select ϕ_p in such a way that it induces an isomorphism $\phi_p : \mathfrak{o}_p \cong M_2(\mathbb{Z}_p)$. Fixing such an isomorphism we obtain an isomorphism $\imath_p : M_n(B_p) \cong M_{2n}(\mathbb{Q}_p)$ defined by $\imath_p((x_{ij})) := (\phi_p(x_{ij}))$. We remark (see [31, page

2488]) that the map $\imath_p$ induces an isomoprhism $\imath_p : G_v \cong SO(2n,2n)(\mathbb{Q}_p)$, where

$$SO(2n,2n)(\mathbb{Q}_p) = \left\{ x \in SL_{4n}(\mathbb{Q}_p) \mid x \begin{pmatrix} 0 & 1_{2n} \\ 1_{2n} & 0 \end{pmatrix} {}^t x = \begin{pmatrix} 0 & 1_{2n} \\ 1_{2n} & 0 \end{pmatrix} \right\}.$$

In the following and until Proposition 3.6 below our considerations are local, so we will be simply writing $\imath$ for $\imath_p$. Our aim now is to relate in Lemma 3.4 below the Siegel series above to the Siegel series corresponding to the orthogonal group $SO(2n,2n)$, which have been studied extensively in [21, Chapter III]. This relation has already been observed in [31, page 2488], and here we just fill in some details. We introduce the notation

$$S_o^{2n} := \left\{ x \in M_{2n}(\mathbb{Q}_p) \mid {}^t x = -x \right\}.$$

Moreover we write S_q^n for S_v and Λ_q^n for Λ_v. Then if we let

$$K_n := \mathrm{diag}[J_1, J_1, \ldots, J_1], \tag{4}$$

with $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we note that ${}^t K_n = -K_n$, $K_n^{-1} = -K_n$, and $K_n^2 = -I_{2n}$. Then we have

$$S_q^n \cong S_o^{2n}, \quad x \mapsto \imath(x) K_n.$$

Indeed, by [22, page 73], we have that $\imath(x^*) = K_n^{-1}\, {}^t\imath(x)\, K_n$. Hence, since K_n is invertible we need only to check that,

$${}^t(\imath(x)K_n) = {}^t K_n {}^t\imath(x) = {}^t K_n K_n \imath(x^*) K_n^{-1} = K_n^2 \imath(x^*) K_n = -\imath(x^*) K_n = -(\imath(x)K_n).$$

We now let $T_o^{2n} := \{x \in S_o^{2n} \mid \mathrm{tr}\,(xy) \subset \mathbb{Z}_p,\ \forall y \in S_o^{2n}(\mathbb{Z}_p)\}$, and for a $\zeta \in T_o^{2n}$ define the orthogonal Siegel-series,

$$\alpha_o(\zeta, s, \chi_v) = \sum_{\sigma \in S_o^{2n}/\Lambda_o^{2n}} e_v(-\mathrm{tr}\,(\zeta\sigma)) \chi_v(\tilde{\nu}_0(\sigma)) \tilde{\nu}(\sigma)^{-s},$$

where $\Lambda_o^{2n} := S^{2n}(\mathbb{Z}_p)$ and $\tilde{\nu}_0(\sigma)$ and $\tilde{\nu}(\sigma)$ for a matrix $\sigma \in S_o^{2n}$ are defined similarly as above and we refer to [21, Chapter III]. One has that $\nu(\sigma) = \tilde{\nu}(\imath(\sigma))$ and $\nu_0(\sigma) = \tilde{\nu}_0(\imath(\sigma))$.

The following lemma shows that for a finite place, where the quaternion algebra B is unramified, we have an equality of the quaternionic Siegel type series with the orthogonal ones.

Lemma 3.4. *Let $\zeta \in T_v^n$. Then we have*

$$\alpha(\zeta, s, \chi_v) = \alpha_o(\frac{1}{2}\imath(\zeta)K_n, s, \chi_v).$$

Proof. We have by definition

$$\alpha_o(\frac{1}{2}\imath(\zeta)K_n, s, \chi_v) = \sum_{\sigma \in S_o^{2n}/\Lambda_o^{2n}} e_v(-\frac{1}{2}\mathrm{tr}\,(\imath(\zeta)K_n\sigma)) \chi_v(\tilde{\nu}_0(\sigma)\tilde{\nu}(\sigma)^{-s} =$$

$$\sum_{\sigma \in S_q^n/\Lambda_q^n} e_v(-\frac{1}{2}\mathrm{tr}\,(\imath(\zeta)K_n\imath(\sigma)K_n)) \chi_v(\tilde{\nu}_0(\imath(\sigma)K_n)\tilde{\nu}(\imath(\sigma)K_n)^{-s} =$$

$$\sum_{\sigma \in S_q^n/\Lambda_q^n} e_v(\frac{1}{2}\mathrm{tr}\,({}^t K_n\imath(\zeta)\imath(\sigma)K_n)) \chi_v(\tilde{\nu}_0(\imath(\sigma)K_n)\tilde{\nu}(\imath(\sigma)K_n)^{-s} =$$

$$\sum_{\sigma\in S_q^n/\Lambda_q^n} e_v(-\frac{1}{2}\mathrm{tr}\,(\imath(\zeta\sigma)))\chi_v(\tilde{\nu}_0(\imath(\sigma)))\tilde{\nu}(\imath(\sigma))^{-s} =$$

$$\sum_{\sigma\in S_q^n/\Lambda_q^n} e_v(-\lambda(\zeta\sigma))\chi_v(\nu_0(\sigma))\nu(\sigma)^{-s} = \alpha(\zeta,\chi_v,s),$$

where we have used the facts that $K_n\sigma K_n^{-1} = {}^t\sigma$, and $\nu(\sigma K_n^{-1}) = \nu(\sigma) = \nu({}^t\sigma)$ and similarly for $\tilde{\nu}_0$. For this we refer to [20, Proposition 3.6 (3)], namely the fact that $\nu(\sigma)$ is determined by the elementary divisors of the matrix σ. $\square$

We set $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\chi)$, the smallest algebraic extension of $\mathbb{Q}$ containing the values of the finite character χ. Then from [21, Theorem 13.6], where the orthogonal Siegel series are studied and the identification established in Lemma 3.4 above, we can obtain the following,

Proposition 3.5. *Let v be a finite place, corresponding to a prime number p and assume that B is unramified at p and p does not divide $\mathfrak{c}$. Let $\zeta \in T_v^n$ and write r for its rank over B_v. Then*

$$\alpha(\zeta,s,\chi_v) = \frac{\prod_{i=1}^{n}(1-p^{2i-2-2s}\chi^*(p)^2)}{\prod_{i=1}^{n-r}(1-p^{4n-r-2i-1-2s}\chi^*(p)^2)} \times P_\zeta(\chi^*(p)p^{-s}),$$

where $P_\zeta(X)\in\mathcal{O}[X]$, $P_\zeta(0)=1$. Moreover if $\det(\imath_p(\zeta))\in 2^{n+1}\mathbb{Z}_p^\times$ then $P_\zeta = 1$.

In particular with notation as in Proposition 3.3 we obtain.

Proposition 3.6. *Let $h \in T^n$ of rank r over B. Then we have that*

$$\alpha_{\mathfrak{c}}(\beta q^* hq,s,\chi) = \frac{\prod_{i=1}^{n-r} L_{\mathfrak{c}}(2s-4n+2r+2i+1,\chi^2)}{\prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s-2i,\chi^2)} \times \prod_{p\in\mathbf{b}} P_{h,q,p}(\chi^*(p)p^{-s}),$$

where $\mathbf{b}$ is a finite set of primes and $P_{h,q,p}(X)\in\mathcal{O}[X]$. Here, for the Dirichlet character χ^2, we write $L_{\mathfrak{c}}(s,\chi^2)$ for the Dirichlet series associated to χ^2 with the Euler factors at the primes dividing $\mathfrak{c}$ removed.

We state one more fact regarding the function $\xi(\cdot)$ above.

Proposition 3.7 (Shimura, page 293 in [17])**.** *For $y \in S_{\mathbf{a}}^+$, $h \in S_{\mathbf{a}}$, the function $\xi(y,h,s,s')$ has a meromorphic continuation in $s,s'\in\mathbb{C}$. In particular we have*

$$\xi(y,h,s,s') = \frac{\Gamma_t(s+s'-2n+1)}{\Gamma_{n-q}(s)\Gamma_{n-p}(s')} \times \widetilde{\omega}(y,h,s,s'),$$

where p is the number of positive eigenvalues of h, q is the number of negative eigenvalues of h, $t := n-p-q$,

$$\Gamma_m(s) := \pi^{m(m-1)}\prod_{i=0}^{m-1}\Gamma(s-2i),\ \ m\in\mathbb{N},$$

and $\widetilde{\omega}(\cdot)$ is holomorphic with respect to $(s,s')\in\mathbb{C}^2$.

Analytic properties of Siegel-type Eisenstein series. We are now ready to state the main result of this section. In the following we use the notation $E(z,s) = E_\ell(z,s;\chi,\Gamma)$ introduced above. The following Theorem should be seen as the analogue in the quaternionic case of [19, Theorem 7.3] (symplectic) and [21, Theorem 19.3] (unitary).

Theorem 3.8. *For $s \in \mathbb{C}$ and $\ell \in \mathbb{Z}$ we define the function $\mathcal{G}^n(s,\ell)$ as*

$$\mathcal{G}^n(s,\ell) := \begin{cases} \Gamma_n(s+\ell), & \text{if } \ell \geq n, \\ \Gamma_\ell(s+\ell)\Gamma(s-\ell)\prod_{i=0}^{n-\ell-2}\Gamma(s-\ell-i), & \text{if } 0 \leq \ell < n. \end{cases}$$

We define the function $\mathcal{P}(s) := \mathcal{G}^n(s,\ell)\Lambda_{\mathfrak{c}}(\chi^2,s)E(z,s)$, where

$$\Lambda_{\mathfrak{c}}(\chi^2,s) := \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s-2i,\chi^2).$$

Then $\mathcal{P}(s)$ is a meromorphic function on $s \in \mathbb{C}$ with finitely many poles all of which are simple. More precisely, the function $\mathcal{P}(s)$ is holomorphic if $\chi^2 \neq 1$, or $\ell \geq n$. If $\chi^2 = 1$ and $0 \leq \ell < n$ then the possible poles are all simple and are in the set:

$$\{j \in \mathbb{Z} \mid n \leq j \leq 2n-\ell\}.$$

Remark 3.9. We note that our result is different in nature from [32, Theorem 3.1] and [33, Proposition 9.1] where the analytic properties of Siegel-type Eisenstein series are considered. The differences are: (1) the inclusion of gamma factors in the normalization of the Eisenstein series, and, (2) the fact that our result is not "generic", that is it does depend on the weight of the Eisenstein series, or better say on the particular section chosen to define the Eisenstein series. Of course the results of [32, 33] apply to more general Eisenstein series, but the results here, in the particular case, allow us to obtain a more precise description of the possible poles. We should note here that this kind of difference also appears between the work of Shimura [19, Theorem 7.3] compared to the approach taken by Kudla and Rallis [14, Theorem 1.0.1] and [15, Theorem 1.1 and Theorem 4.12] in the symplectic case.

Proof. Our proof is modeled to the one of [21, Theorem 19.3], where an analogue statement for unitary Siegel-type Eisenstein series is proved.

The fact that the function $\mathcal{P}(s)$ has a meromorphic continuation to the entire complex plane is well-know by the general theory of Eisenstein series due to Langlands. Here we focus on the location of poles, and their orders. We define $E^*(z,s)$ by $E^*(x\cdot\mathbf{i},s) = j_x(\mathbf{i})^\ell\mathbf{E}^*(x,s)$, where $x \in G_{\mathbf{a}}$ such that $x\cdot\mathbf{i} = z = x+iy \in \mathcal{H}$ and we note that the series $E^*(z,s)$ and $E(z,s)$ have the same analytic properties with respect to the variable s. In particular by Proposition 3.3 above, and an argument similar to [20, Proposition 19.1] it is enough to study the analytic properties with respect to $s \in \mathbb{C}$ of the Fourier coefficients,

$$\mathcal{G}^n(s,\ell)\Lambda_{\mathfrak{c}}(\chi^2,s)c(h,y^{1/2},s) = \mathcal{G}^n(s,\ell) \times A(y^{1/2},n,\chi) \times \left(\prod_{i=1}^{n-r} L_{\mathfrak{c}}(2s-4n+2r+2i+1,\chi^2)\right) \times \prod_{p\in\mathbf{b}} P_{h,1,p}(\chi^*(p)p^{-s}) \times \frac{\Gamma_t(2s-2n+1)}{\Gamma_{n-q}(s+\ell)\Gamma_{n-p}(s-\ell)} \times \widetilde{\omega}(y,h,s+\ell,s-\ell),$$

for any $h \in S$ and $y \in S_{\mathbf{a}}^{+}$, where p, q, t are as in Proposition 3.7, and r is the rank of the matrix h, and so $r = n - t$.

We note that for any integer r, with $0 \leq r \leq n$ the function,

$$\mathcal{L}(s) := \prod_{i=1}^{n-r} L_{\mathfrak{c}}(2s - 4n + 2r + 2i + 1, \chi^2)\Gamma(s - 2n + r + i + 1/2),$$

is holomorphic for $\chi^2 \neq 1$. Moreover if $\chi^2 = 1$ and since $\mathfrak{c} \neq \mathbb{Z}$ we have that it is meromorphic with simple poles at $s = 2n - r - i$ for $i = 1, \ldots, n - r$, or equivalently at $s = n + t - i - 1$ for $i = 0, \ldots, t - 1$, where $t = n - r$.

Our aim is to determine a factor $\mathcal{G}(s)$, which eventually will be our $\mathcal{G}^n(s, \ell)$, which is independent of the matrix h, such that the function

$$\mathcal{F}(s) := \mathcal{G}(s) \times \frac{\Gamma_t(2s - 2n + 1)}{\Gamma_{n-q}(s + \ell)\Gamma_{n-p}(s - \ell)} \times \prod_{i=1}^{n-r} \Gamma(s - 2n + r + i + 1/2)^{-1} \times \mathcal{L}(s),$$

has simple poles, and we can locate them independently of the matrix h.

We now introduce some notation used in [19]. For two meromorphic functions f and g on $\mathbb{C}$ we write $f \succ g$ if f/g is an entire function. We write $f \sim g$ if $f \succ g$ and $g \succ f$ holds.

The case of $\ell \geq n$: We claim that we can choose $\mathcal{G}(s) = \Gamma_n(s + \ell)$. We first assume that $\chi^2 \neq 1$. That is, we know that $\mathcal{L}(s)$ is holomorphic for all $s \in \mathbb{C}$. Since $\Gamma(2s) \sim \Gamma(s)\Gamma(s + 1/2)$, we obtain

$$\frac{\Gamma_t(2s - 2n + 1)}{\prod_{i=1}^{n-r} \Gamma(s - 2n + r + i + 1/2)} \sim \frac{\prod_{i=0}^{t-1} \Gamma(s - n - i + 1/2)\Gamma(s - n - i + 1)}{\prod_{i=1}^{t} \Gamma(s - n - t + i + 1/2)} = \prod_{i=0}^{t-1} \Gamma(s - n - i + 1)$$

That is, we need to study the analytic properties of the exression

$$\frac{\Gamma_n(s + \ell)}{\Gamma_{n-q}(s + \ell)\Gamma_{n-p}(s - \ell)} \times \prod_{i=0}^{t-1} \Gamma(s - n - i + 1).$$

We note that $\Gamma_{m+n}(s) \sim \Gamma_m(s)\Gamma_n(s - 2m)$ and $\Gamma_n(s + m) \succ \Gamma_n(s)$ for $n, m \in \mathbb{Z}_+$. That is, $\Gamma_n(s + \ell) \sim \Gamma_{p+t}(s + \ell)\Gamma_q(s + \ell - 2p - 2t)$ and $\Gamma_{q+t}(s - \ell) \sim \Gamma_q(s - \ell)\Gamma_t(s - \ell - 2q)$ and hence,

$$\frac{\Gamma_n(s + \ell)}{\Gamma_{p+t}(s + \ell)\Gamma_{q+t}(s - \ell)} \times \prod_{i=0}^{t-1} \Gamma(s - n - i + 1) \sim \frac{\Gamma_q(s + \ell - 2p - 2t)}{\Gamma_q(s - \ell)\Gamma_t(s - \ell - 2q)} \times \prod_{i=0}^{t-1} \Gamma(s - n - i + 1) \sim$$

$$\frac{\Gamma_q(s + \ell - 2p - 2t)}{\Gamma_q(s - \ell)} \times \prod_{i=0}^{t-1} \frac{\Gamma(s - n - i + 1)}{\Gamma(s - \ell - 2q - 2i)}. \tag{5}$$

This function is holomorphic if $\ell - 2p - 2t \geq -\ell$ and $-n - i + 1 \geq -\ell - 2q - 2i$ for $i = 0, \ldots, t - 1$. The first gives $\ell \geq p + t$ and the second $n - 1 \leq \ell + 2q + i$. Hence if $\ell > n - 1$ both inequalities hold.

We now consider the case where $\chi^2 = 1$. We claim again that the function $\mathcal{F}(s)$ is holomorphic. In this situation we have seen that the function $\mathcal{L}(s)$ has simple poles at $s = n+t-i-1$ for $i = 0, \dots, t-1$. We now see which of these poles can be cancelled by the expression in Equation (5) above. For $\ell > n-1$ we have that $-n-i+1 > -\ell-2q-2i$ and hence the function $\frac{\Gamma(s-n-i+1)}{\Gamma(s-\ell-2q-2i)}$ has a factor $(s-n-i)$, and so zeros at $s = n+i$, for $i = 0, \dots, t-1$. That is, all the poles are cancelled.

