Global strong solutions in $\mathbb{R}^3$ for ionic Vlasov-Poisson systems

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Abstract

Systems of Vlasov-Poisson type are kinetic models describing dilute plasma. The structure of the model differs according to whether it describes the electrons or positively charged ions in the plasma. In contrast to the electron case, where the well-posedness theory for Vlasov-Poisson systems is well established, the well-posedness theory for ion models has been investigated more recently. In this article, we prove global well-posedness for two Vlasov-Poisson systems for ions, posed on the whole three-dimensional Euclidean space $\mathbb{R}^3$, under minimal assumptions on the initial data and the confining potential.

1 Introduction

In this article, we investigate the well-posedness theory of a kinetic model for the ions in a dilute plasma. Plasma is an ionised gas, which forms when an electrically neutral gas is subjected to a high temperature or a strong electromagnetic field. This causes the gas particles to dissociate: electrons split apart from the rest of the gas particle. A plasma therefore contains two distinguished types of charged particle: negatively charged electrons and positively charged ions.

The Vlasov-Poisson system is a well established kinetic model used to describe plasma. The version of the system that has been most widely discussed in the mathematics literature is a model for the electrons in the plasma, evolving against a background of ions that is presumed to have a given stationary distribution. This model takes the following form:

$$(VP) := \begin{cases} \partial_t f_e + v \cdot \nabla_x f_e + \dfrac{q_e}{m_e} E \cdot \nabla_v f_e = 0, \\ \nabla_x \times E = 0, \quad \epsilon_0 \nabla_x \cdot E = q_i \rho[f_i] + q_e \rho[f_e], \\ \rho[f_e](t,x) := \displaystyle\int_{\mathbb{R}^d} f_e(t,x,v)\,\mathrm{d}v, \\ f_e(0,x,v) = f_{e,0}(x,v) \geq 0. \end{cases} \tag{1.1}$$

Here $f_e(t,x,v)$ represents the phase-space density of electrons, q_e and q_i denote respectively the charge on each electron and each ion, m_e is the mass of an electron, ϵ_0 is the vacuum permittivity, and $\rho[f_i](x)$ denotes the spatial density of ions which is assumed to be given and independent of time. The assumption that the ion distribution is stationary is justified by the fact that the mass of an ion is typically much greater than the mass of an electron. It is therefore common to make the approximation that the ions are stationary and uniformly distributed.

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In this article, we instead consider a Vlasov-Poisson type system, extensively used in physics, describing the ions in a plasma. In analogy with the electron model (1.1), we consider a system of the form

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i + \dfrac{q_i}{m_i} E \cdot \nabla_v f_i = 0, \\ \nabla_x \times E = 0, \quad \epsilon_0 \nabla_x \cdot E = q_i \rho[f_i] + q_e \rho[f_e], \\ \rho[f_i](t,x) := \displaystyle\int_{\mathbb{R}^d} f_i(t,x,v)\,\mathrm{d}v, \\ f_i(0,x,v) = f_{i,0}(x,v) \geq 0. \end{cases}$$

To complete this model, it is necessary to specify the electron distribution $\rho[f_e]$. A widely used assumption is that the electrons are in thermal equilibrium. This is justified by the fact that the electrons are relatively very light and so fast moving, with a significant collision frequency. Thus the equilibrium distribution is a Maxwell-Boltzmann law of the form

$$\rho[f_e] \sim e^{-\beta_e q_e \Phi},$$

where the ambient electrostatic potential Φ is defined to be a function such that $E = -\nabla_x \Phi$, while β_e denotes the inverse electron temperature.

After an appropriate rescaling, this choice of electron distribution results in the following system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \quad \Delta U = A(t,U)\, e^U - \rho[f], \\ \rho[f](t,x) := \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \\ f(0,x,v) = f_0(x,v) \geq 0, \displaystyle\int_{\mathbb{R}^{2d}} f_0\,\mathrm{d}x\,\mathrm{d}v = 1. \end{cases} \tag{1.2}$$

Here $A(t,U) > 0$ is a scaling term in the electron distribution, which we will discuss further below.

It is natural to include a further spatial confinement of the electrons, using an external potential. That is, we assume that the electrons are also subject to a given external potential H. Their thermal equilibrium is then of the form

$$\rho[f_e] \sim e^{-H+U} = g e^U,$$

where the function $g : \mathbb{R}^d \to [0, +\infty)$ is defined by $g := e^{-H}$. We assume throughout the paper a minimal condition on g, namely that g is fixed and belongs to the space $L^1 \cap L^\infty(\mathbb{R}^d)$.

We consider the two most natural versions of the Vlasov-Poisson system for ions. These differ based on the choice of the scaling A. Choosing $A = 1$ results in the following system:

$$(VPME)_V := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla U, \\ \Delta U = g e^U - \rho_f, \\ f|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{R}^{2d}} f_0\, dx\, dv = 1. \end{cases} \tag{1.3}$$

Note that for solutions of (1.3), the total charge is not necessarily conserved and the system therefore may not be globally neutral at all times. An alternative choice is to enforce global neutrality. For this A must be chosen to normalise the electron distribution, that is, $A = \left(\int_{\mathbb{R}^d} g e^U\, \mathrm{d}x \right)^{-1}$, which results in the following alternative system:

$$(VPME)_F := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla U, \\ \Delta U = \dfrac{g e^U}{\int_{\mathbb{R}^d} g e^U\, \mathrm{d}x} - \rho_f, \\ f|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{R}^{2d}} f_0\, dx\, dv = 1. \end{cases} \tag{1.4}$$

Both systems are usually referred to as the Vlasov-Poisson system *with massless electrons*, abbreviated to VPME. This refers to the fact that these systems can be derived from a coupled system of ions and

electrons in the limit of 'massless electrons', in which the ratio of the electron and ion masses, $\frac{m_e}{m_i}$, tends to zero. For example, Bardos, Golse, Nguyen and Sentis [1] discuss this limit for coupled kinetic systems of the form

$$
\begin{cases}
\partial_t f_i + v \cdot \nabla_x f_i + \frac{q_i}{m_i} E \cdot \nabla_v f_i = 0, \\
\partial_t f_e + v \cdot \nabla_x f_e + \frac{q_e}{m_e} E \cdot \nabla_v f_e = C(m_e) Q(f_e), \\
\nabla_x \times E = 0, \quad \epsilon_0 \nabla_x \cdot E = q_i \rho[f_i] + q_e \rho[f_e].
\end{cases}
$$

In the equation above, Q represents a collision operator such as a Boltzmann or BGK operator. Under the assumption that sufficiently regular solutions of this system exist, they identify that in the limit the electrons indeed assume a Maxwell-Boltzmann law, leading to a model for the ions that is similar to (1.3), but with a time-dependent electron temperature. Also, for such a model, they prove a global well-posedness result on the torus.

Systems of the form (1.2) have been used in astrophysics literature, for example in studies of the expansion of plasma into vacuum [13], numerical investigations of the formation of ion-acoustic shocks [12, 16] and of the phase-space vortices that form behind these shocks [3].

In this article, we consider the well-posedness of both (1.3) and (1.4). We remark that the well-posedness theory for Vlasov-Poisson-type systems heavily depends on the dimension d in which the problem is posed and on the boundary conditions imposed on the system. Two frequently considered boundary conditions are the periodic case, in which the system is posed on the d-dimensional flat torus, and the whole space case, in which the problem is posed on all of $\mathbb{R}^d$ with a condition that f and E decay at infinity.

Remark 1.1. *Note that for the Vlasov-Poisson system for ions on the torus, the external confining potential H is not typically used (in other words, $g \equiv 1$). Moreover one may take $A = 1$ without loss of generality, since changing A corresponds to adding a constant to U. On the torus, the Poisson equation*

$$
\Delta U = h
$$

has a solution only if h has total integral zero; it follows that if a solution of (1.3) on the torus exists, it must necessarily be globally neutral at all times. Thus on the torus there is no distinction between the system (1.3) with variable total charge and the system (1.4) with fixed total charge.

In one dimension ($d = 1$), global well-posedness for VPME was proved by Han-Kwan and the second author [7]. In dimension $d = 3$, Bouchut [4] proved that global weak solutions exist for both systems (1.3) and (1.4) on the whole space. In a recent work [5], the authors proved global well-posedness for the Vlasov-Poisson system for ions in dimension $d = 2$ and $d = 3$ in the periodic case, i.e. when the problem is posed on the flat torus, with $g \equiv 1$. However, a similar well-posedness result was not previously available for the whole space case. This is the goal of this work.

