

Simple Games versus Weighted Voting Games: Bounding the Critical Threshold Value*

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Abstract. A simple game (N, v) is given by a set N of n players and a partition of 2^N into a set \mathcal{L} of losing coalitions L with value $v(L) = 0$ that is closed under taking subsets and a set \mathcal{W} of winning coalitions W with value $v(W) = 1$. We let $\alpha = \min_{p \geq 0, p \neq 0} \max_{W \in \mathcal{W}, L \in \mathcal{L}} \frac{p(L)}{p(W)}$. It is known that the subclass of simple games with $\alpha < 1$ coincides with the class of weighted voting games. Hence, α can be seen as a measure of the distance between a simple game and the class of weighted voting games. We prove that $\alpha \leq \frac{1}{4}n$ holds for every simple game (N, v) , confirming the conjecture of Freixas and Kurz (IJGT, 2014). For complete simple games, Freixas and Kurz conjectured that $\alpha = O(\sqrt{n})$. We also prove this conjecture, up to an $\ln n$ factor. Moreover, we prove that for graphic simple games, that is, simple games in which every minimal winning coalition has size 2, the problem of computing α is NP-hard, but polynomial-time solvable if the underlying graph is bipartite. Finally, we show that for every graphic simple game, deciding if $\alpha < \alpha_0$ is polynomial-time solvable for every fixed $\alpha_0 > 0$.

1 Introduction

Cooperative game theory provides a mathematical framework for capturing situations where subsets of agents may form a coalition in order to obtain some collective profit or share some collective cost. Formally, a *cooperative game (with transferable utilities)* consists of a pair (N, v) , where N is a set of n agents called *players* and $v : 2^N \rightarrow \mathbb{R}_+$ is a *value function* that satisfies $v(\emptyset) = 0$. In our context, the value $v(S)$ of a *coalition* $S \subseteq N$ represents the profit for S if all players in S choose to collaborate with (only) each other. The central problem in cooperative game theory is how to allocate the total profit $v(N)$ of the *grand coalition* N to the individual players $i \in N$ in a “fair” way. To this end various *solution concepts* such as the core, the Shapley value or the nucleolus have been

* A partial answer to the conjecture of Freixas and Kurz appeared, together with the results in Sections 3 and 4 of this paper, in an extended abstract published in the proceedings of SAGT 2018 (Hof et al. 2018).

designed; see the book of Peters (2008) for an overview. For example, core solutions try to allocate the total profit in such a way that every coalition $S \subseteq N$ gets at least $v(S)$. This is of course not always possible, that is, the core may be empty. This leads to related questions such as “How much do we need to spend in total if we want to give at least $v(S)$ to each coalition $S \subseteq N$?”, or equivalently, “What is the cost of stability for a cooperative game?” (Bachrach et al. 2018; Nguyen and Zick 2018). In the specific case of simple games (see below) where v takes only values 0 and 1, classifying coalitions into “losing” and “winning” coalitions, one may also ask: “How much do we have to give in the worst case to a losing coalition if we want to give at least $v(S) = 1$ to each winning coalition?”

As mentioned above, we study simple games. Simple games form a classical class of games, which are well studied; see also the book of Taylor and Zwicker (1999). The notion of being simple means that every coalition either has some equal amount of power or no power at all. Formally, a cooperative game (N, v) is *simple* if v is a monotone 0–1 function with $v(\emptyset) = 0$ and $v(N) = 1$, so $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and $v(S) \leq v(T)$ whenever $S \subseteq T$. In other words, if (N, v) is simple, then there is a set $\mathcal{W} \subseteq 2^N$ of *winning coalitions* W that have value $v(W) = 1$ and a set $\mathcal{L} \subseteq 2^N$ of *losing coalitions* L that have value $v(L) = 0$. Note that $N \in \mathcal{W}$, $\emptyset \in \mathcal{L}$ and $\mathcal{W} \cup \mathcal{L} = 2^N$. The monotonicity of v implies that subsets of losing coalitions are losing and supersets of winning coalitions are winning. A winning coalition W is *minimal* if every proper subset of W is losing, and a losing coalition L is *maximal* if every proper superset of L is winning.

A simple game (N, v) is a *weighted voting game* if there exists a payoff vector $p \in \mathbb{R}_+^N$ and some integer q , called the *quota* for (N, v) , such that a coalition S is winning if $p(S) \geq q$ and losing if $p(S) < q$; here, we denote the entries of a vector $x \in \mathbb{R}^N$ by x_i , and for $S \subseteq N$ we use the shorthand notation $x(S) = \sum_{i \in S} x_i$. Weighted voting games are also known as *weighted majority games* (von Neumann and Morgenstern 1944; Shapley 1962) and form one of the most popular classes of simple games. Throughout our paper we assume without loss of generality that weighted voting games are normalized, that is, $q = 1$.

