SIAM J. APPL. MATH.
Vol. 80, No. 5, pp. 2194–2225

EIGENVALUES OF MAGNETOHYDRODYNAMIC MEAN-FIELD DYNAMO MODELS: BOUNDS AND RELIABLE COMPUTATION*

SABINE BOEGLI† AND CHRISTIANE TRETTER‡

Abstract. This paper provides the first comprehensive study of the linear stability of three important magnetohydrodynamic (MHD) mean-field dynamo models in astrophysics, the spherically symmetric α^2-model, the $\alpha^2\omega$-model, and the $\alpha\omega$-model. For each of these highly nonnormal problems, we establish upper bounds for the real part of the spectrum, prove resolvent estimates, and derive thresholds for the helical turbulence function α and the rotational shear function ω below which no MHD dynamo action can occur for the linear models (antidynamo or bounding theorems). In addition, we prove that interval truncation and finite section method, which are employed to regularize the singular differential expressions and the infinite number of coupled equations, are spectrally exact. This means that all spectral points are approximated and no spectral pollution occurs, thus confirming, for the first time, that numerical eigenvalue approximations for the highly nonnormal MHD dynamo problems are reliable.

Key words. eigenvalues, approximation, magnetohydrodynamics, dynamo problem, spectral convergence, resolvent bounds

AMS subject classifications. 35Q86, 47A10, 65N25, 76W05

DOI. 10.1137/19M1286359

1. Introduction.

The existence of magnetic fields of astrophysical bodies like planets, stars and galaxies gives rise to linear stability problems in magnetohydrodynamic (MHD) dynamo theory. They are governed by induction caused by motions of the electrically conducting fluid in the interior. It was only in 1999 that the first successful dynamo action was observed experimentally in two large scale liquid sodium facilities in Riga, Latvia, and Karlsruhe, Germany [7, 18, 22]. Numerical computations were performed by physicists under various (symmetry) assumptions on the magnetic and the velocity field and using necessary simplifications to enable a computer to tackle the problem numerically. The corresponding eigenvalue problems are all highly nonnormal, and hence numerical eigenvalue approximations are prone to spectral pollution and failure of spectral approximation. However, up to now, there has not been *any* mathematical justification guaranteeing that the numerical computations in the physics literature reflect the true spectra of dynamos. The only existing analytical results are eigenvalue estimates confined to the simplest case, the axisymmetric α^2-model with isotropic α-profile [9].

We do not only aim at filling this gap for a variety of dynamo models and under weak assumptions, e.g., *not* assuming axisymmetry of the magnetic field or small deviation from it [11]. In fact, our approach and results are twofold. On the analytical side, we establish eigenvalue bounds for three important dynamo models in astrophysics [17], the spherically symmetric α^2-model, the $\alpha\omega$-model, and the $\alpha^2\omega$-model.

*Received by the editors September 10, 2019; accepted for publication (in revised form) May 20, 2020; published electronically September 30, 2020.
https://doi.org/10.1137/19M1286359

Funding: This work was supported by the Swiss National Science Foundation through grant 200020. The work of the first author was supported by an Early Postdoctoral Mobility grant P2BEP2, and by the Imperial College London through a Chapman Fellowship. The work of the second author was supported by an Early Postdoctoral Mobility grant 169104.

†Department of Mathematical Sciences, Durham University, Durham DH1 3LE, UK (sabine.boegli@durham.ac.uk).

‡Mathematisches Institut, Universität Bern, 3012 Bern, Switzerland (tretter@math.unibe.ch).

Here the first group of our main results are so-called *antidynamo theorems* or *bounding theorems*; see [5] (Theorems 4.2, 4.7, and 4.10). For each of the three models, we derive thresholds for the helical turbulence function α, the rotational shear function ω, and their derivatives α', ω' below which all eigenvalues lie in the linearly stable half-plane ($\mathrm{Re}(\lambda) < 0$) and hence *no* dynamo action can occur for the corresponding linearized models. On the numerical side, for each of the three models, we prove that eigenvalue approximations for the corresponding singular dynamo operators are *spectrally exact*. This is ensured by the second group of our main results (Theorems 6.5 and 6.7), where we show that, upon both regularization by interval truncation and truncation to finite operator matrices (finite section method), all eigenvalues are approximated and no spurious eigenvalues (spectral pollution) occur, thus providing the first reliable eigenvalue approximations for dynamo problems.

The three dynamo models, and their names, have the following origin. Decomposing the magnetic field into poloïdal and toroïdal components, the mean-field induction equation modelling the kinematic mean-field dynamo becomes a coupled system of two coupled partial differential equations. In the three models distinguished in physics, the α^2*-model,* $\alpha\omega$*-model* and $\alpha^2\omega$*-model*, α is a helical turbulence function, ω a shear function representing differential rotation, and the name of the model suggests which effects are taken into account in these two differential equations. Here one α indicates that the α-effect is assumed in the first differential equation, while the rest of the name reflects the effects assumed in the second differential equation, i.e., for the α^2-model only the α-effect, for the $\alpha\omega$-model only the ω-effect, and for the $\alpha^2\omega$-model both the α- and ω-effect. Important examples include the solar magnetic field cycle which is modelled by an $\alpha\omega$-model and geodynamo models which are essentially based on an α^2-mechanism since the differential rotation in the Earth's fluid core is assumed to be weak; see, e.g., [8].

In real cosmic bodies the functions α and ω have a complicated spatial structure. A widely used simpler model, which we also consider here, is the spherically symmetric case with purely radial dependence of α and ω. Expanding the poloïdal and toroïdal components further in spherical harmonics Y_l^m, the dynamo problem turns into a time-independent eigenvalue problem for an *infinite* operator matrix. For the α^2-model, the corresponding dynamo operator decouples into infinitely many pairs of coupled differential equations given by the operator

$$\mathcal{A}^m_{\alpha^2} = \mathrm{diag}\big(\mathcal{A}_{\alpha^2,l} : l \geq k_m\big) - \mathrm{i}m\omega, \quad \mathcal{A}_{\alpha^2,l} = \begin{pmatrix} \partial_r^2 - \frac{l(l+1)}{r^2} & \alpha \\ -\partial_r \alpha \partial_r - \alpha \frac{l(l+1)}{r^2} & \partial_r^2 - \frac{l(l+1)}{r^2} \end{pmatrix},$$

with Robin and Dirichlet boundary conditions $y_1'(1) + l y_1(1) = 0$, $y_2(1) = 0$ imposed for $\mathcal{A}_{\alpha^2,l}$ and with $k_m = \max\{|m|, 1\}$. For the more involved $\alpha\omega$-model and $\alpha^2\omega$-model, the system of infinitely many differential equations remains coupled and the corresponding dynamo operators are infinite tridiagonal operator matrices, e.g., for the $\alpha\omega$-model and $m \neq 0$ given by

$$\mathcal{A}^m_{\alpha^2\omega} := \mathcal{A}^m_{\alpha^2} + \omega' \, \mathcal{C}^m, \quad \mathcal{C}^m := \begin{pmatrix} 0 & -\widetilde{\mathcal{C}}_{k_m+1,m} & 0 & \cdots & \\ \mathcal{C}_{k_m,m} & 0 & -\widetilde{\mathcal{C}}_{k_m+2,m} & 0 & \cdots \\ 0 & \mathcal{C}_{k_m+1,m} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & & \end{pmatrix},$$

where $\mathcal{A}^m_{\alpha^2}$ is the block diagonal operator matrix in the α^2-model above and $\mathcal{C}^m$ is a constant, but unbounded tridiagonal infinite operator matrix whose off-diagonal entries $\mathcal{C}_{l,m}$, $\widetilde{\mathcal{C}}_{l,m}$ are multiplication operators by constant matrices in $L^2(0,1) \oplus L^2(0,1)$

whose norms tend to ∞, $\|\mathcal{C}_{l,m}\|$, $\|\widetilde{\mathcal{C}}_{l,m}\| \to \infty$ as $l \to \infty$. The $\alpha\omega$-model has a similar structure as the $\alpha^2\omega$-model, with the operator $\mathcal{A}_{\alpha^2,l}$ replaced by $\mathcal{A}_{\alpha,l}$, which is obtained from the former by setting the left lower entry equal to 0.

The sparsity or nonexistence of analytical and reliable numerical results for the three dynamo models is due to the multiple problems that they pose. From the analytical point of view, already for the decoupled α^2-model the entries $\mathcal{A}_{\alpha^2,l}$ on the diagonal are highly nonnormal operators and their entries are singular differential expressions at the endpoint $r = 0$. For the $\alpha\omega$-model and $\alpha^2\omega$-model, not only the entries $\mathcal{A}_{\alpha,l}$ and $\mathcal{A}_{\alpha^2,l}$ on the diagonal are highly nonnormal and singular, but the infinite coupling matrix $\mathcal{C}^m$ is both unbounded and also highly nonnormal since $\mathcal{C}_{l,m}$, $\widetilde{\mathcal{C}}_{l,m}$ are far from being adjoint to each other. From the computational point of view, two steps may be needed to obtain numerical eigenvalue approximations, the first one is interval truncation at the singular endpoint $r = 0$ and the second one is truncation of the infinite operator matrices for the $\alpha\omega$-model and $\alpha^2\omega$-model.

There do not exist any abstract perturbation theorems that could be applied here and there are no results guaranteeing that numerical eigenvalue approximations for these nonnormal problems *do* approximate *all* eigenvalues and *do not* produce spurious eigenvalues. The present paper provides a series of results addressing all of these problems, starting with new spectral bounds, resolvent estimates and antidynamo theorems, general perturbation results to establish generalized strong resolvent convergence and discrete compactness of the resolvents, and, finally, the first results on the spectral exactness of interval truncation and finite section method for the three dynamo models.

This paper is organized as follows. In section 2 we present the physical background and, for each of the three mean-field dynamo models, the system of coupled linear differential equations modelling it. In section 3 we establish eigenvalue estimates for the corresponding dynamo operators in an infinite product of L^2-spaces which allows us to write this system of differential equations equivalently as an eigenvalue problem for the operators $\mathcal{A}^m_{\alpha^2}$, $\mathcal{A}^m_{\alpha\omega}$, and $\mathcal{A}^m_{\alpha^2\omega}$ and we prove antidynamo theorems for all three models; see Theorems 4.2, 4.7, and 4.10. In section 6, we show that the regularization process via interval truncation and the truncation to finite operator matrices (finite section method) are spectrally exact. This implies that the numerical eigenvalue approximations computed and illustrated in section 7 for the α^2- and $\alpha^2\omega$-model for different functions α and ω do indeed reflect the true spectra of the dynamos.

We use the following notation. The norm and scalar product of a Hilbert space H are denoted by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$, respectively; if no confusion may arise, we sometimes omit the subscript H. The convergence in H, i.e., $\|x_n - x\|_H \to 0$, is written as $x_n \to x$. For Hilbert spaces H_i, $i \in \mathbb{N}$, define

$$\mathcal{H} = l^2(H_i : i \in \mathbb{N}) := \left\{ (x_i)_{i\in\mathbb{N}} : x_i \in H_i, \sum_{i\in\mathbb{N}} \|x_i\|^2_{H_i} < \infty \right\}, \ \|(x_i)_{i\in\mathbb{N}}\|_{\mathcal{H}} := \left(\sum_{i\in\mathbb{N}} \|x_i\|^2_{H_i} \right)^{\frac{1}{2}};$$

here $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The domain, spectrum, resolvent set and numerical range of a linear operator T in a Hilbert space are denoted by $\mathcal{D}(T)$, $\sigma(T)$, $\varrho(T)$, and $W(T)$, respectively. For bounded linear operators we write $T_n \xrightarrow{s} T$ for strong operator convergence. Identity operators are denoted by I; scalar multiples λI are written as λ. Analogously, the operator of multiplication with a function w in some L^2-space is again denoted by w; if $w \in L^\infty$, then its operator norm is $\|w\| = \|w\|_\infty$.

Throughout the paper, vectors in $\mathbb{R}^3$ and vector-valued functions (vector fields) are set in boldface. We consider all vectors and vector fields in spherical coordinates, with r denoting the radial component, θ the inclination angle, and φ the azimuthal angle. The vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ are the local orthogonal unit vectors in the directions of increasing r, θ, φ, respectively. The partial derivatives are written as ∂_t, ∂_r, ∂_θ, ∂_φ.

2. Physical background of MHD mean-field dynamo models. The motion of an electrically conducting fluid (plasma or liquid metal) in a magnetic field induces an electric current which, in turn, induces a magnetic field. If the inducing and the induced magnetic fields coincide, this effect is termed a self-excited dynamo.

The starting point for the mean-field dynamo equations in the α^2-model, the $\alpha\omega$-model, and the $\alpha^2\omega$-model is the magnetic induction equation, which is obtained from pre-Maxwell's equations and Ohm's law; see [14, Chaps. 11–15]. The velocity field $\mathbf{v}$ is assumed to be steady, i.e., we neglect the back-reaction of the self-excited magnetic field on the flow. Then the ansatz

$$\mathbf{B}(\mathbf{r},t) = \mathrm{e}^{\lambda t}\,\mathbf{b}(\mathbf{r}), \quad t \in [0,\infty),$$

turns the dynamo problem into a time-independent eigenvalue problem for the eigenvalue parameter λ which is, in general, complex.

In order to model highly turbulent flows, the velocity field $\mathbf{v}$ and the magnetic field $\mathbf{B}$ are written as superpositions of mean and fluctuating parts,

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad \mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'.$$

Considering the magnetic field inside a sphere $B_R(0)$ of radius R filled with an electrically conducting fluid, the mean part of the induction equation becomes

$$\partial_t \overline{\mathbf{B}} = \nabla \times (\overline{\mathbf{v}} \times \overline{\mathbf{B}} + \overline{\mathbf{v}' \times \mathbf{B}'}) + \frac{1}{\mu_0 \sigma}\Delta \overline{\mathbf{B}} \quad \text{on} \quad B_R(0) \times [0,\infty).$$

Here, σ is the electrical conductivity of the fluid (assumed to be constant in $B_R(0)$ and zero in the exterior), and μ_0 is the permeability of the vacuum; the case of an ideally conducting fluid corresponds to $\sigma = \infty$; see [6].

The following two assumptions are often used in dynamo theory. (i) The *turbulent electromotive force* $\mathcal{E} := \overline{\mathbf{v}' \times \mathbf{B}'}$ is supposed to be of the form

$$\mathcal{E} = \alpha\,\overline{\mathbf{B}} - \beta\,\nabla \times \overline{\mathbf{B}}, \tag{2.1}$$

where the *helical turbulence function* α and the *turbulent magnetic diffusivity* β are scalar functions. We assume that α depends only on the radius $\alpha : [0,R] \to \mathbb{R}$, $r \mapsto \alpha(r)$, is differentiable, and $\beta \in \mathbb{R}$ is constant. (ii) The mean velocity field is assumed to have the form of a rotation

$$\overline{\mathbf{v}}(\mathbf{r}) := \omega(|\mathbf{r}|)\,\mathbf{e}_\omega \times \mathbf{r} = \omega(|\mathbf{r}|)|\mathbf{r}|\sin\theta\,\mathbf{e}_\varphi \tag{2.2}$$

with a differentiable function $\omega : [0,R] \to \mathbb{R}$, $r \mapsto \omega(r)$, called *rotational shear function*, and $\mathbf{e}_\omega$ denoting the unit vector in the direction of the rotation axis.

From now on, rescaling $\mathbf{r}$ and t, we assume that $R=1$ and $\eta := \beta + \frac{1}{\mu_0\sigma} = 1$ and all involved vector fields are mean fields; we omit the averaging symbol in the following.

If the toroïdal and poloïdal parts of the magnetic field are separated,

$$\mathbf{B} = \mathbf{B}_t + \mathbf{B}_p = -\mathbf{r} \times \nabla T - \nabla \times \mathbf{r} \times \nabla S$$

with scalar-valued functions S, $T : B_1(0) \times [0,\infty) \to \mathbb{R}$, then, under the normalization conditions

$$\int_0^{\pi}\int_0^{2\pi} S(r,\theta,\varphi;t)\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = 0, \quad \int_0^{\pi}\int_0^{2\pi} T(r,\theta,\varphi;t)\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta \quad = 0, \ r \in [0,1], \tag{2.3}$$

the functions S, T are uniquely determined, see [14, sects. 13.3 and 13.4], and satisfy the following system of coupled partial differential equations on $B_1(0) \times [0,\infty)$:

$$\partial_t \begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} \Delta & \alpha \\ -\alpha\Delta - \frac{1}{r}\alpha'\partial_r r + \omega' \sin\theta\partial_\theta & \Delta \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} - \omega\partial_\varphi \begin{pmatrix} S \\ T \end{pmatrix}. \tag{2.4}$$

Note that in models with axisymmetric fields the induction effect of ω is only due to its radial derivative, as it occurs, e.g., in the differentially rotating layers of the Sun.

Classification of the various MHD dynamo models. The acronyms of the different models can be read from (2.4). The so-called *α-effect* creates poloïdal magnetic field from toroïdal field; see the upper right matrix entry α. There are two effects that create toroïdal magnetic field from poloïdal field; see the lower left matrix entry: the *α-effect* (represented by $-\alpha\Delta - \frac{1}{r}\alpha'\partial_r r$) and the *$\omega$-effect* (represented by $\omega' \sin\theta\partial_\theta$).

The first α in the acronym refers to the α-effect in the upper right matrix entry. The remaining parts of the name indicate which effects dominate in the lower left matrix entry: in the *α^2-model* it is assumed that the α-effect dominates and the term $\omega' \sin\theta\partial_\theta$ is neglected; in the *$\alpha\omega$-model* the ω-effect is considered to be stronger and the term $-\alpha\Delta - \frac{1}{r}\alpha'\partial_r r$ is neglected; in the *$\alpha^2\omega$-model* both effects are kept in the lower left matrix entry.

Finally, in the *axisymmetric case* it is assumed that $\partial_\varphi T = 0$, $\partial_\varphi S = 0$; then the last term $-\omega\partial_\varphi (S,T)^t$ in (2.4) vanishes.

In a next step, the functions S and T are expanded in spherical harmonics Y_l^m, $l \in \mathbb{N}_0$, $m = -l, \dots, l$, or, equivalently, $m \in \mathbb{Z}$, $l \geq |m|$, as

$$\begin{aligned} S(r,\theta,\varphi;t) &= \sum_{m=-\infty}^{\infty} \sum_{l=|m|}^{\infty} \frac{x_{l,m}(r)}{r} Y_l^m(\theta,\varphi) \mathrm{e}^{\lambda_{l,m} t}, \\ T(r,\theta,\varphi;t) &= \sum_{m=-\infty}^{\infty} \sum_{l=|m|}^{\infty} \frac{y_{l,m}(r)}{r} Y_l^m(\theta,\varphi) \mathrm{e}^{\lambda_{l,m} t} \end{aligned} \tag{2.5}$$

for $r \in [0,1]$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$, where

$$Y_l^m(\theta,\varphi) := N_l^m P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\varphi}, \quad N_l^m := \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}, \tag{2.6}$$

and $P_l^m \colon [-1,1] \to \mathbb{R}$ are the associated Legendre polynomials; see, e.g., [1, Chap. 8]).