To make a parallel, the classical Vlasov-Poisson system for electrons is known to be globally well-posed in dimension $d \leq 3$. Global existence of classical solutions in the whole space or on the torus was shown in, for example, [18, 15, 10, 2]. Uniqueness for solutions with bounded density was proved by Loeper [11]. For a more detailed account of the development of this theory, see for example [6].

1.1 Main Result

The main result of this paper is a global well-posedness result for the VPME systems (1.3) and (1.4) in $\mathbb{R}^3$, under minimal assumptions on the initial data and the confining potential g. To state the main theorem, we first define the energy functionals associated to each of the systems (1.3) and (1.4). Each of these functionals is conserved by sufficiently regular solutions of the associated system. For system (1.3) where the total charge is variable, we use the following functional:

$$
\mathcal{E}_V[f] := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, \mathrm{d}x \, \mathrm{d}v + \int_{\mathbb{R}^3} |E|^2 \, \mathrm{d}x + 2 \int_{\mathbb{R}^3} (U - 1) g e^U \, \mathrm{d}x.
$$

For system (1.4), with fixed total charge, we use

$$\mathcal{E}_F[f] := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv + \int_{\mathbb{R}^3} |E|^2 \, dx + 2 \int_{\mathbb{R}^3} \phi g e^{\phi} \, dx,$$

where

$$\phi = U - \log \left(\int_{\mathbb{R}^3} g e^U \, dx \right).$$

The following theorem is the main result of this article.

Theorem 1.2 (Global well-posedness). *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be a probability density satisfying*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{m_0} f_0(x, v) \, dx \, dv < +\infty \quad \text{for some } m_0 > 6, \qquad f_0(x, v) \leq \frac{C}{(1 + |v|)^r} \quad \text{for some } r > 3.$$

Assume that $g \in L^1 \cap L^\infty(\mathbb{R}^3)$, with $g \geq 0$ satisfying $\int_{\mathbb{R}^3} g = 1$, and that $\mathcal{E}_V[f_0] \leq C$ (resp. $\mathcal{E}_F[f_0] \leq C$). Then there exists a unique solution $f \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ of (1.3) (resp. (1.4)) with initial datum f_0 such that $\rho_f \in L^\infty([0, T]; L^\infty(\mathbb{R}^3))$.

Remark 1.3. *We refer to the class of solutions constructed here – bounded distributional solutions f of (1.3) and (1.4) whose density ρ_f is uniformly bounded – as strong solutions.*

Uniqueness holds for this class of solutions, as we prove below in Theorem 3.1. Moreover, for strong solutions the electric field has at least a log-Lipschitz regularity, and so the associated characteristic flow is well-defined. In particular it is possible to show that, if the initial datum is additionally C^1, then the corresponding solution is also C^1. Thus our global existence of strong solutions (Theorem 1.2) also includes, as a byproduct, the global existence of classical solutions.

Remark 1.4. *Instead of assuming $f_0(x, v) \leq \frac{C}{(1+|v|)^r}$ for some $r > 3$, one can replace this hypothesis with assumption (10) in [10, Corollary 3].*

Remark 1.5. *Our result is essentially optimal in terms of the assumptions. Indeed, as shown in [10, Equation 16], controlling moments of order larger than 6 is needed to guarantee that our solution is strong (i.e., $\rho_f \in L^\infty([0, T]; L^\infty(\mathbb{R}^3))$). Also, the boundedness of g is needed to ensure that the electric field enjoys at least a log-Lipschitz regularity, so that characteristics exist and are unique.*

1.2 Strategy

The first step in the proof is to obtain regularity and stability estimates on the electrostatic potential U. This is carried out in Section 2. This provides the necessary tools to complete the proof of well-posedness. This is done in two stages. In Section 3 we prove uniqueness for solutions with bounded density. Then, in Section 4, we show the global existence of strong solutions.

1.2.1 Analysis of the Electrostatic Potential

The analysis of the VPME systems (1.3) and (1.4) hinges on an understanding of the electrostatic potential U.

We require two kinds of estimates. On the one hand, we prove regularity estimates that give quantitative bounds on U given certain L^p bounds on ρ_f. On the other hand, we prove stability estimates, controlling the distance between the electric fields for two solutions of the VPME system, as is needed to prove the uniqueness of solutions.

Our strategy is based on the following decomposition of the electric field. We write the electrostatic potential in the form

$$U = \bar{U} + \widehat{U},$$

where $\bar{U}$ satisfies the equation

$$-\Delta \bar{U} = \rho_f, \qquad \lim_{|x| \to 0} \bar{U}(x) = 0.$$

In other words, $\bar{U}$ satisfies the same equation as the electrostatic potential in the Vlasov-Poisson system for electrons. The remainder $\widehat{U}$ must then satisfy either

$$\Delta \widehat{U} = g e^{\bar{U} + \widehat{U}} \quad \text{or} \quad \Delta \widehat{U} = \frac{g e^{\bar{U} + \widehat{U}}}{\int_{\mathbb{R}^3} g e^{\bar{U} + \widehat{U}} \, \mathrm{d}x}. \tag{1.5}$$

We will similarly write

$$E = \bar{E} + \widehat{E}, \qquad \bar{E} := -\nabla \bar{U}, \quad \widehat{E} := -\nabla \widehat{U}.$$

This decomposition was introduced in [7] in order to study the Vlasov-Poisson system for ions in the one dimensional case. It was then used in [5] to study well-posedness in the cases $d = 2, 3$ on the torus.

In Section 2 of this article, we analyse the remainder term $\widehat{U}$. To deal with the nonlinearity in the equation satisfied by $\widehat{U}$, we make use of techniques from the calculus of variations.

In this way we are able to show that, under assumptions on ρ_f that we expect to be satisfied by solutions of the Vlasov-Poisson systems (1.3) and (1.4), the equations (1.5) for $\widehat{U}$ are well-posed and enjoy good regularity estimates. Specifically, we show that $\widehat{U} \in C^{1,\alpha}$ for any $\alpha \in (0, 1)$, with a uniform in time bound on this norm. Thus U is close to $\bar{U}$ up to a smoother correction, which is controlled uniformly in time in a strong norm. These estimates then allow methods developed for the Vlasov-Poisson system for electrons to be adapted to the ion case.

This strategy was previously used in [5] to show well-posedness on the torus in dimension two and three. Here we apply it to the case where $x \in \mathbb{R}^3$. There are two main differences in the whole space case compared to the torus case. One is that the domain is unbounded and we therefore need to account for the decay of the potential at infinity. In practice, this involves identifying an appropriate functional setting for the optimisation problem characterising $\widehat{U}$.

The other is that in the whole space we study two different models, with different nonlinearities. In particular, for the model (1.4) with fixed total charge, the nonlinearity is different from the torus case due to the normalisation of the electron density. Notice, for instance, that this nonlinearity is not monotone in $\widehat{U}$. To handle this nonlinearity with techniques from the calculus of variations, it is necessary to modify the functional used to show the well-posedness of the equation. The natural choice of functional is not bounded below, and so the proof in Subsection 2.3.2 differs from the one in [5]. Moreover, a new stability estimate is needed (Lemma 2.12).

1.2.2 Well-posedness in $\mathbb{R}^3$

The well-posedness consists of two parts: existence and uniqueness of strong solutions. The existence is based on showing the propagation of moments. The idea is to show an a priori estimate on solutions, to the effect that, if the initial datum has a velocity moment of sufficiently high order: if

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{m_0} f_0 \, \mathrm{d}x \, \mathrm{d}v < +\infty,$$

then the velocity moments of the solution can also be controlled:

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{m_0} f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v < +\infty.$$

In Section 4, we prove the propagation of moments in this sense for the VPME systems (1.3) and (1.4). The principle is to follow the approach of Lions and Perthame [10], adapting it to include the extra part of the electrostatic potential $\widehat{U}$. This is possible thanks to the uniform estimates obtained in Section 2.

For the uniqueness part of Theorem 1.2, we use an approach in the style of Loeper [11], who proved uniqueness for solutions of the Vlasov-Poisson system for electrons such that ρ_f is bounded in $L^\infty(\mathbb{R}^d)$.