It is easy to construct simple games that are not weighted voting games. We give an example below, but in fact there are many important simple games that are not weighted voting games. Gvozdeva et al. (2013) introduced a parameter α , called the *critical threshold value*, to measure the “distance” of a simple game to the class of weighted voting games:

$$\alpha = \alpha(N, v) = \min_{p \geq \mathbf{0}, p \neq \mathbf{0}} \max_{\substack{W \in \mathcal{W} \\ L \in \mathcal{L}}} \frac{p(L)}{p(W)}. \quad (1)$$

Here, for an integer r , we let \mathbf{r} denote the vector whose entries are all equal to r , and for two vectors $p, q \in \mathbb{R}^N$ we write $p \geq q$ if $p_i \geq q_i$ for every $i \in \{1, \dots, n\}$.

A simple game (N, v) is a weighted voting game if and only if $\alpha < 1$. This follows from observing that each optimal solution p of (1) can be scaled to satisfy $p(W) \geq 1$ for all winning coalitions W . The scaling enables us to reformulate the critical threshold value as follows:

$$\alpha = \alpha(N, v) = \min_{p \in Q(\mathcal{W})} \max_{L \in \mathcal{L}} p(L),$$

where

$$Q(\mathcal{W}) = \{p \in \mathbb{R}^N \mid p(W) \geq 1 \text{ for every } W \in \mathcal{W}, p \geq \mathbf{0}\}.$$

The following concrete example of a simple game (N, v) that is not a weighted voting game and that has in fact a large value of α was given by Freixas and Kurz (2014):

Example 1 Let $N = \{1, \dots, n\}$ for some even integer $n \geq 4$, and let the minimal winning coalitions be the pairs $\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}$. Then

$$Q(\mathcal{W}) = \{p \in \mathbb{R}^N \mid p_1 + p_2 \geq 1, p_2 + p_3 \geq 1, \dots, p_n + p_1 \geq 1, p \geq \mathbf{0}\}.$$

This means that $p(N) \geq \frac{1}{2}n$ for every $p \in Q(\mathcal{W})$. Then, for every $p \in Q(\mathcal{W})$ and for at least one coalition L of the two losing coalitions $\{2, 4, 6, \dots, n\}$ or $\{1, 3, 5, \dots, n-1\}$, we have $p(L) \geq \frac{1}{4}n$, showing that $\alpha \geq \frac{1}{4}n$. On the other hand, it is easily seen that $p = \frac{1}{2}$ satisfies $p(W) \geq 1$ for all winning coalitions and $p(L) \leq \frac{1}{4}n$ for all losing coalitions, showing that $\alpha \leq \frac{1}{4}n$. Thus $\alpha = \frac{1}{4}n$.

This example led Freixas and Kurz (2014) to the following conjecture:

Conjecture 1 (Freixas and Kurz 2014) For every simple game (N, v) , it holds that $\alpha \leq \frac{1}{4}n$.

1.1 Our Results

Section 2 contains our main result. In this section we reformulate and strengthen Conjecture 1 and then we prove the obtained strengthening.

In Section 3 we consider a subclass of simple games based on a natural desirability order, as introduced by Isbell (1956). A simple game (N, v) is *complete* if the players of N can be ordered by a complete, transitive ordering \succeq , say, $1 \succeq 2 \succeq \dots \succeq n$, indicating that higher ranked players have more "power" than lower ranked players. More precisely, $i \succeq j$ means that $v(S \cup i) \geq v(S \cup j)$ for any coalition $S \subseteq N \setminus \{i, j\}$. The class of complete simple games properly contains all weighted voting games (Freixas and Puente 2008). For complete simple games, we show an asymptotical upper bound on α , namely $\alpha = O(\sqrt{n} \ln n)$. This bound matches, up to a $\ln n$ factor, the lower bound of $\Omega(\sqrt{n})$ that Freixas and Kurz (2014) conjectured to be tight. Intuitively, complete simple games are much closer to weighted voting games than arbitrary simple games. So, from this perspective, our result seems to support the hypothesis that α is indeed a sensible measure for the distance to weighted voting games.

In Section 4 we discuss some algorithmic and complexity issues. We focus on instances where all minimal winning coalitions have size 2. We say that such simple games are *graphic*, as they can be conveniently described by a graph $G = (N, E)$ with vertex set N and edge set $E = \{ij \mid \{i, j\} \text{ is winning}\}$ (here, we denote an edge with endpoints i and j as ij). For graphic simple games we show that computing α is NP-hard in general, but polynomial-time solvable if the underlying graph $G = (N, E)$ is bipartite, or if α is known to be small (less than a fixed number α_0).