Remark 2.1. (i) Since $Y_0^0 \equiv 1/\sqrt{4\pi}$ is constant, the normalization conditions (2.3) imply that, in (2.5),

$$x_{0,0} = y_{0,0} = 0.$$

(ii) Since Y_l^m, $l \geq |m|$, depends on φ for $m \in \mathbb{Z}\backslash\{0\}$, in the axisymmetric case we have

$$\forall\, m \in \mathbb{Z}\backslash\{0\} : \quad x_{l,m} = y_{l,m} = 0, \quad l \geq |m|.$$

In the following we use the physical boundary conditions for S and T to derive the boundary conditions for the coefficients $x_{l,m}$, $y_{l,m}$, $m \in \mathbb{Z}$, $l \geq |m|$, in the expansions (2.5). To this end, we extend S, T to $\mathbb{R}^3 \times [0, \infty)$, and thus $x_{l,m}$, $y_{l,m}$ to $[0, \infty)$.

Toroïdal magnetic field only exists in material, hence at $r = 1$ the boundary condition for $\mathbf{B}_t$ is $T|_{r=1} = 0$, and hence $y_{l,m}(1) = 0$, $m \in \mathbb{Z}$, $l \geq |m|$. The poloïdal magnetic field $\mathbf{B}_p$ is given by a gradient field outside the considered sphere $\overline{B_1(0)}$ and thus satisfies $\Delta S|_{r>1} = 0$; moreover, the magnetic field needs to decay sufficiently fast in the limit $r \to \infty$. This amounts to $x_{l,m}(r) = b_{l,m} r^{-l}$, $r > 1$, for some constant $b_{l,m} \in \mathbb{R}$, and hence, since the function $x_{l,m}$is continuously differentiable at $r = 1$, the boundary condition becomes $x'_{l,m}(1) + l x_{l,m}(1) = 0$, $m \in \mathbb{Z}$, $l \geq |m|$.

Thus, via the expansions (2.5) of S, T in spherical harmonics, the system of partial differential equations (2.4) for S, T is equivalent to a system of infinitely many ordinary differential equations for $x_{l,m}$, $y_{l,m}$, $m \in \mathbb{Z}$, $l \geq |m|$, that only decouples in m but *not* necessarily in l. The coupling constants are as follows.

NOTATION 2.2. *For $l \in \mathbb{N}$, $m \in \mathbb{Z}$, we define*

$$c_{l,m} := \frac{N_l^m}{N_{l+1}^m} \frac{l(l-m+1)}{2l+1} = l \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}, \qquad l \geq \max\{|m|, 1\},$$

$$\widetilde{c}_{l,m} := \frac{N_l^m}{N_{l-1}^m} \frac{(l+1)(l+m)}{2l+1} = (l+1) \sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}, \quad l \geq \max\{|m|, 1\}+1.$$

THEOREM 2.3. *The functions S, T on $[0,1] \times [0,\pi] \times [0,2\pi] \times [0,\infty)$ are solutions of* (2.4) *satisfying the physical boundary conditions if and only if for every $m \in \mathbb{Z}$, the parameter $\lambda_{l,m}$ is independent of l,*

$$\lambda_{l,m} = \lambda_m, \quad l \geq \max\{|m|, 1\} =: k_m,$$

and the functions $x_{l,m}$, $y_{l,m}$, $l \geq k_m$, on $[0,1]$ satisfy the infinite system of differential equations

$$\begin{aligned} (\lambda_m + \mathrm{i}m\omega)\, x_{l,m} &= \left(x''_{l,m} - \frac{l(l+1)}{r^2} x_{l,m} \right) + \alpha\, y_{l,m}, && l \geq k_m, \\ (\lambda_m + \mathrm{i}m\omega)\, y_{l,m} &= \left(y''_{l,m} - \frac{l(l+1)}{r^2} y_{l,m} \right) - \alpha \left(x''_{l,m} - \frac{l(l+1)}{r^2} x_{l,m} \right) \\ &\quad - \alpha' x'_{l,m} + \omega' \left(c_{l-1,m} x_{l-1,m} - \widetilde{c}_{l+1,m} x_{l+1,m} \right), && l \geq k_m, \end{aligned} \tag{2.7}$$

with boundary conditions

$$x'_{l,m}(1) + l x_{l,m}(1) = 0, \quad y_{l,m}(1) = 0, \quad l \geq k_m. \tag{2.8}$$

Remark 2.4. (i) Note that $x_{l,m}$, $y_{l,m}$, $m \in \mathbb{Z}$, $l \geq |m|$ may be complex-valued. However, since α is real-valued and $c_{l,m} = c_{l,-m}$, $\widetilde{c}_{l,m} = \widetilde{c}_{l,-m}$, we have

$$\lambda_m = \overline{\lambda_{-m}}, \quad x_{l,m} = \overline{x_{l,-m}}, \quad y_{l,m} = \overline{y_{l,-m}}, \quad m \in \mathbb{Z}, \quad l \geq |m|,$$

which, together with $Y_{l,m} = \overline{Y_{l,-m}}$, ensures that S, T are real-valued.

(ii) In [16] the expansions in spherical harmonics for various ansätze of mean velocity fields are calculated. The result for a rotation, see (2.2), agrees with the system (2.7); note that in [16] the spherical harmonics are defined without the coefficient N_l^m, cf. (2.6).

The problem (2.7), (2.8) is an eigenvalue problem for an infinite operator matrix in a suitable product Hilbert space. More precisely, $x_{l,m}$, $y_{l,m}$, $l \geq k_m$, are solutions of (2.7), (2.8) if and only if $f_m := \big((x_{l,m}, y_{l,m})^t\big)_{l\geq k_m}$ is a solution of the eigenvalue problem

$$\mathcal{A}_\chi^m f_m = \lambda_m f_m, \quad m \in \mathbb{Z},$$

where χ stands for one of the acronyms α^2, $\alpha\omega$, and $\alpha^2\omega$ denoting the different dynamo models. The corresponding spaces and operators will be introduced in the next section; see Definition 3.3.

The three different dynamo models are obtained from (2.7) as follows. In the second differential equation in (2.7), for the

(i) α^2-*model*, the α-effect is assumed to dominate and the term with ω' is neglected;

(ii) $\alpha\omega$-*model*, the ω-effect is assumed to dominate and the terms with with α, α' are neglected;

(iii) $\alpha^2\omega$-*model*, both effects are kept and no term is neglected.

Remark 2.5. In the literature, "α^2-model" usually refers to the axisymmetric case ($m = 0$) which is described by the dynamo matrix $\mathcal{A}^0_{\alpha^2} = \mathcal{A}_{\alpha^2}$ in Definition 3.3 below.

3. The $\boldsymbol{\alpha^2}$-, $\boldsymbol{\alpha\omega}$-, and $\boldsymbol{\alpha^2\omega}$-MHD dynamo operators. In this section we set up the operator theoretic framework for the different dynamo models and, simultaneously, for the corresponding regularized operators at the singular endpoint 0. The latter will be needed later to show that interval truncation of $(0, 1]$ to $[a_n, 1]$ with $a_n > 0$, $a_n \to 0$, $n \to \infty$, is spectrally exact; see section 6.

In what follows, to ease notation, we will use the same symbols for multiplication operators and first order derivative as operators on $L^2(0, 1)$ and on $L^2(a, 1)$, and we only use different notation for all operators involving Bessel and Bessel type differential expressions.

Definition 3.1. *Let $\alpha : [0, 1] \to \mathbb{R}$ be differentiable with $\alpha' \in L^\infty(0, 1)$. Denote the Bessel and Bessel type differential expressions τ_l, $\tau_{l,\alpha}$, $l \in \mathbb{N}$,*

$$\begin{aligned}(\tau_l x)(r) &:= -\partial_r^2 x(r) + \frac{l(l+1)}{r^2}x(r),\\ (\tau_{l,\alpha} x)(r) &:= -\partial_r \alpha \partial_r x(r) + \alpha\frac{l(l+1)}{r^2}x(r),\end{aligned} \qquad r \in (0, 1]. \tag{3.1}$$

For $l \in \mathbb{N}$, $\vartheta \in \{l, \infty\}$ and $a \in [0, 1)$ we define the Bessel operator $A_l(a, \vartheta)$ and Bessel type differential operator $A_{l,\alpha}(a)$ in $L^2(a, 1)$ as the realizations of τ_l, $\tau_{l,\alpha}$, respectively, with domain

$$\mathcal{D}(A_l(a,\vartheta)) := \left\{ x \in L^2(a,1) : \begin{array}{l} x,\, x' \in \mathrm{AC}_{\mathrm{loc}}((a,1]),\ \tau_l x \in L^2(a,1)\,, \\ \lim_{r\searrow a} x(r) = 0,\ x'(1) + \vartheta x(1) = 0 \end{array} \right\},$$

$$A_l(a,\vartheta)x := \tau_l x, \quad x \in \mathcal{D}(A_l(a,\vartheta))$$

and

$$\begin{aligned}&\mathcal{D}(A_{l,\alpha}(a,\vartheta)) := \mathcal{D}(A_l(a,\vartheta)),\\ &\quad A_{l,\alpha}(a,\vartheta)x := \tau_{l,\alpha}x = \alpha A_l(a,l) - \alpha'\mathrm{D}_r, \quad x \in \mathcal{D}(A_{l,\alpha}(a,\vartheta));\end{aligned}$$

here α, α' denote the (bounded) multiplication operators and D_r the operator of differentiation in $L^2(a, 1)$,

$$\mathrm{D}_r x := \partial_r x, \quad \mathcal{D}(\mathrm{D}_r) := W^{1,2}(a, 1).$$

Remark 3.2. (i) If $\vartheta = \infty$ the boundary condition $x'(1) + \vartheta x(1) = 0$ in the definition of $A_l(a, \vartheta)$ is interpreted as $x(1) = 0$. Note that one can also consider $\vartheta \in [l, \infty]$ as a homotopy parameter; see [13].

(ii) The differential expression τ_l is in limit point case at $r = 0$; see, e.g., [2, App. II, sect. 9.IV]. Hence, for $a = 0$ the boundary condition $\lim_{r \searrow 0} x(r) = 0$ is already implied by $\tau_l x \in L^2(0, 1)$.

DEFINITION 3.3. *Let $\alpha : [0, 1] \to \mathbb{R}$ be differentiable with $\alpha' \in L^\infty(0, 1)$. For $l \in \mathbb{N}$ and $a \in [0, 1)$ we define the operator matrices $\mathcal{A}_{\alpha^2,l}(a)$, $\mathcal{A}_{\alpha,l}(a)$ in $L^2(a, 1) \oplus L^2(a, 1)$ by*

$$\mathcal{A}_{\alpha^2,l}(a) := \begin{pmatrix} \partial_r^2 - \frac{l(l+1)}{r^2} & \alpha \\ -\partial_r \alpha \partial_r - \alpha \frac{l(l+1)}{r^2} & \partial_r^2 - \frac{l(l+1)}{r^2} \end{pmatrix} = \begin{pmatrix} -A_l(a, l) & \alpha \\ A_{l,\alpha}(a, l) & -A_l(a, \infty) \end{pmatrix},$$

$$\mathcal{A}_{\alpha,l}(a) := \begin{pmatrix} \partial_r^2 - \frac{l(l+1)}{r^2} & \alpha \\ 0 & \partial_r^2 - \frac{l(l+1)}{r^2} \end{pmatrix} = \begin{pmatrix} -A_l(a, l) & \alpha \\ 0 & -A_l(a, \infty) \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}_{\alpha^2,l}(a)) = \mathcal{D}(\mathcal{A}_{\alpha,l}(a)) := \mathcal{D}(A_l(a, l)) \oplus \mathcal{D}(A_l(a, \infty)),$$

and the (constant) matrices $\mathcal{C}_{l,m}$, $\widetilde{\mathcal{C}}_{l,m}$ in $L^2(a, 1) \oplus L^2(a, 1)$ by

$$\mathcal{C}_{l,m} := \begin{pmatrix} 0 & 0 \\ c_{l,m} & 0 \end{pmatrix}, \quad \widetilde{\mathcal{C}}_{l,m} := \begin{pmatrix} 0 & 0 \\ \widetilde{c}_{l,m} & 0 \end{pmatrix},$$

$$\mathcal{D}(\mathcal{C}_{l,m}) := \mathcal{D}(\widetilde{\mathcal{C}}_{l,m}) := L^2(a, 1) \oplus L^2(a, 1),$$

with entries $c_{l,m}$, $\widetilde{c}_{l,m}$ as in Notation 2.2. In the Hilbert space

$$\mathcal{H}(a) := l^2\big(L^2(a, 1) \oplus L^2(a, 1) : i \in \mathbb{N}\big) \tag{3.2}$$

we further define the (unbounded) constant coupling matrices

$$\mathcal{C}^m := \begin{pmatrix} 0 & -\widetilde{\mathcal{C}}_{k_m+1,m} & 0 & \cdots & \cdots \\ \mathcal{C}_{k_m,m} & 0 & -\widetilde{\mathcal{C}}_{k_m+2,m} & 0 & \cdots \\ 0 & \mathcal{C}_{k_m+1,m} & \ddots & \ddots & \\ \vdots & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix},$$

$$\mathcal{D}(\mathcal{C}^m) := \{f \in \mathcal{H}(a) : \mathcal{C}^m f \in \mathcal{H}(a)\},$$

with $k_m := \max\{|m|, 1\}$, $m \in \mathbb{Z}$, and the infinite block diagonal operator matrices

$$\mathcal{A}_{\alpha^2}(a,m) := \operatorname{diag}(\mathcal{A}_{\alpha^2,l}(a) : l \geq k_m), \ \mathcal{D}(\mathcal{A}_{\alpha^2}(a,m)) := l^2\big(\mathcal{D}(\mathcal{A}_{\alpha^2,l}(a)) : l \geq k_m\big),$$

$$\mathcal{A}_{\alpha}(a,m) := \operatorname{diag}(\mathcal{A}_{\alpha,l}(a) : l \geq k_m), \quad \mathcal{D}(\mathcal{A}_{\alpha}(a,m)) := l^2\big(\mathcal{D}(\mathcal{A}_{\alpha,l}(a)) : l \geq k_m\big).$$

Then, for $m \in \mathbb{Z}$, the MHD dynamo operator matrices *for the α^2-model, the $\alpha^2\omega$-model, and the $\alpha\omega$-model, respectively, are given by*

$$\mathcal{A}^m_{\alpha^2}(a) := \mathcal{A}_{\alpha^2}(a, m) - \mathrm{i}m\omega, \qquad \mathcal{D}(\mathcal{A}^m_{\alpha^2}(a)) := \mathcal{D}(\mathcal{A}_{\alpha^2}(a, m)),$$

$$\mathcal{A}^m_{\alpha^2\omega}(a) := \mathcal{A}_{\alpha^2}(a, m) - \mathrm{i}m\omega + \omega'\mathcal{C}^m, \quad \mathcal{D}(\mathcal{A}^m_{\alpha^2\omega}(a)) := \mathcal{D}(\mathcal{A}_{\alpha^2}(a, m)),$$

$$\mathcal{A}^m_{\alpha\omega}(a) := \mathcal{A}_{\alpha}(a, m) - \mathrm{i}m\omega + \omega'\mathcal{C}^m, \quad \mathcal{D}(\mathcal{A}^m_{\alpha\omega}(a)) := \mathcal{D}(\mathcal{A}_{\alpha}(a, m)).$$

Remark 3.4. (i) The infinite matrix $\mathcal{C}^m$ is an unbounded operator in $\mathcal{H}(a)$ since

$$\lim_{l\to\infty} c_{l,m} = \infty, \quad \lim_{l\to\infty} \widetilde{c}_{l,m} = \infty;$$

see Notation 2.2. In Lemma 5.7 below we will show that

$$\mathcal{D}(\mathcal{A}_{\alpha^2}(a,m)) \subset \mathcal{D}(\mathcal{C}^m), \quad \mathcal{D}(\mathcal{A}_{\alpha}(a,m)) \subset \mathcal{D}(\mathcal{C}^m),$$

which implies that $\mathcal{C}^m$ is relatively bounded with respect to $\mathcal{A}_{\alpha^2}(a,m)$ and $\mathcal{A}_{\alpha}(a,m)$.

(i) The difference between the $\alpha^2\omega$- and α^2-model is the infinite off-diagonal coupling matrix $\omega'\mathcal{C}^m$,

$$\mathcal{A}^m_{\alpha^2\omega} - \mathcal{A}^m_{\alpha^2} = \omega'\mathcal{C}^m,$$

which represents the ω-effect that creates toroïdal field from poloïdal field. The difference between the $\alpha^2\omega$- and $\alpha\omega$-model is the infinite off-diagonal matrix

$$\mathcal{A}^m_{\alpha^2\omega} - \mathcal{A}^m_{\alpha\omega} = \mathcal{A}_{\alpha^2}(a,m) - \mathcal{A}_{\alpha}(a,m) = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 \\ A_{l,\alpha}(a,l) & 0 \end{pmatrix} : l \geq k_m\right)$$

representing the α-effect which creates toroïdal field from poloïdal field.

The criteria for the nonexistence of dynamo effects for the various models which we present in the next section are formulated in terms of the functions α, ω and of the smallest eigenvalues of the positive operators $A_l(a,\vartheta)$, $\vartheta \in \{l,\infty\}$ on the diagonal of each dynamo model. For this we need the following auxiliary result.

PROPOSITION 3.5. *For $l \in \mathbb{N}$, $\vartheta \in \{l,\infty\}$, and $a \in [0,1)$, the operator $A_l(a,\vartheta)$ is self-adjoint and positive with compact resolvent. Its spectrum $\sigma(A_l(a,\vartheta))$ consists of a strictly increasing sequence $\big(\lambda_{l,j}(a,\vartheta)\big)_{j\in\mathbb{N}}$ of simple eigenvalues satisfying*

$$l(l+1) \leq \lambda_{l,1}(a,l) < \lambda_{l,1}(a,\infty) < \lambda_{l+1,1}(a,l+1) < \lambda_{l+1,1}(a,\infty), \quad l \in \mathbb{N}. \tag{3.3}$$

Moreover, $\lambda_{l,j}(a,\vartheta) \leq \lambda_{l,j}(a',\vartheta)$ for a, $a' \in [0,1)$, $a \leq a'$, i.e., $\lambda_{l,j}(\cdot,\vartheta)$ is monotonically increasing in $[0,1)$ for $l \in \mathbb{N}$, $\vartheta \in \{l,\infty\}$.