Loeper's strategy is to prove a stability property for solutions with respect to the initial data, quantified in the second order Wasserstein distance W_2. In Section 3, we prove an estimate of this type for the VPME systems (1.3) and (1.4) in the whole space. The key here is that we need to prove suitable stability estimates for the smooth part of the potential $\widehat{U}$, in the case of the unbounded domain $\mathbb{R}^3$. We carry this out in Subsection 2.5.

In Subsection 1.3, we show how to use these results to complete the proof of Theorem 1.2 – in particular, to show that under the assumptions of the theorem, the resulting solutions have bounded density so that the uniqueness result may be applied.

1.3 Proof of the Main Result

Proof of Theorem 1.2. Arguing as in [10] and in [5], by approximation one can construct a global solution $f \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ of (1.3) (resp. (1.4)) with uniformly bounded energy. Then, it follows by Proposition 4.1 below that all moments of order less than m_0 are uniformly bounded on every finite time interval.

Next, we show that this implies that the solution has bounded density. We introduce the characteristic system associated to the Vlasov-Poisson system, which is the following ODE system: for $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, x, v) = V(t, x, v), \quad \frac{\mathrm{d}}{\mathrm{d}t} V(t, x, v) = E(X(t, x, v)); \quad X(0, x, v) = 0, V(0, x, v) = v.$$

As in [10], since $m_0 > 6$ this implies that $\bar{E}$ is uniformly bounded (see [10, Equation 16]), while $\widehat{E}$ is uniformly bounded thanks to Propositions 2.5-2.7. This implies that E is uniformly bounded, and therefore the characteristics satisfy the bound

$$|V(t, x, v) - v| \leq C_T \qquad \text{for all } (t, x, v) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Thus

$$f(t, X(t, x, v), V(t, x, v)) = f_0(x, v) \leq \frac{C}{(1 + |v|)^r} \leq \frac{C_T}{(1 + |V(t, x, v)|)^r} \qquad \text{for all } (t, x, v) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3,$$

and so

$$f(t, y, w) \leq \frac{C_T}{(1 + |w|)^r} \qquad \text{for all } (t, y, w) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Since $r > 3$, this yields

$$\rho_f(t, y) \leq C_T \int_{\mathbb{R}^3} \frac{1}{(1 + |w|)^r} \, dw \leq C_T \qquad \text{for all } (t, y) \in [0, T] \times \mathbb{R}^3,$$

and the uniqueness follows by Theorem 3.1.

$\square$

1.4 Energy Functionals

We noted above that each of the VPME systems has an associated energy functional, which we denoted respectively by $\mathcal{E}_V$ and $\mathcal{E}_F$. These energy functionals are formally conserved by their associated systems. The control of these energy functionals implies an integrability bound on the mass density ρ_f.

Lemma 1.6 (Control of the energy implies a moment bound). *Assume one of the conditions*

$$\mathcal{E}_V[f] \leq C_0, \qquad \mathcal{E}_F[f] \leq C_0.$$

Then there exists a constant C depending on C_0 and $\|g\|_{L^1(\mathbb{R}^3)}$ only such that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, \mathrm{d}x \, \mathrm{d}v \leq C.$$

It follows that, if $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, the associated mass density, $\rho_f = \int_{\mathbb{R}^3} f \, \mathrm{d}v$ satisfies

$$\|\rho_f\|_{L^{\frac{5}{3}}(\mathbb{R}^3)} \leq C. \tag{1.6}$$

Proof. Observe that the functions xe^x, $(x-1)e^x$ are bounded from below, uniformly for all $x \in \mathbb{R}$:

$$xe^x \geq -e^{-1}, \qquad (x-1)e^x \geq -1.$$

Therefore, since $g \geq 0$, in the variable charge case we have

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, \mathrm{d}x \, \mathrm{d}v \leq \mathcal{E}_V[f] + 2\|g\|_{L^1(\mathbb{R}^3)} \leq C\left(C_0, \|g\|_{L^1(\mathbb{R}^3)}\right).$$

In the fixed charge case, we have

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, \mathrm{d}x \, \mathrm{d}v \leq \mathcal{E}_V[f] + \frac{2}{e}\|g\|_{L^1(\mathbb{R}^3)} \leq C\left(C_0, \|g\|_{L^1(\mathbb{R}^3)}\right).$$

The estimate (1.6) then follows from a standard interpolation argument; see Lemma A.1 below.

$\square$

Notation. The notation $L^p(g)$ denotes L^p norms taken with respect to the density g:

$$\|f\|_{L^p(g)}^p = \int_{\mathbb{R}^3} |f(x)|^p g(x) \, \mathrm{d}x.$$

2   Electric Field Estimates

2.1   Decomposition

We decompose the electrostatic potential U into the form $U = \bar{U} + \widehat{U}$, where $\bar{U}$ satisfies

$$-\Delta \bar{U} = \rho_f, \qquad \lim_{|x| \to \infty} \bar{U}(x) = 0. \tag{2.1}$$

Thus $\bar{U}$ is exactly the electrostatic potential we would have in the case of the classical Vlasov-Poisson system. The remainder $\widehat{U}$ must satisfy either

$$\Delta \widehat{U} = ge^{\bar{U}+\widehat{U}}, \tag{2.2}$$

in the case of variable total charge, or

$$\Delta \widehat{U} = \frac{ge^{\bar{U}+\widehat{U}}}{\int_{\mathbb{R}^3} ge^{\bar{U}+\widehat{U}} \, \mathrm{d}x},$$

in the case of fixed total charge.

In the rest of this section, we show that $\bar{U}$ and $\widehat{U}$ exist and exhibit regularity estimates for them.

2.2   Singular Part

We recall some basic estimates on $\bar{U}$ satisfying the Poisson equation (2.1), in the case where $\rho_f \in L^1 \cap L^{5/3}(\mathbb{R}^3)$. This is the degree of integrability we expect to have on ρ_f when f is a solution of the VPME system, based on the conservation of mass and energy.

To study $\bar{U}$, we make use of the Green's function for the Laplace equation on $\mathbb{R}^3$, which is the function

$$G(x) = \frac{1}{4\pi|x|}, \quad x \neq 0.$$

The Poisson equation (2.1) has a distributional solution of the form $G * \rho_f$ (see for example [9, Theorem 6.21]). This solution decays at infinity and thus is the unique such solution by Liouville's theorem for harmonic functions.

We have the following integrability estimates on $\bar{U}$, which follow from [8, Section 4.5].

Lemma 2.1. *Let $\rho_f \in L^1 \cap L^{\frac{5}{3}}(\mathbb{R}^3)$. Then $\bar{U} \in L^{3,\infty} \cap L^\infty(\mathbb{R}^3)$ with the estimates*

$$\|\bar{U}\|_{L^{3,\infty}(\mathbb{R}^3)} \leq C\|\rho_f\|_{L^1(\mathbb{R}^3)}, \quad \|\bar{U}\|_{L^\infty(\mathbb{R}^3)} \leq C\|\rho_f\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}}\|\rho_f\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}}, \quad [\bar{U}]_{C^{0,\frac{1}{5}}(\mathbb{R}^3)} \leq C\|\rho_f\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}.$$

Let $\rho_f \in L^1 \cap L^p(\mathbb{R}^3)$, where $p \in (1,3)$. Then

$$\|\bar{E}\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq C\|\rho_f\|_{L^1(\mathbb{R}^3)}, \qquad \|\bar{E}\|_{L^q(\mathbb{R}^3)} \leq C\|\rho_f\|_{L^p(\mathbb{R}^3)},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

Note in particular that for $p = \frac{5}{3}$, we have $q = \frac{15}{4}$. We thus expect to control $\bar{E}$, uniformly in time, in the spaces $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$ and $L^{\frac{15}{4}}(\mathbb{R}^3)$.