1.2 Related Work

Due to their practical applications in voting systems, computer operating systems and resource allocation (see, for example, Bilbao et al. (2002); Chalkiadakis et al. (2011)), both structural and computational aspects of weighted voting games have been thoroughly investigated (see, for example, Axenovich and Roy (2010); Carreras and Freixas (2005); Elkind et al. (2008, 2009); Fishburn and Brams (1996); Freixas et al. (2011); Gvozdeva et al. (2013); Pashkovich (2018)).

Taylor and Zwicker (1993) measured the distance of a simple game to the class of weighted voting games by its dimension instead of by α . To explain this alternative distance measure, the *intersection* of two simple games (N, v_1) and (N, v_2) with sets of winning coalitions \mathcal{W}_1 and \mathcal{W}_2 , respectively, is the simple game (N, v) with set of winning coalitions $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$. The *dimension* of a simple game (N, v) is the smallest number of weighted voting games whose intersection equals (N, v) .⁵ However, computing the dimension of a simple game given as the intersection of a number of weighted voting games is NP-hard, as shown by Deineko and Woeginger (2006). Moreover, the largest dimension of a simple game with n players is $2^{n-o(n)}$, as shown by Kurz et al. (2016), and, α may be arbitrarily large for simple games with dimension larger than 1. Hence, there is no direct relation between the two distance measures. Apart from α , Gvozdeva et al. (2013) introduced two other distance parameters. One measures the power balance between small and large coalitions. The other one allows multiple thresholds instead of threshold 1 only.

For graphic simple games, it is natural to take the number of players n as the input size for answering complexity questions, but in general simple games may have different representations. For instance, one can list all minimal winning coalitions or all maximal losing coalitions. Under these two representations the problem of deciding if $\alpha < 1$, that is, if a given simple game is a weighted voting game, is polynomial-time solvable. This follows from results by Hegedüs and Megiddo (1996) and Peled and Simeone (1985), as shown by Freixas et al. (2011). The latter paper also showed that the same result holds if the representation is given by listing all winning coalitions or all losing coalitions.

As mentioned, a crucial case in our study is when the simple game is graphic, that is, defined on some graph $G = (N, E)$. In the corresponding *matching game* a coalition $S \subseteq N$ has value $v(S)$ equal to the maximum size of a matching in the subgraph of G induced by S . One of the most prominent solution concepts is the *core* of a game, defined by $core(N, v) := \{p \in \mathbb{R}^N \mid p(N) = v(N), p(S) \geq v(S) \forall S \subseteq N\}$. Matching games are not simple games. Yet their core constraints are readily seen to simplify to $p \geq 0$ and $p_i + p_j \geq 1$ for all $ij \in E$. Classical solution concepts, such as the core and core-related ones like the least core,

⁵ A related notion, introduced by Freixas and Marciniak (2010), is that of the *codimension* of a simple game (N, v) , which is the smallest number of weighed voting games whose union equals (N, v) . Here, the *union* of two simple games (N, v_1) and (N, v_2) with sets of winning coalitions \mathcal{W}_1 and \mathcal{W}_2 , respectively, is the simple game (N, v) with set of winning coalitions $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$.

the nucleolus or the nucleon are well studied for matching games; see, for example, Biro et al. (2012); Bock et al. (2015); Faigle et al. (1998); Kern and Paulusma (2003); Könemann et al. (2019); Solymosi and Raghavan (1994).

2 The Proof of the Conjecture

To prove Conjecture 1 we reformulate, strengthen and only then verify it. Our approach is inspired by the work of Abdi, Cornuéjols and Lee on identically self-blocking clutters (Section 3 in the thesis of Abdi (2018)). A coalition $C \subseteq N$ is called a *cover* of \mathcal{W} if C has at least one common player with every coalition in \mathcal{W} . We call the collection of covers of \mathcal{W} the *blocker*⁶ of \mathcal{W} and denote it by $b(\mathcal{W})$. We claim that

$$\mathcal{L} = \{N \setminus C \mid C \in b(\mathcal{W})\}.$$

In order to see this, first suppose that there exists a cover $C \in b(\mathcal{W})$ such that $N \setminus C \notin \mathcal{L}$. As $\mathcal{L} \cup \mathcal{W} = 2^N$, this means that $N \setminus C \in \mathcal{W}$. However, as C contains no player from $N \setminus C$, this contradicts our assumption that $C \in b(\mathcal{W})$. Now suppose that there exists a losing coalition $L \in \mathcal{L}$ such that $C = N \setminus L$ does not belong to $b(\mathcal{W})$. Then, by definition, there exists a winning coalition $W \in \mathcal{W}$ with $C \cap W = \emptyset$. As $C \cap W = \emptyset$, we find that $W \subseteq N \setminus C = L$. Then, by the monotonicity property of simple games, L must be winning as well, a contradiction.