Proof. For $a = 0$ all claims but the lower bound $\lambda_{l,1}(a,\vartheta) \geq l(l+1)$ were proved in [9, Prop. 3.1]. The proofs for $a > 0$ are similar; note that then τ_l is regular at the endpoint a. The lower bound follows from the numerical range estimate

$$\langle A_l(a,\vartheta)x, x\rangle = \beta_\vartheta |x(1)|^2 + \int_a^1 \left(|x'(r)|^2 + \frac{l(l+1)}{r^2}\,|x(r)|^2\right) \mathrm{d}r \geq l(l+1)\|x\|^2$$

for $x \in \mathcal{D}(A_l(a,\vartheta))$ with $\beta_l := l$ and $\beta_\infty := 0$. The last claim follows from the min-max principle if we consider all operators $A_l(a,\vartheta)$ in $L^2(0,1)$ and note that, for a, $a' \in [0,1)$, $a \leq a'$, and $x \in \mathcal{D}(A_l(a',\vartheta)) \subset \mathcal{D}(A_l(a,\vartheta))$,

$$\langle A_l(a',\vartheta)x, x\rangle - \langle A_l(a,\vartheta)x, x\rangle = \int_a^{a'} \left(|x'(r)|^2 + \frac{l(l+1)}{r^2}\,|x(r)|^2\right) \mathrm{d}r \geq 0. \qquad \square$$

Remark 3.6. The eigenvalues $\lambda_{l,s}(0,l)$ and $\lambda_{l,s}(0,\infty)$ are the sth nonzero zeros of the fractional Bessel functions $J_{l-\frac{1}{2}}(\sqrt{\cdot})$ and $J_{l+\frac{1}{2}}(\sqrt{\cdot})$, respectively; see [9, Lemma 3.3], i.e.,

$$\lambda_{l,s}(0,l) = (j_{l-\frac{1}{2},s})^2, \quad \lambda_{l,s}(0,\infty) = (j_{l+\frac{1}{2},s})^2, \quad l \in \mathbb{N},\ s \in \mathbb{N}, \tag{3.4}$$

in the notation of [1, sect. 10.1 and 9.5.14], which can also be used to show that the quotient of the first ($s = 1$) eigenvalues has the asymptotics

$$\frac{j_{l+\frac{1}{2},1}}{j_{l-\frac{1}{2},1}} = 1 + \mathrm{O}(l^{-2/3}), \quad l \to \infty. \tag{3.5}$$

4. Anti-MHD dynamo theorems. In this section, for each of the three dynamo models, we establish criteria for the eigenvalues to lie in the linearly stable left half-plane. These so-called *antidynamo theorems* or bounding theorems provide simple threshold conditions for the functions α and ω under which the on-set of a dynamo effect is impossible.

These criteria involve only the functions α, ω and the first eigenvalues of the Bessel operators $A_l(0,l)$ and $A_l(0,\infty)$, i.e., the smallest nonzero zeros of the spherical Bessel functions $J_{l-\frac{1}{2}}(\sqrt{\cdot})$ and $J_{l+\frac{1}{2}}(\sqrt{\cdot})$.

PROPOSITION 4.1. *Let $\alpha : [0,1] \to \mathbb{R}$ be differentiable with $\alpha' \in L^\infty(0,1)$. For $l \in \mathbb{N}$ and $a \in [0,1)$ define*

$$\gamma_{\alpha^2,l}(a) := \left(\|\alpha\| + \frac{\|\alpha'\|}{\sqrt{\lambda_{l,1}(a,l)}}\right), \quad \gamma_{\alpha,l}(a) := 1 + \frac{\|\alpha\|}{\sqrt{\lambda_{l,1}(a,\infty)}}.$$

Then, for $\chi \in \{\alpha^2, \alpha\}$,

(i) $\gamma_{\chi,l}(a) > \gamma_{\chi,l+1}(a)$;

(ii) $\gamma_{\chi,l}(a) \geq \gamma_{\chi,l}(a')$ *for* $a, a' \in [0,1)$, $a \leq a'$;

(iii) $\gamma_{\chi,l}(0) = \mathrm{O}(1)$, $l \to \infty$.

Proof. Claims (i) and (ii) follow from Proposition 3.5, and claim (iii) is a consequence of the fact that $\sqrt{\lambda_{l,1}(0,l)}$ is the first positive zero $j_{l,1}$ of the spherical Bessel function $J_{l-\frac{1}{2}}$ and the asymptotics of the latter which yield $j_{l,1} = \mathrm{O}(l)$; see [1, sect. 10.1 and 9.5.14]. ∎

THEOREM 4.2 (antidynamo theorem for the α^2-model). *Let $\alpha : [0,1] \to \mathbb{R}$ be differentiable, and let $\alpha' \in L^\infty(0,1)$, $m \in \mathbb{Z}$, and $k_m := \max\{|m|, 1\}$. Then the α^2-dynamo operator $\mathcal{A}_{\alpha^2}(0,m)$ has no eigenvalues with real part > 0 if*

$$\left(\|\alpha\| + \frac{\|\alpha'\|}{j_{k_m-\frac{1}{2},1}}\right)\left(\|\alpha\| + |m|\,\|\omega\|\right) + |m|\,\|\omega\| \frac{(j_{k_m+\frac{1}{2},1})^2}{(j_{k_m-\frac{1}{2},1})^2} \leq (j_{k_m+\frac{1}{2},1})^2, \tag{4.1}$$

where $j_{l\pm\frac{1}{2},1}$ denote the first positive zeros of the spherical Bessel functions $J_{l\pm\frac{1}{2}}(\cdot)$; in particular, for the axisymmetric case $m = 0$, there are no eigenvalues with real part > 0 if

$$\left(\|\alpha\| + \frac{\|\alpha'\|}{j_{\frac{1}{2},1}}\right)\|\alpha\| \leq (j_{\frac{3}{2},1})^2. \tag{4.2}$$

The antidynamo theorem [9, Cor. 4.8] is the special case $m = 0$ in Theorem 4.2, (4.2) above; see also [21, (27)]. Note that a combination of $\|\alpha\|$ and $\|\alpha'\|$ also appears in the *optimal energy bounds* obtained for the spherically symmetric α^2-model in [12]. While they chose the combination $\sqrt{\|\alpha\|^2 + \|\alpha'\|^2}$ where both norms contribute equally, our combination $\|\alpha\| + \frac{1}{j_{\frac{1}{2},1}}\|\alpha'\|$ in (4.2) emerges from the spectral estimates for the α^2-dynamo operator $\mathcal{A}_{\alpha^2}(0,m)$.

COROLLARY 4.3. *None of the α^2-dynamo operators $\mathcal{A}_{\alpha^2}(0,m)$, $m \in \mathbb{Z}$, has eigenvalues with real part > 0 if one of the following two conditions equivalent to condition (4.1) hold:*

$$\left(\|\alpha\| + \frac{\|\alpha'\|}{j_{k_m-\frac{1}{2},1}}\right) \frac{\|\alpha\| + |m|\|\omega\|}{(j_{k_m+\frac{1}{2},1})^2} + \frac{|m|\|\omega\|}{(j_{k_m-\frac{1}{2},1})^2} \leq 1, \quad m \in \mathbb{Z},$$

or equivalently,

(4.3)
$$\left(\|\alpha\|+\frac{\|\alpha'\|}{j_{\frac12,1}}\right)\frac{\|\alpha\|}{(j_{\frac32,1})^2}\leq 1,\quad \|\omega\|\leq \min_{m\in\mathbb{N}}\frac{1}{m}\frac{1-\left(\|\alpha\|+\frac{\|\alpha'\|}{j_{m-\frac12,1}}\right)\frac{\|\alpha\|}{(j_{m+\frac12,1})^2}}{\left(\|\alpha\|+\frac{\|\alpha'\|}{j_{m-\frac12,1}}\right)\frac{1}{(j_{m+\frac12,1})^2}+\frac{1}{(j_{m-\frac12,1})^2}}.$$

Remark 4.4. For fixed α inequality (4.1) in Theorem 4.2 is satisfied for all sufficiently large $|m|$, $m\in\mathbb{Z}$. This follows from (3.3) and Proposition 4.1 (iii) which show that the right-hand side $(j_{k_m+\frac12,1})^2=\lambda_{k_m,1}(0,\infty)\geq|m|(|m|+1)$, while the left-hand side is of order $\mathrm{O}(|m|)$ as $|m|\to\infty$ due to the asymptotics of the Bessel zeros; see [1, sect. 10.1 and 9.5.14] and (3.5).

The following example illustrates this effect and shows which modes m set the thresholds in Theorem 4.2. It turns out that not only the modes $m=0$ and $m=1$ play a role here.

Example 4.5 (constant α, comparing thresholds for $m\in\mathbb{Z}$). We consider constant α and set $\alpha=:\alpha_0$, $\|\omega\|=:\omega_0$. Then the conditions (4.3) become

$$\alpha_0\leq j_{\frac32,1},\quad \omega_0\leq\min_{m\in\mathbb{N}}\frac{1}{m}\frac{1-\frac{\alpha_0^2}{(j_{m+\frac12,1})^2}}{\frac{\alpha_0}{(j_{m+\frac12,1})^2}+\frac{1}{(j_{m-\frac12,1})^2}}=:\min_{m\in\mathbb{N}}b(m,\alpha_0). \tag{4.4}$$

While the function $m\mapsto\frac{m}{(j_{m-\frac12,1})^2}$ is monotonically decreasing for $m=1,2,\dots$, this is not true for the function $m\mapsto\frac{m}{(j_{m+\frac12,1})^2}$ which is monotonically increasing for $m=1,2,3$ and monotonically decreasing for $m=3,4,\dots$, attaining its maximum for $m=3$; this can be proved analytically using a number of different properties of the dependence of the first Bessel zeros on the order in [10]. Therefore, for any $\alpha_0\in[0,j_{\frac32,1}]$, the function $b(\cdot,\alpha_0)$ on the right-hand side above is guaranteed to be monotonically increasing for $m=3,4,\dots$. Moreover, also $b(1,\alpha_0)<b(3,\alpha_0)$ and $b(2,\alpha_0)<b(3,\alpha_0)$. However, it turns out that the graphs of $b(1,\cdot)$ and $b(2,\cdot)$ have two intersection points $\alpha_1<\alpha_2$ in $[0,j_{\frac32,1}]$ with

$$\begin{aligned}b(1,\alpha_0)\leq b(2,\alpha_0)<b(3,\alpha_0)\quad&\text{for}\quad \alpha_0\in[0,\alpha_1]\cup[\alpha_2,j_{\frac32,1}],\\ b(2,\alpha_0)\leq b(1,\alpha_0)<b(3,\alpha_0)\quad&\text{for}\quad \alpha_0\in[\alpha_1,\alpha_2],\end{aligned}$$

where α_1, α_2 are solutions of a cubic equation. Figure 1 shows the graphs of $b(m,\cdot)$ for $m=1,2,3,4$, the condition $\alpha_0\leq j_{\frac32,1}$ for $m=0$, and the corresponding antidynamo region for the full α^2-model in the $\alpha_0\equiv\alpha$, $\omega_0=\|\omega\|$ plane. So depending on the value of $\alpha\equiv\alpha_0$, either the modes $m=0,1$ or the modes $m=0,2$ set the thresholds.

The antidynamo criteria for the $\alpha^2\omega$ model and for the $\alpha\omega$ model are not as simple as (4.1) since they involve infinite operator matrices.

NOTATION 4.6. *For $m\in\mathbb{Z}$, $k_m:=\max\{|m|,1\}$, define*

$$\delta_{l,m}:=\begin{cases}\sqrt{2}\,c_{l,m}, & l=k_m,\\ \sqrt{2c_{l,m}^2+\widetilde{c}_{l,m}^2}, & l=k_m+1,\\ \sqrt{2(c_{l,m}^2+\widetilde{c}_{l,m}^2)}, & l\in\mathbb{N},\ l\geq k_m+2,\end{cases} \tag{4.5}$$

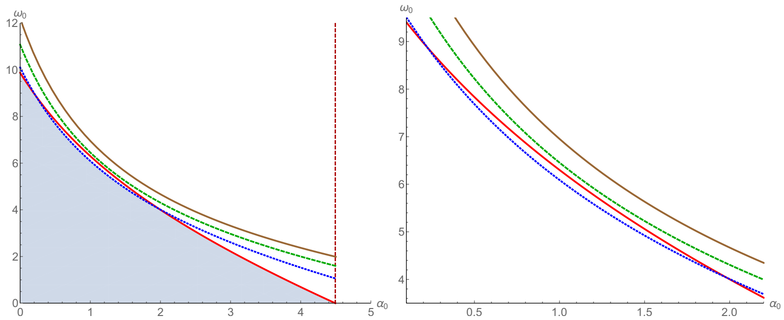

FIG. 1. *Anti-α^2-dynamo region (blue/grey) in the $\alpha \equiv \alpha_0$, $\|\omega\| \equiv \omega_0$ plane according to (4.4); lines for $m=1$ (red/solid), $m=2$ (blue/dotted), $m=3$ (green/dashed), $m=4$ (brown/solid), and zoom into the intersection of the thresholds for $m=1$, $m=2$. (Figure in color online.)*

where $c_{l,m}$ for $l \geq k_m$ and $\widetilde{c}_{l,m}$ for $l \geq k_m+1$ are as in Notation 2.2,

$$c_{l,m} = l\sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}, \quad \widetilde{c}_{l,m} = (l+1)\sqrt{\frac{l^2-m^2}{(2l-1)(2l+1)}}.$$

THEOREM 4.7 (antidynamo theorem for the $\alpha^2\omega$-model). *Let α, $\omega : [0,1] \to \mathbb{R}$ be differentiable with α', $\omega' \in L^\infty(0,1)$. Let $m \in \mathbb{Z}$, $k_m := \max\{|m|,1\}$, and let $\delta_{l,m}$ be as in Notation 4.6. Then the $\alpha^2\omega$-dynamo operator $\mathcal{A}_{\alpha^2\omega}(0,m)$ has no eigenvalues with real part >0 if for some γ_1, $\gamma_2 \in [0,1]$ with $\gamma_1+\gamma_2 \leq 1$*

$$\left(\|\alpha\| + \frac{\|\alpha'\|}{j_{k_m-\frac{1}{2},1}}\right)\left(\|\alpha\| + \frac{|m|\,\|\omega\|}{\gamma_1}\right) + \frac{|m|\,\|\omega\|}{\gamma_1}\frac{(j_{k_m+\frac{1}{2},1})^2}{(j_{k_m-\frac{1}{2},1})^2} \leq (j_{k_m+\frac{1}{2},1})^2, \tag{4.6}$$

$$\left(\|\alpha\| + \frac{\|\alpha'\|}{j_{l-\frac{1}{2},1}}\right)\|\alpha\| + \frac{\|\omega'\|}{\gamma_2}\delta_{l,m}\frac{\sqrt{(j_{l+\frac{1}{2},1})^4+\|\alpha\|^2}}{(j_{l-\frac{1}{2},1})^2} \leq (j_{l+\frac{1}{2},1})^2; \quad l \in \mathbb{N},\ l \geq k_m \tag{4.7}$$

for $m=0$ the conditions simplify to (4.7) with $\gamma_2 = 1$ since then condition (4.6) is automatically satisfied.

Remark 4.8. For given α and ω, the inequalities in Theorem 4.7 are satisfied for all sufficiently large $|m|$, $m \in \mathbb{Z}$. This follows from (3.3), (3.4), and (3.5) which show that the right-hand sides are all $\geq |m|(|m|+1)$, while the left-hand side of (4.6) is of order $\mathrm{O}(|m|)$ and the left-hand side of (4.7) is of order $\mathrm{O}(\sqrt{|m|})$ as $|m| \to \infty$ since $k_m = |m|$ and $c_{l_m,m} = \mathrm{O}(\sqrt{|m|})$, $\widetilde{c}_{l_m,m} = \mathrm{O}(\sqrt{|m|})$ for $l_m = k_m, k_m+1, k_m+2$.

The following example illustrates how the inequalities (4.6), (4.7) can be verified, and sometimes simplified, for concrete functions α, ω.

Example 4.9 (constant α, ω', axisymmetric case $m=0$). Let $\alpha \equiv \alpha_0 \in \mathbb{R}$ and $\omega(r) = \omega_0 r$, $r \in [0,1]$, and $m=0$. We claim that, in this case, the infinitely many anti-$\alpha^2\omega$-dynamo inequalities (4.6), (4.7) are equivalent to the four inequalities

$$\alpha_0 \leq j_{\frac{3}{2},1}, \quad \omega_0 \leq \min_{l=1}^{3} \frac{(j_{l+\frac{1}{2},1})^2 - \alpha_0^2}{\sqrt{(j_{l+\frac{1}{2},1})^4+\alpha_0^2}}\,\frac{(j_{l-\frac{1}{2},1})^2}{\delta_{l,0}}, \tag{4.8}$$

yielding explicit thresholds for α_0, ω_0; here the first condition in (4.8) is the anti-α^2-dynamo condition; see (4.2).

Proof. Since $m = 0$ and $\alpha' \equiv 0$, the conditions (4.6), (4.7) reduce to

$$\alpha_0^2 + \omega_0\, \delta_{l,0} \frac{\sqrt{(j_{l+\frac12,1})^4 + \alpha_0^2}}{(j_{l-\frac12,1})^2} \le (j_{l+\frac12,1})^2; \quad l \in \mathbb{N}. \tag{4.9}$$

To prove the equivalence of (4.9) and (4.8), we first note that (4.9) implies (4.8) due to the fact that the sequence of Bessel zeros $(j_{k+\frac12,1})_{k\in\mathbb{N}}$ is monotonically increasing. For the converse it suffices to show that the values of the product $q_1(l)q_2(l)$ of quotients in (4.8) for $l \in \mathbb{N}$, $l \ge 4$, are larger than the value for $l = 3$ and hence larger than the minimum taken over $l \in \{1, 2, 3\}$. The first quotient $q_1(l)$ in (4.8) is strictly increasing in l due to the monotonicity of the Bessel zeros; see above. By Notation 4.6 with $k_m = 1$ and $l \ge 3$, the second quotient in (4.8) is given by

$$q_2(l) = \frac{(j_{l-\frac12,1})^2}{\delta_{l,0}} = (j_{l-\frac12,1})^2 \frac{\sqrt{(2l-1)(2l+3)}}{2l(l+1)}, \quad l \in \mathbb{N},\ l \ge 3.$$

By [10, Thm. 2], the function $l \mapsto \frac{(j_{l-\frac12,1})^2}{l+\frac12}$ is monotonically increasing for $l \in [-\frac12, \infty)$ and hence so is

$$q_2(l) = \frac{(j_{l-\frac12,1})^2}{(l+\frac12)} \frac{(l+\frac12)\sqrt{(2l-1)(2l+3)}}{2l(l+1)}, \quad l \in \mathbb{N},\ l \ge 3,$$

being the product of monotonically increasing positive functions. Since q_1, q_2 are positive, we conclude $q_1(l)q_2(l) \ge q_1(3)q_2(3)$, $l \in \mathbb{N}$, $l \ge 4$, as required. ∎

THEOREM 4.10 (antidynamo theorem for the $\alpha\omega$-model). *Let α, $\omega : [0,1] \to \mathbb{R}$ be such that ω is differentiable and α, $\omega' \in L^\infty(0,1)$. Let $m \in \mathbb{Z}$, $k_m := \max\{|m|, 1\}$, and let $\delta_{l,m}$ be as in Notation 4.6. Then the $\alpha\omega$-dynamo operator $\mathcal{A}_{\alpha\omega}(0,m)$ has no eigenvalues with real part > 0 if for some γ_1, $\gamma_2 \in [0,1]$ with $\gamma_1 + \gamma_2 \le 1$*

$$\frac{|m|\,\|\omega\|}{\gamma_1} \frac{(j_{k_m+\frac12,1})^2}{(j_{k_m-\frac12,1})^2} \le (j_{k_m+\frac12,1})^2, \tag{4.10}$$

$$\frac{\|\omega'\|}{\gamma_2} \delta_{l,m} \frac{\sqrt{(j_{l+\frac12,1})^4 + \|\alpha\|^2}}{(j_{l-\frac12,1})^2} \le (j_{l+\frac12,1})^2; \quad l \in \mathbb{N},\ l \ge k_m \tag{4.11}$$

for $m = 0$ the conditions simplify to (4.11) with $\gamma_2 = 1$ since then condition (4.10) is automatically satisfied.

Remark 4.11. For fixed α and ω, the inequalities in Theorem 4.10 are satisfied for sufficiently large $|m|$, $m \in \mathbb{Z}$, for analogous reasons as in Remark 4.8.