The case $\ell < n$: We set $\mathcal{G}(s) = \Gamma_\ell(s+\ell)\Gamma(s-\ell)\prod_{i=0}^{n-\ell-2}\Gamma(s-\ell-i)$. Then we see that we need to understand the analytic properties of the expression

$$\frac{\Gamma_\ell(s+\ell)\Gamma(s-\ell)\prod_{i=0}^{n-\ell-2}\Gamma(s-\ell-i)}{\Gamma_{n-q}(s+\ell)\Gamma_{n-p}(s-\ell)} \times \prod_{i=0}^{t-1}\Gamma(s-n-i+1). \tag{6}$$

We first assume that $\ell \geq p+t$, and $\chi^2 \neq 1$. We note that

$$\Gamma_\ell(s+\ell)\Gamma(s-\ell)\prod_{i=0}^{n-\ell-2}\Gamma(s-\ell-i) \succ \Gamma_\ell(s+\ell)\prod_{i=0}^{n-\ell-1}\Gamma(s-\ell-i). \tag{7}$$

Then the expression in Equation (6) above, up to holomorphic functions, is equal to

$$\frac{\Gamma_{\ell-p-t}(s+\ell-2p-2t)\prod_{i=0}^{n-\ell-1}\Gamma(s-\ell-i)}{\Gamma_{q+t}(s-\ell)} \times \prod_{i=0}^{t-1}\Gamma(s-n-i+1) =$$

$$\frac{\Gamma_{\ell-p-t}(s+\ell-2p-2t)\prod_{i=0}^{n-\ell-1}\Gamma(s-\ell-i)}{\Gamma_{q}(s-\ell)\Gamma_t(s-\ell-2q)} \times \prod_{i=0}^{t-1}\Gamma(s-n-i+1) =$$

$$\frac{\Gamma_{\ell-p-t}(s+\ell-2p-2t)\prod_{i=0}^{n-\ell-1}\Gamma(s-\ell-i)}{\Gamma_{q}(s-\ell)} \times \prod_{i=0}^{t-1}\frac{\Gamma(s-n-i+1)}{\Gamma(s-\ell-2q-2i)}. \tag{8}$$

The last factor is holomorphic since we are taking $\ell \geq p+t$. Moreover

$$\frac{\Gamma_{\ell-p-t}(s+\ell-2p-2t)\prod_{i=0}^{n-\ell-1}\Gamma(s-\ell-i)}{\Gamma_{q}(s-\ell)} =$$

$$\frac{\Gamma_{\ell-p-t}(s+\ell-2p-2t)}{\Gamma_{\ell-p-t}(s-\ell)}\prod_{i=0}^{n-\ell-1}\frac{\Gamma(s-\ell-i)}{\Gamma(s-3\ell+2p+2t-2i)},$$

which is holomorphic since $\ell \geq p+t$.

We are still assuming $\ell \geq p+t$ but now we consider the case of $\chi^2 = 1$. We observe that the factor $\prod_{i=0}^{t-1}\frac{\Gamma(s-n-i+1)}{\Gamma(s-\ell-2q-2i)}$ in Equation (8) is independent of the substitution made in Equation (7), in the sense that the zeros produced by the ratio are also zeros appearing in the initial expression in Equation (6). But then arguing exactly as before we see that all the poles of the factor $\mathcal{L}(s)$ are cancelled by these zeroes. So in this case there are no poles.

We now consider the case of $\ell < p+t$ and $\chi^2 \neq 1$. In this case we have $\frac{\Gamma_\ell(s+\ell)}{\Gamma_{p+t}(s+\ell)} \sim \frac{\Gamma_\ell(s+\ell)}{\Gamma_\ell(s+\ell)\Gamma_{p+t-\ell}(s-\ell)} = \frac{1}{\Gamma_{p+t-\ell}(s-\ell)}$. Moreover we have that

$$\Gamma(s-\ell)\prod_{i=0}^{n-\ell-2}\Gamma(s-\ell-i) \times \prod_{i=0}^{t-1}\Gamma(s-n-i+1) = \Gamma(s-\ell)\times\prod_{i=0}^{t-2+n-\ell}\Gamma(s-\ell-i).$$

That is, it is enough to show that

$$\frac{\Gamma(s-\ell)\times\prod_{i=0}^{t-2+n-\ell}\Gamma(s-\ell-i)}{\Gamma_{p+t-\ell}(s-\ell)\Gamma_{n-p}(s-\ell)} \succ 1.$$

We first observe that

$$\Gamma(s-\ell)\times\prod_{i=0}^{t-2+n-\ell}\Gamma(s-\ell-i) \succ \prod_{i=0}^{\nu-1}\Gamma(s-\ell-2i)\prod_{i=0}^{\nu+\mu-1}\Gamma(s-\ell-2i) = \Gamma_\nu(s-\ell)\Gamma_{\nu+\mu}(s-\ell),$$

where we have set $n+t-\ell = 2\nu+\mu$ with ν a positive integer and $\mu \in \{0,1\}$. In particular we need to show

$$\frac{\Gamma_\nu(s-\ell)\Gamma_{\nu+\mu}(s-\ell)}{\Gamma_{p+t-\ell}(s-\ell)\Gamma_{n-p}(s-\ell)} \succ 1.$$

We now employ an idea of Shimura in [21, page 162], and set $\alpha := \max\{p+t-\ell, n-p\}$ and $\beta := \min\{p+t-\ell, n-p\}$. That is, $\alpha+\beta = n+t-\ell = 2\nu+\mu$ and hence also $\beta \leq \nu \leq \nu+\mu \leq \alpha$. In particular we have

$$\Gamma_\nu(s-\ell) \sim \Gamma_\beta(s-\ell)\Gamma_{\nu-\beta}(s-\ell-2\beta),$$

and

$$\Gamma_\alpha(s-\ell) \sim \Gamma_{\nu+\mu}(s-\ell)\Gamma_{\nu-\beta}(s-\ell-2(\nu+\mu)).$$

In particular

$$\frac{\Gamma_\nu(s-\ell)\Gamma_{\nu+\mu}(s-\ell)}{\Gamma_{p+t-\ell}(s-\ell)\Gamma_{n-p}(s-\ell)} = \frac{\Gamma_\nu(s-\ell)\Gamma_{\nu+\mu}(s-\ell)}{\Gamma_\alpha(s-\ell)\Gamma_\beta(s-\ell)} \sim \frac{\Gamma_{\nu-\beta}(s-\ell-2\beta)}{\Gamma_{\nu-\beta}(s-\ell-2(\nu+\mu))} \succ 1,$$

since $\beta \leq \nu+\mu$.

Finally, we now consider the case of $\ell < p+t$ and $\chi^2 = 1$. We recall that the poles of the $\mathcal{L}(s)$ factor are

$$\{j \in \mathbb{Z} \mid n \leq j \leq n+t-1\}.$$

We will show that if $2n-\ell \leq j \leq n+t-1$, then these poles are cancelled. Indeed in this case we have $\ell > n+1-t = p+q+1$. Using the notation above we have that $\alpha = q+t$, $\beta = p+t-\ell$ and $\alpha-\beta > 2q+1$ and so $\nu > \beta$. Moreover we have $2\nu = n+t-\ell-\mu$ and hence $2n-\ell \leq j \leq 2\nu+\ell+\mu-1$. As we have seen above,

$$\frac{\Gamma_\nu(s-\ell)\Gamma_{\nu+\mu}(s-\ell)}{\Gamma_{p+t-\ell}(s-\ell)\Gamma_{n-p}(s-\ell)} \sim \frac{\Gamma_{\nu-\beta}(s-\ell-2\beta)}{\Gamma_{\nu-\beta}(s-\ell-2(\nu+\mu))}.$$

The latter expression has zeros at $\ell+2\beta+i$ for $i = 1,\dots,2(\nu-\beta)$. For j as above we have that

$$0 < 2q < j-\ell-2\beta \leq 2\nu+\mu-2\beta.$$

That is $j = \ell + 2\beta + k$ for $0 < k \le 2\nu - 2\beta$. In particular we can cancel all the poles for $j \ge 2n - \ell$. That is, the remaining poles have to be in the interval

$$\{j \in \mathbb{Z} \mid n \le j \le 2n - \ell\}.$$

□

4. Howe-Weyl Duality

In this section we study the Howe-Weyl duality for the pair $Sp_n(\mathbb{C}) \times GL_{2n}(\mathbb{C})$. Our aim is to prove Theorem 4.2 below, which is an extension of an analogue theorem proved by Kashiwara and Vergne [12] for the pair $O_n(\mathbb{C}) \times GL_n(\mathbb{C})$, or more generally for $O_m(\mathbb{C}) \times GL_n(\mathbb{C})$. In the next section we will use this theorem to construct some vector valued theta series. The presentation of the general setting in this section is taken from [9, Chapter 5].

We first introduce the Weyl algebra following [9, section 5.6.1]. We let V be a complex vector space of dimension n and for each vector $v \in V$ we write $v = \sum_{i=1}^n x_i e_i$, where $x_i \in \mathbb{C}$ and $\{e_i\}$ denotes the standard basis, after we have fixed an isomorphism of V with $\mathbb{C}^n$. We write $\mathcal{P}(V)$ for the polynomial functions on V, that is polynomials in $\mathbb{C}[x_i, i = 1, \dots, n]$. We now define the Weyl algebra $\mathbb{D}(V) \subset \mathrm{End}(\mathcal{P}(V))$ to be the algebra of polynomial coefficient differential operators on V, generated by

$$D_i := \frac{\partial}{\partial x_i} \text{ and } M_i := \text{multiplication by } x_i, \;\; i = 1, \dots, n.$$

We let $\mathrm{GL}(V)$ acting on $\mathcal{P}(V)$ by $\tau(g)f(x) = f(g^{-1}x)$, and we can view $\mathbb{D}(V)$ as a $\mathrm{GL}(V)$ module by $g \cdot T := \tau(g)T\tau(g)^{-1}$, for $T \in \mathbb{D}(V)$ (see [9, page 279]). We now state the following theorem of Weyl as it is presented in [9, Theorem 5.6.1].

Theorem 4.1 (Weyl)**.** *Let G be a reductive algebraic subgroup of* $\mathrm{GL}(V)$*, and let it act on $\mathcal{P}(V)$ by τ. Then we have that*

$$\mathcal{P}(V) \cong \bigoplus_{\lambda \in S} E^\lambda \otimes F^\lambda,$$

as a $\mathbb{D}(V)^G \otimes \mathbb{C}[G]$ module, where S is a set of irreducible representations of G, F^λ an irreducible $\mathbb{C}[G]$-module, and E^λ an irreducible $\mathbb{D}(V)$ module that uniquely determines F^λ.

We now take $G := Sp_n(\mathbb{C})$ and $V := \mathbb{C}^{2n} \oplus \ldots \oplus \mathbb{C}^{2n} = \left(\mathbb{C}^{2n}\right)^{2n}$ (This notation of G is valid for this section only and should not be confused with the general G of the paper. The latter does not appear in this section.). We let G act on V by $g \cdot [x_1, \dots, x_{2n}] = [gx_1, \dots gx_{2n}]$, and we identify V with $M_{2n}(\mathbb{C})$. In particular we have that G acts on $\mathcal{P}(V) = \mathcal{P}(M_{2n})$ by $\tau(g)f(x) = f(g^{-1}x)$, where $x = [x_1, \dots, x_{2n}]$ and $x_i = {}^t(x_{1,i}, \dots, x_{2n,i})$. We now take the symplectic group with respect to the skew-symmetric bilinear form $\omega(x, y) = {}^t x \eta_n y$, where $\eta_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Following [9, section 5.6.5], for $1 \le i, j \le 2n$ we define the following elements in $\mathbb{D}(V)^G$:

$$D_{ij} := \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_{k,i} \partial x_{k+n,j}} - \frac{\partial^2}{\partial x_{k+n,i} \partial x_{k,j}} \right), \tag{9}$$

$$M_{ij} := \text{multiplication by } \sum_{k=1}^{n} \left(x_{k,i} x_{k+n,j} - x_{k+n,i} x_{k,j} \right), \tag{10}$$

$$E_{ij} := \sum_{k=1}^{2n} x_{k,i} \frac{\partial}{\partial x_{k,j}}. \tag{11}$$

We remark here that if we choose to realize $Sp_n(\mathbb{C})$ using the form $\Omega(x,y) = {}^t x J_n y$, where $J_n = \begin{pmatrix} 0 & s_n \\ -s_n & 0 \end{pmatrix}$, where s_n is the anti-diagonal matrix with ones on the anti-diagonal, we have the following elements in $\mathbb{D}(V)^G$:

$$D_{ij} := \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_{k,i} \partial x_{2n+1-k,j}} - \frac{\partial^2}{\partial x_{2n+1-k,i} \partial x_{k,j}} \right), \tag{12}$$

$$M_{ij} := \text{multiplication by } \sum_{k=1}^{n} \left(x_{k,i} x_{2n+1-k,j} - x_{2n+1-k,i} x_{k,j} \right), \tag{13}$$

$$E_{ij} := \sum_{k=1}^{2n} x_{k,i} \frac{\partial}{\partial x_{k,j}}. \tag{14}$$

We now define $\mathfrak{g} := \mathrm{span}\{E_{ij} + n\delta_{ij}, M_{ij}, D_{ij}\} \subset \mathbb{D}(V)^G$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. It is then shown in [9, Theorem 5.6.14] that $\mathfrak{g}$ is a Lie subalgebra of $\mathbb{D}(V)^G$ isomorphic to $\mathfrak{so}(4n, \mathbb{C})$ and generates $\mathbb{D}(V)^G$. In particular, using the general theorem of Weyl above, we have that $\mathcal{P}(V) \cong \bigoplus_{\lambda \in S} E^\lambda \otimes F^\lambda$, where S is a subset of irreducible representations of $Sp_n(\mathbb{C})$ in bijection with irreducible Lie algebra representations E^λ of $\mathfrak{so}(4n, \mathbb{C})$.

Following [10], we now consider the $Sp_n(\mathbb{C})$-harmonic polynomials defined as

$$\mathcal{H} := \mathcal{H}(V) := \{ f \in \mathcal{P}(V) \mid D_{ij} f = 0 \text{ for all } i \text{ and } j \}.$$

(**Note:** We will use the notation $\mathcal{H}$ to denote the harmonic polynomials only in this section, and hence there is no risk of confusing it with the symmetric space of the quaternionic modular forms.)

By [9, Equation (5.94)] we have $[E_{ij} + \delta_{ij}, D_{rs}] = -\delta_{ir} D_{js} + \delta_{is} D_{jr}$. In particular we may view $\mathcal{H}$ as a $\mathfrak{g}' := \mathrm{span}\{E_{ij}\}$ module, and of course we have that $\mathfrak{g}' \cong \mathfrak{gl}(2n, \mathbb{C})$. Actually the action of $\mathfrak{g}'$ on $\mathcal{H}$ is nothing else than the differential of the action $\rho(g) f(x) = f(xg)$ of $g \in GL_{2n}(\mathbb{C})$. We consider $\mathcal{H}^\lambda := \mathcal{H} \cap (E^\lambda \otimes F^\lambda)$. It is then shown by Howe in [10, Theorem 9] that $\mathcal{H}^\lambda$ is an irreducible $GL_{2n}(\mathbb{C}) \otimes Sp_n(\mathbb{C})$ module. In particular we have the $GL_{2n}(\mathbb{C}) \times Sp_n(\mathbb{C})$-module decomposition

$$\mathcal{H} = \bigoplus_{\lambda \in S} \mathcal{H}^\lambda. \tag{15}$$

We now denote with T the set of maximal weights of the irreducible representations of $GL_{2n}(\mathbb{C})$ obtained in this decomposition. It is not hard to see that this is independent on the realization of the group $Sp_n(\mathbb{C})$, i.e. whether we consider the bilinear form ω or Ω above.

We take that $Sp_n(\mathbb{C})$ is realized by the bilinear form Ω defined above. With respect to this realization we have that a maximal torus $H \subset Sp_n(\mathbb{C})$ is given

$$H = \{t = \mathrm{diag}[t_1, \ldots. t_n, t_n^{-1}, \ldots, t_1^{-1}],\ t_i \in \mathbb{C}^\times\},$$

and the corresponding Borel subgroup consists of the upper triangular matrices in $Sp_n(\mathbb{C})$. The irreducible representations of $Sp_n(\mathbb{C})$ are parametrized by the dominant integral weights $\mu = \prod_{i=1}^n \epsilon_i^{m_i}$ where $m_i \in \mathbb{Z}$ with $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, and ϵ_i is the character on H given by $\epsilon_i(t) = t_i$. Similarly for GL_{2n} we take the diagonal matrices to play the role of the selected maximal torus and we also write ϵ_i with $i = 1, \ldots, 2n$ for the multiplicative characters given by mapping an invertible diagonal matrix to the i^{th} element of the diagonal.

Theorem 4.2. *The set T appearing above is given by*

$$T = \{(m_1, m_2, \ldots, m_n, 0, \ldots, 0) \mid m_1 \geq m_2 \geq \ldots \geq m_n \geq 0\}.$$

Proof. We follow an idea of Kashiwara and Vergne in [12] where the pair $O_n(\mathbb{C}) \times GL_n(\mathbb{C})$ was considered.

We first establish that if $f \in \mathcal{H}$ is a highest weight vector for the action of $GL_{2n}(\mathbb{C})$ and a lowest weight vector for the action of $Sp_n(\mathbb{C})$, then it has to be of the form

$$f(x) = \Delta_1(x)^{a_1} \Delta_2(x)^{a_2} \cdots \Delta_n(x)^{a_n},$$

where $a_i \in \mathbb{N} = \{0, 1, \ldots\}$ and $\Delta_\ell(x)$ is the determinant of the $\ell \times \ell$ leading minor of the matrix $x \in M_{2n}$.

Indeed it is easy to see that a function f of the form above (product of $\Delta_\ell(x)$ with $\ell < n+1$) is a harmonic function since we have $\frac{\partial}{\partial x_{k,i}} \Delta_\ell(x) = 0$ for all $k > n$ and $\ell < n+1$. We furthermore note that $\Delta_\ell(x) \in \mathcal{P}(V)^{N_n^- \times N_n^+}$ where N_n^+ (resp. N_n^-) are the upper-triangular (resp. lower-triangular) unipotent matrices of size $2n$, and clearly for a, b diagonal matrices with $a \in Sp_n(\mathbb{C})$ and $b \in GL_{2n}(\mathbb{C})$ we have $\rho(a,b)\Delta_\ell(x) = \lambda_\ell(a)^{-1}\lambda_\ell(b)\Delta_\ell(x)$ where λ_ℓ is the character on the diagonal matrices given by $\lambda_\ell = \prod_{i=1}^\ell \epsilon_i$.

Given now an n-tuple $\mathbf{m} := (m_1, \ldots, m_n)$ with $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$ we define the harmonic function

$$\Delta_{\mathbf{m}}(x) := \prod_{i=1}^n \Delta_i(x)^{m_i - m_{i+1}},$$

where we set $m_{n+1} = 0$. Then $\Delta_{\mathbf{m}}$ is a lowest weight vector for the action of $Sp_n(\mathbb{C})$ of weight $(-m_1, -m_2, \ldots, -m_n)$ and a highest weight vector for $GL_{2n}(\mathbb{C})$ of weight

$(m_1, \ldots, m_n, 0, \ldots, 0)$. That is the irreducible representation of $Sp_n(\mathbb{C})$ of lowest weight $(-m_1, -m_2, \ldots, -m_n)$ pairs with the irreducible representation of $GL_{2n}(\mathbb{C})$ of highest weight vector $(m_1, \ldots, m_n, 0, \ldots, 0)$. But the irreducible representation of $Sp_n(\mathbb{C})$ of lowest weight $(-m_1, \ldots, -m_n)$ has highest weight $(m_1, \ldots, m_n)$. Indeed by Theorem 3.2.13 in [9] the highest and lowest weights are related by λ and $w_0(\lambda)$ where w_0 the unique element in the Weyl group changing positive to negative roots (under our selected torus) and in the case of $Sp_n(\mathbb{C})$ the element w_0 acts as $-I_n$ with resect to the characters ϵ_i (see Lemma 3.1.6 in [9]). In particular the highest weight vector representation of $Sp_n(\mathbb{C})$ of weight $\mathbf{m}$ will pair with the highest weight vector representation $(m_1, \ldots, m_n, 0, \ldots, 0)$ of $GL_{2n}(\mathbb{C})$. Since we have accounted for all possible finite dimensional representations of $Sp_n(\mathbb{C})$ and each of them contributes with multiplicity at most one, we conclude that these are all possible pairs. □

We will denote by $\mathcal{T}_n$ the set of irreducible polynomial representations (ρ, V) of $GL_{2n}(\mathbb{C})$ with maximal weight belonging to the set T. That is,

$$\mathcal{T}_n := \{\rho : GL_{2n}(\mathbb{C}) \to \mathrm{Aut}(V) \mid \rho \text{ irreducible and polynomial with } \mathbf{m}_\rho \in T\}, \tag{16}$$

where $\mathbf{m}_\rho \in \mathbb{Z}^{2n}$ denotes the maximal weight of the irreducible representation ρ.