2.3 Existence of the Smooth Part

2.3.1 Variable Total Charge

We prove the existence of $\widehat{U}$ by making use of techniques from the calculus of variations. Consider the functional

$$J_V[h] := \int_{\mathbb{R}^3} |\nabla h(x)|^2 + g(x)e^{h(x)+\bar{U}(x)} \, dx \geq 0.$$

The idea is to minimise J_V over those functions h decaying at infinity for which $\nabla h \in L^2(\mathbb{R}^3)$. Note that, by a Sobolev inequality, these functions belong to $L^6(\mathbb{R}^3)$. Hence we introduce the following classical notation:

$$\dot{W}^{1,2}(\mathbb{R}^3) := \{h : \mathbb{R}^3 \to \mathbb{R} \, : \, h \in L^6(\mathbb{R}^3), \nabla h \in L^2(\mathbb{R}^3)\}.$$

Lemma 2.2. *Assume that $\bar{U} \in \dot{W}^{1,2}(\mathbb{R}^3)$. There exists a unique minimiser of J_V over $\dot{W}^{1,2}(\mathbb{R}^3)$.*

Proof. Consider a minimising sequence $(h_n)_n \subset \dot{W}^{1,2}(\mathbb{R}^3)$. For sufficiently large n we have the bound

$$J_V[h_n] \leq J_V[-\bar{U}] = \int_{\mathbb{R}^3} |\nabla\bar{U}|^2 \, dx + \int_{\mathbb{R}^3} g(x) \, dx.$$

It follows that $(\nabla h_n)_n$ is uniformly bounded in $L^2(\mathbb{R}^3)$. We may therefore pass to a subsequence such that $h_n \rightharpoonup h$ in $L^6(\mathbb{R}^3)$ and $\nabla h_n \rightharpoonup \nabla h$ in $L^2(\mathbb{R}^3)$. Also, by the Rellich-Kondrachov theorem, for any bounded set A the sequence $h_n \mathbb{1}_A$ converges to $h \mathbb{1}_A$ strongly in $L^p(\mathbb{R}^3)$ for any $p < 6$. Hence, by a diagonal argument, it follows that (by passing to a further subsequence) we may assume that h_n converges to h almost everywhere on $\mathbb{R}^3$.

By lower semi-continuity of the norm under weak convergence, we have

$$\int_{\mathbb{R}^3} |\nabla h|^2 \, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla h_n|^2 \, dx.$$

By Fatou's lemma, we have

$$\int_{\mathbb{R}^3} ge^{h+\bar{U}} \, dx = \int_{\mathbb{R}^3} \lim_{n\to\infty} ge^{h_n+\bar{U}} \, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} ge^{h_n+\bar{U}} \, dx.$$

It follows that

$$J_V[h] \leq \liminf_{n\to\infty} J_V[h_n] = \inf_\phi J_V[\phi].$$

Thus h is a minimiser. The uniqueness of h follows from the convexity of J_V. $\square$

We now show that the smooth part of the potential $\widehat{U}$ can be taken to be the minimiser of J_V. Let $\widehat{U}$ denote the minimiser of J_V and note that

$$\int_{\mathbb{R}^3} ge^{\widehat{U}+\bar{U}}\,\mathrm{d}x \le J_V[\widehat{U}] \le J_V[-\bar{U}]$$

and thus $ge^{\widehat{U}+\bar{U}}$ is a function in $L^1(\mathbb{R}^3)$.

It is then possible to show that $\widehat{U}$ satisfies

$$\Delta\widehat{U} = ge^{\bar{U}+\widehat{U}},$$

which is the Euler-Lagrange equation associated to the minimisation problem above (see Appendix B).

2.3.2 Fixed Total Charge

In this subsection we prove the existence of $\widehat{U}$ in the case of fixed total charge. We will use an estimate due to Bouchut [4, Lemma 2.6], which is used to obtain lower bounds on the integral

$$\int_{\mathbb{R}^3} ge^U\,\mathrm{d}x.$$

This will provide upper bounds on the nonlinearity in the Poisson equation in the fixed total charge case.

Lemma 2.3. *Let $g \in L^1 \cap L^\infty(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} g\,\mathrm{d}x = 1$. Then, for $U \in L^{3,\infty}(\mathbb{R}^3)$, the following estimate holds:*

$$\int_{\mathbb{R}^3} ge^{-|U|}\,\mathrm{d}x \ge Ce^{-C\|U\|_{L^{3,\infty}(\mathbb{R}^3)}\|g\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{3}}}.$$

We recall that $\bar{U}$ has the representation $G * \rho_f$ and is therefore non-negative in the cases we consider $(d = 3)$.

Lemma 2.4 (Existence of $\widehat{U}$). *Let $\bar{U} \in \dot{W}^{1,2}(\mathbb{R}^3)$ be non-negative. Then there exists a unique solution $\widehat{U} \in \dot{W}^{1,2}(\mathbb{R}^3)$ satisfying*

$$\Delta\widehat{U} = \frac{ge^{\widehat{U}+\bar{U}}}{\int_{\mathbb{R}^3} ge^{\bar{U}+\widehat{U}}\,\mathrm{d}x}.$$

For this $\widehat{U}$, we have

$$0 < \int_{\mathbb{R}^3} ge^{\bar{U}+\widehat{U}}\,\mathrm{d}x < +\infty.$$

Proof. The uniqueness of solutions in the class $\dot{W}^{1,2}(\mathbb{R}^3)$ follows from [4, Lemma 2.5]. To construct a solution, we look for a minimiser of

$$J_F[h] := \int_{\mathbb{R}^3} |\nabla h|^2\,\mathrm{d}x + \log\left(\int_{\mathbb{R}^3} ge^{\bar{U}+h}\,\mathrm{d}x\right).$$

The difficulty in this case compared to the variable charge case is that this functional is not bounded below. We therefore introduce an approximating functional J_K, defined by

$$J_K[h] := \int_{\mathbb{R}^3} |\nabla h|^2\,\mathrm{d}x + L_K\left(\int_{\mathbb{R}^3} ge^{\bar{U}+h}\,\mathrm{d}x\right).$$

The function L_K is a smooth and non-decreasing approximation of the logarithm function, satisfying

$$L_K(x) := \begin{cases} \log x & x > e^{-(K-1)} \\ -K & x \le e^{-K} \end{cases}, \qquad |L_K'(x)| \le \frac{1}{x} \wedge e^{K-1} \qquad \|L_K''\|_{L^\infty} \le C_K.$$

We minimise J_K over the space $\dot{W}^{1,2}(\mathbb{R}^3)$. First, note that

$$\inf J_K[h] \le J_K[-\bar{U}] = \|\nabla\bar{U}\|_{L^2(\mathbb{R}^3)}^2 + L_K\left(\|g\|_{L^1(\mathbb{R}^3)}\right).$$

Let $(h_n)_n$ be a minimising sequence. Since L_K is bounded from below by $-K$, we have the uniform estimates

$$\|\nabla h_n\|_{L^2(\mathbb{R}^3)}^2 \le \|\nabla\bar{U}\|_{L^2(\mathbb{R}^3)}^2 + L_K(\|g\|_{L^1(\mathbb{R}^3)}) + K$$

$$\int_{\mathbb{R}^3} g e^{\bar{U}+h_n} \, \mathrm{d}x \le e^{-(K-1)} \vee \exp\left[\|\nabla\bar{U}\|_{L^2(\mathbb{R}^3)}^2 + L_K(\|g\|_{L^1(\mathbb{R}^3)})\right].$$

As in the proof of Lemma 2.2, we may pass to a subsequence such that h_n converges almost everywhere to some $h^{(K)}$, with ∇h_n converging weakly in $L^2(\mathbb{R}^3)$ to $\nabla h^{(K)}$. Therefore

$$\|\nabla h^{(K)}\|_{L^2(\mathbb{R}^3)} \le \liminf_{n\to\infty} \|\nabla h_n\|_{L^2(\mathbb{R}^3)}, \qquad \int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x \, \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} g e^{\bar{U}+h_n} \, \mathrm{d}x.$$

Since L_K is smooth and increasing, we have that

$$J_K[h^{(K)}] \le \liminf_{n\to\infty} J_K[h_n] = \inf_h J_K[h].$$

Hence $h^{(K)}$ is a minimiser of J_K. It follows that $h^{(K)}$ is a solution of the associated Euler-Lagrange equation

$$\Delta h^{(K)} = g e^{\bar{U}+h^{(K)}} L_K'\left(\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x\right). \tag{2.3}$$

The right hand side of the approximating Poisson equation (2.3) is non-negative and its L^1 norm satisfies

$$\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} L_K'\left(\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x\right) \, \mathrm{d}x \le M_K\left(\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x\right),$$

where M_K denotes the function

$$M_K(x) = x L_K'(x).$$

By assumption on L_K, $|M_K| \le 1$. Therefore $\Delta h^{(K)} \in L^1(\mathbb{R}^3)$ with

$$\|\Delta h^{(K)}\|_{L^1(\mathbb{R}^3)} \le 1.$$

It follows that there exists C independent of K such that

$$\|h^{(K)}\|_{L^{3,\infty}(\mathbb{R}^3)} \le C.$$

Therefore, by Lemma 2.3,

$$\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x \ge \int_{\mathbb{R}^3} g e^{h^{(K)}} \, \mathrm{d}x \ge C_g > 0,$$

where C_g depends only on g, and in particular is independent of K. We may choose K sufficiently large such that $e^{-(K-1)} < C_g$. This implies that

$$L_K'\left(\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x\right) = \frac{1}{\int_{\mathbb{R}^3} g e^{\bar{U}+h^{(K)}} \, \mathrm{d}x},$$

so that for this choice of K, $h^{(K)}$ is in fact a solution of (2.7). We let $\widehat{U} = h^{(K)}$.