5. Proofs of the antidynamo theorems. In this section we give the proofs of the three different antidynamo theorems presented in the previous section. To this end we need a number of auxiliary results and estimates which we present first.

We begin with results on the diagonal entries of the infinite operator matrices describing the $\alpha^2\omega$ model and the $\alpha\omega$ model, the 2×2 operator matrices $\mathcal{A}_{\alpha^2,l}(a)$, and $\mathcal{A}_{\alpha,l}(a)$, $l \ge k_m = \max\{|m|, 1\}$, respectively, in $L^2(a,1) \oplus L^2(a,1)$ where $a \in [0,1)$; see Definition 3.3. Here the functions α, $\omega : [0,1] \to \mathbb{R}$ are supposed to satisfy the assumptions of the corresponding dynamo model.

Recall that the diagonal entries of the latter are the selfadjoint Bessel operators $-A_l(a,\vartheta)$, $\vartheta \in \{l, \infty\}$, for which $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge -\lambda_{l,1}(a,\vartheta)\} \subset \varrho(-A_l(a,\vartheta))$, where $\lambda_{l,1}(a,\vartheta) > 0$ is the smallest eigenvalue of $A_l(a,\vartheta)$.

LEMMA 5.1. *For* $l\in\mathbb{N}$, $a\in[0,1)$ *and* $\lambda\in\mathbb{C}$,

$$\left\|(-A_l(a,\vartheta)-\lambda)^{-1}\right\| \le \frac{1}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,\vartheta)}, \qquad \mathrm{Re}\,\lambda \ge -\lambda_{l,1}(a,\vartheta),$$

$$\left\|A_{l,\alpha}(a,l)(-A_l(a,l)-\lambda)^{-1}\right\| \le \|\alpha\| + \frac{\|\alpha'\|}{\sqrt{\lambda_{l,1}(a,l)}} = \gamma_{\alpha^2,l}(a), \qquad \mathrm{Re}\,\lambda \ge 0.$$

Proof. The claims for $a=0$ are immediate from [9, Lemmas 3.7 and 3.5]; the proof for $a>0$ is analogous, noting that the norm of the multiplication operator $\alpha\cdot$ in $L^2(a,1)$ is bounded by its norm in $L^2(0,1)$. ∎

The following proposition and corollary concern the operator matrices $\mathcal{A}_{\alpha^2,l}(a)$, $l\in\mathbb{N}$, and the infinite operator matrix $\mathcal{A}_{\alpha^2}(a,m)$; see Definition 3.3.

PROPOSITION 5.2. *Let* $a\in[0,1)$ *and* $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\ge 0$.

(i) *Let* $l\in\mathbb{N}$. *If* $\mathrm{Re}(\lambda)>-\lambda_{l,1}(a,\infty)+\gamma_{\alpha^2,l}(a)\|\alpha\|$, *then* $\lambda\in\varrho(\mathcal{A}_{\alpha^2,l}(a))$ *and*

$$\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\| \le \frac{1}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\left(\frac{\lambda_{l,1}(a,\infty)}{\lambda_{l,1}(a,l)}+\gamma_{\alpha^2,l}(a)\right);$$

further, $\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\|<\kappa$ *if* $\kappa>0$ *is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{l,1}(a,\infty)+\gamma_{\alpha^2,l}(a)\|\alpha\|+\frac{1}{\kappa}\left(\frac{\lambda_{l,1}(a,\infty)}{\lambda_{l,1}(a,l)}+\gamma_{\alpha^2,l}(a)\right).$$

(ii) *Let* $m\in\mathbb{Z}$, $k_m:=\max\{|m|,1\}$. *If* $\mathrm{Re}(\lambda)>-\lambda_{k_m,1}(a,\infty)+\gamma_{\alpha^2,k_m}(a)\|\alpha\|$, *then* $\lambda\in\varrho(\mathcal{A}_{\alpha^2}(a,m))$ *and*

$$\begin{aligned}&\left\|(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\right\|\\ &\le \frac{1}{\mathrm{Re}(\lambda)+\lambda_{k_m,1}(a,\infty)-\gamma_{\alpha^2,k_m}(a)\|\alpha\|}\left(\frac{\lambda_{k_m,1}(a,\infty)}{\lambda_{k_m,1}(a,k_m)}+\gamma_{\alpha^2,k_m}(a)\right);\end{aligned}$$

further, $\left\|(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\right\|<\kappa$ *if* $\kappa>0$ *is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{k_m,1}(a,\infty)+\gamma_{\alpha^2,k_m}(a)\|\alpha\|+\frac{1}{\kappa}\left(\frac{\lambda_{k_m,1}(a,\infty)}{\lambda_{k_m,1}(a,k_m)}+\gamma_{\alpha^2,k_m}(a)\right).$$

Proof. Throughout this proof, let $a\in[0,1)$ and $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\ge 0$.

(i) Let $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>-\lambda_{l,1}(a,\infty)+\gamma_{\alpha^2,l}(a)\|\alpha\|$. For $\vartheta\in\{l,\infty\}$ we set

$$d_{\lambda,l}(a,\vartheta) := \mathrm{Re}(\lambda)+\lambda_{l,1}(a,\vartheta) > \gamma_{\alpha^2,l}(a)\|\alpha\|. \tag{5.1}$$

The resolvent $(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}$ admits a matrix representation in terms of the inverses $S_{2,l}^{\alpha^2}(a,\lambda)^{-1}$ of the second Schur complement of $\mathcal{A}_{\alpha^2,l}(a)$, see [24, Thm. 2.3.3 i)], which is given by

$$\begin{aligned}S_{2,l}^{\alpha^2}(a,\lambda) &:= -A_l(a,\infty)-\lambda-A_{l,\alpha}(a,l)(-A_l(a,l)-\lambda)^{-1}\alpha,\\ \mathcal{D}(S_{2,l}^{\alpha^2}(a,\lambda)) &:= \mathcal{D}(A_l(a,\infty));\end{aligned}$$

see [24, Def. 2.2.12]. By Lemma 5.1 and (5.1), we have

$$\begin{aligned}&\left\|(-A_l(a,\infty)-\lambda)^{-1}A_{l,\alpha}(a,l)(-A_l(a,l)-\lambda)^{-1}\alpha\right\|\\ &\le \left\|(-A_l(a,\infty)-\lambda)^{-1}\right\|\left\|A_{l,\alpha}(a,l)(-A_l(a,l)-\lambda)^{-1}\right\|\|\alpha\|\\ &\le \frac{1}{d_{\lambda,l}(a,\infty)}\left(\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{\lambda_{l,1}(a,l)}}\right)\|\alpha\| = \frac{\gamma_{\alpha^2,l}(a)\|\alpha\|}{d_{\lambda,l}(a,\infty)} < 1,\end{aligned} \tag{5.2}$$

and hence

$$S_{2,l}(a,\lambda)=(-A_l(a,\infty)-\lambda)\left(I-(-A_l(a,\infty)-\lambda)^{-1}A_{l,\alpha}(a,l)(-A_l(a,l)-\lambda)^{-1}\alpha\right)$$

is boundedly invertible with

$$\left\|S_{2,l}(a,\lambda)^{-1}\right\| \leq \frac{1}{d_{\lambda,l}(a,\infty)}\frac{1}{1-\frac{\gamma_{\alpha^2,l}(a)\|\alpha\|}{d_{\lambda,l}(a,\infty)}} = \frac{1}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}. \tag{5.3}$$

If we estimate the norm of the inverse of $(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1} =: \left(R_{ij}\right)_{i,j=1}^2$ by

$$\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\| \leq \max\{\|R_{11}\|,\|R_{22}\|\}+\max\{\|R_{12}\|,\|R_{21}\|\},$$

use its representation [24, Thm. 2.3.3 i)] and combine it with the estimates in Lemma 5.1 and in (5.3), we find

$$\begin{aligned}
&\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\| \\
&\leq \max\left\{\frac{1}{d_{\lambda,l}(a,l)}\left(1+\frac{\|\alpha\|\gamma_{\alpha^2,l}(a)}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\right), \frac{1}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\right\} \\
&\quad + \max\left\{\frac{1}{d_{\lambda,l}(a,l)}\frac{\|\alpha\|}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}, \frac{\gamma_{\alpha^2,l}(a)}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\right\} \\
&= \frac{1}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\left(\max\left\{\frac{d_{\lambda,l}(a,\infty)}{d_{\lambda,l}(a,l)},1\right\}+\max\left\{\frac{\|\alpha\|}{d_{\lambda,l}(a,l)},\gamma_{\alpha^2,l}(a)\right\}\right) \\
&= \frac{1}{d_{\lambda,l}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\left(\frac{d_{\lambda,l}(a,\infty)}{d_{\lambda,l}(a,l)}+\max\left\{\frac{\|\alpha\|}{d_{\lambda,l}(a,l)},\gamma_{\alpha^2,l}(a)\right\}\right).
\end{aligned}$$

Since $\lambda_{l,1}(a,\infty)-\lambda_{l,1}(a,l)>0$ and $\mathrm{Re}(\lambda)\geq 0$, we can estimate

$$\frac{d_{\lambda,l}(a,\infty)}{d_{\lambda,l}(a,l)} = \frac{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,\infty)}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,l)} \leq \frac{\lambda_{l,1}(a,\infty)}{\lambda_{l,1}(a,l)}.$$

Further we use that $\mathrm{Re}(\lambda)\geq 0$ and $\lambda_{l,1}(a,l)\geq l(l+1)>1$ by (3.3) to estimate

$$\frac{\|\alpha\|}{d_{\lambda,l}(a,l)} = \frac{\|\alpha\|}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,l)} \leq \|\alpha\| \leq \gamma_{\alpha^2,l}(a).$$

Altogether this proves the first estimate in (i). The second claim in (i) is immediate from the first one.

(ii) Let $l \geq k_m$. By (3.3) we have $\lambda_{l,1}(a,\infty)\geq\lambda_{k_m,1}(a,\infty)$, $\lambda_{l,1}(a,l)\geq\lambda_{k_m,1}(a,k_m)$ and hence $\gamma_{\alpha^2,l}(a)\leq\gamma_{\alpha^2,k_m}(a)$ by (4.1). Thus, $\mathrm{Re}(\lambda)>-\lambda_{k_m,1}(a,\infty)+\gamma_{\alpha^2,k_m}(a)\|\alpha\|$ implies that $\mathrm{Re}(\lambda)>-\lambda_{l,1}(a,\infty)+\gamma_{\alpha^2,l}(a)\|\alpha\|$ and hence $\lambda\in\varrho(\mathcal{A}_{\alpha^2,l}(a))$ by (i) for every $l\geq k_m$. Moreover, the resolvent norm estimate in (i) shows that

$$\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\| \longrightarrow 0, \quad l\to\infty.$$

All this implies that the infinite operator matrix $\mathcal{A}_{\alpha^2}(a,m)-\lambda = \mathrm{diag}\big(\mathcal{A}_{\alpha^2,l}(a)-\lambda : l\geq k_m\big)$ is boundedly invertible, and so $\lambda\in\varrho(\mathcal{A}_{\alpha^2}(a,m))$, with $(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1} = \mathrm{diag}\big((\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1} : l\geq k_m\big)$ and

$$\|(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\| = \sup_{l\geq k_m}\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\| = \max_{l\geq k_m}\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\|.$$

Now the first estimate in (ii) follows from the monotonicity of $\lambda_{l,1}(a,l)$ and $\gamma_{\alpha^2,l}(a)$, see above, which implies that the norm bound in (i), which can also be written as

$$\frac{1}{\lambda_{l,1}(a,l)}+\frac{1}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,\infty)-\gamma_{\alpha^2,l}(a)\|\alpha\|}\left(\frac{-\mathrm{Re}(\lambda)+\gamma_{\alpha^2,l}(a)\|\alpha\|}{\lambda_{l,1}(a,l)}+\gamma_{\alpha^2,l}(a)\right), \tag{5.4}$$

is monotonically decreasing in l for $l \geq k_m$. The second claim in (ii) is immediate from the first one. ∎

Remark 5.3. If, in the proof above, we would use the first Schur complement for the representation of the inverse $(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}$, we arrive at the same estimate.

COROLLARY 5.4. *Let* $a\in[0,1)$, $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\geq 0$, *and let* $m\in\mathbb{Z}$, $k_m:=\max\{|m|,1\}$. *Then* $\left\|\mathrm{i}m\omega(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\right\|<\gamma_1$ *if* $\gamma_1>0$ *is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{k_m,1}(a,\infty)+\gamma_{\alpha^2,k_m}(a)\|\alpha\| + \frac{|m|\,\|\omega\|}{\gamma_1}\left(\frac{\lambda_{k_m,1}(a,\infty)}{\lambda_{k_m,1}(a,k_m)}+\gamma_{\alpha^2,k_m}(a)\right).$$

Proof. There is nothing to prove for $m=0$. For $m\neq 0$, the claim is immediate from Proposition 5.2 (ii) applied with $\kappa=\frac{\gamma_1}{|m|\,\|\omega\|}$. ∎

The following proposition and corollary concern the operator matrices $\mathcal{A}_{\alpha,l}(a)$, $l\in\mathbb{N}$, and the infinite operator matrix $\mathcal{A}_\alpha(a,m)$; see Definition 3.3.

PROPOSITION 5.5. *Let* $a\in[0,1)$ *and* $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\geq 0$.

(i) *Let* $l\in\mathbb{N}$. *Then* $\lambda\in\varrho(\mathcal{A}_{\alpha,l}(a))$ *and*

$$\left\|(\mathcal{A}_{\alpha,l}(a)-\lambda)^{-1}\right\| \leq \frac{1}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,l)}\left(1+\frac{\|\alpha\|}{\lambda_{l,1}(a,\infty)}\right);$$

further, $\left\|(\mathcal{A}_{\alpha,l}(a)-\lambda)^{-1}\right\|<\kappa$ *if* $\kappa>0$ *is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{l,1}(a,l)+\frac{1}{\kappa}\left(1+\frac{\|\alpha\|}{\lambda_{l,1}(a,\infty)}\right).$$

(ii) *Let* $m\in\mathbb{Z}$, $k_m:=\max\{|m|,1\}$. *Then* $\lambda\in\varrho(\mathcal{A}_\alpha(a,m))$ *and*

$$\|(\mathcal{A}_\alpha(a,m)-\lambda)^{-1}\| \leq \frac{1}{\mathrm{Re}(\lambda)+\lambda_{k_m,1}(a,k_m)}\left(1+\frac{\|\alpha\|}{\lambda_{k_m,1}(a,\infty)}\right);$$

further, $\left\|(\mathcal{A}_\alpha(a,m)-\lambda)^{-1}\right\|<\kappa$ *if* $\kappa>0$ *is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{k_m,1}(a,k_m)+\frac{1}{\kappa}\left(1+\frac{\|\alpha\|}{\lambda_{k_m,1}(a,\infty)}\right).$$

Proof. Let $a\in[0,1)$ and $\lambda\in\mathbb{C}$. (i) It is easy to see that

$$(\mathcal{A}_{\alpha,l}(a)-\lambda)^{-1}=\begin{pmatrix}(-A_l(a,l)-\lambda)^{-1} & -(-A_l(a,l)-\lambda)^{-1}\alpha(-A_l(a,\infty)-\lambda)^{-1}\\ 0 & (-A_l(a,\infty)-\lambda)^{-1}\end{pmatrix}. \tag{5.5}$$

Since $\mathrm{Re}(\lambda)\geq 0 > -\lambda_l(a,l) > -\lambda_l(a,\infty)$, we can use Lemma 5.1 to estimate

$$\begin{aligned}\left\|(\mathcal{A}_l(a,l)-\lambda)^{-1}\right\| &\leq \max\left\{\left\|(-A_l(a,l)-\lambda)^{-1}\right\|,\left\|(-A_l(a,\infty)-\lambda)^{-1}\right\|\right\}\\ &\quad+\left\|(-A_l(a,l)-\lambda)^{-1}\alpha(-A_l(a,\infty)-\lambda)^{-1}\right\|\\ &\leq \frac{1}{\mathrm{Re}(\lambda)+\lambda_{l,1}(a,l)}\left(1+\frac{\|\alpha\|}{\lambda_{l,1}(a,\infty)}\right),\end{aligned}$$

where we used $\mathrm{Re}(\lambda) \geq 0$ to estimate $\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty) \geq \lambda_{l,1}(a,\infty)$ in the last step. The second claim is immediate from this estimate.

(ii) The proof is analogous to the proof of Proposition 5.2 (ii) if we use the equality $\|(\mathcal{A}_\alpha(a,m) - \lambda)^{-1}\| = \max_{l \geq k_m} \big\|(\mathcal{A}_{\alpha,l}(a) - \lambda)^{-1}\big\|$. ∎

Corollary 5.6. *Let $a \in [0,1)$, $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$, and $m \in \mathbb{Z}$, $k := \max\{|m|, 1\}$. Then $\big\|\mathrm{i}m\omega(\mathcal{A}_\alpha(a,m) - \lambda)^{-1}\big\| < \gamma_1$ if $\gamma_1 > 0$ is such that*

$$\mathrm{Re}(\lambda) > -\lambda_{k_m,1}(a,k_m) + \frac{|m|\,\|\omega\|}{\gamma_1}\left(1 + \frac{\|\alpha\|}{\lambda_{k_m,1}(a,\infty)}\right).$$

Proof. There is nothing to prove for $m = 0$. For $m \neq 0$, the claim is immediate from Proposition 5.5 (ii) applied with $\kappa = \frac{\gamma_1}{|m|\,\|\omega\|}$. ∎

For the $\alpha^2\omega$ and $\alpha\omega$ dynamo models the main task is to study the interaction of the unbounded infinite coupling matrices $\mathcal{C}^m$ with the resolvents of $\mathcal{A}_{\alpha^2}(a,m)$ or $\mathcal{A}_\alpha(a,m)$; see Definition 3.3.

Lemma 5.7. *Let $m \in \mathbb{Z}$, $k_m := \max\{|m|, 1\}$, and $a \in [0,1)$. For $\chi \in \{\alpha^2, \alpha\}$ let*

$$\mathcal{A}_\chi(a,m) := \mathrm{diag}(\mathcal{A}_{\chi,l}(a) : l \geq k_m), \quad \mathcal{D}(\mathcal{A}_\chi(a,m)) := l^2\big(\mathcal{D}(\mathcal{A}_{\chi,l}(a)) : l \geq k_m\big)$$

be one of the operator matrices $\mathcal{A}_{\alpha^2}(a,m)$ or $\mathcal{A}_\alpha(a,m)$, and let $\mathcal{C}^m$ be given as in Definition 3.3. Then $\mathcal{D}(\mathcal{A}_\chi(a,m)) \subset \mathcal{D}(\mathcal{C}^m)$ and hence, for $\lambda \in \mathbb{C}$ with

$$\mathrm{Re}(\lambda) > \begin{cases} \max\{0, -\lambda_{k_m,1}(a,\infty) + \gamma_{\alpha^2,k_m}(a)\|\alpha\|\} & \text{if } \chi = \alpha^2, \\ 0 & \text{if } \chi = \alpha, \end{cases}$$

the operator product $\mathcal{C}^m(\mathcal{A}_\chi(a,m) - \lambda)^{-1}$ is bounded with

$$\big\|\mathcal{C}^m(\mathcal{A}_\chi(a,m) - \lambda)^{-1}\big\| \leq \max_{l=k_m}^{k_m+2} \left\{\delta_{l,m}\sqrt{1 + \frac{\|\alpha\|^2}{(\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty))^2}}\, s^\chi_{1,l}(a,\lambda)\right\},$$

where $\delta_{l,m} > 0$ are given as in Notation 4.6 and $s^\chi_{1,l}(a,\lambda)$ are norm bounds for the inverses of the first Schur complements of $\mathcal{A}_{\chi,l}(a)$ given by

$$\|S^{\alpha^2}_{1,l}(a,\lambda)^{-1}\| \leq s^{\alpha^2}_{1,l}(a,\lambda) := \frac{1}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,l)} \frac{1}{1 - \frac{\gamma_{\alpha^2,l}(a)\|\alpha\|}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty)}}, \tag{5.6}$$

$$\|S^{\alpha}_{1,l}(a,\lambda)^{-1}\| \leq s^{\alpha}_{1,l}(a,\lambda) := \frac{1}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,l)}, \tag{5.7}$$

with $\gamma_{\alpha^2,l}$ defined as in (4.1) for $l = k_m,\ k_m + 1,\ k_m + 2$.