As $\mathcal{L} = \{N \setminus C \mid C \in b(\mathcal{W})\}$, the critical threshold value can be reformulated as follows:

$$\alpha = \min_{p \in Q(\mathcal{W})} \max_{L \in \mathcal{L}} p(L) = \min_{p \in Q(\mathcal{W})} \max_{C \in b(\mathcal{W})} p(N \setminus C) = \min_{p \in Q(\mathcal{W})} \max_{\substack{q \in Q(\mathcal{W}) \\ q \in \{0,1\}^N}} \langle p, \mathbf{1} - q \rangle.$$

Here, $\langle p, q \rangle$ stands for the scalar product of two vectors p and q . The last equality can be justified as follows. For every cover $C \in b(\mathcal{W})$, we can define a vector $q \in \{0,1\}^N \cap Q(\mathcal{W})$ by setting, for each $i \in N$, $q_i = 1$ if $i \in C$ and $q_i = 0$ otherwise. Similarly, for every vector $q \in \{0,1\}^N \cap Q(\mathcal{W})$ we can define a cover $C = \{i \in N \mid q_i = 1\}$. Hence, there is a 1-to-1 correspondence between the covers in $b(\mathcal{W})$ and the vectors in $\{0,1\}^N \cap Q(\mathcal{W})$.

We can now reformulate Conjecture 1 of Freixas and Kurz (2014) as follows:

Conjecture 1 (reformulated) *For a simple game (N, v) with n players and set of winning coalitions \mathcal{W} , we have*

$$\min_{p \in Q(\mathcal{W})} \max_{\substack{q \in Q(\mathcal{W}) \\ q \in \{0,1\}^N}} \langle p, \mathbf{1} - q \rangle \leq n/4.$$

In order to prove Conjecture 1 we need the following observation. Here, we write $\|p\|_2 = \sqrt{p_1^2 + \dots + p_n^2}$ for a vector $p \in \mathbb{R}^N$.

⁶ The notion of a blocker was originally defined by Edmonds and Fulkerson (1970) as the collection of minimal covers, but for simplicity of exposition, we define it as the collection of all covers.

Observation 2 Let P be a nonempty polyhedron and let p^* be the (unique⁷) optimal solution of the program $\min\{\|p\|_2 \mid p \in P\}$. Then p^* is a (not necessarily unique) optimal solution of the linear program $\min\{\langle p^*, q \rangle \mid q \in P\}$.

Proof. Assume to the contrary that there exists some $q \in P$ with $\langle p^*, q \rangle < \langle p^*, p^* \rangle$, implying that $\langle p^*, q - p^* \rangle < 0$. For any $0 < \epsilon \leq 1$, we define the point $p_\epsilon := p^* + \epsilon(q - p^*) \in P$ and note that

$$\|p_\epsilon\|_2 = \langle p_\epsilon, p_\epsilon \rangle = \langle p^*, p^* \rangle + 2\epsilon \langle p^*, q - p^* \rangle + \epsilon^2 \langle q - p^*, q - p^* \rangle.$$

Then, as $\langle p^*, q - p^* \rangle < 0$, we can take $\epsilon > 0$ to be sufficiently small in order to obtain: $\|p_\epsilon\|_2 < \langle p^*, p^* \rangle = \|p^*\|_2$, a contradiction. \square

We are now ready to prove Conjecture 1; in fact we show a slightly stronger statement.

Theorem 3 (Strengthening of Conjecture 1) For a simple game (N, v) with n players and set of winning coalitions \mathcal{W} , we have

$$\min_{p \in Q(\mathcal{W})} \max_{q \in Q(\mathcal{W})} \langle p, \mathbf{1} - q \rangle \leq n/4.$$

In particular, if p^* is the (unique) optimal solution for the program

$$\min\{\|p\|_2 \mid p \in Q(\mathcal{W})\},$$

then

$$\max_{q \in Q(\mathcal{W})} \langle p^*, \mathbf{1} - q \rangle \leq n/4.$$

Proof. Let p^* be the unique optimal solution of $\min\{\|p\|_2 \mid p \in Q(\mathcal{W})\}$. By Observation 2, p^* is an optimal solution for the program $\min\{\langle p^*, q \rangle \mid q \in Q(\mathcal{W})\}$. As $\langle p^*, \mathbf{1} - q \rangle = p^*(N) - \langle p^*, q \rangle$, we find that p^* is also an optimal solution for the program $\max\{\langle p^*, \mathbf{1} - q \rangle \mid q \in Q(\mathcal{W})\}$. Thus, we have

$$\max_{q \in Q(\mathcal{W})} \langle p^*, \mathbf{1} - q \rangle = \langle p^*, \mathbf{1} - p^* \rangle = \sum_{i=1}^n p_i^*(1 - p_i^*) \leq \sum_{i=1}^n \frac{1}{4} = \frac{n}{4},$$

finishing the proof. \square

We note that for a simple game (N, v) that has a singleton winning coalition $W = \{i\}$, we may put, without loss of generality, $p_i = 1$; doing this does not affect $\alpha = \min_{p \in Q(\mathcal{W})} \max_{L \in \mathcal{L}} p(L)$ since no losing coalition L contains i . Hence, in our context, we may assume that all winning coalitions have size at least 2, and we call simple games with this property *non-singular*.