We note here that we can realize such a representation ρ by taking right translates by $GL_{2n}(\mathbb{C})$ of an element $\Delta_{\mathbf{m}}$ above, with $\mathbf{m}$ the corresponding tuple in T. As it is explained in [6, page 59] in this way we obtain a polynomial map $P : M_{2n}(\mathbb{C}) \to V$ with the property that:

(1) if we write $P(x) = {}^t(p_1(x), \ldots, p_d(x))$ with $d = \dim_{\mathbb{C}}(V)$, then all coordinates $p_j(x) \in \mathcal{H}$, $j = 1, \ldots, d$,

(2) $P(x\, {}^t g) = \rho(g) P(x)$ for all $g \in GL_{2n}(\mathbb{C})$, and $x \in M_{2n}(\mathbb{C})$.

The polynomial map $P(x)$ as well as the polynomials $p_j(x)$ will be called "pluriharmonic".

5. Quaternionic Theta Series.

The main aim of this section is to construct some vector valued theta series, which we will use later. For this contruction we will use the pluriharmonic polynomials whose existence we proved in the previous section. Another vital component of our construction is a quaternionic Jacobi theta series constructed by Krieg in [13, Chapter IV]. We first fix some notation.

We recall that we write $\imath : M_n(\mathbb{H})_{\mathbb{C}} \to M_{2n}(\mathbb{C})$ for the isomorphism induced from the embedding $\imath : M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})$, where $\imath(A + \mathbf{j}B) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$, with $A, B \in M_n(\mathbb{C})$.

For $M = A + iB \in M_n(\mathbb{H})_{\mathbb{C}} = M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ with $A, B \in M_n(\mathbb{H})$ we define $M^* := A^* + iB^*$. It is important to note here that this notation will be valid in this section only, as it differs from the notation introduced before where we had set $M^* = A^* - iB^*$.

We then introduce the pairing $\tau : M_n(\mathbb{H})_{\mathbb{C}} \times M_n(\mathbb{H})_{\mathbb{C}} \to \mathbb{C}$ by

$$\tau(M_1, M_2) := \frac{1}{2} \mathrm{tr}\,(M_1 M_2^* + M_2 M_1^*) \in \mathbb{C}.$$

In the above definition we do indeed mean the sum of the diagonal entries of the matrix and not the "reduced" trace introduced before. A priori $\tau(M_1, M_2) \in \mathbb{H}$ but since $(M_1 M_2^* + M_2 M_1^*)^* = M_1 M_2^* + M_2 M_1^*$ it is easy to see that this is actually in $\mathbb{C}$. We now list some further properties of this pairing that we will use repeatedly later, and refer to [13, Chapter IV, Proposition 1.1] for a proof. For $A, B, C \in M_n(\mathbb{H})_{\mathbb{C}}$, we have

(1) $\tau(A, B) = \tau(B, A) = \tau(A^*, B^*)$ and
(2) $\tau(A, B\,C) = \tau(A, C\,B^*) = \tau(B, A^*\,C)$.

Moreover we note that the map is $\mathbb{C}$-linear, that is for a constant $c \in \mathbb{C}$, and $A, B \in M_n(\mathbb{H})_{\mathbb{C}}$ we have $c\,\tau(A, B) = \tau(cA, B) = \tau(A, cB)$ as for example follows from [13, Chapter IV, Equations (2) and (3)].

For $\xi = (\xi_{ij}) \in M_n(\mathbb{H})$ we write $\xi = v + \mathbf{j}w$, with $v = (v_{ij}), w = (w_{ij}) \in M_n(\mathbb{C})$ and its image $\imath(\xi) \in M_{2n}(\mathbb{C})$ is given by $\begin{pmatrix} v & -\bar{w} \\ w & \bar{v} \end{pmatrix}$. For a polynomial map $p : M_{2n}(\mathbb{C}) \to \mathbb{C}$, we write $p(D)$ for the differential operator obtained by

$$p\begin{pmatrix} \frac{\partial}{\partial v_{ij}} & -\frac{\partial}{\partial \overline{w}_{ij}} \\ \frac{\partial}{\partial w_{ij}} & \frac{\partial}{\partial \overline{v}_{ij}} \end{pmatrix}, \quad i, j = 1, \ldots, n.$$

Our first step towards the construction of the vector valued theta series is to construct some differential operators. We first prove the following Proposition, in the statement of which the notion of a "pluriharmonic" polynomial is understood in the sense of the last section, where we consider the bilinear form ω as the symplectic form for the definition of $Sp_n(\mathbb{C})$ there. In particular a pluriharmonic polynomial p is annihilated by the operators D_{ij} as given in Equation (9).

Proposition 5.1. *Let $p(x)$ be a pluriharmonic polynomial, and let $\phi \in C^\infty(M_n(\mathbb{H}))$. Then for an $a \in M_n(\mathbb{H})_{\mathbb{C}}$, we have that*

(1) $p(D)e^{\tau(a,\,\xi^*\xi)}\phi(\xi)|_{\xi=0} = p(D)\phi(\xi)|_{\xi=0}$, if $a^* = a$,
(2) Let $c \in \mathbb{C}$. Then we have

$$p(D)e^{c\,\tau(a,\,\xi)}|_{\xi=0} = p\left(\frac{c}{2}\,\widetilde{\imath(a)}\right),$$

where if $a = x + iy \in M_n(\mathbb{H})_{\mathbb{C}}$ we write $\widetilde{\imath(a)} := \overline{\imath(x)} + i\,\overline{\imath(y)} \in M_{2n}(\mathbb{C})$.

Proof. We first establish the second part of the Proposition. Since ξ occurs linearly in the exponent, by the properties of the exponential function it is easy to see that it is enough to establish the statement for the monomials $\frac{\partial}{\partial v_{ij}}, \frac{\partial}{\partial w_{ij}}, \frac{\partial}{\partial \overline{v}_{ij}}, \frac{\partial}{\partial \overline{w}_{ij}}$. We demonstrate the proof for $\frac{\partial}{\partial v_{ij}}$ and similarly one can argue for the rest. We have,

$$\tau(a,\,\xi) = \frac{1}{2}\mathrm{tr}\,(a\,\xi^* + \xi\,a^*) = \frac{1}{2}\mathrm{tr}\,((x+iy)\xi^* + \xi(x^* + iy^*)) =$$

$$\frac{1}{2}(\mathrm{tr}\,(x\,\xi^* + \xi\,x^*) + i\mathrm{tr}\,(y\,\xi^* + \xi\,y)) = \frac{1}{2}(\mathrm{tr}\,(\imath(x^*\xi)) + i\mathrm{tr}\,(\imath(y^*\xi))).$$

If we now write $x = a + \mathbf{j}b$ with $a, b \in M_n(\mathbb{C})$ then $\imath(x^*) = \begin{pmatrix} {}^t\overline{a} & {}^tb \\ -{}^tb & {}^ta \end{pmatrix}$. In particular we have that $\mathrm{tr}\,(\imath(x^*\xi)) = \mathrm{tr}\,({}^t\,\overline{a}v + {}^tb\overline{w} + {}^tb\overline{w} + {}^ta\overline{v})$. Similarly we obtain $\mathrm{tr}\,(\imath(y^*\xi)) = \mathrm{tr}\,({}^t\overline{c}v + {}^td\overline{w} + {}^td\overline{w} + {}^tc\overline{v})$, where $y = c + \mathbf{j}d$. In particular

$$\frac{\partial}{\partial v_{ij}} e^{c\,\tau(a,\xi)} = \frac{c}{2} e^{\tau(a,\xi)} \frac{\partial}{\partial v_{ij}} \left(\mathrm{tr}\,({}^t\overline{a}\,v) + i\mathrm{tr}\,({}^t\overline{c}\,v)\right)$$

But $\mathrm{tr}\,({}^t\overline{a}\,v) = \sum_{ij} {}^t\overline{a}_{ji}v_{ij}$ and similarly $\mathrm{tr}\,({}^t\overline{c}\,v) = \sum_{ij} {}^t\overline{c}_{ji}v_{ij}$, and hence

$$\frac{\partial}{\partial v_{ij}} \left(\mathrm{tr}\,({}^t\overline{a}\,v) + i\mathrm{tr}\,({}^t\overline{c}\,v)\right) = {}^t\overline{a}_{ji} + i\,{}^t\overline{c}_{ji} = \overline{a}_{ij} + i\,\overline{c}_{ij},$$

which is exactly what is stated in the proposition.

We now prove the first statement. Our proof is inspired by the proof of [24, Lemma A3.6], where the case of $O_n(\mathbb{C})$-harmonic polynomials is considered.

By writing p as a sum of homogeneous polynomials we may assume without loss of generality that p is homogeneous. We now observe that if $q(x) = q(x_1, \ldots, x_m)$ is some homogeneous polynomial in the variables $x_1, \ldots, x_m$, then if we set $q_{x_i}(x) := \frac{\partial q}{\partial x_i}$ then we have that

$$[q(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})(x_i\phi)](0) = [q_{x_i}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})\phi](0), \tag{17}$$

for every i and every C^∞ function ϕ. Moreover if q is of degree d then we have that

$$dq(x) = \sum_{i=1}^m x_i q_{x_i}(x). \tag{18}$$

We now note that since $a^* = a$ we have that $\tau(a, \xi^*\xi) = \lambda(a\,\xi^*\xi) = \lambda(x\,\xi^*\xi) + i\lambda(y\,\xi^*\xi)$ for $a = x + iy$ with $x, y \in M_n(\mathbb{H})$. But then

$$\mathrm{tr}\,(x\,\xi^*\xi) = \mathrm{tr}\,\left(\imath(x)\begin{pmatrix} v^*v + w^*w & w^*\bar{v} - v^*\bar{w} \\ -\,{}^tw\,v + {}^tvw & {}^tw\bar{w} + {}^tv\bar{v} \end{pmatrix}\right) =$$

$$\mathrm{tr}\,\left(x_1(v^*v + w^*w) + x_2(-\,{}^tw\,v + {}^tvw) + x_3(w^*\bar{v} - v^*\bar{w}) + x_4({}^tw\bar{w} + {}^tv\bar{v})\right),$$

where $\imath(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ with $x_i \in M_n(\mathbb{C})$. And a similar expression holds for $\lambda(y\,\xi^*\xi)$. Putting these together it is then enough to show the following statements:

$$[p(D)(\omega\phi)](0) = [p(D)\phi](0),$$

where

$$\omega = \omega(v, w) = \exp(\sum_{hk} c_{hk}(\sum_{i=1}^n \bar{v}_{ih}v_{ik} + \sum_{i=1}^n \bar{w}_{ih}w_{ik})), \tag{19}$$

or

$$\omega = \omega(v, w) = \exp(\sum_{hk} c_{hk}(\sum_{i=1}^n v_{ih}w_{ik} - \sum_{i=1}^n w_{ih}v_{ik})), \tag{20}$$

for $c_{hk} \in \mathbb{C}$. We will perform induction on the degree d of p. We first consider the case of Equation (19). We note that

$$\frac{\partial \omega}{\partial v_{\alpha\beta}} = \omega \sum_h c_{h\beta} \bar{v}_{\alpha h}, \quad \frac{\partial \omega}{\partial w_{\alpha\beta}} = \omega \sum_h c_{h\beta} \bar{w}_{\alpha h},$$

and

$$\frac{\partial \omega}{\partial \bar{v}_{\alpha\beta}} = \omega \sum_k c_{\beta k} v_{\alpha k}, \quad \frac{\partial \omega}{\partial \bar{w}_{\alpha\beta}} = \omega \sum_k c_{\beta k} w_{\alpha k}.$$

Then by (17) and (18) above we may write

$$[dp(D)(\omega\phi)](0) = \left[\sum_{i,h=1}^{n} \left(p_{v_{ih}}(D)\frac{\partial}{\partial v_{ih}} + p_{\bar{v}_{ih}}(D)\frac{\partial}{\partial \bar{v}_{ih}} + p_{w_{ih}}(D)\frac{\partial}{\partial w_{ih}} + p_{\bar{w}_{ih}}(D)\frac{\partial}{\partial \bar{w}_{ih}}\right)(\omega\phi)\right](0) =$$

$$\sum_{i,h=1}^{n} \left([p_{v_{ih}}(D)(\omega\frac{\partial \phi}{\partial v_{ih}})](0) + [p_{\bar{v}_{ih}}(D)(\omega\frac{\partial \phi}{\partial \bar{v}_{ih}})](0) + [p_{w_{ih}}(D)(\omega\frac{\partial \phi}{\partial w_{ih}})](0) + [p_{\bar{w}_{ih}}(D)(\omega\frac{\partial \phi}{\partial \bar{w}_{ih}})](0)\right) +$$

$$\left[\sum_{i,h,k} c_{kh}\left(\frac{p_{v_{ih}}}{\partial \bar{v}_{ik}} + \frac{p_{w_{ih}}}{\partial \bar{w}_{ik}}\right)(D)(\omega\phi)\right](0) + \left[\sum_{i,h,k} c_{hk}\left(\frac{p_{\bar{v}_{ih}}}{\partial v_{ik}} + \frac{p_{\bar{w}_{ih}}}{\partial w_{ik}}\right)(D)(\omega\phi)\right](0).$$

Since $p_{v_{ih}}, p_{w_{ih}}, p_{\bar{v}_{ih}}$ and $p_{\bar{w}_{ih}}$ are all of degree $d-1$, by the induction hypothesis and (18) the first term is equal to

$$\sum_{i,h=1}^{n} \left([p_{v_{ih}}(D)(\frac{\partial \phi}{\partial v_{ih}})](0) + [p_{\bar{v}_{ih}}(D)(\frac{\partial \phi}{\partial \bar{v}_{ih}})](0) + [p_{w_{ih}}(D)(\frac{\partial \phi}{\partial w_{ih}})](0) + [p_{\bar{w}_{ih}}(D)(\frac{\partial \phi}{\partial \bar{w}_{ih}})](0)\right) =$$

$$[dp(D)\phi](0).$$

But the last term in the summand is zero, since it is equal to

$$\left[\sum_{h,k} c_{kh}\sum_i\left(\frac{p_{v_{ih}}}{\partial \bar{v}_{ik}} + \frac{p_{w_{ih}}}{\partial \bar{w}_{ik}}\right)(D)(\omega\phi)\right](0) + \left[\sum_{h,k} c_{hk}\sum_i\left(\frac{p_{\bar{v}_{ih}}}{\partial v_{ik}} + \frac{p_{\bar{w}_{ih}}}{\partial w_{ik}}\right)(D)(\omega\phi)\right](0),$$

and we have that $\sum_i \frac{p_{v_{ih}}}{\partial \bar{v}_{ik}} + \frac{p_{w_{ih}}}{\partial \bar{w}_{ik}} = \sum_i \frac{p_{\bar{v}_{ih}}}{\partial v_{ik}} + \frac{p_{\bar{w}_{ih}}}{\partial w_{ik}} = 0$ since p is harmonic. Indeed in the notation of Section 4, this is nothing else than $D_{h,k+n}p = 0$ for $1 \leq h, k \leq n$.

We now consider the case of Equation (20). As before we have,

$$\frac{\partial \omega}{\partial v_{\alpha\beta}} = \omega\left(\sum_k c_{\beta k} w_{\alpha k} - \sum_h c_{h\beta} w_{\alpha h}\right),$$

and

$$\frac{\partial \omega}{\partial w_{\alpha\beta}} = \omega\left(\sum_h c_{h\beta} v_{\alpha h} - \sum_k c_{\beta k} v_{\alpha k}\right),$$

and $\frac{\partial \omega}{\partial \bar{v}_{\alpha\beta}} = \frac{\partial \omega}{\partial \bar{w}_{\alpha\beta}} = 0$. We then have

$$[dp(D)(\omega\phi)](0) = \left[\sum_{i,h=1}^{n} \left(p_{v_{ih}}(D)\frac{\partial}{\partial v_{ih}} + p_{\bar{v}_{ih}}(D)\frac{\partial}{\partial \bar{v}_{ih}} + p_{w_{ih}}(D)\frac{\partial}{\partial w_{ih}} + p_{\bar{w}_{ih}}(D)\frac{\partial}{\partial \bar{w}_{ih}}\right)(\omega\phi)\right](0) =$$

$$\sum_{i,h=1}^{n}\left([p_{v_{ih}}(D)(\omega\frac{\partial\phi}{\partial v_{ih}})](0)+[p_{\bar{v}_{ih}}(D)(\omega\frac{\partial\phi}{\partial \bar{v}_{ih}})](0)+[p_{w_{ih}}(D)(\omega\frac{\partial\phi}{\partial w_{ih}})](0)+[p_{\bar{w}_{ih}}(D)(\omega\frac{\partial\phi}{\partial \bar{w}_{ih}})](0)\right)+$$

$$\left[\sum_{i,h,k}\left(c_{hk}\frac{p_{v_{ih}}}{\partial w_{ik}}-c_{kh}\frac{p_{v_{ih}}}{\partial w_{ik}}\right)(D)(\omega\phi)\right](0)+\left[\sum_{i,h,k}\left(c_{kh}\frac{p_{w_{ih}}}{\partial v_{ik}}-c_{hk}\frac{p_{w_{ih}}}{\partial v_{ik}}\right)(D)(\omega\phi)\right](0)$$

But the last term can be rewritten as

$$\left[\sum_{i,h,k}c_{hk}\left(\frac{p_{v_{ih}}}{\partial w_{ik}}-\frac{p_{w_{ih}}}{\partial v_{ik}}\right)(D)(\omega\phi)\right](0)+\left[\sum_{i,h,k}c_{kh}\left(\frac{p_{w_{ih}}}{\partial v_{ik}}-\frac{p_{v_{ih}}}{\partial w_{ik}}\right)(D)(\omega\phi)\right](0),$$

which is equal to zero since $\sum_i\left(\frac{p_{v_{ih}}}{\partial w_{ik}}-\frac{p_{w_{ih}}}{\partial v_{ik}}\right)=\sum_i\left(\frac{p_{w_{ih}}}{\partial v_{ik}}-\frac{p_{v_{ih}}}{\partial w_{ik}}\right)=0$ as p is pluri-harmonic. Indeed, in the notation of section 4, this is nothing else than $D_{hk}p=0$ for $1\leq h,k\leq n$. □

Following Krieg as in [13] we now introduce a Jacobi theta series associated to an *even* hermitian positive definite matrix $s\in M_n(\mathbb{H})$. Here *even* means that $s_{jj}\in 2\mathbb{Z}$ and $s_{ij}\in\mathfrak{o}^{\sharp}$ for $i\neq j$, where

$$\mathfrak{o}^{\sharp}:=\{x\in B\mid \mathrm{Re}(xy^{\rho})\in\mathbb{Z},\ \ \forall y\in\mathfrak{o}\}.$$

The level of an even matrix s is defined as the smallest positive integer q such that $\frac{q}{\mathrm{vol}(\mathfrak{o})}s^{-1}$ is again an even hermitian matrix, where $\mathrm{vol}(\mathfrak{o})$ is defined as the Euclidean volume of a fundamental parallelotope of the lattice $\mathfrak{o}$.