Proof. Let $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$, be as above. In what follows, since a is fixed, we abbreviate $\mathcal{A}_{\chi,l} := \mathcal{A}_{\chi,l}(a)$, $l \geq k_m$, in some intermediate steps.

Then, for $x = (x_l)_{l \geq k_m} \in \mathcal{H}(a) = l^2\big(L^2(a,1) \oplus L^2(a,1) : i \in \mathbb{N}\big)$ with $\|x\| = 1$,

$$\begin{aligned} &\mathcal{C}^m(\mathcal{A}_\chi(a,m) - \lambda)^{-1}x \\ &= \begin{pmatrix} 0 & -\widetilde{\mathcal{C}}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1} - \lambda)^{-1} & 0 & \cdots \\ \mathcal{C}_{k_m,m}(\mathcal{A}_{\chi,k_m} - \lambda)^{-1} & 0 & -\widetilde{\mathcal{C}}_{k_m+2,m}(\mathcal{A}_{\chi,k_m+2} - \lambda)^{-1} & \\ 0 & \mathcal{C}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1} - \lambda)^{-1} & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix} x, \end{aligned}$$

and hence

$$
\begin{aligned}
&\left\|\mathcal{C}^m(\mathcal{A}_\chi(a,m)-\lambda)^{-1}x\right\|^2\\
&=\left\|\widetilde{\mathcal{C}}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}x_{k_m+1}\right\|^2\\
&+\sum_{l=k_m+1}^{\infty}\left\|\mathcal{C}_{l-1,m}(\mathcal{A}_{\chi,l-1}-\lambda)^{-1}x_{l-1}-\widetilde{\mathcal{C}}_{l+1,m}(\mathcal{A}_{\chi,l+1}-\lambda)^{-1}x_{l+1}\right\|^2\\
&\le\left\|\widetilde{\mathcal{C}}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}\right\|^2\|x_{k_m+1}\|^2\\
&+\sum_{l=k_m+1}^{\infty}2\Big(\left\|\mathcal{C}_{l-1,m}(\mathcal{A}_{\chi,l-1}-\lambda)^{-1}\right\|^2\|x_{l-1}\|^2+\left\|\widetilde{\mathcal{C}}_{l+1,m}(\mathcal{A}_{\chi,l+1}-\lambda)^{-1}\right\|^2\|x_{l+1}\|^2\Big)\\
&=2\left\|\mathcal{C}_{k_m,m}(\mathcal{A}_{\chi,k_m}-\lambda)^{-1}\right\|^2\|x_k\|^2\\
&+\sum_{l=k_m+2}^{\infty}2\Big(\left\|\mathcal{C}_{l,m}(\mathcal{A}_{\chi,l}-\lambda)^{-1}\right\|^2+\left\|\widetilde{\mathcal{C}}_{l,m}(\mathcal{A}_{\chi,l}-\lambda)^{-1}\right\|^2\Big)\|x_l\|^2\\
&\quad+\Big(2\left\|\mathcal{C}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}\right\|^2+\left\|\widetilde{\mathcal{C}}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}\right\|^2\Big)\|x_{k_m+1}\|^2\\
&\le\max\left\{\begin{matrix}2\left\|\mathcal{C}_{k_m,m}(\mathcal{A}_{\chi,k_m}-\lambda)^{-1}\right\|^2,\\ 2\left\|\mathcal{C}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}\right\|^2+\left\|\widetilde{\mathcal{C}}_{k_m+1,m}(\mathcal{A}_{\chi,k_m+1}-\lambda)^{-1}\right\|^2,\\ \sup_{l\ge k_m+2}2\Big(\left\|\mathcal{C}_{l,m}(\mathcal{A}_{\chi,l}-\lambda)^{-1}\right\|^2+\left\|\widetilde{\mathcal{C}}_{l,m}(\mathcal{A}_{\chi,l}-\lambda)^{-1}\right\|^2\Big)\end{matrix}\right\}. \qquad (5.8)
\end{aligned}
$$

The inclusions $\mathcal{D}(\mathcal{A}_{\alpha^2}(a,m))\subset\mathcal{D}(\mathcal{C}^m)$, $\mathcal{D}(\mathcal{A}_\alpha(a,m))\subset\mathcal{D}(\mathcal{C}^m)$ follow from the fact that the above supremum is finite in both cases. The latter is a consequence of the following two properties: first, the asymptotic behavior $c_{l,m}=\mathrm{O}(l)$, $\widetilde{c}_{l,m}=\mathrm{O}(l)$, $l\to\infty$, see Notation 2.2, of the only nonzero entry of the 2×2 matrices $\mathcal{C}_{l,m}$, $\widetilde{\mathcal{C}}_{l,m}$, see Definition 3.3; second, the decay of the resolvents proved in Propositions 5.2 i) and 5.5 i), which amounts to

$$
\left\|(\mathcal{A}_{\alpha^2,l}(a)-\lambda)^{-1}\right\|=\mathrm{O}(l^{-2}),\quad \left\|(\mathcal{A}_{\alpha,l}(a)-\lambda)^{-1}\right\|=\mathrm{O}(l^{-2}),\quad l\to\infty; \qquad (5.9)
$$

here we have used $\lambda_l(\vartheta)\ge l(l+1)$, $\vartheta\in\{l,\infty\}$, by Proposition 3.5 and $\gamma_{\alpha^2,l}=\mathrm{O}(1)$, $\gamma_{\alpha,l}=\mathrm{O}(1)$, $l\to\infty$; see (4.1).

The first Schur complement $S_{1,l}^{\chi}(a,\lambda)$ of $\mathcal{A}_{\chi,l}(a)$ is given by

$$
S_{1,l}^{\alpha^2}(a,\lambda):=-A_l(a,l)-\lambda-\alpha(-A_l(a,\infty)-\lambda)^{-1}A_{l,\alpha}(a,l),\quad S_{1,l}^{\alpha}(a,\lambda):=-A_l(a,l)-\lambda.
$$

The assumptions on λ ensure that $S_{1,l}^{\chi}(a,\lambda)$ is boundedly invertible and that Lemma 5.1 is applicable, which implies the estimates for $\|S_{1,l}^{\chi}(a,\lambda)^{-1}\|$ in (5.6), (5.7) in a straightforward way.

Using the special form of $\mathcal{C}_{l,m}$, $\widetilde{\mathcal{C}}_{l,m}$, see Definition 3.3, and the representation of the resolvent of $\mathcal{A}_{\chi,l}(a)$ in terms of its first Schur complement, see [24, Thm. 2.3.3 ii)], we find that, for $l\ge k_m$,

$$
\begin{pmatrix}0&0\\1&0\end{pmatrix}(\mathcal{A}_{\chi,l}(a)-\lambda)^{-1}=\begin{pmatrix}0&0\\ S_{1,l}^{\chi}(a,\lambda)^{-1}&-S_{1,l}^{\chi}(a,\lambda)^{-1}\alpha(-A_l(a,\infty)-\lambda)^{-1}\end{pmatrix}
$$

and hence, again by Lemma 5.1,

$$
\left\|\begin{pmatrix}0&0\\1&0\end{pmatrix}(\mathcal{A}_{\chi,l}(a)-\lambda)^{-1}\right\|\le s_{1,l}^{\chi}(a,\lambda)\sqrt{1+\frac{\|\alpha\|^2}{(\mathrm{Re}(\lambda)+\lambda_{l,1}(a,\infty))^2}}. \qquad (5.10)
$$

This, together with the estimates (5.8), proves the claimed bound. ∎

LEMMA 5.8. *Let $m \in \mathbb{Z}$, $k_m := \max\{|m|, 1\}$, $a \in [0,1)$, $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$. Then $\|\omega' \mathcal{C}^m (\mathcal{A}_{\alpha^2}(a,m) - \lambda)^{-1}\| < \gamma_2$ for $\gamma_2 > 0$ if*

$$\mathrm{Re}(\lambda) > \max_{l=k_m}^{k_m+2} \left\{ -\lambda_{l,1}(a,\infty) + \gamma_{\alpha^2,l}(a)\|\alpha\| + \delta_{l,m} \frac{\sqrt{\lambda_{l,1}(a,\infty)^2 + \|\alpha\|^2}}{\lambda_{l,1}(a,l)} \frac{\|\omega'\|}{\gamma_2} \right\} \tag{5.11}$$

with $\delta_{l,m}$ given by (4.5).

Proof. If $\|\omega'\| = 0$, there is nothing to prove, so we assume that $\|\omega'\| \neq 0$. Let $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$, be as above. Condition (5.11) on $\mathrm{Re}(\lambda)$ implies that $\mathrm{Re}(\lambda) > -\lambda_{k_m,1}(a,\infty) + \gamma_{\alpha^2,k_m}(a)\|\alpha\|$, and hence the assumptions on λ in Lemma 5.7 for $\chi = \alpha^2$ are satisfied. The claim is proved if we show, by means of Lemma 5.7, that

$$\left\|\mathcal{C}^m (\mathcal{A}_{\alpha^2}(a,m) - \lambda)^{-1}\right\| < \frac{\gamma_2}{\|\omega'\|}. \tag{5.12}$$

By Lemma 5.7 with (5.6), we have to show that, for $l = k_m$, k_m+1, k_m+2,

$$\delta_{l,m} \sqrt{1 + \frac{\|\alpha\|^2}{(\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty))^2}} \frac{1}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,l)} \frac{1}{1 - \frac{\gamma_{\alpha^2,l}(a)\|\alpha\|}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty)}} < \frac{\gamma_2}{\|\omega'\|},$$

with $\delta_{l,m}$ given by (4.5) or, equivalently,

$$\delta_{l,m} \frac{\sqrt{(\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty))^2 + \|\alpha\|^2}}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,l)} \frac{1}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty) - \gamma_{\alpha^2,l}(a)\|\alpha\|} < \frac{\gamma_2}{\|\omega'\|}.$$

For $\mathrm{Re}(\lambda) \geq 0$ we can estimate

$$\frac{\sqrt{(\mathrm{Re}(\lambda) + \lambda_{l,1}(a,\infty))^2 + \|\alpha\|^2}}{\mathrm{Re}(\lambda) + \lambda_{l,1}(a,l)} \leq \frac{\sqrt{\lambda_{l,1}(a,\infty)^2 + \|\alpha\|^2}}{\lambda_{l,1}(a,l)};$$

in fact, it is easy to check that, since $\lambda_{l,1}(a,\infty) > \lambda_{l,1}(a,l)$, the left-hand side is monotonically decreasing for $\mathrm{Re}(\lambda) \in [0,\infty)$. Altogether, this shows that (5.12) is satisfied if (5.11) holds. ∎

LEMMA 5.9. *Let $m \in \mathbb{Z}$, $k_m := \max\{|m|, 1\}$, $a \in [0,1)$, and $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$. Then $\|\omega' \mathcal{C}^m (\mathcal{A}_{\alpha}(a,m) - \lambda)^{-1}\| < \gamma_2$ for $\gamma_2 > 0$ if*

$$\mathrm{Re}(\lambda) > \max_{l=k_m}^{k_m+2} \left\{ -\lambda_{l,1}(a,l) + \delta_{l,m} \sqrt{1 + \frac{\|\alpha\|^2}{\lambda_{l,1}(a,\infty)^2}} \frac{\|\omega'\|}{\gamma_2} \right\} \tag{5.13}$$

with $\delta_{l,m}$ given by (4.5).

Proof. The proof is analogous to the proof of Lemma 5.8 if we use Lemma 5.7 with (5.7). ∎

Now we are ready to combine the above results to prove the three antidynamo theorems in section 3.

Proof of Theorem 4.2 (α^2*-model*). Let $m \in \mathbb{Z}$. We apply Corollary 5.4 with $a = 0$ and $\gamma_1 = 1$, using the definition of γ_{α^2,k_m} in Proposition 4.1 and noting that the eigenvalues $\lambda_{l,1}(0,\vartheta)$, $\vartheta \in \{k_m, \infty\}$, are given by Bessel zeros; see (3.4). Assumption (4.1) ensures that the lower bound for $\mathrm{Re}(\lambda)$ in Corollary 5.4 is ≤ 0 so that $\|\mathrm{i}m\omega(\mathcal{A}_{\alpha^2}(0,m) - \lambda)^{-1}\| < 1$ for all $\mathrm{Re}(\lambda) > 0$ and thus

$$\mathcal{A}_{\alpha^2}(0,m) - \lambda - \mathrm{i}m\omega = \left(I - \mathrm{i}m\omega(\mathcal{A}_{\alpha^2}(0,m) - \lambda)^{-1}\right)(\mathcal{A}_{\alpha^2}(0,m) - \lambda)$$

is boundedly invertible for $\mathrm{Re}(\lambda) > 0$. Due to Definition 3.3, this shows that

$$\sigma(\mathcal{A}^m_{\alpha^2}(0)) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} = \emptyset,$$

and hence the claim for $m \in \mathbb{Z}$; the claim for the special case $m = 0$ is an immediate consequence if we note that then $k_m = \max\{|m|, 1\} = 1$. ◻

Proof of Theorem 4.7 ($\alpha^2\omega$*-model*). Let $m \in \mathbb{Z}$. We apply Corollary 5.4 and Lemma 5.8 with $a=0$ and $\gamma_1, \gamma_2 \in [0,1]$ such that $\gamma_1+\gamma_2 \le 1$, using the definition of γ_{α^2,k_m} in Proposition 4.1 and noting that the eigenvalues $\lambda_{l,1}(0,\vartheta)$, $\vartheta \in \{k_m, \infty\}$, are given by Bessel zeros; see (3.4). Assumption (4.6) ensures that the lower bound for $\mathrm{Re}(\lambda)$ in Corollary 5.4 is ≤ 0, while assumptions (4.7) ensure that the lower bound for $\mathrm{Re}(\lambda)$ in Lemma 5.8 is ≤ 0. Together this yields $\big\|im\omega(\mathcal{A}_{\alpha^2}(0,m)-\lambda)^{-1} - \omega'\mathcal{C}^m(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\big\| < 1$ for all $\mathrm{Re}(\lambda) > 0$, and thus

$$\mathcal{A}_{\alpha^2\omega}(0,m)-\lambda-\mathrm{i}m\omega+\omega'\mathcal{C}^m = \big(I-(\mathrm{i}m\omega-\omega'\mathcal{C}^m)(\mathcal{A}_{\alpha^2\omega}(0,m)-\lambda)^{-1}\big)(\mathcal{A}_{\alpha^2\omega}(0,m)-\lambda)$$

is boundedly invertible for $\mathrm{Re}(\lambda) > 0$. Due to Definition 3.3, this shows that

$$\sigma(\mathcal{A}^m_{\alpha^2\omega}(0)) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} = \emptyset,$$

and thus the claim for $m\in\mathbb{Z}$; the claim for the special case $m=0$ is obvious. ◻

Proof of Theorem 4.10 ($\alpha^2\omega$*-model*). The proof is analogous to the proof of Theorem 4.7 for the $\alpha^2\omega$-model if we use Lemma 5.6 instead of Lemma 5.4 and Lemma 5.9 instead of Lemma 5.8. ◻

6. Spectral exactness of interval truncation and finite section method.

Since all three dynamo models are described by nonselfadjoint linear operators, numerical computations are prone to two undesirable effects, spectral pollution and failure of spectral inclusion. The former means that numerically computed approximations of eigenvalues converge to points that are no true eigenvalues; such points are called spurious eigenvalues. The latter means that not all true eigenvalues are obtained as limits of numerical approximations. Both effects may be fatal since spectral pollution may wrongly indicate the existence of a dynamo effect and failure of spectral inclusion may wrongly exclude a dynamo effect.

Numerical eigenvalue approximations for all three dynamo models require interval truncation at the singular endpoint $r = 0$ as well as truncation of the infinite operator matrices to finite sections. However, results guaranteeing that these two approximation schemes are spectrally exact, i.e., that they do not exhibit spectral pollution and spectral inclusion prevails, are lacking.

In this section we close this gap and show that, for all dynamo models, interval truncation and finite section method are spectrally exact. In general, spectral exactness is equivalent to convergence of the spectra in the Attouch–Wets metric, cf. [4, p. 28]; since both approximating operators and limit operator have compact resolvents, we can use locally uniform convergence here.

To establish spectral exactness, we prove that the corresponding approximating operator sequences have discretely compact resolvents and converge in generalized strong resolvent sense, allowing us to employ a powerful spectral convergence result; see [3, Thm. 2.19]. For the reader's convenience, we briefly recall the notion of discrete compactness due to Stummel; see [23, Def. 3.1.(k)] or [3, Def. 2.5].

Definition 6.1. *Let E_0 be a Banach space, E, $E_n \subset E_0$, $n \in \mathbb{N}$, closed complemented subspaces, and D_n arbitrary Banach spaces. A sequence $(A_n)_{n\in\mathbb{N}}$ of bounded* *linear operators* $A_n \in L(D_n, E_n)$ *is called* discretely compact *if for each infinite subset* $I \subset \mathbb{N}$ *and each bounded sequence of elements* $x_n \in D_n$*,* $n \in I$*, there exist* $y \in E$ *and an infinite subset* $J \subset I$ *so that* $\|A_n x_n - y\|_{E_0} \to 0$ *for* $n \in J$*,* $n \to \infty$*.*

The generality of this concept makes it applicable to different situations such as interval truncation and finite section method. In our case E_0 is the Hilbert space $\mathcal{H}(0) = l^2\big(L^2(0,1) \oplus L^2(0,1) : i \in \mathbb{N}\big)$. For interval truncation, we choose $E = E_0$ and $E_n = \mathcal{H}(a_n) = l^2\big(L^2(a_n,1) \oplus L^2(a_n,1) : i \in \mathbb{N}\big)$, $n \in \mathbb{N}$, with $a_n \searrow 0$. For the finite section method, we take $E = \mathcal{H}(a) = l^2\big(L^2(a,1) \oplus L^2(a,1) : i \in \mathbb{N}\big)$ with $a \in [0,1)$ and $E_n = \mathcal{H}_n(a) := \bigoplus_{i=1,\dots,n} L^2(a,1) \oplus L^2(a,1)$.

Note that we tacitly regard $L^2(a,1)$ as a subspace of $L^2(0,1)$, whence $\mathcal{H}(a)$ as a subspace of $\mathcal{H}(0)$, and $\mathcal{H}_n(a) = \bigoplus_{i=1,\dots,n} L^2(a,1) \oplus L^2(a,1)$ as a subspace of $\mathcal{H}(a) = l^2\big(L^2(a,1) \oplus L^2(a,1) : i \in \mathbb{N}\big)$.