$\square$

2.4 Regularity of the Smooth Part

In this subsection, we prove regularity estimates on $\widehat{U}$.

2.4.1 Variable Total Charge

We prove the following regularity estimates on the function $\widehat{U}$, constructed above as the unique minimiser of J_V over $\dot{W}^{1,2}(\mathbb{R}^3)$.

Proposition 2.5. *Let* $\rho \in L^1 \cap L^{\frac{5}{3}}(\mathbb{R}^3)$. *Let* $\bar{U} = G * \rho$. *Then there exists* $\widehat{U}$ *satisfying* (2.2) *and the estimates*

$$\|\widehat{U}\|_{L^{3,\infty}} \leq C\|g\|_{L^1(\mathbb{R}^3)} \exp\left\{C\|\rho\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}}\|\rho\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}}\right\}$$

$$\|\nabla\widehat{U}\|_{L^{\frac{3}{2},\infty}} \leq C\|g\|_{L^1(\mathbb{R}^3)} \exp\left\{C\|\rho\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}}\|\rho\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}}\right\}$$

$$\|\widehat{U}\|_{C^{1,\alpha}} \leq C\|g\|_{L^\infty(\mathbb{R}^3)} \exp\left\{C\|\rho\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}}\|\rho\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}}\right\}, \qquad \text{for all } \alpha \in (0,1).$$

These estimates will follow from standard regularity theory for the Poisson equation, provided that we can prove suitable integrability estimates on $ge^{\bar{U}+\widehat{U}}$. To do this, we first find a representation for $\widehat{U}$ in terms of the Green's function G. First recall that $\widehat{U}$ satisfies the equation

$$\Delta\widehat{U} = ge^{\bar{U}+\widehat{U}}.$$

Then note that the following convolution with G is a solution of the same equation:

$$-G * (ge^{\bar{U}+\widehat{U}}).$$

Since $ge^{\bar{U}+\widehat{U}} \in L^1$, this convolution belongs to the space $L^{3,\infty}(\mathbb{R}^3)$. Thus the difference $-G*(ge^{\bar{U}+\widehat{U}})-\widehat{U}$ is a harmonic function decaying at infinity. Then by Liouville's theorem

$$\widehat{U} = -G * (ge^{\bar{U}+\widehat{U}}). \tag{2.4}$$

From this representation it follows that $\widehat{U} \leq 0$. In particular,

$$ge^{\bar{U}+\widehat{U}} \leq ge^{\bar{U}}.$$

Then, for all $p \in [1,+\infty]$,

$$\|ge^{\bar{U}+\widehat{U}}\|_{L^p(\mathbb{R}^3)} \leq \|ge^{\bar{U}}\|_{L^p(\mathbb{R}^3)} \tag{2.5}$$

$$\leq e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^\infty(\mathbb{R}^3)}^{1-1/p}\|g\|_{L^1(\mathbb{R}^3)}^{1/p} < +\infty.$$

Using this, we may deduce the following lemma.

Lemma 2.6. *Assume that* $\bar{U} \in L^\infty(\mathbb{R}^3)$. *Then* $\widehat{U} \in L^{3,\infty} \cap C^{1,\alpha}(\mathbb{R}^3)$ *for all* $\alpha \in (0,1)$, *with the estimates*

$$\|\widehat{U}\|_{L^{3,\infty}(\mathbb{R}^3)} \leq Ce^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^1(\mathbb{R}^3)}, \qquad \|\widehat{U}\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq C\|g\|_{L^\infty(\mathbb{R}^3)}e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}.$$

Proof. We use the representation (2.4) in combination with the L^p estimates (2.5).

In the case $p = 1$, we have

$$\|\Delta\widehat{U}\|_{L^1(\mathbb{R}^3)} \leq e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^1(\mathbb{R}^3)}$$

By [8, Section 4.5], $\widehat{U} \in L^{3,\infty}(\mathbb{R}^3)$ and $\widehat{E} \in L^{\frac{3}{2},\infty} \cap L^\infty(\mathbb{R}^3)$, with

$$\|\widehat{U}\|_{L^{3,\infty}(\mathbb{R}^3)} \leq Ce^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^1(\mathbb{R}^3)}, \quad \|\widehat{E}\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq Ce^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^1(\mathbb{R}^3)}.$$

In the case $p = \infty$, we have

$$\|\Delta\widehat{U}\|_{L^\infty(\mathbb{R}^3)} \leq e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}\|g\|_{L^\infty(\mathbb{R}^3)}. \tag{2.6}$$

By [8, Section 4.5], $\widehat{E} \in C^{0,\alpha}(\mathbb{R}^3)$ for all $\alpha \in (0,1)$, with

$$\|\widehat{E}\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C_g e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}.$$

$\square$

2.4.2 Fixed Total Charge

In this case, $\widehat{U}$ satisfies

$$\Delta\widehat{U} = \frac{ge^{\widehat{U}+\bar{U}}}{\int_{\mathbb{R}^3} ge^{\widehat{U}+\bar{U}}\,\mathrm{d}x}. \tag{2.7}$$

We will perform a similar analysis as in the variable charge case above. The idea is to prove integrability estimates for $\Delta\widehat{U}$. In the fixed charge case, we always have

$$\|\Delta\widehat{U}\|_{L^1(\mathbb{R}^3)} = 1.$$

This implies that $\widehat{U} \in L^{3,\infty}(\mathbb{R}^3)$ and that for some universal constant C,

$$\|\widehat{U}\|_{L^{3,\infty}(\mathbb{R}^3)} \leq C, \quad \|\widehat{E}\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq C. \tag{2.8}$$

We next consider an L^∞ estimate. Once again, we have the representation of $\widehat{U}$ in terms of a convolution with the fundamental solution G. This representation implies that $\widehat{U} \leq 0$, and so

$$ge^{\bar{U}+\widehat{U}} \leq ge^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}.$$

In order to prove an L^∞ estimate on $\Delta\widehat{U}$, the remaining step is to find a lower bound for the integral

$$\int_{\mathbb{R}^3} ge^{\widehat{U}+\bar{U}}\,\mathrm{d}x.$$

To do this, we use the fact that $\bar{U} \geq 0$ to deduce that

$$\int_{\mathbb{R}^3} ge^{\widehat{U}+\bar{U}}\,\mathrm{d}x \geq \int_{\mathbb{R}^3} ge^{\widehat{U}}\,\mathrm{d}x.$$

Then, by estimate (2.8) and Lemma 2.3, there exists a constant $C_g > 0$ depending on g only such that

$$\int_{\mathbb{R}^3} ge^{\widehat{U}+\bar{U}}\,\mathrm{d}x \geq C_g > 0.$$

Thus

$$\|\Delta\widehat{U}\|_{L^\infty(\mathbb{R}^3)} \leq C_g e^{\|\bar{U}\|_{L^\infty(\mathbb{R}^3)}}. \tag{2.9}$$

From these estimates we deduce the following proposition.

Proposition 2.7. *Let $\rho \geq 0$ satisfy $\|\rho\|_{L^1(\mathbb{R}^3)} = 1$ and $\rho \in L^{\frac{5}{3}}(\mathbb{R}^3)$. Let $\bar{U}$ be the unique $\dot{W}^{1,2}(\mathbb{R}^3)$ solution of (2.1). Then there exists a solution of (2.7), which satisfies for all $\alpha \in (0,1)$,*

$$\|\widehat{U}\|_{L^{3,\infty}(\mathbb{R}^3)} \leq C, \quad \|\widehat{E}\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq C, \quad \|\widehat{U}\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq \exp\left[C_{\alpha,g}\left(\|\rho\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}}\right)\right].$$

This is proved using the same Sobolev embedding estimates as in the variable charge case, using the corresponding L^p estimates on $\Delta\widehat{U}$ proved above.