We define the lattice $\Lambda:=M_n(\mathfrak{o})\hookrightarrow M_n(\mathbb{H})\hookrightarrow M_n(\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C}=M_{2n}(\mathbb{C})$. Then for $z\in\mathcal{H}$, and $u,w\in M_n(\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C}$, we define the Jacobi theta series,

$$\Theta_s(z;u,w):=\Theta(z;u,w)=\sum_{\lambda\in\Lambda}\mathbf{e}(\tau(z,s[\lambda-w])+2\tau(u,\lambda-w)),$$

where $\mathbf{e}(x):=\exp(\pi ix)$, for $x\in\mathbb{C}$. This is the definition in [13, page 101] where we have set $P=-w$ and $Q=u$ with the notation used there. By [13, Theorem 3.6] we then have,

Theorem 5.2 (Krieg)**.** *Suppose that s is of level $q>1$, and write $\mathfrak{q}$ for the ideal generated by q. Then for any $\gamma\in\Gamma_0(\mathfrak{q})$ we have*

$$\Theta(z;u,w)=j(\gamma,z)^{-n}\psi(\gamma)\kappa(\gamma;(u,w))\Theta(\gamma(z);\gamma*(u,w)),$$

where $\gamma*(u,w):=(ua_{\gamma}^*+swb_{\gamma}^*,wd_{\gamma}^*+s^{-1}uc_{\gamma}^*)$ and

$$\kappa(\gamma;(u,w))=\exp(\pi i\tau(s[w],d_{\gamma}^*b_{\gamma})+\pi i\tau(s^{-1}[u],a_{\gamma}^*c_{\gamma})+2\pi i\tau(w^*u,b_{\gamma}^*c_{\gamma})).$$

Moreover ψ is a character defined as

$$\psi(\gamma)=t^{-4n^2}\det(\alpha_{\gamma})^n\sum_{k\in M_n(\mathfrak{o})/tM_n(\mathfrak{o})}\exp(-\pi i\tau(s[k],a_{\gamma}^{-1}b_{\gamma})),$$

with $t\in\mathbb{N}$ such that $ta_{\gamma}^{-1}\in M_n(\mathfrak{o})$.

Remark 5.3. We make the following two remarks,

(1) Even though Krieg's book [13] is written for a specific quaternion algebra (the quaternions of Hurwitz) it is easy to see that at least the theorem stated above and its proof is valid for any quaternion algebra and order.
(2) We note here that in the notation of [13] we have that $\det(\hat{A}) = \det(A)^2$, which explains the different power of the (reduced) determinant.

We now consider a bit in more detail the character ψ above. In particular we show,

Lemma 5.4. *The character ψ is trivial if we restrict it to the subgroup $\Gamma_0(\mathfrak{m}\mathfrak{q})$ where $\mathfrak{m}$ is the ideal generated by the product of all finite primes of $\mathbb{Z}$ where B is ramified and do not divide q.*

Proof. We first note that we since $\alpha_\gamma \in GL_n(B)$ we have that $\det(\alpha_\gamma) \in \mathbb{Q}_+^\times$. In particular we may rewrite the above expression as

$$\psi(\gamma) = t^{-4n^2} \left(\prod_{p \in \mathbf{h}} |\det(\alpha_\gamma)|_p^{-n} \right) \sum_{k \in M_n(\mathfrak{o})/tM_n(\mathfrak{o})} \exp(-\pi i \tau(s[k], a_\gamma^{-1} b_\gamma)) \tag{21}$$

Moreover for any prime p we have that $\mathrm{ord}_p(t) = 1$ if p divides mq. Indeed since $\gamma \in \Gamma_0(\mathfrak{m}\mathfrak{q})$ we have that $a_\gamma d_\gamma^* - b_\gamma c_\gamma^* = 1_n$ and since $c_\gamma \in p\mathfrak{o}$ we have that $\det(a_\gamma d_\gamma^*) \equiv 1 \pmod p$ which gives that $\det(a_\gamma) \in \mathbb{Z}_p^\times$. After writing $t = \prod_i p_i^{a_i}$ and use the Chinese Remainder Theorem we can restrict to the case where $t = p^a$ for some prime number p which is relative prime to mq. Using the identification $\imath : M_n(\mathfrak{o}_p) = M_{2n}(\mathbb{Z}_p)$ we see that for any matrix $\sigma \in M_n(\mathfrak{o}_p)$ with $\sigma^* = \sigma$ we have with t equal the denominator of σ,

$$\sum_{k \in M_n(\mathfrak{o}_p)/tM_n(\mathfrak{o}_p)} e_p(\pi i \tau(s[k], \sigma)) = \sum_{k \in M_n(\mathfrak{o}_p)/tM_n(\mathfrak{o}_p)} e_p(\pi i \mathrm{tr}\,(\imath(k^* s k \sigma))) =$$
$$\sum_{k \in M_n(\mathfrak{o}_p)/tM_n(\mathfrak{o}_p)} e_p(\pi i \mathrm{tr}\,(K_n^{-1}\, {}^t\imath(k)\, K_n \tilde{s} K_n^{-1} \imath(k) \tilde{\sigma} K_n^{-1})),$$

for some skew-symmetric matrices $\tilde{s}, \tilde{\sigma} \in M_{2n}(\mathbb{Q}_p)$. We can rewrite the sum as

$$\sum_{k \in M_{2n}(\mathbb{Z}_p)/tM_{2n}(\mathbb{Z}_p)} e_p(-\pi i \mathrm{tr}\,({}^t k K_n \tilde{s} K_n^{-1} k \tilde{\sigma})) = \sum_{k \in M_{2n}(\mathbb{Z}_p)/tM_{2n}(\mathbb{Z}_p)} e_p(\pi i \mathrm{tr}\,({}^t k \Psi k \tilde{\sigma})),$$

where we have set $\Psi := K_n \tilde{s} K_n$, a skew-symmetric matrix in $M_{2n}(\mathbb{Q}_p)$ and K_n was defined in Equation (4). This kind of exponential sums have been computed in [21, page 115, proof of Lemma 14.8]. Indeed it is shown there that,

$$\sum_{k \in M_{2n}(\mathbb{Z}_p)/tM_{2n}(\mathbb{Z}_p)} e_p(\pi i \mathrm{tr}\,({}^t k \Psi k \tilde{\sigma})) = \nu[\tilde{\sigma}]^{-n} t^{4n^2}.$$

We now choose $\sigma = \alpha_\gamma^{-1} b_\gamma$ to evaluate (21) above. The lemma follows, since $\nu[\alpha_\gamma^{-1} b_\gamma] = |\det(a_\gamma)|_p^{-1}$ by definition and $\nu[\tilde{\sigma}] = \nu[\tilde{\sigma} K_n^{-1}]$. $\square$

Assume now that we are given a polynomial representation $(\rho, V) \in \mathcal{T}_n$ of $GL_{2n}(\mathbb{C})$, where $\mathcal{T}_n$ was defined in (16) above. In particular, as it was explained at the end of the previous section, we can find a polynomial map $p : M_{2n}(\mathbb{C}) \to V$ such that $p(x^t g) = \rho(g) p(x)$ for all $x \in M_{2n}(\mathbb{C})$ and $g \in GL_{2n}(\mathbb{C})$. Furthermore if we write

$p(x) = {}^t(p_1(x),\dots,p_d(x))$ for $d := \dim_{\mathbb{C}} V$, then we have that the $p_i(x)$ are pluriharmonic in the sense discussed above. Moreover given such a polynomial map $p(x) = {}^t(p_1(x),\dots,p_d(x))$ we define a differential operator $p(D) := {}^t(p_1(D),\dots,p_d(D))$, with $p_i(D)$ as above. This operator maps scalar valued functions to V-valued functions by $p(D)f := {}^t(p_1(D)f,\dots,p_d(D)f)$.

We consider now a finite Hecke character χ of $\mathbb{Q}$ of conductor some integral ideal $\mathfrak{n}$ generated by some $\nu \in \mathbb{N}$ and a polynomial map $p(x) : M_{2n}(\mathbb{C}) \to V$ as above. For an element $y \in M_n(\mathbb{H})_{\mathbb{C}}$ we set $p(y) := p(\imath(y))$, and $p(\overline{y}) := p(\overline{\imath(y)})$. We then define the series,

$$\theta(z;\chi,p) := \sum_{\lambda\in\Lambda} \chi^*(\det(\lambda))p\left(\overline{s^{1/2}\lambda}\right)\mathbf{e}(\tau(z,s[\lambda])),$$

where we recall that $\mathbf{e}(x) := \exp(\pi i x)$, for $x \in \mathbb{C}$, $\Lambda = M_n(\mathfrak{o})$, and χ^* is the associated ideal character to χ (see page 3). We then prove:

Theorem 5.5. *Assume that $\mathfrak{m}$ divides $\mathfrak{n}$. Then we have that,*

$$\theta(z) := \theta(z;\chi,p) \in M_{\rho_n}(\Gamma_0(\mathfrak{n}^2\mathfrak{q}),\chi),$$

where $\mathfrak{q}$ denotes the ideal generated by q. Moreover we have

$$\theta(z) = \sum_{\sigma\in S_+} c_\theta(\sigma)e_{\mathbf{a}}(\lambda(\sigma z)),$$

where

$$c_\theta(\sigma) = \sum_{\xi\in\Xi_\sigma} \chi^*(\det(\xi))p\left(\overline{s^{1/2}\xi}\right),$$

and $\Xi_\sigma := \{\xi \in M_n(\mathfrak{o}) \mid \sigma = \xi^* s\xi\}$.

Proof. The rest of this section is devoted to proving this theorem. We will apply the differential operators discussed above to the Jacobi-theta series. This idea goes back to Andrianov and Maloletkin [2] (see also [26]), where scalar valued symplectic theta series were considered. In the following calculation we will repeatedly use the properties (1) and (2) of the pairing $\tau : M_n(\mathbb{H})_{\mathbb{C}} \times M_n(\mathbb{H})_{\mathbb{C}} \to \mathbb{C}$ above.

We note that we may write

$$\theta(z;\chi,p) = \sum_{\widetilde{\lambda}\in\Lambda/\mathfrak{n}\Lambda} \chi^*(\det(\widetilde{\lambda}))\theta_{\widetilde{\lambda}}(z;\mathfrak{n},p), \tag{22}$$

where $\theta_{\widetilde{\lambda}}(z;\mathfrak{n},p) := \sum_{\lambda\in\Lambda_{\widetilde{\lambda}}} p\left(\overline{s^{1/2}\lambda}\right)\mathbf{e}(\tau(z,s[\lambda]))$, and here we have set

$$\Lambda_{\widetilde{\lambda}} := \{\lambda \in \Lambda \mid \lambda \equiv \widetilde{\lambda} \pmod{\mathfrak{n}\Lambda}\}.$$

We now claim that for a variable $\xi \in M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ we have

$$\rho(A)\theta_{\widetilde{\lambda}}(z;\mathfrak{n},p) = p(D)\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda} - \nu^{-1}s^{-1/2}\xi)_{|\xi=0},$$

where $A := \pi\, i\, {}^t\widetilde{\imath(z)} \in GL_{2n}(\mathbb{C})$ (see Proposition (5.1) for the notation) and in the right hand side of the equation we identify the class $\tilde{\lambda}$ with one of its representatives. We remark here that the equation holds independently of the choice of this representative. Indeed we have

$$\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)=\sum_{\lambda\in\Lambda_{\widetilde{\lambda}}}\mathbf{e}(\tau(z,s[\lambda+s^{-1/2}\xi])=$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}}}\mathbf{e}(\tau(z,s[\lambda])+\tau(z,\xi^*\xi)+2\tau(z,\lambda^* s^{1/2}\xi))=$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}}}\mathbf{e}(\tau(z,s[\lambda])+\tau(z,\xi^*\xi)+2\tau(s^{1/2}\lambda z,\xi)),$$

and hence there is no dependency on the choice of the representative $\tilde{\lambda}$. Now using Proposition 5.1, and the notation there, we conclude that

$$\rho(A)\theta_{\widetilde{\lambda}}(z;\mathfrak{n},p)=p(D)\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)_{|\xi=0},$$

where we have used the fact that

$$p\left(\pi i\widetilde{\imath(s^{1/2}\lambda z)}\right)=p\left(\widetilde{\imath(s^{1/2}\lambda)}\pi i^{\,t}\left(\widetilde{{}^t\imath(z)}\right)\right)=\rho(A)p\left(\widetilde{\imath(s^{1/2}\lambda)}\right).$$

Moreover for $\gamma\in\Gamma_0(\mathfrak{n}^2 q)$ we have that

$$\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)=$$
$$j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(s[\widetilde{\lambda}+s^{-1/2}\xi)],d_\gamma^* b_\gamma))\Theta_{f_\chi^2 s}(\gamma(z);\gamma*(0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)).$$

Moreover, we have that

$$\Theta_{\nu^2 s}(\gamma(z);\gamma*(0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi))=$$
$$\Theta_{\nu^2 s}(\gamma(z);f_\chi s(-\widetilde{\lambda}-s^{-1/2}\xi)b_\gamma^*,(-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)d_\gamma^*)=$$
$$\sum_{\lambda\in\Lambda}\mathbf{e}(\tau(\gamma(z),s[\mathfrak{n}\lambda+(\widetilde{\lambda}+s^{-1/2}\xi)d_\gamma^*])+2\tau(s(-\widetilde{\lambda}-s^{-1/2}\xi)b_\gamma^*,\nu\lambda+(\widetilde{\lambda}+s^{-1/2}\xi)d_\gamma^*))=$$
$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}}\mathbf{e}(\tau(\gamma(z),s[\lambda+s^{-1/2}\xi d_\gamma^*])+2\tau(s(-\widetilde{\lambda}-s^{-1/2}\xi)b_\gamma^*,\lambda+s^{-1/2}\xi d_\gamma^*))=$$
$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}}\mathbf{e}(\tau(\gamma(z),s[\lambda+s^{-1/2}\xi d_\gamma^*])+2\tau(s(-\widetilde{\lambda}-s^{-1/2}\xi)b_\gamma^*,\lambda+s^{-1/2}\xi d_\gamma^*))=$$
$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}}\mathbf{e}(\tau(\gamma(z),\lambda^* s\lambda+2d_\gamma\xi^* s^{1/2}\lambda+d_\gamma\xi^*\xi d_\gamma^*)-2\tau(s(\widetilde{\lambda}+s^{-1/2}\xi)b_\gamma^*,\lambda+s^{-1/2}\xi d_\gamma^*)).$$

We now note that

$$\mathbf{e}(\tau(s[\widetilde{\lambda}+s^{-1/2}\xi)],d_\gamma^* b_\gamma))=\mathbf{e}(\tau(s[\widetilde{\lambda}]+s[s^{-1/2}\xi],d_\gamma^* b_\gamma)+2\tau(s\widetilde{\lambda},s^{-1/2}\xi d_\gamma^* b_\gamma))=$$
$$\mathbf{e}(\tau(s[\widetilde{\lambda}]+\xi^*\xi,d_\gamma^* b_\gamma)+2\tau(s^{1/2}\widetilde{\lambda}b_\gamma^*,\xi d_\gamma^*))$$

That is,

$$\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda}-\nu^{-1}s^{-1/2}\xi)=j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(s[\widetilde{\lambda}]+\xi^*\xi,d_\gamma^* b_\gamma))\times$$
$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}}\mathbf{e}(\tau(\gamma(z),\lambda^* s\lambda+2d_\gamma\xi^* s^{1/2}\lambda+d_\gamma\xi^*\xi d_\gamma^*)-2(\tau(s\widetilde{\lambda}b_\gamma^*,\lambda)+\tau(\xi^*\xi,d_\gamma^* b_\gamma)+\tau(s^{1/2}\xi b_\gamma^*,\lambda))$$

We now use the identity: $z(c_\gamma z+d_\gamma)^{-1}d_\gamma = d_\gamma^*\gamma(z)d_\gamma - b_\gamma^* d_\gamma$ to obtain the equality

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(\gamma(z),\lambda^* s\lambda + 2d_\gamma\xi^* s^{1/2}\lambda + d_\gamma\xi^*\xi d_\gamma^*) - 2(\tau(s\widetilde{\lambda}b_\gamma^*,\lambda) + \tau(\xi^*\xi, d_\gamma^* b_\gamma) + \tau(s^{1/2}\xi b_\gamma^*,\lambda)) =$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(z(c_\gamma z+d_\gamma)^{-1}d_\gamma,\xi^*\xi) + 2\tau(z(c_\gamma z+d_\gamma)^{-1},\xi^* s^{1/2}\lambda) + \tau(\gamma(z),\lambda^* s\lambda) - 2(\tau(s\widetilde{\lambda}b_\gamma^*,\lambda)),$$

and hence we obtain,

$$\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda} - \nu^{-1}s^{-1/2}\xi) = j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(s[\widetilde{\lambda}] + \xi^*\xi, d_\gamma^* b_\gamma))\times$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(z(c_\gamma z+d_\gamma)^{-1}d_\gamma,\xi^*\xi) + 2\tau(z(c_\gamma z+d_\gamma)^{-1},\xi^* s^{1/2}\lambda) + \tau(\gamma(z),\lambda^* s\lambda) - 2(\tau(s\widetilde{\lambda}b_\gamma^*,\lambda)).$$