Definition 6.2. *For* $a \in [0,1)$ *and* $j \in \mathbb{N}$ *define*

$$
\begin{aligned}
&P(a) : L^2(0,1) \to L^2(a,1), \quad P(a)x := x|_{[a,1]},\\
&\mathcal{P}(a) : L^2(0,1) \oplus L^2(0,1) \to L^2(a,1) \oplus L^2(a,1), \quad \mathcal{P}(a) := \operatorname{diag}\big(P(a), P(a)\big),\\
&\mathcal{P}^\infty(a) : \mathcal{H}(0) \to \mathcal{H}(a), \quad \mathcal{P}^\infty(a) := \operatorname{diag}\big(\mathcal{P}(a) : i \in \mathbb{N}\big),
\end{aligned}
$$

where $\mathcal{H}(a) := l^2\big(L^2(a,1) \oplus L^2(a,1) : i \in \mathbb{N}\big)$*, see* (3.2)*, and*

$$
\mathcal{Q}_j(a) : \mathcal{H}(a) \to \mathcal{H}_j(a), \quad \mathcal{Q}_j(a)((x_i, y_i))_{i \in \mathbb{N}} := ((x_i, y_i))_{i=1}^j,
$$

where $\mathcal{H}_j(a) := \bigoplus_{i=1,\dots,j} L^2(a,1) \oplus L^2(a,1)$*.*

Throughout this section, let $(a_n)_{n \in \mathbb{N}} \subset (0,1)$, $a_n \searrow 0$ as $n \to \infty$, be a decreasing sequence.

Remark 6.3. It is easy to see that $P(a_n) \xrightarrow{s} I_{L^2(0,1)}$, $\mathcal{P}(a_n) \xrightarrow{s} I_{L^2(0,1) \oplus L^2(0,1)}$, and $\mathcal{P}^\infty(a_n) \xrightarrow{s} I_{\mathcal{H}(0)}$ as $n \to \infty$, see [3, Cor. 3.15], and that $\mathcal{Q}_j(a) \xrightarrow{s} I_{\mathcal{H}(a)}$ as $j \to \infty$ for every $a \in [0,1)$.

Lemma 6.4. *For every* $l \in \mathbb{N}$ *and* $\vartheta \in \{l, \infty\}$*, the subspace*

$$
\Phi_l(\vartheta) := \big\{ x \in \mathcal{D}(A_l(0,\vartheta)) : \exists\, \varepsilon \in (0,1)\ x|_{[0,\varepsilon)} \equiv 0 \big\} \subset L^2(0,1)
$$

is a core of $A_l(0,\vartheta)$*; moreover, for all* $x \in \Phi_l(\vartheta)$ *there exists* $n_0(x) \in \mathbb{N}$ *with*

$$
\begin{aligned}
&P(a_n)x \in \mathcal{D}(A_l(a_n,\vartheta)), \quad n \ge n_0(x),\\
&\big\|A_l(a_n,\vartheta)P(a_n)x - A_l(0,\vartheta)x\big\| \longrightarrow 0, \quad n \to \infty.
\end{aligned}
\tag{6.1}
$$

Proof. Since the Bessel differential expression τ_l is in limit point case at the singular endpoint $r = 0$, the core property of $\Phi_l(\vartheta)$ is immediate from Sturm–Liouville theory; see, e.g., the proof of [25, Satz 14.12].

For $x \in \Phi_l(\vartheta)$ let $n_0(x) \in \mathbb{N}$ be such that $x(r) = 0$ for $r \in [0, a_{n_0(x)}]$. This implies $P(a_n)x \in \mathcal{D}(A_l(a_n,\vartheta))$ for $n \ge n_0(x)$. The convergence (6.1) follows from $A_l(a_n,\vartheta)P(a_n)x = P(a_n)A_l(0,\vartheta)x$, $n \ge n_0(x)$, and from the strong convergence $P(a_n) \xrightarrow{s} I_{L^2(0,1)}$, $n \to \infty$. ∎

First, we establish spectral exactness of interval truncation to $(a_n, 1]$, $a_n \searrow 0$ for the 2×2 operator matrices $\mathcal{A}_{\alpha^2,l}(0)$, $\mathcal{A}_{\alpha,l}(0)$, $l \in \mathbb{N}$, which are the leading operators on the diagonals of the three infinite dynamo operator matrices $\mathcal{A}^m_{\alpha^2}(0)$, $\mathcal{A}^m_{\alpha^2\omega}(0)$, and $\mathcal{A}^m_{\alpha\omega}(0)$, $m \in \mathbb{Z}$, respectively; see Definition 3.3.

THEOREM 6.5. *Let $l \in \mathbb{N}$ be fixed, and let $\mathcal{A}_{\chi,l}(0)$, $\chi \in \{\alpha^2, \alpha\}$, denote one of the 2×2 operator matrices $\mathcal{A}_{\alpha^2,l}(0)$ or $\mathcal{A}_{\alpha,l}(0)$ in $L^2(0,1)\oplus L^2(0,1)$, and let $\big(\mathcal{A}_{\chi,l}(a_n)\big)_{n\in\mathbb{N}}$ denote its approximating sequence in $L^2(a_n,1)\oplus L^2(a_n,1)$ with $a_n \searrow 0$. Then there exists $u_0 \in \mathbb{R}$, independent of $l \in \mathbb{N}$, so that, for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > u_0$,*

(a) $\lambda \in \bigcap_{n\in\mathbb{N}} \varrho\big(\mathcal{A}_{\chi,l}(0)\big) \cap \varrho\big(\mathcal{A}_{\chi,l}(a_n)\big)$ *and* $\sup_{n\in\mathbb{N}} \big\|\big(\mathcal{A}_{\chi,l}(a_n) - \lambda\big)^{-1}\big\| < \infty$*;*

(b) $(\mathcal{A}_{\chi,l}(0) - \lambda)^{-1}$*,* $\big(\mathcal{A}_{\chi,l}(a_n) - \lambda\big)^{-1}$*,* $n \in \mathbb{N}$*, are compact;*

(c) $\big((\mathcal{A}_{\chi,l}(a_n) - \lambda)^{-1}\big)_{n\in\mathbb{N}}$ *is discretely compact;*

(d) $(\mathcal{A}_{\chi,l}(a_n) - \lambda)^{-1}\mathcal{P}(a_n) \xrightarrow{s} (\mathcal{A}_{\chi,l}(0) - \lambda)^{-1}$*,* $n \to \infty$*.*

Hence the approximation $\big(\mathcal{A}_{\chi,l}(a_n)\big)_{n\in\mathbb{N}}$ of $\mathcal{A}_{\chi,l}(0)$ is spectrally exact, i.e., no spectral pollution occurs and spectral inclusion prevails. Moreover,

(i) *if $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_{\chi,l}(0)$ of algebraic multiplicity κ_0, then, for n large enough, $\mathcal{A}_{\chi,l}(a_n)$ has exactly κ_0 eigenvalues λ_i^n, $i = 1, 2, \dots, \kappa_0$, (counted with algebraic multiplicities) in a neighborhood of λ_0 that converge to λ_0 as $n \to \infty$,*

(ii) *all normalized eigenvectors and associated vectors for λ_i^n, $i = 1, 2, \dots, \kappa_0$, viewed as elements of $L^2(0,1)\oplus L^2(0,1)$ converge to eigenvectors and associated vectors for λ_0.*

Proof. Let $l \in \mathbb{N}$ be fixed. We show that in each claim (a), (b), (c), (d) there exists a $u \in \mathbb{R}$, independent of $l \in \mathbb{N}$, such that the respective assertions hold for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq u$.

(a) The claims follow from Propositions 5.2 (i) and 5.5 (i) if we use the lower bounds $\lambda_l(a,\vartheta) \geq l(l+1) \geq 2$ for $a \in [0,1)$, $\vartheta \in \{l, \infty\}$; see (3.3). More precisely, if $\chi = \alpha^2$, we can use Proposition 5.2 (i) and the equivalent form (5.4) of the bound therein to estimate that, for $\mathrm{Re}(\lambda) > u_{\alpha^2} := \max\big\{0, -2 + \big(\|\alpha\| + \frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|\big\}$,

$$\sup_{n\in\mathbb{N}} \big\|(\mathcal{A}_{\alpha^2,l}(a) - \lambda)^{-1}\big\| \leq \frac{1}{\mathrm{Re}(\lambda) + 2 - \big(\|\alpha\| + \frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|}\left(1 + \left(\|\alpha\| + \frac{\|\alpha'\|}{\sqrt{2}}\right)\|\alpha\|\right);$$

if $\chi = \alpha$, we use Proposition 5.5 (i) for $\mathrm{Re}(\lambda) > u_\alpha := 0$.

(b) In [9, Thm. 4.1] it was proved that the operator $\mathcal{A}_{\alpha^2,l}(0)$ has compact resolvent; the proof for $\mathcal{A}_{\alpha^2,l}(a_n)$, $n \in \mathbb{N}$, is analogous. The compactness of $(\mathcal{A}_{\alpha,l}(a) - \lambda)^{-1}$, $n \in \mathbb{N}$, for $\mathrm{Re}(\lambda) > u_\alpha$ follows from (5.5) because $A_l(a_n, l)$, $A_l(a_n, \infty)$ have compact resolvents and α is bounded.

(c) We prove the claim for $\mathcal{A}_{\alpha^2,l}(a_n)$, $n \in \mathbb{N}$; the proof for $\mathcal{A}_{\alpha,l}(a_n)$, $n \in \mathbb{N}$, is analogous. To this end, we show that the operators $\mathcal{A}_{\alpha^2,l}(a_n)$, $n \in \mathbb{N}$, satisfy the criteria (i), (ii), (iii) for discrete compactness derived in [3, Prop. 4.6].

(i) By [3, Prop. 2.12 ii) and Ex. 4.17], the diagonal entries $A_l(a_n, l)$, $A_l(a_n, \infty)$ of $\mathcal{A}_{\alpha^2,l}(a_n)$, $n \in \mathbb{N}$, have discretely compact resolvents.

(ii) The operator matrices $\mathcal{A}_{\alpha^2,l}(a_n)$, $n \in \mathbb{N}$, are diagonally dominant since α is bounded and $\mathcal{D}(A_l(a_n, l)) = \mathcal{D}(A_{l,\alpha}(a_n, l))$ which implies that $A_{l,\alpha}(a_n, l)$ is $A_l(a_n, l)$-bounded.

(iii) Let $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > u_{\alpha^2} = \max\big\{0, -2 + \big(\|\alpha\| + \frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|\big\}$. Then, by Lemma 5.1,

$$\lambda \in \left(\bigcap_{n\in\mathbb{N}} \varrho(-A_l(a_n, l))\right) \cap \left(\bigcap_{n\in\mathbb{N}} \varrho(-A_l(a_n, \infty))\right),$$

both $\sup_{n\in\mathbb{N}} \big\|(-A_l(a_n, l) - \lambda)^{-1}\big\|$, $\sup_{n\in\mathbb{N}} \big\|(-A_l(a_n, \infty) - \lambda)^{-1}\big\|$ are finite and, for all $n \in \mathbb{N}$,

$$\left\|A_{l,\alpha}(a_n,l)(-A_l(a_n,l)-\lambda)^{-1}\right\| \leq \|\alpha\| + \frac{\|\alpha'\|}{\sqrt{2}} =: \gamma_\lambda^1,$$

$$\|\alpha(-A_l(a_n,\infty)-\lambda)^{-1}\| \leq \frac{\|\alpha\|}{\mathrm{Re}(\lambda)+2} =: \gamma_\lambda^2,$$

where $\gamma_\lambda^1\gamma_\lambda^2<1$ due to the assumption on $\mathrm{Re}(\lambda)$.

Now [3, Prop. 4.6] yields that $\big((\mathcal{A}_{\alpha^2,l}(a_n)-\lambda)^{-1}\big)_{n\in\mathbb{N}}$ is discretely compact for each $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>u_{\alpha^2}$.

(d) We prove the claim for the operators $\mathcal{A}_{\alpha^2,l}(0)$, $\mathcal{A}_{\alpha^2,l}(a_n)$, $n\in\mathbb{N}$; the proof for $\mathcal{A}_{\alpha,l}(0)$, $\mathcal{A}_{\alpha,l}(a_n)$, $n\in\mathbb{N}$, is analogous. To this end, we show that the operators $\mathcal{A}_{\alpha^2,l}(0)$, $\mathcal{A}_{\alpha^2,l}(a_n)$, $n\in\mathbb{N}$, satisfy the assumptions (i) to (iv) in [3, Prop. 3.9].

(i) The operator matrices $\mathcal{A}_{\alpha^2,l}(0)$, $\mathcal{A}_{\alpha^2,l}(a_n)$, $n\in\mathbb{N}$, are diagonally dominant by (ii) in the proof of (c).

(ii) and (iii) By Lemma 6.4, for $\vartheta\in\{l,\infty\}$, there exists a core $\Phi_l(\vartheta)\subset\mathcal{D}(A_l(0,\vartheta))$ of $A_l(0,\vartheta)$ so that for all $x\in\Phi_l(\vartheta)$ there exists $n_0(x)\in\mathbb{N}$ with

$$\begin{aligned}
&P(a_n)x\in\mathcal{D}(A_l(a_n,\vartheta)), \quad n\geq n_0(x),\\
&\|A_l(a_n,\vartheta)P(a_n)x-A_l(0,\vartheta)x\| = \left\|(I_{L^2(0,1)}-P(a_n))A_l(a_n,\vartheta)x\right\|\to 0,\\
&\|A_{l,\alpha}(a_n,l)P(a_n)x-A_{l,\alpha}(0,l)x\| = \|(I_{L^2(0,1)}-P(a_n))A_{l,\alpha}(a_n,l)x\|\to 0,\\
&\|\alpha P(a_n)x-\alpha x\| = \|(I-P(a_n))\alpha x\|\to 0
\end{aligned}$$

for $n\to\infty$ since $P(a_n)\xrightarrow{s} I_{L^2(0,1)}$, $n\to\infty$.

(iv) This assumption follows since (a) holds for $\mathrm{Re}(\lambda)>u_\chi$ if we observe [3, Def. 2.1 (iii)].

Now [3, Prop. 3.9], together with [3, Prop. 2.11 i)] and (a), yields that (d) holds for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>u_\chi$, where u_χ is as defined in the proof of (a).

Altogether, we proved that all assumptions of [3, Thm. 2.6] are satisfied and the latter yields all the claims. ∎

LEMMA 6.6. *Let $m\in\mathbb{Z}$, $a_1\in(0,1)$, and $\chi\in\{\alpha^2,\alpha\}$. Then, for every $\gamma>0$, there exists $u_{\gamma,\chi}\in\mathbb{R}$ independent of $a\in[0,a_1]$, so that, for $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>u_{\gamma,\chi}$, we have $\lambda\in\varrho(\mathcal{A}_\chi(a,m))$ with*

$$\begin{aligned}
\left\|(\mathcal{A}_\chi(a,m)-\lambda)^{-1}\right\| &< \gamma,\\
\left\|\mathrm{i}m\omega(\mathcal{A}_\chi(a,m)-\lambda)^{-1}\right\| &< \gamma,\\
\left\|\omega'\mathcal{C}^m(\mathcal{A}_\chi(a,m)-\lambda)^{-1}\right\| &< \gamma.
\end{aligned}$$

Proof. It suffices to show that each estimate holds for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>u$ with $u\in\mathbb{R}$ sufficiently large and independent of $a\in[0,a_1]$. For this we use $1<l(l+1)\leq\lambda_l(a,\vartheta)\leq\lambda_l(a_1,\vartheta)$ for $l\in\mathbb{N}$, $\vartheta\in\{l,\infty\}$ and $a\in[0,a_1]$; see Proposition 3.5. Then the claims follow from Proposition 5.2 (ii), Corollary 5.4, and Lemma 5.8 if $\chi=\alpha^2$ and from Proposition 5.5 (ii), Corollary 5.6, and Lemma 5.9 if $\chi=\alpha$. ∎

The following comprehensive theorem combines both necessary approximation schemes, interval truncation $a_n\searrow 0$ for $n\to\infty$, and truncation to finite $2j\times 2j$ operator matrices for $j\to\infty$, by letting $j=n\to\infty$. It provides spectral exactness for this combined approximation scheme and for all three dynamo models.

THEOREM 6.7. *Let $m\in\mathbb{Z}$ be fixed, and let $\mathcal{A}_\chi^m(0)$, $\chi\in\{\alpha^2,\alpha^2\omega,\alpha\omega\}$, denote one of the infinite dynamo operator matrices $\mathcal{A}_{\alpha^2}^m(0)$, $\mathcal{A}_{\alpha^2\omega}^m(0)$, $\mathcal{A}_{\alpha\omega}^m(0)$ in $\mathcal{H}(0)=l^2\big(L^2(0,1)\oplus L^2(0,1) : i\in\mathbb{N}\big)$, and consider the combined approximating sequence*

$\big(\mathcal{A}^m_{\chi,n}(a_n)\big)_{n\in\mathbb{N}}$ in $\mathcal{H}_n(a_n) = \bigoplus_{i=1,\dots,n} L^2(a_n,1)\oplus L^2(a_n,1)$ with $a_n \searrow 0,\ n\to\infty$, given by

$$\mathcal{A}^m_{\chi,n}(a_n) := \mathcal{Q}_n(a_n)\mathcal{A}^m_\chi(a_n), \quad \mathcal{D}\big(\mathcal{A}^m_{\chi,n}(a_n)\big) := \mathcal{D}\big(\mathcal{A}^m_\chi(a_n)\big)\cap\mathcal{H}_n(a_n), \quad n\in\mathbb{N}.$$

Then there exists $u_0\in\mathbb{R}$ such that, for every $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > u_0$,

(a) $\lambda\in\bigcap_{n\in\mathbb{N}}\varrho\big(\mathcal{A}^m_{\chi,n}(0)\big)\cap\varrho\big(\mathcal{A}^m_\chi(a_n)\big)$ *and* $\sup_{n\in\mathbb{N}}\big\|\big(\mathcal{A}^m_{\chi,n}(a_n)-\lambda\big)^{-1}\big\| < \infty$*;*

(b) $(\mathcal{A}^m_\chi(0)-\lambda)^{-1}$, $\big(\mathcal{A}^m_{\chi,n}(a_n)-\lambda\big)^{-1}$, $n\in\mathbb{N}$, *are compact;*

(c) $\Big(\big(\mathcal{A}^m_{\chi,n}(a_n)-\lambda\big)^{-1}\Big)_{n\in\mathbb{N}}$ *is discretely compact;*

(d) $\big(\mathcal{A}^m_{\chi,n}(a_n)-\lambda\big)^{-1}\mathcal{Q}_n(a_n)\xrightarrow{s}(\mathcal{A}^m_\chi(0)-\lambda)^{-1}, \quad n\to\infty.$

Hence $\big(\mathcal{A}^m_{\chi,n}(a_n)\big)_{n\in\mathbb{N}}$ is a spectrally exact approximation of $\mathcal{A}^m_\chi(0)$, i.e., no spectral pollution occurs and spectral inclusion prevails. Moreover,

(i) *if $\lambda_0\in\mathbb{C}$ is an eigenvalue of $\mathcal{A}^m_\chi(0)$ of algebraic multiplicity κ_0, then, for n large enough, $\mathcal{A}^m_{\chi,n}(a_n)$ has exactly κ_0 eigenvalues λ^n_i, $i=1,2,\dots,\kappa_0$, (counted with algebraic multiplicities) in a neighborhood of λ_0 that converge to λ_0 as $n\to\infty$,*

(ii) *all normalized eigenvectors and associated vectors for λ^n_i, $i=1,2,\dots,\kappa_0$, when viewed as elements of $\mathcal{H}(0)=l^2\big(L^2(0,1)\oplus L^2(0,1) : i\in\mathbb{N}\big)$, converge to eigenvectors and associated vectors for λ_0.*

Remark 6.8. Note that $\mathcal{Q}_n(a_n) = \mathcal{Q}_n(0)\mathcal{P}^\infty(a_n) = \mathcal{P}^\infty(a_n)\mathcal{Q}_n(0)$, $n\in\mathbb{N}$. Then $\mathcal{Q}_n(a_n)\xrightarrow{s} I_{\mathcal{H}(0)}$ follows using $\mathcal{Q}^\infty(a_n)\xrightarrow{s} I_{\mathcal{H}(0)}$ and $\mathcal{Q}_n(0)\xrightarrow{s} I_{\mathcal{H}(0)}$.