2.5 Stability estimates

We want to extend to the VPME setting the uniqueness results in the style of Loeper for the case of solutions with $\rho_f \in L^\infty(\mathbb{R}^3)$. For this, we will need some stability estimates for the electrostatic potential with respect to the charge density. The aim of this section is to prove the following results.

Proposition 2.8 (Stability estimates: variable total charge). *Let $\rho_1, \rho_2 \in L^\infty(\mathbb{R}^3)$ be probability densities on $\mathbb{R}^3$. Let $\bar{U}_i \in \dot{W}^{1,2} \cap L^\infty(\mathbb{R}^3)$ solve respectively for $i = 1, 2$*

$$-\Delta \bar{U}_i = \rho_i.$$

Let $\widehat{U}_i \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$ satisfy

$$\Delta \widehat{U}_i = g e^{\widehat{U}_i + \bar{U}_i}.$$

Then

$$\|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{R}^3)} \le \max_i \|\rho_i\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} W_2(\rho_1, \rho_2),$$

$$\|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|_{L^2(\mathbb{R}^3)} \le C \max_i \|\rho_i\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} W_2(\rho_1, \rho_2),$$

where

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{1}{2}} \exp\left\{ C_0 \left[1 + \max_i \|\bar{U}_i\|_{L^\infty(\mathbb{R}^3)} + \max_i \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right] \right\}.$$

Proposition 2.9 (Stability estimates: fixed total charge). *Let $\rho_1, \rho_2 \in L^\infty(\mathbb{R}^3)$ be probability densities on $\mathbb{R}^3$. Let $\bar{U}_i \in \dot{W}^{1,2} \cap L^\infty(\mathbb{R}^3)$ solve respectively for $i = 1, 2$*

$$-\Delta \bar{U}_i = \rho_i.$$

Let $\widehat{U}_i \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$ satisfy

$$\Delta \widehat{U}_i = \frac{g e^{\widehat{U}_i + \bar{U}_i}}{\int_{\mathbb{R}^3} g e^{\widehat{U}_i + \bar{U}_i} \, \mathrm{d}x}.$$

Then

$$\|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{R}^3)} \le \max_i \|\rho_i\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} W_2(\rho_1, \rho_2),$$

$$\|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|_{L^2(\mathbb{R}^3)} \le C \max_i \|\rho_i\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} W_2(\rho_1, \rho_2),$$

where

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{1}{2}} \exp\left\{ C_0 \left[1 + \max_i \|U_i\|_{L^\infty(\mathbb{R}^3)} + \max_i \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right] \right\}.$$

To prove these results, we first recall the following estimate from [11, Theorem 2.9]. Note that in the original statement it is assumed that the densities have finite second moments. However, by approximation, this assumption can be dropped.

Lemma 2.10 (Stability for $\bar{U}$). *Let $\rho_1, \rho_2 \in L^\infty(\mathbb{R}^3)$ be probability densities on $\mathbb{R}^3$. Let $\bar{U}_i$ solve respectively for $i = 1, 2$*

$$-\Delta \bar{U}_i = \rho_i, \qquad \bar{U}_i(x) \to 0 \text{ as } |x| \to \infty.$$

Then

$$\|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{R}^3)} \le \max_i \|\rho_i\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} W_2(\rho_1, \rho_2).$$

The next step is to control the smoother part of the potential in terms of the singular part.

2.5.1 Variable Total Charge

Lemma 2.11 (Stability for $\widehat{U}$: variable total charge). *Let $\phi_1, \phi_2 \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$ be given non-negative functions. Let $\widehat{U}_1, \widehat{U}_2 \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$ satisfy*

$$\Delta \widehat{U}_i = g e^{\widehat{U}_i + \phi_i}, \ i = 1, 2.$$

Then

$$\|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|_{L^2(\mathbb{R}^3)}^2 \leq C \, \|\nabla \phi_1 - \nabla \phi_2\|_{L^2(\mathbb{R}^3)}^2,$$

where, for some uniform constant C_0,

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \exp \left\{ C_0 \left[1 + \max_{i=1,2} \|\phi_i\|_{L^\infty(\mathbb{R}^3)} + \max_{i=1,2} \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right] \right\}.$$

Proof. Consider the difference $\widehat{U}_1 - \widehat{U}_2$, which satisfies the equation

$$\Delta(\widehat{U}_1 - \widehat{U}_2) = g \left(e^{\widehat{U}_1 + \phi_1} - e^{\widehat{U}_2 + \phi_2} \right). \tag{2.10}$$

Using $\widehat{U}_1 - \widehat{U}_2$ as a test function in the weak form of (2.10), we find that

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} g \left(e^{\widehat{U}_2 + \phi_2} - e^{\widehat{U}_1 + \phi_1} \right) (\widehat{U}_1 - \widehat{U}_2) \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} g e^{\widehat{U}_2} \left(e^{\phi_2} - e^{\phi_1} \right) (\widehat{U}_1 - \widehat{U}_2) \, \mathrm{d}x + \int_{\mathbb{R}^3} g e^{\phi_1} \left(e^{\widehat{U}_2} - e^{\widehat{U}_1} \right) (\widehat{U}_1 - \widehat{U}_2) \, \mathrm{d}x.$$

It is valid to use $\widehat{U}_1 - \widehat{U}_2$ as a test function since $\widehat{U}_1 - \widehat{U}_2 \in \dot{W}^{1,2}(\mathbb{R}^3)$ and $g \left(e^{\widehat{U}_2 + \phi_2} - e^{\widehat{U}_1 + \phi_1} \right) \in L^1 \cap L^\infty(\mathbb{R}^3)$.

For all $x, y \in \mathbb{R}$, by the Mean Value Theorem there exists $\xi \in (x, y)$ such that

$$e^x - e^y = (x - y)e^\xi.$$

We therefore have the two inequalities

$$(e^x - e^y)(x - y) \geq |x - y|^2 e^{\min\{x, y\}} \tag{2.11}$$

and

$$|e^x - e^y| \leq |x - y| e^{\max\{x, y\}}. \tag{2.12}$$

Since $\widehat{U}_i \leq 0$, we have the estimate

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C_{\phi_1, \phi_2}}{C_{\widehat{U}_1, \widehat{U}_2}} \int_{\mathbb{R}^3} g e^{\widehat{U}_2} |\phi_1 - \phi_2| |\widehat{U}_1 - \widehat{U}_2| \, \mathrm{d}x - \frac{C_{\widehat{U}_1, \widehat{U}_2}}{C_{\phi_1, \phi_2}} \int_{\mathbb{R}^3} g e^{\phi_1} |\widehat{U}_1 - \widehat{U}_2|^2 \, \mathrm{d}x,$$

where

$$C_{\phi_1, \phi_2} = \exp \left(\max_{i=1,2} \|\phi_i\|_{L^\infty(\mathbb{R}^3)} \right), \qquad C_{\widehat{U}_1, \widehat{U}_2} = \exp \left(-\max_{i=1,2} \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right).$$

Using Young's inequality for products, with a small parameter, we obtain for any $\eta > 0$

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C_{\phi_1, \phi_2}}{4\eta} \|\phi_1 - \phi_2\|_{L^2(g)}^2 + \left(\eta C_{\phi_1, \phi_2} - C_{\widehat{U}_1, \widehat{U}_2} \right) \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(g)}^2.$$

Taking η such that $\eta C_{\phi_1, \phi_2} = C_{\widehat{U}_1, \widehat{U}_2}$, we conclude that

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{4} \frac{C_{\phi_1, \phi_2}^3}{C_{\widehat{U}_1, \widehat{U}_2}^2} \|\phi_1 - \phi_2\|_{L^2(g)}^2.$$

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We may then apply Hölder and Sobolev inequalities to obtain

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|^2_{L^2(\mathbb{R}^3)} \leq C \, \|\nabla \phi_1 - \nabla \phi_2\|^2_{L^2(\mathbb{R}^3)}.$$

where

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \exp \left\{ C_0 \left[1 + \max_{i=1,2} \|\phi_i\|_{L^\infty(\mathbb{R}^3)} + \max_{i=1,2} \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right] \right\}.$$

$\square$

2.5.2 Fixed Total Charge

Lemma 2.12 (Stability for $\widehat{U}$: fixed total charge). *Let* $\phi_1, \phi_2 \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$, $\phi_1, \phi_2 \geq 0$ *be given. Let* $\widehat{U}_1, \widehat{U}_2 \in L^{3,\infty} \cap L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^3)$ *satisfy*

$$\Delta \widehat{U}_i = \frac{g e^{\widehat{U}_i + \phi_i}}{\int_{\mathbb{R}^3} g e^{\widehat{U}_i + \phi_i} \, \mathrm{d}x}. \tag{2.13}$$