⁷ For every nonempty polyhedron P , the program $\min\{\|p\|_2^2 \mid p \in P\}$ is a feasible convex quadratic program with a strictly convex objective function, where all values of the function are bounded from below by 0. Hence, $\min\{\|p\|_2^2 \mid p \in P\}$, and consequently $\min\{\|p\|_2 \mid p \in P\}$, has a unique optimal solution (see, for example, Borwein and Lewis (2000)).

Lemma 1 *Let (N, v) be a non-singular simple game with $\alpha = n/4$. Then $p^* = \frac{1}{2}\mathbf{1}$ is an optimal solution for $\min\{\langle p^*, q \rangle \mid q \in Q(\mathcal{W})\}$ with value $n/4$.*

Proof. We first observe that $p^* \in Q(\mathcal{W})$, as every winning coalition has size at least 2. Now, if we choose $q = p^*$, then the inequality

$$\langle p^*, \mathbf{1} - p^* \rangle = \sum_{i=1}^n p_i^*(1 - p_i^*) \leq \sum_{i=1}^n \frac{1}{4} = \frac{n}{4}$$

in the proof of Theorem 3 becomes tight, as $\langle p^*, \mathbf{1} - p^* \rangle = \frac{n}{4}$. This implies that p^* is an optimal solution for the program $\max\{\langle p^*, \mathbf{1} - q \rangle \mid q \in Q(\mathcal{W})\}$. Then, as $\langle p^*, \mathbf{1} - q \rangle = p^*(N) - \langle p^*, q \rangle$, we find that p^* is also an optimal solution for the program $\min\{\langle p^*, q \rangle \mid q \in Q(\mathcal{W})\}$, with value $\frac{n}{4}$. \square

We now discuss when Conjecture 1 provides a tight upper bound for the critical threshold value. In order to do this we first recall some standard terminology. A set X of points is *convex* if for every $x, y \in X$ and $0 \leq \tau \leq 1$, the point $\tau x + (1 - \tau)y$ belongs to X . The *convex hull* of a set Y of points is the smallest convex set that contains Y . The *characteristic vector* of a coalition $S \subseteq N$ is the vector $x \in \mathbb{R}^N$ with, for every $i \in N$, $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$.

Our next theorem shows that if the upper bound in Conjecture 1 is tight, then this fact can be certified in the same way as in Example 1. Thus, if we were to convince someone that $\alpha = n/4$ is best possible for a given instance, then we could prove this by revealing that specific vectors, namely $\frac{2}{n}\mathbf{1}$ and $\frac{1}{2}\mathbf{1}$, are in the convex hull of the set of characteristic vectors of winning and losing coalitions, respectively. By Caratheodory's Theorem (see, for example, Faigle et al. (2002)), every vector in the convex hull of some set can be written as a convex combination of at most $n + 1$ vectors from that set. Hence, giving an appropriate set of at most $2(n + 1)$ characteristic vectors would provide a succinct proof for $\alpha = n/4$.

Theorem 4 *For a non-singular simple game (N, v) with n players and sets of winning coalitions \mathcal{W} and losing coalitions \mathcal{L} , we have*

$$\alpha = \min_{p \in Q(\mathcal{W})} \max_{L \in \mathcal{L}} p(L) = n/4$$

if and only if $\frac{2}{n}\mathbf{1}$ lies in the convex hull of the set of characteristic vectors of winning coalitions and $\frac{1}{2}\mathbf{1}$ lies in the convex hull of the set of characteristic vectors of losing coalitions.

Proof. First suppose that $\frac{2}{n}\mathbf{1}$ lies in the convex hull of the characteristic vectors of winning coalitions and that $\frac{1}{2}\mathbf{1}$ lies in the convex hull of the characteristic vectors of losing coalitions. Then for every $p \in Q(\mathcal{W})$ we have

$$\max_{L \in \mathcal{L}} p(L) \geq \langle p, \frac{1}{2}\mathbf{1} \rangle = \frac{n}{4} \langle p, \frac{2}{n}\mathbf{1} \rangle \geq \frac{n}{4},$$

showing that $\alpha \geq n/4$ and hence $\alpha = n/4$ by Theorem 3.