Proof of Theorem 6.7. The proof will be given in two steps. In the first step we establish spectral exactness of interval truncation to $(a_n,1]$, $a_n\searrow 0$, for the three infinite dynamo operator matrices $\mathcal{A}^m_{\alpha^2}(0)$, $\mathcal{A}^m_{\alpha^2\omega}(0)$, and $\mathcal{A}^m_{\alpha\omega}(0)$, $m\in\mathbb{Z}$; in the second step we prove spectral exactness of the combination of interval truncation and finite section methods.

Step 1. *Spectral exactness of interval truncation to $(a_n,1]$, $a_n\searrow 0$.* Here we show that all claims in Theorem 6.7 hold for the approximating sequence $\big(\mathcal{A}^m_\chi(a_n)\big)_{n\in\mathbb{N}}$ in $\mathcal{H}(a_n) = l^2\big(L^2(a_n,1)\oplus L^2(a_n,1) : i\in\mathbb{N}\big)$ of $\mathcal{A}^m_\chi(0)$ where, in (d), we have to replace $\mathcal{Q}_n(a_n) = \mathcal{P}^\infty(a_n)\mathcal{Q}_n(0)$ by $\mathcal{P}^\infty(a_n)$ alone for $n\in\mathbb{N}$.

Let $m\in\mathbb{Z}$ be fixed, and let $k_m := \max\{|m|,1\}$. We prove the claims for the α^2-model and the $\alpha^2\omega$-model; the proof for the $\alpha\omega$-model is analogous, e.g., we have to replace $\mathcal{A}_{\alpha^2}(a,m)$ by $\mathcal{A}_\alpha(a,m)$.

We show that in each claim (a), (b), (c), (d) there exists a $u\in\mathbb{R}$ such that the assertions hold for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > u$. To this end, we first consider the infinite block diagonal operators $\mathcal{A}_{\alpha^2}(a,m)$, $\mathcal{A}_{\alpha^2}(a_n,m)$, $n\in\mathbb{N}$, and then we employ perturbation results.

(a) For $\chi\in\{\alpha^2,\alpha^2\omega\}$ and $a\in[0,1)$, we write

$$\mathcal{A}^m_\chi(a)-\lambda \;= \big(I - K_\chi(a,m)(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\big)(\mathcal{A}_{\alpha^2}(a,m)-\lambda), \tag{6.2}$$

where

$$K_\chi(a,m) := \begin{cases} im\omega & \text{if } \chi=\alpha^2,\\ im\omega-\omega'\mathcal{C}^m & \text{if } \chi=\alpha^2\omega.\end{cases}$$

By Lemma 6.6 applied with $\gamma=\frac12$, there exists $u_{\frac12,\alpha^2}\in\mathbb{R}$ so that for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > u_{\frac12,\alpha^2}$ and $a\in[0,a_1]$, we have $\lambda\in\varrho\big(\mathcal{A}_{\alpha^2}(a,m)\big)$ and

$$\|K_\chi(a,m)(\mathcal{A}_{\alpha^2}(a,m)-\lambda)^{-1}\| < 1. \tag{6.3}$$

This, together with (6.2) and a Neumann series argument, implies that $\lambda \in \varrho\big(\mathcal{A}^m_\chi(a)\big)$ for $\mathrm{Re}(\lambda) > u_{\frac{1}{2},\alpha^2}$.

(b) For $a \in [0,1)$, $j \in \mathbb{N}$ let $\mathcal{Q}_j(a)$ be the mapping of $\mathcal{H}(a)$ onto the first $2j$ components, see Definition 6.2, and let $u_{\alpha^2} := \max\big\{0, -2+\big(\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|\big\}$ be as in the proof of Theorem 6.5. Then, by Theorem 6.5 (b), for $\mathrm{Re}(\lambda) > u_{\alpha^2}$, all operators $\mathcal{Q}_j(a)(\mathcal{A}_\chi(a_n,m)-\lambda)^{-1}$, $j \in \mathbb{N}$, are compact. Using Proposition 5.2 (i), the equivalent form (5.4) of the upper bound therein and $\lambda_l(a,\vartheta) \geq l(l+1)$ for $l \in \mathbb{N}$, $\vartheta \in \{l,\infty\}$, see (3.3), we conclude that, for $\mathrm{Re}(\lambda) > u_{\alpha^2}$,

$$\begin{aligned}\big\|\mathcal{Q}_j(a)(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}-(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\big\| &= \max_{l\geq k_m+j}\big\|(\mathcal{A}_{\alpha^2,l}(a_n)-\lambda)^{-1}\big\| \\ \leq \frac{1+\Big(\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{2}}\Big)\|\alpha\|}{\mathrm{Re}(\lambda)+(k_m+j+1)(k_m+j+2)-\big(\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|} &\longrightarrow 0, \quad j\to\infty. \end{aligned} \tag{6.4}$$

Therefore, being the limit of compact operators, $(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}$ is compact for $\mathrm{Re}(\lambda) > u_{\alpha^2}$. If, for $\mathrm{Re}(\lambda) > \max\{u_{\alpha^2}, u_{\frac{1}{2},\alpha^2}\}$ with $u_{\frac{1}{2},\alpha^2}$ as in (a), we take inverses in (6.2), we conclude that $(\mathcal{A}^m_{\alpha^2}(a_n)-\lambda)^{-1}$ is compact.

(c) In the same way as in (b), cf. (6.4), we obtain that, for $\mathrm{Re}(\lambda) > u_{\alpha^2}$,

$$\sup_{n\in\mathbb{N}}\big\|(\mathcal{A}_{\alpha^2,l}(a_n)-\lambda)^{-1}\big\| \leq \frac{1+\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{2}}}{\mathrm{Re}(\lambda)+l(l+1)-\big(\|\alpha\|+\frac{\|\alpha'\|}{\sqrt{2}}\big)\|\alpha\|} \longrightarrow 0, \quad l\to\infty.$$

This and Theorem 6.5 (c) imply that $\big((\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\big)_{n\in\mathbb{N}}$ is discretely compact for $\mathrm{Re}(\lambda) > u_{\alpha^2}$ by [3, Thm. 4.8]. For $\mathrm{Re}(\lambda) > \max\{u_{\alpha^2}, u_{\frac{1}{2},\alpha^2}\}$ the claim now follows from the perturbation result [3, Thm. 4.2] using (6.2) and (6.3); see the proof of (a).

(d) Due to Theorem 6.5 (d) for $\mathrm{Re}(\lambda) > u_{\alpha^2}$ and by Lemma 6.6 (i), e.g., for $\mathrm{Re}(\lambda) > u_{\frac{1}{2},\alpha^2}$, which yields that the sequence $\big(\|(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\|\big)_{n\in\mathbb{N}}$ is bounded, we can apply [3, Thm. 3.15] (with $\mathcal{S}=0$, $\mathcal{S}^{(n)}=0$ therein) to the infinite operator matrices $\mathcal{A}_{\alpha^2}(0,m)$, $\mathcal{A}_{\alpha^2}(a_n,m)$, $n\in\mathbb{N}$; note that, since the latter are diagonal, we are in case (a) of [3, Thm. 3.15] with $N = M = 1$. Hence we obtain that, for $\mathrm{Re}(\lambda) > \max\{u_{\alpha^2}, u_{\frac{1}{2},\alpha^2}\}$,

$$\big(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda\big)^{-1}\mathcal{P}^\infty(a_n) \xrightarrow{s} (\mathcal{A}_{\alpha^2}(0,m)-\lambda)^{-1}, \quad n\to\infty, \tag{6.5}$$

and claim (d) follows from the perturbation result [3, Thm 3.3] using (6.2) and (6.3), see the proof of (a).

Altogether, we have proved that all assumptions of [3, Thm. 2.6] are satisfied and the latter yields all the claims for interval truncation.

Step 2. *Spectral exactness of combined interval truncation and finite section method*. First, we prove the claims for the leading operator in the α^2- and $\alpha^2\omega$-models, $\mathcal{A}_{\alpha^2}(0,m) = \mathrm{diag}(\mathcal{A}_{\alpha^2,l}(0) : l \geq k_m)$, and its approximating sequence given by

$$\begin{aligned}\mathcal{A}_{\alpha^2,n}(a_n,m) &:= \mathcal{Q}_n(a_n)\mathcal{A}_{\alpha^2}(a_n,m) = \mathrm{diag}(\mathcal{A}_{\alpha^2,l}(a_n,m) : l = k_m,\dots,k_m+n-1),\\ \mathcal{D}(\mathcal{A}_{\alpha^2,n}(a_n,m)) &:= \mathcal{D}(\mathcal{A}_{\alpha^2}(a_n,m)) \cap \mathcal{H}_n(a_n), \quad n\in\mathbb{N}.\end{aligned}$$

The claims for $\mathcal{A}^m_{\alpha^2}(0)$, $\mathcal{A}^m_{\alpha^2\omega}(0)$ then follow by corresponding perturbation arguments, including [3, Thms. 3.3 and 4.2], that rely on inequalities analogous to those in the proof of Step 1, cf. (6.2) and (6.3).

Again we show that each of the claims (a), (b), (c), (d) is satisfied for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda)$ sufficiently large.

(a) By Lemma 6.6, if $\mathrm{Re}(\lambda)$ is sufficiently large, then $\lambda \in \bigcap_{n\in\mathbb{N}} \varrho\big(\mathcal{A}_{\alpha^2}(0,m)\big) \cap \varrho\big(\mathcal{A}_{\alpha^2,n}(a_n,m)\big)$ since $\mathcal{A}_{\alpha^2}(a_n,m)$ is block diagonal and

$$(\mathcal{A}_{\alpha^2,n}(a_n,m)-\lambda)^{-1} = \mathcal{Q}_n(a_n)(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\big|_{\mathcal{H}_n(a_n)}, \quad n \in \mathbb{N}. \tag{6.6}$$

(b) The compactness of $(\mathcal{A}_{\alpha^2}(0,m)-\lambda)^{-1}$, $(\mathcal{A}_{\alpha^2,n}(a_n,m)-\lambda)^{-1}$, $n \in \mathbb{N}$, follows from Theorem 6.5 (b) and from (6.6).

(c) The claim is a direct consequence of the discrete compactness of the sequence $\big((\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\big)_{n\in\mathbb{N}}$ due to Theorem 6.5 (c) and of (6.6).

(d) First we note that $\mathcal{Q}_n(a_n) = \mathcal{Q}_n(0)\mathcal{P}^\infty(a_n) = \mathcal{P}^\infty(a_n)\mathcal{Q}_n(0)$, $n \in \mathbb{N}$; see Remark 6.8. As $(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}$ is block diagonal, Theorem 6.5 (iv) and [3, Prop. 3.12 (a)] yield $(\mathcal{A}_{\alpha^2}(a_n,m)-\lambda)^{-1}\mathcal{P}^\infty(a_n) \xrightarrow{s} (\mathcal{A}_{\alpha^2}(0,m)-\lambda)^{-1}$ as $n \to \infty$, cf. Step 1, and thus

$$\begin{aligned}&\big(\mathcal{A}_{\alpha^2,n}(a_n,m)-\lambda\big)^{-1}\mathcal{Q}_n(a_n)\\ &\quad = \big(\mathcal{A}_{\alpha^2,n}(a_n,m)-\lambda\big)^{-1}\mathcal{P}^\infty(a_n)\mathcal{Q}_n(0) \xrightarrow{s} (\mathcal{A}_{\alpha^2}(0,m)-\lambda)^{-1}, \quad n \to \infty.\end{aligned}$$

Now all claims for $\mathcal{A}_{\alpha^2}(0,m)$ follow from [3, Thm. 2.6].

To prove the claims for the $\alpha\omega$-model, we replace $\mathcal{A}_{\alpha^2}(0,m)$ by $\mathcal{A}_\alpha(0,m)$ in the above proof and proceed analogously as for the $\alpha^2\omega$-model. □

Remark 6.9. Note that exact a priori error estimates for eigenvalue approximations cannot be derived in the nonnormal case. For differential operators with compact resolvents that are truncated to finite sections/subintervals, it is a classical result that asymptotic error estimates depend on the behavior of the eigenfunctions truncated to finite sections/subintervals; see, e.g., [15] or [4, Thm. 5.2]. However, a priori information about the behavior of the eigenfuntions is, in general, not available.

7. Numerical examples.

In this section we apply and illustrate the results of the previous section by numerical computations of eigenvalues for different dynamo models. Particular attention will be paid to the effect of interval truncation. We consider several concrete functions α and ω, including the helical turbulence function α due to Stefani and Gerbeth [19], for which the existence of an oscillatory dynamo effect for the α^2-model was first suggested by numerical approximations, then still without a theoretical result ensuring convergence to a true eigenvalue with real part >0.

As in [19] we choose $m = 0$, although our results cover all $m \in \mathbb{Z}$. However, we consider not only α^2-dynamos but also $\alpha^2\omega$-models where, in addition to interval truncation, taking finite sections is required for numerical eigenvalue approximations.

The numerical computations underlying the plots in this subsection use a code written in Wolfram Mathematica 9 software and were operated on a standard dual-core Linux machine. After applying finite section/domain truncation, the finite system of regularized ODEs on the interval $(a,1)$ is solved using a shooting method. This amounts to finding the zeros of a holomorphic function, which is done using the argument principle. We use the numerical integrator built into Mathematica to calculate the contour integral. The precision of the numerically found zero depends on the precision of the contour integral, which has to be an integer by the argument principle. Therefore, the distance to the integers is an indicator for numerical precision.

Example 7.1 (α from [19] in the α^2-model, interval truncation). The first example of a helical turbulence function α for which numerical computations of physicists indicated the existence of supercritical modes, i.e., of eigenvalues λ with $\mathrm{Re}(\lambda) > 0$, and hence the existence of an oscillatory dynamo effect for the α^2-model, is due to Stefani and Gerbeth [19]. They also conjectured that polarity reversals of the magnetic field, which are known to occur irregularly, e.g., for the Earth, are related to the existence of oscillatory modes close to criticality, i.e., $\mathrm{Im}(\lambda) \neq 0$, $\mathrm{Re}(\lambda) \approx 0$; see [20].

In [19] an evolutionary strategy was used to construct helical turbulence functions α such that, for the dominating mode $l = 1$, there is a pair of numerical eigenvalue approximations converging to some $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \neq 0$, $\mathrm{Re}(\lambda) \approx 0$, and for $l \geq 2$ the real parts of all numerical eigenvalue approximations are negative. One of these functions is given by

$$\alpha(r) := C(-21.46 + 426.41r^2 - 806.73r^3 + 392.28r^4), \quad r \in [0,1]. \tag{7.1}$$

For $C = 1$ the critical numerical values found in [19] for $l = 1$ are

$$\lambda^{\pm} = 0.01 \pm 7.24\mathrm{i}; \tag{7.2}$$

if the scaling C is increased beyond 1, the positive real part increases further; see [19, Fig. 3].

However, the α^2-dynamo operator is not selfadjoint and, up to now, there was no guarantee whatsoever that these numerical values lie near a pair of true eigenvalues of the α^2-dynamo problem. The results of this paper do not only fill this gap for the particular helical turbulence function (7.1), but for arbitrary differentiable α with essentially bounded α'; see Theorem 6.5 and 6.7.

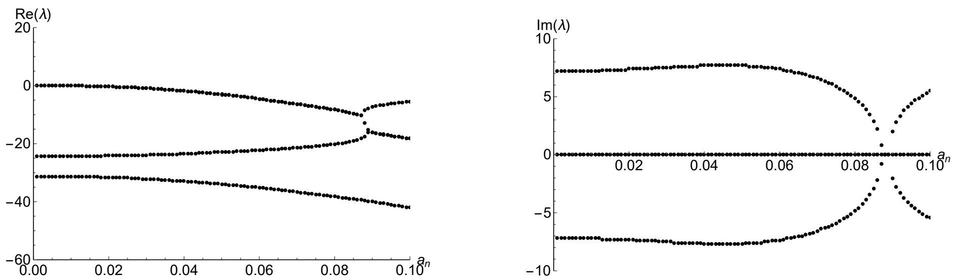

FIG. 2. *Eigenvalues λ_n of $\mathcal{A}_{\alpha^2,l}(a_n)$ for $l=1$, $a_n \in (0, 0.1]$ if $\alpha(r) = -21.46+426.41r^2-806.73r^3+392.28r^4$, $r \in [0,1]$.*

Figures 2 and 3 show our numerically computed eigenvalues of $\mathcal{A}_{\alpha^2,l}(a_n)$ for $l = 1$ as functions of $a_n \in (0, 0.1]$, for $C = 1$ and for $C = 1.05$. In Figure 2 for $C = 1$, the two upper curves in the left picture start, for $a_n = 0.1$, with a single, real eigenvalue at -5.4 and a complex, conjugate eigenvalue pair at $-18.3 \pm 5.4\mathrm{i}$; there is another eigenvalue curve with smaller real part. As a_n decreases, the single eigenvalue moves left on the real axis, while the pair starts to move right and at the same time down towards the real axis. There the pair finally meets and splits into two real eigenvalues moving apart to the left and to the right on the real axis. The eigenvalue moving right then hits the single real eigenvalue which had been continuously moving left. The newly formed pair then splits into a complex conjugate pair, while the other eigenvalue continues to move right on the real axis. As a_n decreases further, the complex conjugate pair moves to the right towards the imaginary axis and slightly beyond with real part converging to 0.01 and imaginary parts converging to ± 7.2, in agreement with (7.2).

In Figure 3 for $C = 1.05$, as a_n decreases from 0.1 to zero, the eigenvalue pair with largest real part behaves similarly as in Figure 2 for $C=1$, but it moves much further

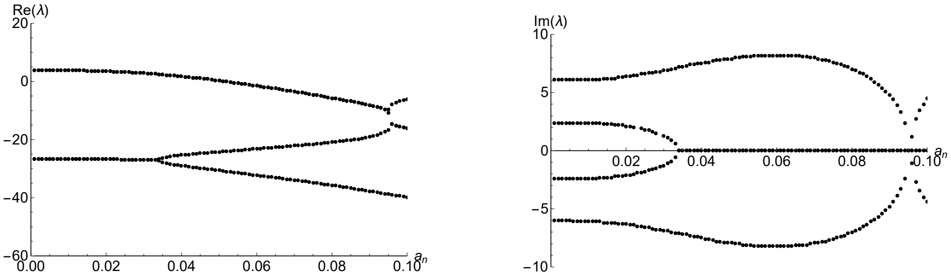

FIG. 3. *Eigenvalues* λ_n *of* $\mathcal{A}_{\alpha^2,l}(a_n)$ *for* $l=1$, $a_n\in(0,0.1]$ *if* $\alpha(r)=1.05\,(-21.46+426.41r^2-806.73r^3+392.28r^4)$, $r\in[0,1]$.

into the complex right half-plane, converging to $3.9\pm 6.1\mathrm{i}$. In contrast to $C=1$, see Figure 2, the next two eigenvalues also form a complex conjugate pair which remains in the left half-plane.