Then

$$\|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|^2_{L^2(\mathbb{R}^3)} \leq C \, \|\nabla \phi_1 - \nabla \phi_2\|^2_{L^2(\mathbb{R}^3)},$$

where, for some uniform constant C_0,

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \exp \left\{ C_0 \left[1 + \max_{i=1,2} \left\{ \|\phi_i\|_{L^\infty(\mathbb{R}^3)}, \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right\} \right] \right\}.$$

Proof. The difference $\widehat{U}_1 - \widehat{U}_2$ satisfies

$$\Delta(\widehat{U}_1 - \widehat{U}_2) = \frac{g e^{\widehat{U}_1 + \phi_1}}{\int_{\mathbb{R}^3} g e^{\widehat{U}_1 + \phi_1} \, \mathrm{d}x} - \frac{g e^{\widehat{U}_2 + \phi_2}}{\int_{\mathbb{R}^3} g e^{\widehat{U}_2 + \phi_2} \, \mathrm{d}x}. \tag{2.14}$$

We introduce the notation

$$m_i := \int_{\mathbb{R}^3} g e^{\widehat{U}_i + \phi_i} \, \mathrm{d}x.$$

We have the estimates

$$\max_{i=1,2} \|\widehat{U}_i\|_{L^{3,\infty}(\mathbb{R}^3)} \leq C,$$

since the right hand sides of the equations (2.13) each have total integral equal to one. By Lemma 2.3, and since $\widehat{U}_i \leq 0$ and $\phi_i \geq 0$, we have uniform upper and lower bounds

$$\|g\|_{L^1(\mathbb{R}^3)} e^{\|\phi_i\|_{L^\infty(\mathbb{R}^3)}} \geq m_i \geq e^{-C}.$$

From the weak form of equation (2.14), for all $\chi \in C_c^\infty(\mathbb{R}^3)$,

$$-\int_{\mathbb{R}^3} \nabla \chi \cdot \nabla(U_1 - U_2) \, \mathrm{d}x = \int_{\mathbb{R}^3} \chi \left[\frac{g e^{\widehat{U}_1 + \phi_1}}{m_1} - \frac{g e^{\widehat{U}_2 + \phi_2}}{m_2} \right] \mathrm{d}x \tag{2.15}$$

From the assumptions on $\widehat{U}_i$, we deduce that the right hand side of (2.14) is uniformly bounded in L^∞ and L^1. We can therefore extend the weak form (2.15) to test functions $\chi \in \dot{W}^{1,2}(\mathbb{R}^3)$. We may therefore choose $\chi = \widehat{U}_1 - \widehat{U}_2$, which results in the identity

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (\widehat{U}_1 - \widehat{U}_2) \left[\frac{g e^{\widehat{U}_2 + \phi_2}}{m_2} - \frac{g e^{\widehat{U}_1 + \phi_1}}{m_1} \right] \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} g \left[(\widehat{U}_1 + \phi_1 - \log m_1) - (\widehat{U}_2 + \phi_2 - \log m_2) \right] \left[\frac{e^{\widehat{U}_2 + \phi_2}}{m_2} - \frac{e^{\widehat{U}_1 + \phi_1}}{m_1} \right] \mathrm{d}x$$

$$- \int_{\mathbb{R}^3} g \, (\phi_1 - \phi_2) \left[\frac{e^{\widehat{U}_2 + \phi_2}}{m_2} - \frac{e^{\widehat{U}_1 + \phi_1}}{m_1} \right] \mathrm{d}x - \log \left(\frac{m_2}{m_1} \right) \int_{\mathbb{R}^3} g \left[\frac{e^{\widehat{U}_2 + \phi_2}}{m_2} - \frac{e^{\widehat{U}_1 + \phi_1}}{m_1} \right] \mathrm{d}x.$$

$\blacksquare$

The final term is equal to zero, by definition of m_i. Applying the inequalities (2.11) and (2.12) above results in the inequality

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq -c_1 \|(\widehat{U}_1 + \phi_1 - \log m_1) - (\widehat{U}_2 + \phi_2 - \log m_2)\|_{L^2(g)}^2$$
$$+ C_1 \int_{\mathbb{R}^3} g \, |\phi_1 - \phi_2| \, \left| (\widehat{U}_1 + \phi_1 - \log m_1) - (\widehat{U}_2 + \phi_2 - \log m_2) \right| \, \mathrm{d}x,$$

where

$$C_1 = \frac{e^{\max_{i=1,2} \|\widehat{U}_i + \phi_i\|_{L^\infty(\mathbb{R}^3)}}}{\min_{i=1,2} m_i}, \qquad c_1 = \frac{e^{-\max_{i=1,2} \|\widehat{U}_i + \phi_i\|_{L^\infty(\mathbb{R}^3)}}}{\max_{i=1,2} m_i}.$$

Young's inequality for products, with a parameter, then implies the following estimate for any $\alpha > 0$:

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq \left(\frac{C_1}{4\alpha} - c_1 \right) \|(\widehat{U}_1 + \phi_1 - \log m_1) - (\widehat{U}_2 + \phi_2 - \log m_2)\|_{L^2(g)}^2 + C_1 \alpha \|\phi_1 - \phi_2\|_{L^2(g)}^2.$$

Choosing $\alpha = \frac{C_1}{4c_1}$ gives

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C_1^2}{4c_1} \|\phi_1 - \phi_2\|_{L^2(g)}^2.$$

Then, since $g \in L^1 \cap L^\infty$, we deduce that

$$\|\nabla(\widehat{U}_1 - \widehat{U}_2)\|_{L^2(\mathbb{R}^3)}^2 \leq C_g \|\phi_1 - \phi_2\|_{L^6(\mathbb{R}^3)}^2 \leq C \|\nabla\phi_1 - \nabla\phi_2\|_{L^2(\mathbb{R}^3)}^2,$$

where, for some universal constant $C_0 > 0$,

$$C = \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \exp \left\{ C_0 \left[1 + \max_{i=1,2} \left\{ \|\phi_i\|_{L^\infty(\mathbb{R}^3)}, \|\widehat{U}_i\|_{L^\infty(\mathbb{R}^3)} \right\} \right] \right\}.$$

$\square$

3   Uniqueness

In this section we prove the uniqueness and stability in W_2 of solutions to $(VPME)_V$ and $(VPME)_F$ with bounded density. Recall that, given two non-negative measures on $\mathbb{R}^d$ with the same mass, one defines

$$W_2^2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(\mathrm{d}x \, \mathrm{d}y),$$

where $\pi \in \Pi(\mu, \nu)$ denotes the set of all probability measures in $\mathbb{R}^{2d}$ that have marginals μ and ν.

Although the strategy of proof is very similar to the one used in our paper [5], the fact of working in the whole space requires some modifications. The proof will be identical for the two models (1.3) and (1.4), so we state it as a single theorem.

Theorem 3.1 (Uniqueness for solutions with bounded density). *Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ be a probability density with $\rho_{f_0} \in L^\infty(\mathbb{R}^3)$. Fix a final time $T > 0$, and assume that $g \in L^1 \cap L^\infty(\mathbb{R}^3)$, with $g \geq 0$ satisfying $\int_{\mathbb{R}^3} g = 1$. Then there exists at most one solution $f \in C([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ of (1.3) (resp. (1.4)) with initial datum f_0 such that $\rho_f \in L^\infty([0,T]; L^\infty(\mathbb{R}^3))$.*

Moreover, the following quantitative stability estimate holds. Let f_i, $i = 1,2$, be two solutions of (1.3) (resp. (1.4)) with $\rho_{f_i} \in L^\infty([0,T]; L^\infty(\mathbb{R}^3))$. Then there exists a constant C, depending only on g and on $\sup_{t \in [0,T]} \left(\|\rho_{f_i}(t)\|_{L^1(\mathbb{R}^3)} + \|\rho_{f_i}(t)\|_{L^\infty(\mathbb{R}^3)} \right)$ $(i = 1,2)$, such that for all $t \in [0,T]$ the following bound holds:

1. If $W_2(f_1(0), f_2(0)) > 1/2$ then

$$W_2(f_1(t), f_2(t)) \leq W_2(f_1(0), f_2(0)) e^{Ct}.$$

2. If $W_2(f_1(0), f_2(0)) \leq 1/2$, let $t_0 > 0$ be such that $\log[W_2(f_1(0), f_2(0))]e^{-Ct_0} = \log(1/2)$. Then

$$W_2(f_1(t), f_2(t)) \leq \begin{cases} \exp\left[\log[W_2(f_1(0), f_2(0))] e^{-Ct}\right] & \text{for } t \in [0, t_0] \\ \frac{1}{2}e^{C(t-t_0)} & \text{for } t \in [t_0, T]. \end{cases}$$

Proof. Let $f_1, f_2 \in C([0, T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ be two solutions of (1.3) (resp. (1.4)) such that $\rho_{f_1}, \rho_{f_2} \in L^\infty([0, T]; L^\infty(\mathbb{R}^3))$.