Now suppose that $\alpha = n/4$. We first show that $\frac{2}{n}\mathbf{1}$ lies in the convex hull of the characteristic vectors of winning coalitions. From Lemma 1 we know that if $\alpha = n/4$, then $p^* = \frac{1}{2}\mathbf{1}$ is an optimal solution for $\min\{\langle p^*, q \rangle \mid q \in Q(\mathcal{W})\}$ with value $n/4$. We denote the characteristic vector of a winning coalition $W \in \mathcal{W}$ by χ_W . Then we can write the dual program as

$$\max\{\langle \mathbf{1}, y \rangle \mid \sum_{W \in \mathcal{W}} y_W \chi_W \leq p^*, y \geq 0\}.$$

Let y^* be an optimal solution of the dual program. By linear duality, the primal and dual program define the same value $\langle \mathbf{1}, y^* \rangle = \langle p^*, p^* \rangle = \frac{1}{4}n$. Moreover, as $p_i^* = \frac{1}{2} > 0$ for every $i \in N$, we find that $\sum_{W \in \mathcal{W}} y_W^* \chi_W = p^*$ by complementary slackness (see, for example, the text-book of Schrijver (1998)). Consequently, $\lambda := \frac{4}{n}y^* \geq 0$ not only satisfies $\sum_{W \in \mathcal{W}} \lambda_W = 1$ but also $\sum_{W \in \mathcal{W}} \lambda_W \chi_W = \frac{4}{n}p^* = \frac{2}{n}\mathbf{1}$. This means that $\frac{2}{n}\mathbf{1}$ is indeed a convex combination of characteristic vectors of winning coalitions.

We now show that $\frac{1}{2}\mathbf{1}$ lies in the convex hull of the characteristic vectors of losing coalitions. Let q^* be the optimal solution for the program

$$\min\{\|q\|_2 \mid q \in \text{conv}\{r \in \{0, 1\}^N \mid r \in Q(\mathcal{W})\}\},$$

where conv denotes the convex hull operator. Note that, as $Q(\mathcal{W})$ is convex, $q^* \in Q(\mathcal{W})$. Recall that $\alpha = \min_{p \in Q(\mathcal{W})} \max\{\langle p, \mathbf{1} - q \rangle \mid q \in \{0, 1\}^N \cap Q(\mathcal{W})\}$. Hence, as $q^* \in Q(\mathcal{W})$, we find that $\alpha \leq \max\{\langle q^*, \mathbf{1} - q \rangle \mid q \in \{0, 1\}^N \cap Q(\mathcal{W})\}$. In the same way as in the proof of Theorem 3, we can show that

$$\alpha \leq \max_{\substack{q \in Q(\mathcal{W}) \\ q \in \{0, 1\}^N}} \langle q^*, \mathbf{1} - q \rangle = \langle q^*, \mathbf{1} - q^* \rangle = \sum_{i=1}^n q_i^*(1 - q_i^*) \leq \frac{n}{4},$$

As $\alpha = \frac{n}{4}$, the above inequality becomes tight, meaning that $q^* = \frac{1}{2}\mathbf{1}$ and thus $\frac{1}{2}\mathbf{1}$ lies in $\text{conv}\{r \in \{0, 1\}^N \mid r \in Q(\mathcal{W})\}$, and hence $\mathbf{1} - q^* = \frac{1}{2}\mathbf{1}$ lies in the convex hull of the characteristic vectors of losing coalitions. This completes the proof. \square

3 Complete Simple Games

Recall that a simple game (N, v) is complete if the players in N can be ordered by a complete, transitive ordering \succeq such that for all $i, j \in N$ with $i \succeq j$ and every $S \subseteq N \setminus \{i, j\}$, we have $v(S \cup i) \geq v(S \cup j)$. As a consequence, in complete simple games we have $v(S) \geq v(S')$ whenever S dominates $S' = \{j_1, \dots, j_s\}$ in the sense that S contains elements i_1, \dots, i_s with $i_p \succeq j_p, p = 1, \dots, s$.

Intuitively, the class of complete simple games is “closer” to weighted voting games than general simple games. The result of this section quantifies this expectation.⁸

⁸ This result and a proof have appeared in an extended abstract published in the proceedings of SAGT 2018 (Hof et al. 2018). As outlined by one of the referees, the proof given there contained a mistake and we include a correct proof in this paper resulting in a slightly weaker upper bound.

Theorem 5 For a complete simple game (N, v) , it holds that $\alpha \leq \sqrt{n}(1 + \ln n)$.

Proof. Let $N = \{1, \dots, n\}$ be the set of players and assume without loss of generality that $1 \geq 2 \geq \dots \geq n$. Let $k \in N$ be the largest number such that $\{k, \dots, n\}$ is winning. For $i = 1, \dots, k$, let s_i denote the size of a smallest winning coalition in $\{i, \dots, n\}$. We define $p_i := 1/s_i$ for $i = 1, \dots, k$, and if $k < n$, $p_i := \frac{1}{n}$ for $i = k + 1, \dots, n$. We note that $p_1 \geq \dots \geq p_k \geq \dots \geq p_n \geq \frac{1}{n}$.