Next, we consider the $\alpha^2\omega$-model. Here the corresponding operator $\mathcal{A}^0_{\alpha^2\omega}(0)$ is an infinite operator matrix which, unlike the α^2-operator, does not decouple. Therefore, in addition to cutting the singular interval $(0,1]$ at the left at a_n, truncating $\mathcal{A}^0_{\alpha^2\omega}(a_n)$ to its first n rows and columns may also lead to spectral pollution and failure of approximation of true eigenvalues. Theorem 6.7 is the first result guaranteeing spectral exactness for either of these truncation processes.

We combine the simplest example of a function ω such that $\omega'\neq 0$,

$$\omega(r):=\omega_0 r,\quad r\in[0,1],\quad \omega_0 \text{ constant,} \tag{7.3}$$

with two different helical turbulence functions α: in Example 7.2 with constant $\alpha\equiv\alpha_0$ and in Example 7.3 with the polynomial function α in (7.1) due to Stefani and Gerbeth; see [19]. In both cases, we again consider the axisymmetric case, i.e., $m=0$, and, in view of the good agreement with [19] in the α^2-dynamo model, we choose $a_n=0.001$.

Since all numerical calculations are computationally expensive, we perform the computations for selected values of α_0 and ω_0; to illustrate the dependence on α_0 or ω_0 in the plots, we interpolate the eigenvalue paths linearly for better visibility.

Example 7.2 (constant α, ω' in the $\alpha^2\omega$-model, interval truncation, and finite section). In the simplest case $\omega_0=0$, i.e., $\omega\equiv 0$, the $\alpha^2\omega$-model reduces to the α^2-model. Even in this case, the eigenvalues are only implicitly known as the solutions $\lambda\in(-\infty,\alpha_0^2/4)$ of the equation

$$J_{l-\frac12}\big(k_+(\lambda)\big)J_{l+\frac12}\big(k_-(\lambda)\big)-J_{l+\frac12}\big(k_+(\lambda)\big)J_{l-\frac12}\big(k_-(\lambda)\big)=0, \tag{7.4}$$

where $J_{l\pm\frac12}$ are Bessel functions of fractional order and $k_\pm(\lambda):=\frac{\alpha_0}{2}\pm\sqrt{\frac{\alpha_0^2}{4}-\lambda}$; see [9, eqs. (7.10) and (7.11)]. Note that, for all $\alpha_0\in\mathbb{R}$, the point $\lambda=\alpha_0^2/4$ is a trivial solution of (7.4) since the left-hand side is identically zero then, but it is not an eigenvalue of $\mathcal{A}_{\alpha^2,l}(0)$. This follows because, for $\alpha_0=0$, the point $\lambda=\alpha_0^2/4=0$ does not belong to $\sigma(\mathcal{A}_{\alpha^2,l}(0))=\sigma(-A_l(0,l))\cup\sigma(-A_l(0,\infty))\subset(-\infty,-\lambda_{l,1}(0,l))$ and the eigenvalues of $\mathcal{A}_{\alpha^2,l}(0)$ depend continuously on α_0. The latter is a consequence of [9, Prop. 5.1 and Thm. 5.3], which shows that they coincide with the eigenvalues of a selfadjoint operator $S=D+\alpha_0 B$, where $D=\operatorname{diag}(-A_l(0,l),-A_l(0,\infty))$ and B is

D-bounded with D-bound 0. In addition, for all $\alpha_0 \in \mathbb{R}$, the point $\lambda = 0$ is another trivial solution of (7.4) but it is not always an eigenvalue of $\mathcal{A}_{\alpha^2,l}(0)$, as the following numerical computations show.

Figure 4 for $\alpha \equiv \alpha_0$, $\omega \equiv 0$ also serves as a reference plot. It shows the numerically computed eigenvalues of the $\alpha^2\omega$-problem obtained for $a_n = 0.001$ and different $\alpha \equiv \alpha_0 \in [0, 5]$. For $\alpha \equiv \alpha_0 = 5$ an $\alpha^2\omega$-dynamo effect is possible since $\|\alpha\|_\infty = 5 > j_{\frac{3}{2},1} \approx 4.493$ violates the anti-$\alpha^2\omega$-dynamo condition (4.8). The choice of a_n was motivated by a separate calculation for $a_n \in [0.001, 0.1]$, where the eigenvalues changed by less than 0.5. The critical value $\alpha_0 = j_{\frac{3}{2},1} \approx 4.493$, where the largest eigenvalue passes into the linearly unstable half-plane $\mathrm{Re}(\lambda) > 0$ in Figure 4 seems to agree with the one obtained from Theorem 4.7, which means that the anti-$\alpha^2\omega$-dynamo theorem is sharp in this case. This was already pointed out in [21, Ex. 1].[1]

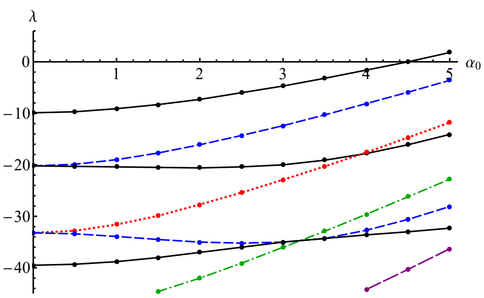

FIG. 4. *Eigenvalue approximations for $\mathcal{A}_{\alpha^2\omega}(0)$ with $\omega \equiv 0$ as functions of $\alpha \equiv \alpha_0 = 0, 0.5, 1, \dots, 5$, lines for $l=1$ (black/solid), $l=2$ (blue/dashed), $l=3$ (red/dotted), $l=4$ (green/dashed-dotted), $l=5$ (purple/long dashed). (Figure in color online.)*

For $\omega_0 > 0$, due to Example 4.9, (4.8), the anti-$\alpha^2\omega$-dynamo criterion in Theorem 4.7 amounts to

$$\alpha_0 \le j_{\frac{3}{2},1}, \quad \omega_0 \le \min_{l=1}^{3} \frac{(j_{l+\frac{1}{2},1})^2 - \alpha_0^2}{\sqrt{(j_{l+\frac{1}{2},1})^4 + \alpha_0^2}} \, \frac{(j_{l-\frac{1}{2},1})^2}{\delta_{l,0}}. \tag{7.5}$$

The explicit values of the constants involved are $\delta_{1,0} = 2\sqrt{\frac{2}{15}}$, $\delta_{2,0} = 2\sqrt{\frac{39}{35}}$, $\delta_{3,0} = \frac{8}{\sqrt{5}}$ and $j_{\frac{1}{2},1} \approx 3.142$, $j_{\frac{3}{2},1} \approx 4.493$, $j_{\frac{5}{3},1} \approx 5.763$, $j_{\frac{7}{3},1} \approx 6.988$. Figure 5 shows the anti-$\alpha^2\omega$-region (in grey) in the α_0, ω_0-plane where no dynamo action can take place due to Theorem 4.7.

First, we performed numerical computations with all combinations of

$$n = 3, 6, 10, \quad \alpha_0 = 0, 0.5, 1, \dots, 10, \quad \omega_0 = 0, 5, 10, \dots, 40.$$

Since the eigenvalue approximations in the box $[-37, 5] + [-5, 5]\,\mathrm{i}$ was not changed by more than the given precision 0.1 between $n = 6$ and $n = 10$, we set $n = 10$ and $a_n = 0.001$ for the following calculations.

In Figure 6 we illustrate the behavior of the eigenvalue approximations if we set $\alpha_0 = 5$, which violates the anti-$\alpha^2\omega$-dynamo criterion $\alpha_0 \le j_{\frac{3}{2},1}$ in (7.5) as required, and increase ω_0. We see that, as ω_0 grows, the real parts of the eigenvalue approximations do not vary much, but there is one pair of eigenvalue approximations that

[1] We note that [21, eq. (27)] contains a misprint; the squares in the numerator should be erased, cf. [9, eq. (6.3)] or Theorem 4.2.

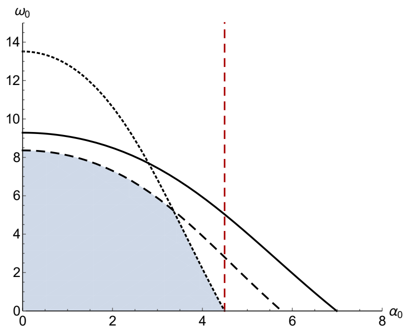

FIG. 5. *Anti-$\alpha^2\omega$-dynamo region (grey) in the $\alpha \equiv \alpha_0$, $\omega' \equiv \omega_0$ plane according to* (7.5)*; lines for $l=1$ (dotted), $l=2$ (dashed), $l=3$ (solid).*

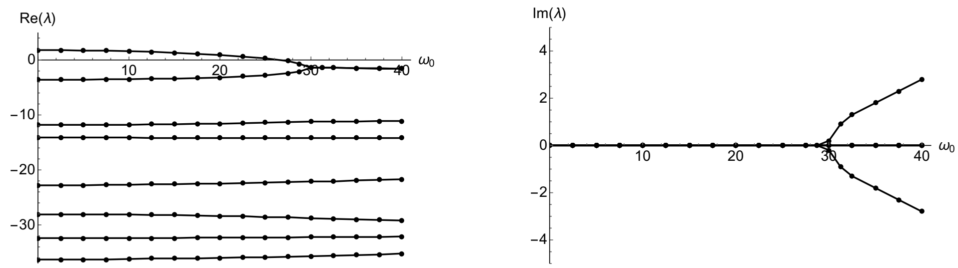

FIG. 6. *Eigenvalue approximations for $\mathcal{A}_{\alpha^2\omega}(a_n, 0)$ with $n = 10$, $a_n = 0.001$ for $\alpha \equiv \alpha_0 = 5$, $\omega' \equiv \omega_0 = 0, 5, 10, \ldots, 40$.*

meet and form a complex conjugate pair for which the moduli of the imaginary parts seem to grow.

In Figure 7 we have set $\omega_0 = 40$ and vary α_0. There are various pairs of eigenvalue approximations that coalesce, some of which separate again when α_0 grows further. The real part of the largest eigenvalue pair is monotonically growing; approximately at $\alpha_0 = 5.5$ the pair crosses the imaginary axis. So, even in this simple example of constant α_0 and ω_0, we observe oscillatory modes close to criticality!

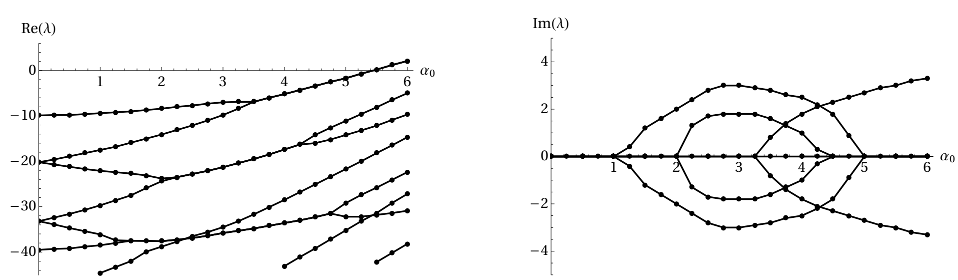

FIG. 7. *Eigenvalue approximations for $\mathcal{A}_{\alpha^2\omega}(a_n, 0)$ with $n = 10$, $a_n = 0.001$ for $\omega' \equiv \omega_0 = 40$, and different $\alpha \equiv \alpha_0$.*

Example 7.3 (α in (7.1) from [19] with $C = 1$, ω' constant in the $\alpha^2\omega$-model, interval truncation, and finite section). No numerical computations are available yet for the combination of the polynomial α studied by Stefani and Gerbeth in [19], see (7.1), and linear ω with $\omega' \equiv \omega_0$; see (7.3). The results presented here were performed for $n = 10$ and $a_n = 0.001$.

Figure 8 shows the dependence of the eigenvalue approximations on ω_0. As in Example 7.2, the real parts of the eigenvalue approximations do not vary much as ω_0 increases. Apart from the oscillatory pair close to criticality (both eigenvalues correspond to $l = 1$), two additional pairs meet as ω_0 grows; the moduli of the imaginary parts increase monotonically up to $\omega_0 = 70$.

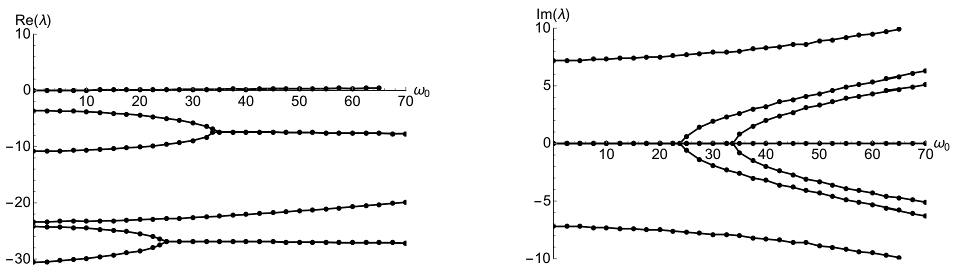

Fig. 8. *Eigenvalue approximations for* $\mathcal{A}_{\alpha^2\omega}(a_n, 0)$ *with* $n = 10$, $a_n = 0.001$, *for* $\alpha(r) = -21.46 + 426.41r^2 - 806.73r^3 + 392.28r^4$, $r \in [0, 1]$, *and different* $\omega' \equiv \omega_0$.

Acknowledgments. Both authors thank Dr. Frank Stefani, Helmholtz-Zentrum Dresden-Rossendorf, for his constructive comments which, in particular, helped us to spot a typo in our code. They are also indebted to the valuable and insightful comments and questions of the anonymous referees.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ninth Dover printing, Dover Publications, New York, 1964.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, translated from the Russian, reprint of the 1961 and 1963 translations, two volumes bound as one, Dover Publications, New York, 1993

[3] S. Bögli, *Convergence of sequences of linear operators and their spectra*, Integral Equations Operator Theory, 88 (2017), pp. 559–599, https://doi.org/10.1007/s00020-017-2389-3.

[4] S. Bögli, P. Siegl, and C. Tretter, *Approximations of spectra of Schrödinger operators with complex potentials on* $\mathbb{R}^d$, Comm. Partial Differential Equations, 42 (2017), pp. 1001–1041, https://doi.org/10.1080/03605302.2017.1330342.

[5] F. Busse and M. Proctor, *Antidynamo and bounding theorems*, Springer, Dordrecht, 2007, pp. 21–23, https://doi.org/10.1007/978-1-4020-4423-6_8.

[6] C. Chicone, Y. Latushkin, and S. Montgomery-Smith, *The spectrum of the kinematic dynamo operator for an ideally conducting fluid*, Comm. Math. Phys., 173 (1995), pp. 379–400, http://projecteuclid.org/euclid.cmp/1104274734.

[7] A. Gailitis, O. Lielausis, E. Platacis, G. Gerbeth, and F. Stefani, *Colloquium: Laboratory experiments on hydromagnetic dynamos*, Rev. Mod. Phys., 74 (2002), p. 973.

[8] A. Giesecke, F. Stefani, and G. Gerbeth, *Spectral properties of oscillatory and non-oscillatory* α^2*-dynamos*, Geophys. Astrophys. Fluid Dyn., 107 (2013), pp. 45–57, https://doi.org/10.1080/03091929.2012.668543.

[9] U. Günther, H. Langer, and C. Tretter, *On the spectrum of the magnetohydrodynamic mean-field* α^2*-dynamo operator*, SIAM J. Math. Anal., 42 (2010), pp. 1413–1447, https://doi.org/10.1137/100782024.

[10] M. E. H. Ismail and M. E. Muldoon, *On the variation with respect to a parameter of zeros of Bessel and q-Bessel functions*, J. Math. Anal. Appl., 135 (1988), pp. 187–207, https://doi.org/10.1016/0022-247X(88)90148-5.

[11] R. Kaiser, *Approximately axisymmetric antidynamo theorems*, SIAM J. Appl. Math., 78 (2018), pp. 3188–3212, https://doi.org/10.1137/18M1173174.

[12] R. Kaiser and A. Tilgner, *Optimal energy bounds in spherically symmetric α^2-dynamos*, Quart. Appl. Math., 76 (2018), pp. 437–461, https://doi.org/10.1090/qam/1501.

[13] O. N. Kirillov, U. Günther, and F. Stefani, *Determining role of Krein signature for three-dimensional Arnold tongues of oscillatory dynamos*, Phys. Rev. E, 79 (2009), 016205, https://doi.org/10.1103/PhysRevE.79.016205.

[14] F. Krause and K.-H. Rädler, *Mean-field Magnetohydrodynamics and Dynamo Theory*, Akademie-Verlag, Berlin, 1980.

[15] J. E. Osborn, *Spectral approximation for compact operators*, Math. Comput. Simulation, 29 (1975), pp. 712–725, http://www.jstor.org/stable/2005282.

[16] K.-H. Rädler, *Zur Dynamotheorie kosmischer Magnetfelder.* I. *Gleichungen für sphärische Dynamomodelle*, Astron. Nachr., 294 (1973), pp. 213–223.

[17] P. H. Roberts, *Kinematic dynamo models*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 272 (1972), pp. 663–698, http://www.jstor.org/stable/74140.

[18] F. Stefani, A. Gailitis, and G. Gerbeth, *Magnetohydrodynamic experiments on cosmic magnetic fields*, ZAMM Z. Angew. Math. Mech., 88 (2008), pp. 930–954.

[19] F. Stefani and G. Gerbeth, *Oscillatory mean-field dynamos with a spherically symmetric, isotropic helical turbulence parameter α*, Phys. Rev. E, 67 (2003), 027302, https://doi.org/10.1103/PhysRevE.67.027302.

[20] F. Stefani and G. Gerbeth, *Asymmetric polarity reversals, bimodal field distribution, and coherence resonance in a spherically symmetric mean-field dynamo model*, Phys. Rev. Lett., 94 (2005), 184506, doi:10.1103/PhysRevLett.94.184506.

[21] F. Stefani and C. Tretter, *On a spectral problem in magnetohydrodynamics and its relevance for the geodynamo*, GAMM-Mitt., 41 (2018), e201800012, https://doi.org/10.1002/gamm.201800012.

[22] R. Stieglitz and U. Müller, *Experimental demonstration of a homogeneous two-scale dynamo*, Phys. Fluids, 13 (2001), pp. 561–564, https://doi.org/10.1063/1.1331315.

[23] F. Stummel, *Diskrete Konvergenz linearer Operatoren.* I, Math. Ann., 190 (1970), pp. 45–92.

[24] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008, https://doi.org/10.1142/9781848161122.

[25] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil* II [Linear operators in Hilbert spaces. Part II], Anwendungen [Applications], Mathematische Leitfäden [Mathematical Textbooks], B. G. Teubner, Stuttgart, 2003.