We now note that $s[\widetilde{\lambda}]$ is even (see [13, page 108]) and hence since $d_\gamma^* b_\gamma = b_\gamma^* d_\gamma$ we have that $\mathbf{e}(\tau(s[\widetilde{\lambda}], d_\gamma^* b_\gamma) = 1$ by [13, page 107, Proposition 1.10 (ii)]. Moreover we have that $\mathbf{e}(2\tau(s\widetilde{\lambda}b_\gamma^*,\lambda) = 1$ since $\tau(s\widetilde{\lambda}b_\gamma^*,\lambda) = \tau(s,\lambda b_\gamma\widetilde{\lambda}^*) \in \mathbb{Z}$ since $s \in M_n(\mathfrak{o}^\sharp)$ and $b_\gamma\widetilde{\lambda}\lambda \in M_n(\mathfrak{o})$. That is,

$$\Theta_{\nu^2 s}(z;0,-\nu^{-1}\widetilde{\lambda} - \nu^{-1}s^{-1/2}\xi) = j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(\xi^*\xi, d_\gamma^* b_\gamma))\times$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(z(c_\gamma z+d_\gamma)^{-1}d_\gamma,\xi^*\xi) + \tau(z(c_\gamma z+d_\gamma)^{-1},\xi^* s^{1/2}\lambda) + \tau(\gamma(z),\lambda^* s\lambda)).$$

This gives,

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}}} \mathbf{e}(\tau(z,s[\lambda]) + \tau(z,\xi^*\xi) + 2\tau(s^{1/2}\lambda z,\xi)) = j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(\xi^*\xi, d_\gamma^* b_\gamma))\times$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(z(c_\gamma z+d_\gamma)^{-1}d_\gamma,\xi^*\xi) + 2\tau(z(c_\gamma z+d_\gamma)^{-1},\xi^* s^{1/2}\lambda) + \tau(\gamma(z),\lambda^* s\lambda)).$$

Or equivalently

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}}} \mathbf{e}(\tau(z,s[\lambda]) + \tau(z,\xi^*\xi) + 2\tau(s^{1/2}\lambda z,\xi)) = j(\gamma,z)^{-n}\psi(\gamma)\mathbf{e}(\tau(\xi^*\xi, d_\gamma^* b_\gamma))\times$$

$$\sum_{\lambda\in\Lambda_{\widetilde{\lambda}d_\gamma^*}} \mathbf{e}(\tau(z(c_\gamma z+d_\gamma)^{-1}d_\gamma,\xi^*\xi) + 2\tau(s^{1/2}\lambda(c_\gamma z+d_\gamma)^{-*}z,\xi) + \tau(\gamma(z),\lambda^* s\lambda)).$$

We now apply the operator $p(D)|_{\xi=0}$ to both sides and we have seen that the left hand side is nothing else than $\rho(A)\theta_{\widetilde{\lambda}}(z;\mathfrak{n},p)$, where we recall from above that $A = \pi\, i\, {}^t\widetilde{\imath(z)} \in GL_{2n}(\mathbb{C})$. Similarly the right hand is equal to

$$j(\gamma,z)^{-n}\psi(\gamma)\rho(A)\rho(c_\gamma z+d_\gamma)^{-1}\theta_{\widetilde{\lambda}d_\gamma^*}(\gamma(z);\mathfrak{n},p).$$

For this one uses that

$$p\left(\pi i\,\widetilde{\imath(s^{1/2}\lambda)}\widetilde{\imath(c_\gamma z+d_\gamma)^{-*}}\widetilde{\imath(z))}\right) = p\left(\widetilde{\imath(s^{1/2}\lambda)}\ {}^t\left({}^t\left(\widetilde{\imath(c_\gamma z+d_\gamma)^{-*}}\right)\right)\ {}^t\left(\pi i\,{}^t\widetilde{\imath(z)}\right)\right) =$$

$$\rho(A)p\left(\widetilde{\imath(s^{1/2}\lambda)}\ {}^t\imath(c_\gamma z+d_\gamma)^{-1}\right) = \rho(A)\rho((c_\gamma z+d_\gamma)^{-1})p\left(\widetilde{\imath(s^{1/2}\lambda)}\right).$$

Putting everything together we get,

$$\theta_{\tilde{\lambda}}(z;\mathfrak{n},p) = j(\gamma,z)^{-n}\psi(\gamma)\rho(c_\gamma z + d_\gamma)^{-1}\theta_{\tilde{\lambda}d_\gamma^*}(\gamma(z);\mathfrak{n},p),$$

or equivalently, using Equation (22),

$$\chi^*(\det(d_\gamma))\psi(\gamma)\rho_n(J(\gamma,z))\theta(z;\chi,p) = \theta(\gamma(z);\chi,p),\quad \gamma\in\Gamma_0(\mathfrak{n}^2q).$$

By Lemma 5.4 we have that $\psi(\gamma) = 1$. The last statement regarding the Fourier expansion of the theta series follows directly from the definition. $\square$

6. The standard L-function

In this section we introduce the standard L-function attached to a quaternionic modular form. Our approach follows [24, Section 19], where the cases of Siegel and Hermitian modular forms are considered.

We let $\mathfrak{c}$ be an ideal of $\mathbb{Z}$ and we assume that $v \mid \mathfrak{c}$ for every finite place v where the division algebra B is ramified. We define the groups

$$E := \prod_{v\in\mathbf{h}} GL_n(\mathfrak{o}_v),\quad M := \{x\in GL_n(B)_{\mathbf{h}} \mid x_v\in M_n(\mathfrak{o}_v),\ \forall v\in\mathbf{h}\},$$

and $\mathfrak{Q} := \{\mathrm{diag}[\hat{r}, r]\ \ r\in M\}$. Then for every $v\in\mathbf{h}$ we have $G_v = D[\mathfrak{b}^{-1},\mathfrak{b}]_v\mathfrak{Q}_v D[\mathfrak{b}^{-1},\mathfrak{b}]_v$. Indeed, if $v\in\mathbf{h}$ is a place where the division algebra B is unramified then this follows from [25, Theorem 6.10], and the identification of G_v with $SO(2n,2n)_v$. If on the other hand v is a finite place where B is ramified, then this has been established in [28, Theorem 6].

Lemma 6.1. *Let $D := D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}]$ and for a finite place $v\,|\,\mathfrak{c}$, let $\sigma = \mathrm{diag}[\hat{q}, q]\in\mathfrak{Q}_v$. Then we have that*

$$D_v\sigma D_v = \bigsqcup_{d,b} D_v\begin{pmatrix}\hat{d} & \hat{d}b\\ 0 & d\end{pmatrix},$$

where $d\in E_v\setminus E_v q E_v$ and $b\in S(\mathfrak{b}_v^{-1})/d^*S(\mathfrak{b}_v^{-1})d$.

Proof. This can be shown exactly as [24, Lemma 19.2]. $\square$

We now set $S' := \{\sigma\in S_{\mathbf{h}} \mid \sigma_v\in S(\mathfrak{b}_v^{-1})\ \forall v\,|\,\mathfrak{c}\}$. We let $\beta\in\mathbb{A}_{\mathbf{h}}^\times$ such that $\mathfrak{b} = \beta\mathbb{Z}$.

Lemma 6.2. *Let χ be a character, $\zeta\in S_{\mathbf{h}}$ and $g\in M$ such that $g^*\beta^{-1}\zeta g\in T_{\mathbf{h}}$. If we set $X := S'/gS_{\mathbf{h}}(\mathfrak{b}^{-1})g^*$, then,*

$$\sum_{x\in X}\mathbf{e}_{\mathbf{h}}(-\lambda(\zeta x))\chi^*(\nu_0(\beta x))\nu(\beta x)^{-s} = \begin{cases}|\det(g)|_{\mathbf{h}}^{-(2n-1)}\alpha_{\mathfrak{c}}(\beta^{-1}\zeta,\chi,s), & \textit{if } \beta^{-1}\zeta\in T_{\mathbf{h}}\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. This can be proven similarly to [24, Lemma 19.6] as soon as we establish that $[S_{\mathbf{h}}(\mathfrak{b}^{-1}) : gS_{\mathbf{h}}(\mathfrak{b}^{-1})g^*] = |\det(g)|_{\mathbf{h}}^{-(2n-1)}$. As in [21, Lemma 13.2] this can be reduced to the case of g being diagonal. Then if v is a place where the algebra B splits the statement is done in that lemma. If v is a place where B ramifies then it follows from the facts (see [22, page 79]) that $[\mathfrak{o} : \mathfrak{o}\alpha] = |\det(\alpha)|_v^{-2}$ and that $\alpha^*\mathfrak{o}\alpha = \mathfrak{o}\alpha^*\alpha$. $\square$

For D as above we define the set $\mathfrak{A} := D\mathfrak{Q}D$, and consider the Hecke algebra $\mathbb{T} := \mathbb{T}(\mathfrak{A}, D)$ as the $\mathbb{Q}$-linear span of $D\alpha D$ for $\alpha \in \mathfrak{A}$. Given an $\mathbf{f} \in \mathcal{S}_\rho(D, \psi)$ we have the usual action of $D\xi D \in \mathbb{T}$ by $(\mathbf{f}|D\xi D)(x) = \sum_{y\in Y} \psi_{\mathfrak{c}}(\det(a_y))^{-1}\mathbf{f}(xy^{-1})$, where Y is a finite subset of $G_{\mathbf{h}}$ such that $D\xi D = \sqcup_{y\in Y} Dy$. We now assume that $\mathbf{f} \neq 0$ is an eigenform, that is $\mathbf{f}|D\xi D = \lambda_{\mathbf{f}}(\xi)\mathbf{f}$, $\lambda_{\mathbf{f}}(\xi) \in \mathbb{C}$ for all $\xi \in \mathfrak{A}$. For a character χ we define the series

$$D(s, \mathbf{f}, \chi) = \sum_{\xi \in D\backslash \mathfrak{A}/D} \lambda_{\mathbf{f}}(\xi)\chi^*(\mathfrak{a}_\xi)N(\mathfrak{a}_\xi)^{-s}, \quad Re(s) \gg 0,$$

where $\mathfrak{a}_\xi := \det(r)\mathbb{Z}$ if $\xi \in D[\mathfrak{b}^{-1}, \mathfrak{b}]\mathrm{diag}[r^{-1}, r^*]D[\mathfrak{b}^{-1}, \mathfrak{b}]$, with $r \in M$. We further set:

$$L(s, \mathbf{f}, \chi) := \left(\prod_{i=1}^{2n} L_{\mathfrak{c}}(2s - 2i + 2, \chi^2)\right) D(s, \mathbf{f}, \chi).$$

We then have,

Proposition 6.3. *We have the Euler product expression*

$$L(s, \mathbf{f}, \chi) = \prod_p L_p(s, \mathbf{f}, \chi),$$

where

$$L_p(s, \mathbf{f}, \chi) = \begin{cases} (1 - p^{4n-2}\chi^*(p)^2)^{-1} \prod_{i=1}^{2n} \left((1 - \mu_{i,p}\chi^*(p)p^{-s})(1 - \mu'_{i,p}\chi^*(p)p^{-s})\right)^{-1}, & \text{if } p \nmid \mathfrak{c}, \\ \prod_{i=1}^{2n}(1 - \mu_{i,p}\chi^*(p)p^{2n-s})^{-1}, & \text{if } p | \mathfrak{c} \text{ and } B_p \text{ splits}, \\ \prod_{i=1}^{n}(1 - \mu_{i,p}\chi^*(p)p^{2n-3-s})^{-1}, & \text{otherwise}, \end{cases}$$

where $\mu_{i,p}, \mu'_{i,p}$ are complex numbers such that $\mu_{i,p}\mu'_{i,p} = p^{4n-2}$ if p does not divide $\mathfrak{c}$.

Proof. The case where p does not divide $\mathfrak{c}$ follows from [25] where the orthogonal case is considered. Indeed in this case, and thanks to our assumption that that $\mathfrak{c}$ is divisible by all primes where B is ramified allows us to identify G_p with $SO(2n, 2n)_p$. Then the Euler factor is given in [25, Proposition 17.14].

We next consider the case where p divides $\mathfrak{c}$, and B_p is a division algebra. We write v for the corresponding place, and δ a prime element of B_v. We first remark that the number of all different cosets $E_v\alpha$ with α an upper triangular matrix in M_v of diagonal entries $\delta^{e_1}, \ldots, \delta^{e_n}$ where $e_i \in \mathbb{Z}$, is equal to $p^{2\sum_{i=1}^n (i-1)e_i}$. Indeed this follows similar to [21, Lemma 3.12] after observing that $[\mathfrak{o}_v : \mathfrak{o}_v\delta] = p^2$. We define a map $\omega_0 : \mathbb{T}(GL_n(B_v), E_v) \to \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$ by $\omega_0(E_v x E_v) = \sum_y \omega_0(E_v y)$ and $\omega_0(E_v y) = \prod_{i=1}^n (p^{-2i}t_i)^{e_i}$ if $E_v y = E_v\alpha$ where α an upper triangular with diagonal entries $\delta_1^{e_1}, \ldots .\delta_n^{e_n}$. We then define an map $\omega : \mathbb{T}_v \to \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$ by $\omega(D_v\sigma D_v) = \sum_\xi \omega_0(d_\xi)$ where we have used the decomposition $D_v\sigma D_v = \bigsqcup_\xi D_v\xi$ and Lemma 6.1.

By [23, Lemma 7] we have that the map ω is an injective ring homomorphism. Using Lemma 6.1 and (the proof of) Lemma 6.2 above we have

$$\omega\left(\sum_{\xi \in D_v\backslash \mathfrak{A}_v/D_v} D_v\xi D_v \nu(\xi)^{-s}\right) = \sum_{d \in E_v\backslash M_v} \omega_0(E_v d)[S(\mathfrak{b}^{-1})_v : d^* S(\mathfrak{b}^{-1})_v d]|\det(d)|_v^s =$$

$$\sum_{d\in E_v\backslash M_v}\omega_0(E_vd)|\det(d)|_v^{-(2n-1)}|\det(d)|_v^s=$$

$$\sum_{e_1=0}^{\infty}\cdots\sum_{e_n=0}^{\infty}p^{\sum_{i=1}^n(2i-2)e_i}\prod_{i=1}^n(p^{-2i}t_i)^{e_i}p^{(2n-1)e_i}p^{-se_i}=$$

$$\prod_{i=1}^n(1-t_ip^{2n-3-s})^{-1}.$$

Then an argument as in [24, Lemma 19.9] establishes the existence of the numbers $\mu_{i,p}\in\mathbb{C}$ as in the statement of the Proposition.

Finally in the case of p dividing $\mathfrak{c}$ and $B_p\cong M_2(\mathbb{Q}_p)$ we argue as above and consider the map $\omega_0:\mathbb{T}(GL_{2n}(\mathbb{Q}_p),GL_{2n}(\mathbb{Z}_p))\to\mathbb{Q}[t_1^{\pm},\ldots,t_{2n}^{\pm}]$ defined in by $\omega_0(E_vxE_v)=\sum_y\omega_0(E_vy)$ and $\omega_0(E_vy)=\prod_{i=1}^{2n}(p^{-i}t_i)^{e_i}$ if $E_vy=E_v\alpha$ where α an upper triangular with diagonal entries $\delta_1^{e_1},\ldots.\delta_{2n}^{e_{2n}}$. Then by [21, Lemma 16.2 and Lemma 16.3] we have that the Euler factor is equal to $\prod_{i=1}^{2n}(1-p^{2n}\mu_{i,p}p^{-s})^{-1}$. □

The proof above gives also that the function $L(s,\mathbf{f},\chi)$ is absolutely convergent for $\mathrm{Re}(s)>4n-1$, since up to finitely many primes its absolute convergence can be reduced to the case of the standard L-function attached to cusp forms of $SO(2n,2n)$, and these are convergent for $\mathrm{Re}(s)>4n-1$ (see for example [25]).

When the nebentype ψ of $\mathbf{f}$ is not the trivial character we define yet another L-series. We set $L_\psi(s,\mathbf{f},\chi):=\prod_p L_{\psi,p}(s,\mathbf{f},\chi)$, where

$$L_{\psi,p}(s,\mathbf{f},\chi)=\begin{cases}L_p(s,\mathbf{f},\chi\psi), & \text{if } p \text{ does not divide } \mathfrak{c},\\ L_p(s,\mathbf{f},\chi), & \text{otherwise.}\end{cases}$$

We note that $L(s,\mathbf{f},\chi)$ and $L_\psi(s,\mathbf{f},\chi\psi^{-1})$ differ only at the primes dividing $\mathfrak{c}$. Given now a $\tau\in S_+$, and a $q\in GL_n(B)_{\mathbf{h}}$ we define:

$$D_{\tau,q}(s,\mathbf{f},\chi):=\sum_{x\in M/E}\psi(\det(qx))\chi(\det(x))c(\tau,qx;\mathbf{f})|\det(x)|_{\mathbf{h}}^{s-(2n-1)}.$$

Following [24, Chapter V], where the symplectic and unitary situations were considered, we will relate the series $D_{\tau,q}(s,\mathbf{f},\chi)$ to $L_\psi(s,\mathbf{f},\chi)$ above through what may be called an *Andrianov-Kalinin* type equation in our setting. This is Theorem 6.5 below which should be seen as the analogue in the quaternionic case of [24, Theorem 20.4]

Similarly to the notation introduced in [24, page 168], we let $\mathcal{L}$ denote all the $\mathfrak{o}$-lattices in B^n, and we define

$$\mathcal{L}_\tau:=\{yL_0\ |\ y\in GL_n(B)_{\mathbf{h}}\text{ s.t }\beta^{-1}y^*\tau y\in T_{\mathbf{h}}\},$$

where we set $L_0:=\mathfrak{o}^n$. Moreover for two lattices $L_1,L_2\in\mathcal{L}$ we write $L_1<L_2$ if $L_1\subseteq L_2$ and $L_{1,v}=L_{2,v}$ for all v dividing $\mathfrak{c}$. Given a $y\in GL_n(B)_{\mathbf{h}}$ such that $yL_0\in\mathcal{L}_\tau$ we set

$$a_\tau(s,y,\chi):=|\det(y)|_{\mathbf{h}}^{2n-1-2s}\alpha_{\mathfrak{c}}(\beta^{-1}y^*\tau y,s,\psi\chi).$$

Further for a lattice $N = qL_0 \in \mathcal{L}_\tau$ we set

$$A_\tau(s, N, \chi) := |\det(q)|_{\mathbf{h}}^{-(2n-1-2s)} \sum_{N < yL_0 \in \mathcal{L}_\tau} \mu(yL_0/N) a_\tau(s, y, \chi).$$

Here, for a finitely generated torsion $\mathfrak{o}$-module A, we define $\mu(A) \in \mathbb{Z}$ inductively by setting $\mu(\{0\}) = 1$ and $\mu(A) := -\sum_{B \subset A} \mu(B)$, where B runs over all proper finitely generated $\mathfrak{o}$-submodules of A. As in [24, Lemma 19.10] we have,

$$\sum_{B \subset A} \mu(A/B) = \begin{cases} 1, & \text{if } A = \{0\} \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

The following Lemma is an extension of [24, Lemma 20.5] to the quaternionic case.