Consider a winning coalition $W \subseteq N$ and let i be the first player in W (with respect to \succeq). If $|W| \leq \sqrt{n}$, then $s_i \leq |W| \leq \sqrt{n}$ and hence $p(W) \geq p_i = \frac{1}{s_i} \geq \frac{1}{\sqrt{n}}$. On the other hand, if $|W| > \sqrt{n}$, then $p(W) > \sqrt{n} \frac{1}{n} = \frac{1}{\sqrt{n}}$. We conclude that for every winning coalition W , it holds that $p(W) \geq \frac{1}{\sqrt{n}}$.

For a losing coalition $L \subseteq N$, we observe that, for $i = 1, \dots, k$, $|L \cap \{1, \dots, i\}| \leq s_i - 1$; otherwise L would dominate the winning coalition of size s_i in $\{i, \dots, n\}$ implying $v(L) = 1$, a contradiction. Let y be the characteristic vector of L . Then $p(L \cap \{1, \dots, k\}) = \sum_{i=1}^k y_i p_i = \sum_{i=1}^k y_i \frac{1}{s_i}$ is bounded by

$$\max \sum_{i=1}^k x_i \frac{1}{s_i} \quad \text{subject to} \quad \sum_{j=1}^i x_j \leq s_i - 1, \quad \text{for } i = 1, \dots, k.$$

The optimal solution of this maximization problem is $x_1 = s_1 - 1$ and $x_i = s_i - s_{i-1}$ for $i = 2, \dots, k$ with corresponding value

$$(s_1 - 1) \frac{1}{s_1} + (s_2 - s_1) \frac{1}{s_2} + \dots + (s_k - s_{k-1}) \frac{1}{s_k} \leq \frac{1}{2} + \dots + \frac{1}{s_k} \leq \ln n,$$

where we use the facts that $(s_1 - 1) \frac{1}{s_1} \leq \frac{1}{2} + \dots + \frac{1}{s_1}$ and that for $i \geq 2$, $(s_i - s_{i-1}) \frac{1}{s_i} \leq \frac{1}{s_{i-1} + 1} + \dots + \frac{1}{s_i}$. Hence, we obtain $p(L) \leq p(L \cap \{1, \dots, k\}) + p(L \cap \{k + 1, \dots, n\}) \leq \ln n + (n - k) \frac{1}{n} \leq 1 + \ln n$.

From the above it follows that for every winning coalition W and every losing coalition L , $p(L)/p(W) \leq \sqrt{n}(1 + \ln n)$, as claimed. \square

Freixas and Kurz (2014) conjectured that $\alpha = O(\sqrt{n})$ holds for complete simple games. In the same paper they gave a lower bound of order \sqrt{n} , as well as some specific subclasses of complete simple games for which $\alpha = O(\sqrt{n})$ can be proven.

4 Algorithmic Aspects

A fundamental question concerns the complexity of our original problem (1), that is, the complexity of computing the critical threshold value of a simple game. For general simple games this depends on how the game in question is given, and we refer to Section 1.2 for a discussion.

In this section, we concentrate on the “graphic” case and present three results. These results and their proofs have appeared in an extended abstract published

in the proceedings of SAGT 2018 (Hof et al. 2018). In order to keep our paper self-contained we also include the proofs of these results.

For a graphic simple game (N, v) defined on a graph $G = (N, E)$, we write $\alpha_G = \alpha(N, v)$.

Proposition 1. *For a bipartite graph G , the quantity α_G can be computed in polynomial time.*

Proof. Let $P \subseteq \mathbb{R}^N$ be the set of feasible payoffs (satisfying $p \geq 0$ and $p_i + p_j \geq 1$ for $ij \in E$). For $\alpha \in \mathbb{R}$, let $P_\alpha := \{p \in P \mid p(L) \leq \alpha \text{ for all independent } L \subseteq N\}$. Thus $\alpha_G = \min\{\alpha \mid P_\alpha \neq \emptyset\}$. The separation problem for P_α (for any given α) is efficiently solvable. Given $p \in \mathbb{R}^N$, we can check feasibility and whether $\max\{p(L) \mid L \subseteq N \text{ independent}\} \leq \alpha$ by solving a corresponding maximum weight independent set problem in the bipartite graph G . Thus we can, for any given $\alpha \in \mathbb{R}$, apply the ellipsoid method to either compute some $p \in P_\alpha$ or conclude that $P_\alpha = \emptyset$. Binary search then exhibits the minimum value for which P_α is non-empty; binary search works indeed in polynomial time as the optimal α has size polynomially bounded in n , which follows from observing that

$$\alpha = \min\{a \mid p_i + p_j \geq 1 \ \forall ij \in E, p(L) - a \leq 0 \ \forall L \subseteq N \text{ independent}, p \geq 0\} \quad (2)$$

can be computed by solving a linear system of n constraints defining an optimal basic solution of the above linear program. \square

The proof of Proposition 1 also applies to other classes of graphs, for which finding a maximum weight independent set is polynomial-time solvable, such as the class of claw-free graphs, as shown by Brandstädt and Mosca (2018). In general, the problem of computing a maximum independent set in a graph is NP-hard (see Garey and Johnson (1979)).