Lemma 6.4. *Let $\tau \in S^+$ and $N = qL_0 \in \mathcal{L}_\tau$. Denote by $\mathbf{b}$ the set of finite places p which do not divide $\mathfrak{c}$ and $\det(2\beta^{-1}q^*\tau q) \notin \mathbb{Z}_p^\times$. Then*

$$A_\tau(s, N, \chi) = \prod_{p \in \mathbf{b}} g_p(\chi(p)p^{-s}) \left(\prod_{i=1}^{n} L_{\mathfrak{c}}(2s - 2i + 2, \psi^2\chi^2) \right)^{-1},$$

where g_p are some polynomials.

Proof. The proof is similar to the one of [24, Lemma 20.5]. We first note that for for each L such that $N < L \in \mathcal{L}_\tau$ we have $L = q\hat{x}L_0$ for some $x \in M$. In particular we have that

$$A_\tau(s, N, \chi) = \prod_{v \nmid \mathfrak{c}} \left(\sum_{x} \mu(L_0/x^*L_0) |\det(x)|_v^{-(2n-1-2s)} \alpha(x^{-1}(\beta^{-1}q^*\tau q)_v \hat{x}, s, \psi_v\chi_v) \right)$$

where the sum runs over all $x \in M_v/E_v$ such that $x^{-1}(\beta^{-1}q^*\tau q)_v\hat{x} \in T_v \cap GL_n(B_v)$. If v is not in the set $\mathbf{b}$ then we have that the above inner sum consists of a single term and is equal to $\alpha_v((\beta^{-1}q^*\tau q)_v, s, \chi_v\psi_v)$, which, by Proposition 3.5, is nothing else than $\prod_{i=1}^n L_v(2s - 2i + 2, \psi_v^2\chi_v^2)^{-1}$. If on the other hand v is in the set $\mathbf{b}$, then the above is a finite sum where each term $\alpha_v(x^{-1}(\beta^{-1}q^*\tau q)_v\hat{x}, s, \psi_v\chi_v)$ is a polynomial times the factor $\prod_{i=1}^n L_v(2s - 2i + 2, \psi_v^2\chi_v^2)^{-1}$, which proves the lemma. ☐

We now have the following Theorem which may be called the *Andrianov-Kalinin* identity in our setting.

Theorem 6.5. *With notation as above (i.e. q, τ fixed and $N := qL_0 \in \mathcal{L}_\tau$) we have*

$$\left(\prod_{i=1}^{2n} L_{\mathfrak{c}}(2s - 2i + 2, \psi^2\chi^2) \right) \times A_\tau(s, N, \chi) \times D_{\tau,q}(s, \mathbf{f}, \chi) =$$

$$L_\psi(s, \mathbf{f}, \chi) \sum_{L < N \in \mathcal{L}_\tau} \mu(N/L)\psi_{\mathfrak{c}}(\det(y)) |\det(q^*\widehat{y})|_{\mathbf{h}}^s c(\tau, y; \mathbf{f}),$$

where the $y \in GL_n(B)_{\mathbf{h}}$ are such that $L = yL_0$. In particular by Lemma 6.4 above, we have that,

$$\prod_{p \in \mathbf{b}} g_p(\chi(p)p^{-s}) \times \prod_{i=n+1}^{2n} L_{\mathfrak{c}}(2s - 2i + 2, \psi^2\chi^2) \times D_{\tau,q}(s, \mathbf{f}, \chi) =$$

$$L_\psi(s,\mathbf{f},\chi)\sum_{L<N\in\mathcal{L}_\tau}\mu(N/L)\psi_{\mathfrak{c}}(\det(y))|\det(q^*\widehat{y})|_{\mathbf{h}}^s c(\tau,y;\mathbf{f}).$$

Proof. This can be proven in a similar manner as [19, Theorem 5.1], where the symplectic case was considered or more generally [24, Theorem 20.4] where also the unitary case is included. Strictly speaking both [19] and [24] consider scalar weight modular forms but it is easy to see that the proofs of [19, Theorem 5.1] or [24, Theorem 20.4] hold also for vector valued once (see [3]). These proofs can also be extended to the quaterionic case considered here. Indeed for the proof of Theorem 5.1 in [19] one needs Lemma 2.6, Lemma 2.7 and Lemma 2.8 of [19], and the property of the function $\mu(\cdot)$ given in Equation (23) above. All these Lemmata are local in nature and have analogues in the quaternionic case. Indeed the analogue of Lemma 2.6 in the quaternionic case is nothing else than Lemma 6.1 above, when v divides $\mathfrak{c}$. On the other hand if v does not divide $\mathfrak{c}$, then G_v is isomorphic to $SO(2n,2n)_v$, and the analogue of Lemma 2.6 is done in [21, paragraph 16.12]. Similarly Lemma 2.7 of [19] is done in [21, Proposition 16.10] if v divides $\mathfrak{c}$ and in [23, Lemma 3.3] if v does not divide $\mathfrak{c}$. Finally the analogue of Lemma 2.8 of [19] in the quaternionic case is Lemma 6.2 above. $\square$

As in [24, Theorem 20.9] one can show that we can select a $\tau\in S^+$ and a $r\in GL_n(B)_{\mathbf{h}}$ such that $c(\tau,r;\mathbf{f})\neq 0$ and we have that the only lattice L such that $L<N$ is the lattice N itself. In particular with such choices,

(24)

$$\prod_{p\in\mathbf{b}}g_p(\chi(p)p^{-s})\times\prod_{i=n+1}^{2n}L_{\mathfrak{c}}(2s-2i+2,\psi^2\chi^2)\times D_{\tau,r}(s,\mathbf{f},\chi)=\psi_{\mathfrak{c}}(\det(r))c(\tau,r;\mathbf{f})L_\psi(s,\mathbf{f},\chi),$$

with $c(\tau,r;\mathbf{f})\neq 0$. We set $\mathcal{D}(s,\mathbf{f},\chi):=D_{\tau,r}(s,\mathbf{f},\chi)$, and hence,

$$\mathcal{D}(s,\mathbf{f},\chi)=\sum_{x\in M/E}\psi(\det(rx))\chi^*(\det(x))c(\tau,rx;\mathbf{f})|\det(x)|_{\mathbf{h}}^{-(2n-1)}|\det(x)|_{\mathbf{h}}^s.$$

As we discussed in Section 2, we have that

$$GL_n(B)_{\mathbb{A}}=GL_n(B)\,E\,GL_n(\mathbb{H}).$$

In particular we may as well assume that the element $r\in GL_n(B)_{\mathbf{h}}\cap GL_n(B)$, since the clearly $eL_0=L_0$ for any $e\in E$. We now set $X_r:=GL_n(B)\cap rM\,GL_n(\mathbb{H})$ and $U:=GL_n(B)\cap EGL_n(\mathbb{H})$. Then we have

$$\mathcal{D}(s,\mathbf{f},\chi)=c(r,\chi,s)\sum_{\xi\in X_r/U}c(\tau,\xi;\mathbf{f})\chi^*(\det(\xi))\det(\xi)^{(2n-1)-s},\tag{25}$$

where $c(r,\chi,s):=|\det(r)|_{\mathbf{h}}^{(2n-1)-s}\chi_{\mathbf{h}}(\det(r))^{-1}$. Since $r\in GL_n(B)$ and since $\det(r)\in\mathbb{R}_+$ we have that $c(r,\chi,s)=\det(r)^{s-(2n-1)}$. That is,

$$\mathcal{D}(s,\mathbf{f},\chi)=c(r,\chi,s)\rho(\hat{r})\sum_{\xi\in X/U}c(r^*\tau r,\xi;\mathbf{f})\chi^*(\det(\xi))\det(\xi)^{(2n-1)-s},\tag{26}$$

where $X:=X_1=GL_n(B)\cap M\,GL_n(\mathbb{H})$. In particular we have that

$$\prod_{i=n+1}^{2n}L_{\mathfrak{c}}(2s-2i+2,\psi^2\chi^2)\sum_{\xi\in X/U}c(r^*\tau r,\xi;\mathbf{f})\chi^*(\det(\xi))\det(\xi)^{(2n-1)-s}=\tag{27}$$

$$\left(c(r,\chi,s)\prod_{p\in\mathbf{b}} g_p(\chi(p)p^{-s})\right)^{-1} \times \psi^*(\det(r))^{-1})\rho(r^*)c(\tau,r;\mathbf{f})L_\psi(s,\mathbf{f},\chi).$$

7. The Rankin-Selberg Method

In this section we will obtain an integral expression, see Equation (31) below, for the standard L-function introduced above. It relates the L-function to a Petersson inner product between the corresponding cusp form, and a product of a vector valued theta series constructed above and a Siegel-type Eisenstein series. A similar expression for vector valued Siegel modular forms has been obtained in [3], extending the scalar weight case obtained originally in [1], and generalized in [19] and further in [24, Section 22]. Our computations below generalize these ideas to the case of vector valued quaternionic modular forms.

We first introduce an operator which has been studied quite extensively by Godement in [8], in the case of vector valued Siegel modular forms. For a rational representation (ρ, V) of $GL_{2n}(\mathbb{C})$, and a positive definite matrix $\tau \in S^+$ we define the operator,

$$H_{\rho,\tau}(s) := \int_{S_{\mathbf{a}}^+} \rho(y)\exp(-4\pi\lambda(\tau y))\delta(y)^{s-(2n-1)}dy \in \mathrm{End}(V), \quad Re(s) >> 0, \tag{28}$$

where we recall here $\delta(y) = \det(y)^{1/2}$. Here we view $\imath : S^+ \hookrightarrow GL_{2n}(\mathbb{C})$ and we abuse the notation and write $\rho(y)$ for $\rho(\imath(y))$. We set $H_\rho(s) := H_{\rho,1_n}(s)$. As in [8] one can show that

$$H_{\rho,\tau}(s) = \rho(\tau^{-1/2})H_\rho(s)\rho(\tau^{-1/2})\delta(\tau)^{-s}.$$

Indeed all we need to do is to perform the change of variable $y \mapsto \tau^{-1/2}y\tau^{-1/2}$. Furthermore, similar to [8], we have that this operator is hermitian with respect to the inner product $\prec \cdot,\cdot \succ$ on V.

For $k,\ell \in \mathbb{Z}$ we consider an $\mathbf{f} \in \mathcal{S}_{\rho_k}(D_1,\psi_1)$ and $\mathbf{g} \in \mathcal{M}_{\rho_\ell}(D_2,\psi_2)$, where $D_1 = D[\mathfrak{b}_1^{-1},\mathfrak{b}_1\mathfrak{c}_1]$ and $D_2 = D[\mathfrak{b}_2^{-1},\mathfrak{b}_2\mathfrak{c}_2]$. We let $f := f_{1_\mathbf{h}} \in S_{\rho_k}(\Gamma_1,\psi_1)$ and $g = g_{1_\mathbf{h}} \in M_{\rho_\ell}(\Gamma_2,\psi_2)$ to be the 1-component of $\mathbf{f}$ and $\mathbf{g}$ in the notation of Definition 2.3. Then $\Gamma_1 = \Gamma[\mathfrak{b}_1^{-1},\mathfrak{b}_1\mathfrak{c}_1], \Gamma_2 = \Gamma[\mathfrak{b}_2^{-1},\mathfrak{b}_2\mathfrak{c}_2]$ are the corresponding congruence subgroups. We write $f(z) = \sum_{\sigma\in S^+} c_\mathbf{f}(\sigma)e_\mathbf{a}(\lambda(\sigma z))$ and $g(z) = \sum_{\sigma\in S^+} c_\mathbf{g}(\sigma)e_\mathbf{a}(\lambda(\sigma z))$ for their Fourier expansions, where for simplicity we have set $c_\mathbf{f}(\sigma) := c_\mathbf{f}(\sigma,1)$ and similarly for $c_\mathbf{g}(\sigma)$. Note that $c_\mathbf{f}(\sigma)$ (resp. $c_\mathbf{g}(\sigma)$) are indeed the Fourier coefficients of f (resp. g) thanks to Equation (1), since $s_\mathbf{h} = 0$ in the notation there.

Remark 7.1. We remark here that we impose no condition on the weights k and ℓ and the sign of the characters ψ_1 and ψ_2. This is different to the situation of Siegel modular forms (compare for example with [24, page 177]). The reason for this, as it will become clear a little bit later, is that we needed to impose no condition on the sign of the character and the weight of the Siegel-type Eisenstein series in section 3 (see Remark 3.1).

We now apply the Rankin-Selberg method to obtain Equation (30) below generalizing the exposition in [24, page 178-179], where the case of scalar valued Siegel (and Hermitian) modular forms is considered. We let $X := S_{\mathbf{a}}/S(\mathfrak{z}^{-1})$, where $\mathfrak{z} := \mathfrak{b}_1 + \mathfrak{b}_2$. Furthermore we define the quotient set $Y := S^+_{\mathbf{a}}/U$ where $U := GL_n(\mathfrak{o})$ acts on $S^+_{\mathbf{a}}$ by conjugation, i.e. $s \mapsto g^* s g$. We now note that by considering the Fourier expansions of f and g we obtain for any $y \in S^+_{\mathbf{a}}$:

$$\int_X \prec \rho(\sqrt{y}) f(x+iy), \rho(\sqrt{y}) g(x+iy) \succ dx =$$

$$\mathrm{vol}(X) \sum_{\sigma \in S^+} \prec \rho(\sqrt{y}) c_{\mathbf{f}}(\sigma), \rho(\sqrt{y}) c_{\mathbf{g}}(\sigma) \succ \exp(-4\pi\lambda(\sigma y)).$$

We set $h := k + \ell$, and for $s \in \mathbb{C}$ with $Re(s) >> 0$ we consider the integral

$$I(s) := \int_Y \int_X \prec \rho(\sqrt{y}) f(x+iy), \rho(\sqrt{y}) g(x+iy) \succ \delta(y)^{h+s-(2n-1)} dx dy,$$

where we recall that $\delta(y) = \det(y)^{1/2}$. For an $\sigma \in S^+$ we define

$$\nu_\sigma := \frac{1}{|\{u \in U \mid u^* \sigma u = \sigma\}|}.$$

We then have that,

$$I(s) = \int_Y \mathrm{vol}(X) \sum_{\sigma \in S^+} \prec \rho(y) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ \exp(-4\pi\lambda(\sigma y)) \delta(y)^{h+s-(2n-1)} dy =$$

$$\mathrm{vol}(X) \int_Y \sum_{\sigma \in \mathcal{T}} \sum_{u \in U} \nu_\sigma \prec \rho(y) c_{\mathbf{f}}(u^*\sigma u), c_{\mathbf{g}}(u^*\sigma u) \succ e_{\mathbf{a}}(2i\lambda(u^*\sigma u y)) \delta(y)^{s+h} \delta(y)^{-(2n-1)} dy,$$

where $\mathcal{T}$ denotes a fixed set of representatives of S^+/U. Using properties (2), (3) and (4) of Proposition 2.4, and the fact that $\det(u) = 1$ for $u \in U$ and hence also $\psi_{1,\mathbf{h}}(\det(u)) = \psi_{1,\mathbf{a}}(\det(u)) = 1$ we obtain $c_{\mathbf{f}}(u^*\sigma u) = \rho_k(u^*) c_{\mathbf{f}}(\sigma) = \rho(u^*) c_{\mathbf{f}}(\sigma)$ and similarly for $c_{\mathbf{g}}(\cdot)$. In particular we obtain

$$\mathrm{vol}(X)^{-1} I(s) =$$

$$\sum_{\sigma \in \mathcal{T}} \int_Y \sum_{u \in U} \nu_\sigma \prec \rho(uyu^*) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ e_{\mathbf{a}}(2i\lambda(\sigma u y u^*)) \delta(uyu^*)^{s+h} \delta(uyu^*)^{-(2n-1)} d(uyu^*) =$$

$$\sum_{\sigma \in \mathcal{T}} \nu_\sigma \int_{S^+_{\mathbf{a}}} \prec \rho(y) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ e_{\mathbf{a}}(2i\lambda(\sigma y)) \delta(y)^{s+h} \delta(y)^{-(2n-1)} dy =$$

$$\sum_{\sigma \in \mathcal{T}} \nu_\sigma \int_{S^+_{\mathbf{a}}} \prec \rho(y) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ \exp(-4\pi\lambda(\sigma y)) \delta(y)^{s+h-(2n-1)} dy =$$

$$\sum_{\sigma \in \mathcal{T}} \nu_\sigma \prec H_{\rho,\sigma}(s+h) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ .$$

We set

$$D(s, f, g) := \sum_{\sigma \in \mathcal{T}} \nu_\sigma \prec H_{\rho,\sigma}(s+h) c_{\mathbf{f}}(\sigma), c_{\mathbf{g}}(\sigma) \succ =$$

$$\sum_{\sigma \in \mathcal{T}} \nu_\sigma \prec H_\rho(s+h) \rho(\sigma^{-1/2}) c_{\mathbf{f}}(\sigma), \rho(\sigma^{-1/2}) c_{\mathbf{g}}(\sigma) \succ \delta(\sigma)^{-s-h}$$

We now compute the same integral $I(s)$ in a different way. In particular we show that we have the equality

$$I(s) = \mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\langle f, gE_{k-\ell}(\bar{s}+2n-1,\phi)\rangle, \tag{29}$$

where,

$$E_{k-\ell}(s,\phi) := E^n_{k-\ell}(z,s;\phi,\Gamma) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \phi_{\mathfrak{y}}(\det(d_\gamma))^{-1} j_\gamma(z)^{-(k-\ell)}\delta(\gamma\cdot z)^{s-(k-\ell)},$$

where $\Gamma := \Gamma[\mathfrak{z}^{-1},\mathfrak{z}\mathfrak{y}]$, $\mathfrak{z} = \mathfrak{b}_1+\mathfrak{b}_2$, $\mathfrak{y} = \mathfrak{z}^{-1}(\mathfrak{b}_1\mathfrak{c}_1\cap\mathfrak{b}_2\mathfrak{c}_2)$, and $\phi := \bar{\psi}_1\psi_2$.

Substituting the definition of the Eisenstein series we have,

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\,\langle f, gE_{k-\ell}(\bar{s}+2n-1,\phi)\rangle =$$

$$\int_{\Gamma\backslash\mathcal{H}_n}\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \prec \rho_k(y)f(z), g(z)(\bar{\psi}_1\psi_2)_{\mathfrak{y}}(\det(d_\gamma))^{-1}j_\gamma(z)^{-(k-\ell)} \succ \delta(\gamma\cdot z)^{s+2n-1-k+\ell}d\nu(z),$$

where we recall $d\nu(z) = \delta(y)^{-2(2n-1)}dxdy$, and we used that $\prec \rho_k(\sqrt{y})v_1, v_2 \succ = \prec v_1, \rho_k(\sqrt{y})v_2 \succ$ for $v_1, v_2 \in V$. Note that for any $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma$ we have (see [13, Theorem 1.7]) $\mathrm{Im}(z) = (cz+d)^*\mathrm{Im}(\gamma\cdot z)(cz+d)$. Using this, and the fact that $f\in S_{\rho_k}(\Gamma,\psi_1)$ and $g\in M_{\rho_\ell}(\Gamma,\psi_2)$ we obtain:

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\,\left\langle f, gE_{k-\ell}(s+\tfrac{n+1}{2},\phi)\right\rangle =$$

$$\int_{\Gamma\backslash\mathcal{H}_n}\left[\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \prec \rho_k((cz+d)^*)\rho_k(\mathrm{Im}(\gamma\cdot z))f(\gamma\cdot z), \rho_\ell((cz+d)^{-1})g(\gamma\cdot z)j_\gamma(z)^{-(k-\ell)} \succ \right.$$
$$\left. \times\, \delta(\gamma\cdot z)^{s+2n-1+\ell-k}\right]d\nu(z) =$$
$$\int_{\Gamma\backslash\mathcal{H}_n}\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \prec \rho_k(\mathrm{Im}(\gamma\cdot z))f(\gamma\cdot z), g(\gamma\cdot z) \succ \delta(\gamma\cdot z)^{s+2n-1+\ell-k}d\nu(z),$$

where we have used that $\langle \rho_k((cz+d)^*)v_1, v_2 \succ = \prec v_1, \rho_k(cz+d)v_2\rangle$ for any $v_1, v_2\in V$. Now we note that the expression $\prec \rho_k(y)f(z), g(z) \succ \delta(y)^{s+2n-1-k+\ell}$ is $P\cap\Gamma$-invariant. Indeed, for a $\gamma\in P$ we have

$$\rho_k(\mathrm{Im}(\gamma z)) = \rho_k(\widehat{d}_\gamma)\rho_k(y)\rho_k(d_\gamma^{-1}), \quad \delta(\gamma z) = \det(d_\gamma)^{-1}\delta(y).$$

Moreover for $\gamma\in P\cap\Gamma$ we have that $\det(d_\gamma) = 1$, and hence $f(\gamma z) = \rho_k(d_\gamma)f(z)$ and $g(\gamma z) = \rho_\ell(d_\gamma)g(z) = \rho_k(d_\gamma)g(z)$ since $\rho_k(d_\gamma) = \rho_\ell(d_\gamma)\det(d_\gamma)^{k-\ell}$. Putting this together, for any $\gamma\in P\cap\Gamma$, we have

$$\prec \rho_k(\mathrm{Im}(\gamma\cdot z))f(\gamma\cdot z), g(\gamma\cdot z) \succ \delta(\gamma\cdot z)^{s+2n-1+\ell-k} =$$
$$\prec \rho_k(y)\rho_k(d_\gamma^{-1})f(\gamma z), \rho_k(d_\gamma^{-1})g(\gamma z) \succ \delta(\gamma z)^{s+2n-1+\ell-k} =$$
$$\prec \rho_k(y)f(z), g(z) \succ \delta(y)^{s+2n-1-k+\ell}.$$

Using this, we obtain,

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\langle f, gE_{k-\ell}(s+2n-1,\phi)\rangle = \int_{P\cap\Gamma\backslash\mathcal{H}_n} \prec \rho_k(y)f(z), g(z) \succ \delta(z)^{s+2n-1+\ell-k} d\nu(z) =$$

$$\int_{P\cap\Gamma\backslash\mathcal{H}_n} \prec \rho(\sqrt{y})f(z), \rho(\sqrt{y})g(z) \succ \delta(z)^{s-(2n-1)+\ell+k} dxdy,$$

since $\det(y) = \delta(y)^2$, and $d\nu(z) = \delta(z)^{-2(2n-1)}dxdy$. We note that the set $X \times Y$ is a set of representatives of $P \cap \Gamma \setminus \mathcal{H}_n$, which gives Equality (29) above. That is,

$$D(s, f, g) = \mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\langle f, gE_{k-\ell}(\bar{s} + 2n - 1, \bar{\psi}_1\psi_2)\rangle. \tag{30}$$

Now we are assuming that $\mathbf{f} \in \mathcal{S}_{\rho_k}(D, \psi)$ is a non-zero eigenform, with $D := D[\mathfrak{b}^{-1}, \mathfrak{bc}]$. Thanks to the remark at the end of Section 2 we may now assume without loss of generality that $\mathfrak{b} = \mathbb{Z}$. We moreover assume that $(\rho, V) \in \mathcal{T}_n$, where we remind the reader our notation $\rho_k = \rho \otimes \det^k$, and we let $P : M_{2n}(\mathbb{C}) \to V$ be a pluriharmonic map with the properties described at the end of section 4 with respect to the representation (ρ, V). We will consider $D(s, f, g)$ above for a particular g. Given a Dirichlet character χ of some conductor $\mathfrak{f}_\chi$, and for a fixed $\tau \in S_+$ and $r \in GL_n(B)$ with $c_{\mathbf{f}}(r^*\tau r) \neq 0$ we consider the theta series $\theta := \theta(z, \chi^{-1}, P) \in M_{\rho_n}(\Gamma', \chi^{-1})$ obtained in Theorem 5.5 where we set $s := r^*2\tau r$ there. We note here that since $c_{\mathbf{f}}(r^*\tau r) \neq 0$ we have (see for example [13, Theorem 1.2]) that s is even hermitian positive definite. Moreover we have that $\Gamma' = \Gamma_0(\mathfrak{n}^2\mathfrak{q})$, where $\mathfrak{q}$ is the ideal of $\mathbb{Q}$ generated by the level of s, and $\mathfrak{n} = \mathfrak{f}_\chi \cap \mathfrak{m}$, with $\mathfrak{m}$ as in Lemma 5.4.

We then have $D(s, f, \theta) = \sum_{\sigma\in\mathcal{T}} \nu_\sigma \prec H^n_{\rho,\sigma}(s + h)c_{\mathbf{f}}(\sigma), c_\theta(\sigma) \succ$ in which

$$c_\theta(\sigma) = \sum_{\xi\in X_\sigma} \chi^*(\det(\xi))^{-1} P\left(\overline{\sqrt{r^*2\tau r}\xi}\right),$$

and $X_\sigma := \{\xi \in GL_n(B) \cap M_n(\mathfrak{o})) \mid \sigma = \xi^* r^* 2\tau r\xi\}$. In this setting we have that $h = k + n$. We have,

$$D(s, f, \theta) = \sum_{\sigma\in\mathcal{T}} \nu_\sigma \prec H_{\rho,\sigma}(s)c_{\mathbf{f}}(\sigma), \sum_{\xi\in X_\sigma} \chi^*(\det(\xi))^{-1} P\left(\overline{\sqrt{r^*2\tau r}\xi}\right) \succ=$$

$$\sum_{\sigma\in\mathcal{T}} \nu_\sigma \prec c_{\mathbf{f}}(\sigma), \sum_{\xi\in X_\sigma} \chi^*(\det(\xi))^{-1} H_{\rho,\sigma}(\bar{s} + h) P\left(\overline{\sqrt{r^*2\tau r}\xi}\right) \succ,$$

where we have used the fact that $H_{\rho,\sigma}(s)$ is hermitian. In the inner sum we have that $\sigma = \xi^* r^* 2\tau r\xi$, and hence we have that the above expression is equal to

$$\sum_{\sigma\in\mathcal{T}} \nu_\sigma \sum_{\xi\in X_\sigma} \chi^*(\det(\xi)) \prec c_{\mathbf{f}}(\xi^* r^* 2\tau r\xi), H_{\rho,\xi^* r^* 2\tau r\xi}(\bar{s} + h) P\left(\overline{\sqrt{r^*\tau r}\xi}\right) \succ=$$

$$\sum_{\xi\in X} \chi^*(\det(\xi)) \prec c_{\mathbf{f}}(\xi^* r^* 2\tau r\xi), \rho(\xi^{-1}\sqrt{r^*2\tau r}^{-1}) H_\rho(\bar{s} + h) P(1) \succ \delta(\xi^* r^* 2\tau r\xi)^{-(s+h)}$$

where $X = (GL_n(B) \cap M_n(\mathfrak{o}))/GL_n(\mathfrak{o})$, and we write $P(1)$ for $P(I_n)$. Indeed we have that

$$H_{\rho,\xi^* r^*\tau r\xi}(s) = \rho(\xi^{-1}\sqrt{r^*2\tau r}^{-1}) H_\rho(s) \rho(\widehat{\sqrt{r^*2\tau r}^{-1}\xi}) \delta(\xi^* r^* 2\tau r\xi)^{-s}.$$

This follows by considering the change of variable $y \mapsto \xi^{-1}(r^*2\tau r)^{-1/2}y(r^*2\tau r)^{-1/2}\widehat{\xi}$ in the definition of $H_{\rho,\xi^*r^*\tau r\xi}(s)$. That is,

$$D(s,f,\theta) = \sum_{\xi \in X} \chi^*(\det(\xi)) \prec c_{\mathbf{f}}(\xi^* r^* 2\tau r\xi), \rho(\xi^{-1}\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ \delta(\xi^* r^* 2\tau r\xi)^{-(s+k+n)} =$$

$$\sum_{\xi \in X} \chi^*(\det(\xi)) \prec \rho_k(\xi^*)^{-1} c_{\mathbf{f}}(\xi^* r^* 2\tau r\xi), \rho(\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ \delta(\xi^* r^* 2\tau r\xi)^{-(s+n)} =$$

$$\delta(r^*2\tau r)^{-s} \sum_{\xi \in X} \chi^*(\det(\xi)) \prec c_{\mathbf{f}}(r^*2\tau r, \xi), \rho(\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ \det(\xi)^{-(s+n)}.$$

Putting all the above together we obtain

$$D(s,f,\theta) = \delta(r^*\tau r)^{-s} \sum_{\xi \in X} \chi^*(\det(\xi)) \prec c_{\mathbf{f}}(r^*2\tau r, \xi), \rho(\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ \det(\xi)^{-(s+n)}.$$

In particular from Equations (26) and (30) we obtain

$$\delta(r^*2\tau r)^{-s} \prec \mathcal{D}(s+3n-1, \mathbf{f}, \chi), \rho(\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ = \mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\langle f, \theta E_{k-n}(s+2n-1, \phi)\rangle,$$

where $\phi = (\psi\chi)^{-1}$. We remark also here that the Eisenstein series are of level $\Gamma_0(\mathfrak{h})$, where is the ideal $\mathfrak{h} := \mathfrak{c} \cap \mathfrak{n}^2\mathfrak{q}$.

After multiplying both sides of the above equation with

$$\prod_{p \in \mathbf{b}} g_p(\chi(p)p^{-(s+3n-1)}) \prod_{i=1}^{n} L_{\mathfrak{h}}(2s+4n-2i, \psi^2\chi^2),$$

and employing Equation (27) above, we obtain,

(31)

$$\delta(r^*2\tau r)^{-s} \prec \rho(r^*)c(\tau, r; \mathbf{f}), \rho(\sqrt{r^*2\tau r}^{-1})H_\rho(\bar{s}+h)P(1) \succ \psi_{\mathfrak{c}}(\det(r))L_\psi(s+3n-1, \mathbf{f}, \chi) =$$

$$c(r,\chi,s')\,\mathrm{vol}(\Gamma\backslash\mathcal{H}_n)\left(\prod_{i=1}^{n}\prod_{p|\mathfrak{t}} L_p(2s+4n-2i, \psi^2\chi^2)\right)\prod_{p\in\mathbf{b}} g_p(\chi(p)p^{-s'})\langle f, \theta\mathcal{E}_{k-n}(\bar{s}+2n-1, \overline{\psi\chi})\rangle,$$

where $s' := s+3n-1$,

$$\mathcal{E}_{k-n}(\bar{s}+2n-1, \overline{\psi\chi}) := \prod_{i=1}^{n} L_{\mathfrak{h}}(2\bar{s}+4n-2i, (\overline{\psi\chi})^2)E_{k-n}(\bar{s}+2n-1, \overline{\psi\chi}),$$

and $\mathfrak{t} := \prod_{i=1}^{k} \mathfrak{p}_i$, where $\mathfrak{p}_i$ are prime ideals dividing $\mathfrak{h}$ but not $\mathfrak{c}$.

8. Analytic Properties of the standard L-function

By the expression obtained in Equation (31) above, we see that the analytic properties of $L_\psi(s, \mathbf{f}, \chi)$ can be understood from the analytic properties of the Siegel-type Eisenstein series $\mathcal{E}_{k-n}$ and the quantity $H_\rho(s)P(1)$, which we now see can be computed explicitly. We note here that a similar integral as in the Proposition below was computed in [3], where the case of Siegel modular forms is considered, and the corresponding pair of groups from the Howe-Weyl duality is $O_n(\mathbb{C}) \times GL_n(\mathbb{C})$.

Proposition 8.1. *Let* $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0 \ldots, 0)$ *be the maximal weight of* $(\rho, V) \in \mathcal{T}_n$. *Then*

$$H_\rho(s)P(1) = \mathbf{\Gamma}_\rho(s)P(1),$$

where,

$$\mathbf{\Gamma}_\rho(s) := (2\pi)^{-(ns+\sum_{j=1}^n \lambda_j)} \prod_{j=1}^n \Gamma(s + \lambda_j - 2(j-1)).$$

Proof. We recall that by the definition of the operator $H_\rho(s)$ in (28) we have

$$H_\rho(s)P(1) = \int_{S^+_{\mathbf{a}}} P({}^t y) e^{-4\pi\lambda(y)} \delta(y)^{s-(2n-1)} dy = \int_{S^+_{\mathbf{a}}} P(y) e^{-4\pi\lambda(y)} \delta(y)^{s-(2n-1)} dy.$$

We write (τ, W) for the irreducible representation of $Sp_n(\mathbb{C})$ associated with ρ by the Howe-Weyl duality discussed in section 4 above. Then we identify $V \otimes W = V \otimes W^* = \mathrm{Hom}(V, W) = M_{d,r}(\mathbb{C})$ where $d = \dim(V)$ and $r = \dim(W)$. As in [12, page 16] we can find pluriharmonic polynomial maps $\mathbf{P} : M_{2n}(\mathbb{C}) \to M_{d,r}(\mathbb{C})$ such that for all $(g_1, g_2) \in Sp_n(\mathbb{C}) \times GL_{2n}(\mathbb{C})$ and $x \in M_{2n}(\mathbb{C})$ we have $\mathbf{P}(g_1 x g_2) = \rho({}^t g_2)\mathbf{P}(x)\tau({}^t g_1)$. Notice that each such polynomial $\mathbf{P}$ consists of polynomial maps (columns) P_j, $j = 1, \ldots, r$, that are pluriharmonic and $P_j(xg) = \rho({}^t g)P_j(x)$. In particular we may choose our polynomial P above to be one of the columns of a polynomial $\mathbf{P}$. So it is enough to show that

$$\int_{S^+_{\mathbf{a}}} \mathbf{P}(y) e^{-4\pi\lambda(y)} \delta(y)^{s-(2n-1)} dy = \mathbf{\Gamma}_\rho(s)\mathbf{P}(1).$$

An argument similar to the one used in [3] shows that we may select $\mathbf{P}$ such that $\mathbf{P}(1)$ is a highest weight vector for the representation $\tau \otimes \rho$.

We now use the identification $M_n(\mathbb{H}) = \{z \in M_{2n}(\mathbb{C}) \mid \eta_n z = \bar{z}\eta_n\}$ where $\eta_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, and hence every such matrix can be written as $z = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, with $a, b \in M_n(\mathbb{C})$. In particular we have that if $z \in M_{2n}(\mathbb{C})$ is $\mathbb{H}$-Hermitian then $a^* = a$ and ${}^t b = -b$. We now use the polar decomposition for $\mathbb{H}$-hermitian matrices (embedded in $M_{2n}(\mathbb{C})$) as explained in [5, Chapter VI]. In particular we have that any $\mathbb{H}$-hermitian matrix y can be written in the form $y = a^*\epsilon a$ with $a \in USp_n := Sp_n(\mathbb{C}) \cap U(2n)$ and $\epsilon = \mathrm{diag}[\epsilon_1, \ldots, \epsilon_n, \epsilon_1, \ldots, \epsilon_n]$ with $\epsilon_i \in \mathbb{R}$. We let $D := \{\epsilon = \mathrm{diag}[\epsilon_1, \ldots, \epsilon_n, \epsilon_1, \ldots, \epsilon_n] \mid \epsilon_i \in \mathbb{R}\}$ and $D_+ = \{\epsilon \in D \mid \epsilon_1 < \epsilon_2 < \ldots < \epsilon_n\}$. Then by [5, Theorem VI 2.3] we have that there exists a constant $c_0 \in \mathbb{C}^\times$ such that

$$\begin{aligned}
&\int_{S^+_{\mathbf{a}}} \mathbf{P}(y) e^{-4\pi\lambda(y)} \delta(y)^{s-(2n-1)} dy \\
&\quad = c_0 \int_{USp_n} \int_{D_+} \mathbf{P}(a^*\epsilon a) e^{-2\pi \mathrm{tr}\,(\epsilon)} \det(\epsilon)^{s/2} \left(\prod_{j<k} (\epsilon_k - \epsilon_j)^4 \right) d\epsilon da \\
&\quad = c_0 \int_{USp_n} \rho({}^t a) \int_{D_+} \mathbf{P}(a^*\epsilon) e^{-2\pi \mathrm{tr}\,(\epsilon)} \det(\epsilon)^{s/2} \left(\prod_{j<k} (\epsilon_k - \epsilon_j)^4 \right) d\epsilon da.
\end{aligned}$$

Since $\mathbf{P}(1) \in V \otimes W$ is a highest weight vector we know that $\mathbf{P}(a^*\epsilon) = [\rho(\epsilon)\mathbf{P}(1)]\tau(\bar{a}) = \epsilon_1^{\lambda_1} \cdots \epsilon_n^{\lambda_n}\mathbf{P}(a^*)$, where we recall the maximal weight of ρ is $(\lambda_1, \dots, \lambda_n, 0, \dots, 0)$. That is, the above integral reads

$$c_0 \int_{USp_n} \rho({}^t a) \int_{D_+} \mathbf{P}(a^*)\epsilon_1^{\lambda_1} \cdots \epsilon_n^{\lambda_n} e^{-2\pi \mathrm{tr}\,(\epsilon)} \det(\epsilon)^{s/2} \left(\prod_{j<k} (\epsilon_k - \epsilon_j)^4\right) d\epsilon da$$

$$= \mathbf{P}(1)c_0 \int_{USp_n} \left(\int_{D_+} \epsilon_1^{s+\lambda_1} \cdots \epsilon_n^{s+\lambda_n} e^{-2\pi \mathrm{tr}\,(\epsilon)} \left(\prod_{j<k} (\epsilon_k - \epsilon_j)^4\right) d\epsilon\right) da,$$

where we have used the fact that $\rho({}^t a)\mathbf{P}(a^*) = \mathbf{P}(a^*a) = \mathbf{P}(1)$ since $a \in USp_n \subset U(2n)$. But the last integral is a genaralized Siegel integral and by [5, Proposition VII.1.2] we have

$$c_0 \int_{USp_n} \left(\int_{D_+} \epsilon_1^{s+\lambda_1} \cdots \epsilon_n^{s+\lambda_n} e^{-\mathrm{tr}\,(\epsilon)} \left(\prod_{j<k} (\epsilon_k - \epsilon_j)^4\right) d\epsilon\right) da = \prod_{j=1}^{n} \Gamma(s + \lambda_j - 2(j-1)).$$