Proposition 2. *Computing α_G for arbitrary graphs G is NP-hard.*

Proof. Let $G' = (N', E')$ and $G'' = (N'', E'')$ be two disjoint copies of a graph $G = (N, E)$ with independence number k . For each $i' \in N'$ and $j'' \in N''$ add an edge $i'j''$ if and only if $i = j$ or $ij \in E$ and call the resulting graph $G^* = (N^*, E^*)$. We claim that $\alpha_{G^*} = k/2$ (thus computing α_{G^*} is as difficult as computing k).

First note that the independent sets in G^* are exactly the sets $L^* \subseteq N^*$ that arise from an independent set $L \subseteq N$ in G by splitting L into two complementary sets L_1 and L_2 and defining $L^* := L'_1 \cup L''_2$. Hence, $p = \frac{1}{2}$ on N^* yields $\max p(L^*) = k/2$ where the maximum is taken over all independent sets $L^* \subseteq N^*$ in G^* . This shows that $\alpha_{G^*} \leq k/2$.

Conversely, let p^* be any feasible payoff in G^* , that is, $p^* \geq 0$ and $p^*_i + p^*_j \geq 1$ for all $ij \in E^*$. Let $L \subseteq N$ be a maximum independent set of size k in G and construct L^* by including for each $i \in L$ either i' or i'' in L^* , whichever has p -value at least $\frac{1}{2}$. Then, by construction, L^* is an independent set in G^* with $p^*(L^*) \geq k/2$, showing that $\alpha_{G^*} \geq k/2$. \square

Summarizing, for graphic simple games, computing α_G is as least as hard as computing the size of a maximum independent in G .

For our last result we assume that α_0 is a fixed number, that is, α_0 is not part of the input.

Proposition 3. *For every fixed $\alpha_0 > 0$, it is possible to decide if $\alpha_G \leq \alpha_0$ in polynomial time for an arbitrary graph $G = (N, E)$.*

Proof. Let $k = 2(\lfloor \alpha_0 \rfloor + 1)$. By brute-force, we can check in $O(n^{2k})$ time if N contains $2k$ vertices $\{u_1, \dots, u_k\} \cup \{v_1, \dots, v_k\}$ that induce k disjoint copies of P_2 , that is, paths $P_i = u_i v_i$ of length 2 for $i = 1, \dots, k$ with no edges joining any two of these paths. If so, then the condition $p(u_i) + p(v_i) \geq 1$ implies that one of u_i, v_i , say u_i , must receive a payoff $p(u_i) \geq \frac{1}{2}$, and hence $U = \{u_1, \dots, u_k\}$ has $p(U) \geq k/2 > \alpha_0$. As U is an independent set, $\alpha(G) > \alpha_0$.

Now assume that G does not contain k disjoint copies of P_2 as an induced subgraph, that is, G is kP_2 -free. For every $s \geq 1$, the number of maximal independent sets in a sP_2 -free graphs is $n^{O(s)}$ due to a result of Balas and Yu (1989). Tsukiyama et al. (1977) show how to enumerate all maximal independent sets of a graph G on n vertices and m edges using time $O(nm)$ per independent set. Hence we can find all maximal independent sets of G and thus solve, in polynomial time, the linear program (2) in the proof of Proposition 1. Then it remains to check if the solution found satisfies $\alpha \leq \alpha_0$. \square

Note that Proposition 3 immediately implies that the problem of deciding if a graphic simple game (N, v) , given as a graph $G = (N, E)$, is a weighted voting game is polynomial-time solvable.

5 Conclusions

We have strengthened and proven the conjecture of Freixas and Kurz (2014) on simple games (Conjecture 1) and showed a number of computational complexity results for graphic simple games. Moreover, we considered complete simple games and proved a stronger upper bound for this class of games. It remains to tighten the upper bound for complete simple games to $O(\sqrt{n})$ if possible. In order to classify simple games, many more subclasses of simple games have been identified in the literature. Besides the two open problems, no optimal bounds for α are known for other subclasses of simple games, such as *strong*, *proper*, or *constant-sum* games, that is, where $v(S) + v(N \setminus S) \geq 1$, $v(S) + v(N \setminus S) \leq 1$, or $v(S) + v(N \setminus S) = 1$ for all $S \subseteq N$, respectively.

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