□

Using Proposition (8.1) and Equation (31) we obtain

$$\delta(r^* 2\tau r)^{-s} \mathbf{\Gamma}_\rho(s+k+n)\mathcal{G}^n(s+2n-1, k-n) \times$$
$$\prec \rho(r^*)c(\tau, r; \mathbf{f}), \rho(\sqrt{r^* 2\tau r}^{-1})P(1) \succ L_\psi(s+3n-1, \mathbf{f}, \chi) =^{\mathbb{C}^\times}$$
$$c(r, \chi, s) \left(\prod_{i=1}^{n} \prod_{p|\mathfrak{t}} L_p(2s+4n-2i, \psi^2\chi^2)\right) \prod_{p \in \mathbf{b}} g_p(\chi(p)p^{-s}) \langle f, \theta\mathcal{P}(\bar{s}+2n-1)\rangle,$$

where

$$\mathcal{P}(\bar{s}+2n-1) := \mathcal{G}^n(\bar{s}, k-n) \prod_{i=1}^{n} L_{\mathfrak{c}}(2\bar{s}+4n-2i, (\overline{\psi\chi})^2) E_{k-n}(\bar{s}+2n-1, \overline{\psi\chi}),$$

and $\mathcal{G}^n(\bar{s}, k-n)$ was defined in Theorem 3.8 and the symbol "$=^{\mathbb{C}^\times}$" means that we have an equality up to a non-zero complex number. Now we note that there is always a choice of τ and r such that

$$\prec \rho(r^*)c(\tau, r; \mathbf{f}), \rho(\sqrt{r^* 2\tau r}^{-1})P(1) \succ = \prec c(\tau_r, 1; \mathbf{f}), \rho(\sqrt{2\tau_r}^{-1})P(1) \succ \neq 0, \quad \tau_r := r^*\tau r.$$

Indeed, since (ρ, V) is irreducible, by an argument as in [30, Lemma 4] one has that the image of $\mathbf{f} : G_{\mathbb{A}} \to V$ is dense in V and hence in particular, after perhaps a change of basis, we may assume, that there exists a positive definite hermitian matrix τ_r such that $c(\tau_r, 1; \mathbf{f})$ is a maximal vector for (ρ, V). Moreover by Gauss decomposition (see for example [9, Lemma B.2.6]) we may write $\rho(\sqrt{2\tau_r}) = \rho(u_- d u_+)$ for u_- a lower triangle matrix with diagonal entries equal to 1, u_+ an upper diagonal with diagonal entries equal to 1, and d a diagonal matrix. This gives,

$$\prec c(\tau_r, 1; \mathbf{f}), \rho(\sqrt{2\tau_r}^{-1})P(1) \succ = \prec \rho(u_-^*)c(\tau_r, 1; \mathbf{f}), \rho(d)\rho(u_+)P(1) \succ =$$
$$\prec c(\tau_r, 1; \mathbf{f}), \rho(d)P(1) \succ = \alpha \prec c(\tau_r, 1; \mathbf{f}), P(1) \succ \neq 0,$$

for some $\alpha \neq 0$ since both $P(1)$ and $c(\tau_r, 1; \mathbf{f})$ are of maximum weight. That is, for such a choice of τ and r we have,

$$\delta(r^*2\tau r)^{-s}\mathbf{\Gamma}_\rho(s+k+n)\mathcal{G}^n(s+2n-1,k-n)L_\psi(s+3n-1,\mathbf{f},\chi) =^{\mathbb{C}^\times} \tag{32}$$

$$c(r,\chi,s')\left(\prod_{i=1}^{n}\prod_{p|\mathfrak{t}} L_p(2s+4n-2i,\psi^2\chi^2)\right)\prod_{p\in\mathbf{b}} g_p(\chi(p)p^{-s'})\langle f, \theta\mathcal{P}(\bar{s}+2n-1)\rangle,$$

where we recall $s' = s+3n-1$. Since the expression

$$\delta(r^*2\tau r)^s c(r,\chi,s')\prod_{p\in\mathbf{b}} g_p(\chi(p)p^{-s'}),$$

defines a holomorphic function in $s \in \mathbb{C}$, we have that the analytic properties of

$$\Lambda_\mathfrak{t}(s,\mathbf{f},\chi) :=$$

$$\mathbf{\Gamma}_\rho(s+k-2n+1)\mathcal{G}^n(s-n,k-n)\left(\prod_{i=1}^{n}\prod_{p|\mathfrak{t}} L_p(2s-2n+2-2i,\psi^2\chi^2)\right)^{-1} L_\psi(s,\mathbf{f},\chi),$$

are the same as the analytic properties of $\mathcal{P}(s-n)$. In particular we have

Theorem 8.2. *Let $\mathbf{f} \in \mathcal{S}_{\rho_k}(\Gamma_0(\mathfrak{c}),\psi)$ be an eigenform, and assume that $k \geq n$, $\rho \in \mathcal{T}_n$ (see (16) for notation) and all the primes where B is ramified divide the ideal $\mathfrak{c}$. The function $\Lambda_\mathfrak{t}(s,\mathbf{f},\chi)$ has a meromorphic continuation to the entire complex plane. Moreover it is analytic if $\psi^2\chi^2 \neq 1$ or $k \geq 2n$. If $\psi^2\chi^2 = 1$ and $n \leq k < 2n$ then the possible poles are all simple and they can appear at the interval,*

$$\{j \in \mathbb{Z} \mid n \leq j \leq 3n-k\},$$

Proof. This follows from the Equation (32) above and Theorem 3.8. □

Remark 8.3. We are now making some remarks regarding the theorem above.

(1) The definition of the function $\Lambda_\mathfrak{t}(s,\mathbf{f},\chi)$ depends on the ideal $\mathfrak{t}$, which in turn depends on the possible choices of τ and r above. This is similar to the situation of Siegel modular forms (for the scalar weight see [19] and the vector valued [3]). We also remark here that the condition on the choices of τ and r depend only on $\mathbf{f}$. Furthermore there are, at least two, situations where the factor $\left(\prod_{i=1}^{n}\prod_{p|\mathfrak{t}} L_p(2s-2n+2-2i,\psi^2\chi^2)\right)^{-1}$ will not show up: (1) since $\mathcal{S}_{\rho_k}(\Gamma_0(\mathfrak{c}),\psi) \subseteq \mathcal{S}_{\rho_k}(\Gamma_0(\mathfrak{h}),\psi)$, by adding some "bad" Euler factors to $\mathbf{f}$, we can make the ideal $\mathfrak{t}$ trivial, and (2) if the conductor of the twisting character χ is divisible by the ideal $\mathfrak{t}$.

(2) Our theorem is different than the one proved by Yamana [33, Theorem 9.1]. First of all we should stress that it is a rather delicate matter to compare our L-function with the one in [33] since the "bad" Euler factors may not be the same. However, looking beyond this issue of the "bad" Euler factors, we remark that the theorem in [33] is more in the spirit of the theorems proved in [14, 15] for Siegel modular forms whereas ours is in the spirit of [19] (scalar weight) and

[3] (vector valued). The theorem in [33] is "generic" in the sense that does not take under consideration the type at the archimedean place of the corresponding automorphic representation of $G_{\mathbb{A}}$. It does of course cover more cases than our theorem above (for example representations $\rho \notin \mathcal{T}_n$ or non-holomorphic quaternionic modular forms), but on the other hand for the cases where our theorem is applicable, then Theorem 8.2 gives more precise information. Indeed the theorem in [33] gives a range of 4n many points as possible poles in the case of twists with trivial or quadratic characters. In our approach not only we show that there exist no poles when the "weight" k is large enough, but even when k is smaller than $2n$ the set of possible poles is at most of size $2n-k$.

(3) We expect that our method can be used to obtain non-vanishing results of the standard L-function by extending the range of absolute convergence beyond $Re(s) > 4n-1$. Indeed the use of the Rankin-Selberg method to obtain such results has been illustrated in the case of scalar weight Siegel modular forms in [20], and for vector valued in [3], where the "obvious" bound of $\mathrm{Re}(s) > 2n+1$ (here the center of the critical strip is taken as $n+\frac{1}{2}$) is extended to $\mathrm{Re}(s) > (3n/2)+1$ where n is the degree of the symplectic group, improving a previous result of [4]. Such a non-vanishing result has important implication for the theory of Siegel modular forms. For example it is used to establish algebraicity results for special L-values of Siegel modular forms of small weights (see [24, Theorem 28.5 and Theorem 28.8]), or to the study of the space of holomorphic Eisenstein series, [24, Theorem 27.13].

An approach similar to the one of [3] should allow us to improve the range of absolute convergence in the case of quaternionic modular forms, and in particular to establish that the range of absolute convergence is $Re(s) > 3n-1$. This will require to establish the existence of some cuspidal theta series as was done in [3] for the Siegel modular forms case. We note here that in the case of quaternionic modular forms, even the scalar weight case requires such a different approach to the one taken in [20] for scalar weight Siegel modular forms, since the determinant is not a ($Sp_n(\mathbb{C})$-) pluriharmonic polynomial.

We aim to explore these questions of non-vanishing of the standard L-function of a quaternionic modular form and their implication to the algebraicity of special L-values in the future.

9. Appendix

We provide a sketch of the proof of Proposition 3.3. As we indicated there we will only discuss the needed modifications to the proof of [21, Proposition 18.14], where the symplectic and the unitary case are considered. We keep the notation as in section 3.

We have

$$c(h,q,s) = \int_{S_{\mathbb{A}}/S} \mathbf{E}^* \left(\begin{pmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{pmatrix}, s \right) e_{\mathbb{A}}(-\lambda(h\sigma)) d\sigma,$$

where $d\sigma$ is the Haar measure on $S_{\mathbb{A}}/S$ normalized such that $\int_{S_{\mathbb{A}}/S} d\sigma = 1$. For a finite place $v \in \mathbf{h}$ we define a Haar measure $d\sigma_v$ on S_v/Λ_v by taking $\int_{S_v/\Lambda_v} d\sigma_v = 1$. For the archimedean place we may identify $S_{\mathbf{a}}$ with $\mathbb{R}^n \times \mathbb{H}^{n(n-1)/2} = \mathbb{R}^{n(2n-1)}$ and take $d\sigma_{\mathbf{a}}$ to be the Lebesgue measure on $\mathbb{R}^{n(2n-1)}$. Then by [31, Lemma 3.1] we have that

$$d\sigma = c(S) d\sigma_{\mathbf{a}} \prod_{v \in \mathbf{h}} d\sigma_v,$$

where $c(S)^{-1} := \mathrm{vol}(S_{\mathbf{a}}/S(\mathbb{Z})) = (4D_B^{-1})^{-n(n-1)/2}$, where D_B is the product of the primes where B ramifies.

Since we are assuming that the ideal $\mathfrak{c}$ is divisible by all finite primes where B is ramified we have in particular $\mathfrak{c} \neq \mathbb{Z}$. That is, for $x \in P_{\mathbb{A}} G_{\mathbf{a}}$,

$$\mathbf{E}^*(x, s) = \sum_{\gamma \in \eta R} \phi(\gamma x \eta_{\mathbf{h}}^{-1}, s).$$

Indeed this is the analogue of [21, Lemma 18.8 (3)] in our case. This is still true, since it is essentially a direct consequence of [21, Lemma 2.12 (1)], which includes also the quaternionic case (in the notation of [21] we take $K = B$ and $\epsilon = -1$). In particular we obtain

$$c(h, q, s) = \int_{S_{\mathbb{A}}} \phi(\eta\tau(\sigma)\mathrm{diag}[q, \hat{q}]\eta_{\mathbf{h}}^{-1}, s) e_{\mathbb{A}}(-\lambda(h\sigma)) d\sigma =$$

$$c(S) \int_{S_{\mathbf{h}}} \phi(\eta\tau(\sigma_{\mathbf{h}})\mathrm{diag}[q_{\mathbf{h}}, \hat{q}_{\mathbf{h}}]\eta_{\mathbf{h}}^{-1}, s) e_{\mathbf{h}}(-\lambda(h\sigma_{\mathbf{h}})) d\sigma_{\mathbf{h}} \times \int_{S_{\mathbf{a}}} \phi(\eta\tau(\sigma_{\mathbf{a}})\mathrm{diag}[q_{\mathbf{a}}, \hat{q}_{\mathbf{a}}], s) e_{\mathbf{a}}(-\lambda(h\sigma_{\mathbf{a}})) d\sigma_{\mathbf{a}}.$$

We first consider the archimedean integral

$$c_{\mathbf{a}}(h, q, s) = \int_{S_{\mathbf{a}}} \phi(\eta\tau(\sigma_{\mathbf{a}})\mathrm{diag}[q_{\mathbf{a}}, \hat{q}_{\mathbf{a}}], s) e_{\mathbf{a}}(-\lambda(h\sigma_{\mathbf{a}})) d\sigma_{\mathbf{a}}.$$

We then have,

$$\phi(\eta\tau(\sigma_{\mathbf{a}})\mathrm{diag}[q_{\mathbf{a}}, \hat{q}_{\mathbf{a}}], s) = \det(q_{\mathbf{a}}\mathbf{i} + \sigma_{\mathbf{a}}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}q_{\mathbf{a}}q_{\mathbf{a}}^* + \sigma_{\mathbf{a}})|^{\ell-s} =$$

$$\det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} \det(q_{\mathbf{a}}q_{\mathbf{a}}^* - \sigma_{\mathbf{a}}\mathbf{i})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} |\det(q_{\mathbf{a}}q_{\mathbf{a}}^* - \sigma_{\mathbf{a}}\mathbf{i})|^{\ell-s} =$$

$$\det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} \det(q_{\mathbf{a}}q_{\mathbf{a}}^* - \sigma_{\mathbf{a}}\mathbf{i})^{-\ell} (\det(q_{\mathbf{a}}q_{\mathbf{a}}^* - \sigma_{\mathbf{a}}\mathbf{i}))^{\ell/2-s/2} \det(q_{\mathbf{a}}q_{\mathbf{a}}^* + \sigma_{\mathbf{a}}\mathbf{i})^{\ell/2-s/2} =$$

$$\det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} (\det(q_{\mathbf{a}}q_{\mathbf{a}}^* - \sigma_{\mathbf{a}}\mathbf{i}))^{-(s/2+\ell/2)} \det(q_{\mathbf{a}}q_{\mathbf{a}}^* + \sigma_{\mathbf{a}}\mathbf{i})^{-(s/2-\ell/2)}$$

where we have used the equality $\overline{\det(x + \mathbf{i}y)} = \overline{\det({}^t x + \mathbf{i}^t y)} = \det(x^* - \mathbf{i}y^*) = \det(x - \mathbf{i}y)$, for $x, y \in S_{\mathbf{a}}$. That is,

$$\phi(\eta\tau(\sigma_{\mathbf{a}})\mathrm{diag}[q_{\mathbf{a}}, \hat{q}_{\mathbf{a}}], s) = \det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} \det(\sigma_{\mathbf{a}} + q_{\mathbf{a}}q_{\mathbf{a}}^*\mathbf{i})^{-\frac{s+\ell}{2}} \det(\sigma_{\mathbf{a}} - \mathbf{i}q_{\mathbf{a}}q_{\mathbf{a}}^*)^{-\frac{s-\ell}{2}}.$$

In particular we have,

$$c_{\mathbf{a}}(h, q, s) = \det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} \xi(q_{\mathbf{a}}q_{\mathbf{a}}^*, h, s+\ell, s-\ell).$$

Moreover we note that $\det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} = \left(\frac{\det(\mathbf{i}\hat{q}_{\mathbf{a}})}{|\det(\mathbf{i}\hat{q}_{\mathbf{a}})|}\right)^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{-s} = (-1)^{n\ell} |\det(\hat{q}_{\mathbf{a}})|^{-s} = (-1)^{n\ell} \det(\hat{q}_{\mathbf{a}})^{-s}$, since $\det(\hat{q}_{\mathbf{a}}) > 0$. Hence we conclude that

$$\det(\mathbf{i}\hat{q}_{\mathbf{a}})^{-\ell} |\det(\mathbf{i}\hat{q}_{\mathbf{a}})|^{\ell-s} = (-1)^{n\ell} \det(\hat{q}_{\mathbf{a}})^{-s}.$$

We now turn to the finite places. We fix a finite place $v \in \mathbf{h}$. We observe that for $x_v = \begin{pmatrix} \hat{q}_v & 0 \\ -\sigma_v \hat{q}_v & q_v \end{pmatrix}$ with $\sigma_v \in S_v$ and $q_v \in GL(B_v)$ if we write $x_v = pw$ with $p \in P_v$ and $w \in D_v$ then we have the equality of ideals,

$$\det(d_p^{-1} d_{x_v})\mathbb{Z}_v = \nu_0(\beta_v^{-1} d_{x_v}^{-1} c_{x_v}), \quad (*).$$

Indeed to see this we fix an embedding $M_n(B_v) \hookrightarrow M_{2n}(K_v)$ for some extension K_v of $\mathbb{Q}_v$, with $[K_v : \mathbb{Q}_v] \leq 2$. Actually $K_v = \mathbb{Q}_v$ if v is a place where B is unramified, otherwise we can pick K_v to be an unramified extansion of $\mathbb{Q}_v$ (see for example [23]). Then we can apply [21, Lemma 9.4(2)] taking K there to be our K_v here. Then we conclude the above identity, but as ideals of K_v. But then using the fact that both sides are actually ideals of $\mathbb{Q}_v$, we conclude that their actually equal as ideals over $\mathbb{Q}_v$.

Using $(*)$ above one can now conclude exactly in the same way the analogue of [21, Lemma 18.13] in the quaternionic setting. That is, for a finite place $v \in \mathbf{h}$ we have that,

$$\phi(x_v, s) = \chi(\det(q)_v)^{-1} |\det(q_v)|_v^{-s} \times$$

$$\begin{cases} \nu(b_v^{-1} q_v^{-1} \sigma_v \hat{q_v}) \chi(\nu_0(\beta_v^{-1} q_v^{-1} \sigma_v \hat{q_v})) & \text{if } v \not| \mathfrak{c} \\ 1 & \text{if } v | \mathfrak{c} \text{ and } (q^{-1}\sigma\hat{q})_v \in M_n(\mathfrak{b}_v \mathfrak{c}_v) \\ 0 & \text{otherwise} \end{cases}.$$

The rest now follows exactly as in the proof [21, Proposition 18.14]. One only needs to observe that in our case we set $\lambda := 2n-1$ and by [17] we have that for all $q \in GL_n(B)_{\mathbb{A}}$,

$$d(q\sigma q^*) = |\det(q)|_{\mathbb{A}}^{\lambda} d\sigma.$$

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