We will prove the result by means of a Gronwall type estimate. To do this, we note that as in [5], thanks to our assumptions on the density, the electric field is log-Lipschitz and therefore our solutions are transported by their respective characteristics, that we denote by $(X^{(1)}, V^{(1)})$ and $(X^{(2)}, V^{(2)})$.

Fix an arbitrary initial coupling $\pi_0 \in \Pi(f_1(0), f_2(0))$ and consider the quantity

$$D(t) := \int |X_t^{(1)} - X_t^{(2)}|^2 + |V_t^{(1)} - V_t^{(2)}|^2 \, d\pi_0. \tag{3.1}$$

As in [5], it follows from the definition of Wasserstein distance that

$$W_2^2\left(\rho_{f_1}(t), \rho_{f_2}(t)\right) \leq W_2^2\left(f_1(t), f_2(t)\right) \leq D(t). \tag{3.2}$$

Moreover, since π_0 was arbitrary, we have

$$W_2^2\left(f_1(0), f_2(0)\right) = \inf_{\pi_0} D(0).$$

Hence, it suffices to control $D(t)$. This amounts to performing a Gronwall estimate along the trajectories of the characteristic flow.

Differentiating with respect to t gives

$$\dot{D}(t) = 2 \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} (X_t^{(1)} - X_t^{(2)}) \cdot (V_t^{(1)} - V_t^{(2)}) + (V_t^{(1)} - V_t^{(2)}) \cdot \left[E_{1,t}(X_t^{(1)}) - E_{2,t}(X_t^{(2)})\right] d\pi_0 \tag{3.3}$$

We split the electric field into four parts:

$$E_{1,t}(X_t^{(1)}) - E_{2,t}(X_t^{(2)}) = \left[\bar{E}_{1,t}(X_t^{(1)}) - \bar{E}_{1,t}(X_t^{(2)})\right] + \left[\bar{E}_{1,t}(X_t^{(2)}) - \bar{E}_{2,t}(X_t^{(2)})\right]$$
$$+ \left[\widehat{E}_{1,t}(X_t^{(1)}) - \widehat{E}_{1,t}(X_t^{(2)})\right] + \left[\widehat{E}_{1,t}(X_t^{(2)}) - \widehat{E}_{2,t}(X_t^{(2)})\right].$$

Applying Hölder's inequality to (3.3), we obtain

$$\dot{D} \leq D + 2\sqrt{D} \sum_{i=1}^{4} I_i^{1/2},$$

where

$$I_1(t) := \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |\bar{E}_{1,t}(X_t^{(1)}) - \bar{E}_{1,t}(X_t^{(2)})|^2 \, d\pi_0, \quad I_2(t) := \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |\bar{E}_{1,t}(X_t^{(2)}) - \bar{E}_{2,t}(X_t^{(2)})|^2 \, d\pi_0;$$
$$\tag{3.4}$$
$$I_3(t) := \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |\widehat{E}_{1,t}(X_t^{(1)}) - \widehat{E}_{1,t}(X_t^{(2)})|^2 \, d\pi_0, \quad I_4(t) := \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |\widehat{E}_{1,t}(X_t^{(2)}) - \widehat{E}_{2,t}(X_t^{(2)})|^2 \, d\pi_0.$$

We estimate the above terms in Lemmas 3.3-3.6 below. Altogether we obtain

$$\dot{D} \leq \begin{cases} CD|\log(D)| & \text{if } D < 1/2 \\ CD & \text{if } D \geq 1/2. \end{cases}$$

Therefore

$$D(t) \leq \exp\left[\log(D(0))e^{-Ct}\right]$$

as long as $D(t) \leq 1/2$, while once $D(t)$ reaches $1/2$ (say at some time $\bar{t} \geq 0$) then we have the alternative bound

$$D(t) \leq \frac{1}{2}e^{C(t-\bar{t})}.$$

From these bounds, the stability follows. $\qquad \square$

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In the remainder of this section, we prove Lemmas 3.3-3.6. We shall need the regularity estimates on $\bar{E}$ provided by the boundedness of the density. It will be convenient to state them in a rather unusual but compact form, for later use in Lemmas 3.3 and 3.5.

Lemma 3.2. *Let* $\bar{U} := G * \rho$, *where* $G = -\frac{1}{4\pi|x|}$ *is the Green function, and assume that* $\|\rho\|_{L^1(\mathbb{R}^3)} + \|\rho\|_{L^\infty(\mathbb{R}^3)} \leq M$ *for some* $M \geq 1$. *Let* $H : \mathbb{R}^+ \to \mathbb{R}^+$ *denote the function defined as*

$$H(s) := \begin{cases} s (\log s)^2 & \text{if } s \leq e^{-2} \\ 4e^{-2} & \text{if } s > e^{-2}. \end{cases}$$

Then there exists a universal constant C *such that*

$$|\nabla\bar{U}(x) - \nabla\bar{U}(y)|^2 \leq C M^2 H(|x-y|^2) \qquad \text{for all } x, y \in \mathbb{R}^3.$$

Proof. Let $\bar{U}_M := \frac{1}{M}\bar{U}$ and $\rho_M := \frac{1}{M}\rho$, and note that $\bar{U}_M := G * \rho_M$ with $\|\rho_M\|_{L^1(\mathbb{R}^3)} + \|\rho_M\|_{L^\infty(\mathbb{R}^3)} \leq 1$. Hence, applying [11, Lemma 3.1] to the function $\bar{U}_M$ we deduce that

$$\|\nabla\bar{U}_M\|_{L^\infty(\mathbb{R}^3)} \leq C, \quad |\nabla\bar{U}_M(x) - \nabla\bar{U}_M(y)| \leq C|x-y|\big|\log|x-y|\big| \quad \text{for all } x, y \in \mathbb{R}^3 \text{ with } |x-y| \leq e^{-1}.$$

This estimate implies that

$$|\nabla\bar{U}_M(x) - \nabla\bar{U}_M(y)|^2 \leq C H(|x-y|^2) \qquad \text{for all } x, y \in \mathbb{R}^3,$$

and recalling that $\bar{U}_M = \frac{1}{M}\bar{U}$, this concludes the proof. $\qquad\square$

In all the following lemmas, $D(t)$ is defined as in (3.1).

Lemma 3.3 (Control of I_1). *Let* I_1 *be defined as in* (3.4). *Then*

$$I_1(t) \leq C H(D(t)),$$

where H *is defined in Lemma 3.2.*

Proof. Since the density associated to ρ_{f_1} is uniformly bounded, we can apply Lemma 3.2 to bound

$$I_1(t) \leq C \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} H\left(|X_t^{(1)} - X_t^{(2)}|^2\right) d\pi_0.$$

Also, one can check that the function H is concave on $\mathbb{R}^+$. Thus, since π_0 is a probability measure, we may apply Jensen's inequality to deduce that

$$I_1(t) \leq C H\left(\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |X_t^{(1)} - X_t^{(2)}|^2 d\pi_0\right) \leq C H(D(t)),$$

where the last inequality follows from the fact that H is non-decreasing. $\qquad\square$

Lemma 3.4 (Control of I_2). *Let* I_2 *be defined as in* (3.4). *Then*

$$I_2(t) \leq C D(t).$$

Proof. One can note that, for any test function ϕ,

$$\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} \phi(X_t^{(i)}) d\pi_0 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(x) f_i(t,x,v) dx\, dv = \int_{\mathbb{R}^3} \phi(x) \rho_{f_i}(t,x) dx. \qquad (3.5)$$

Thus

$$I_2(t) = \int_{\mathbb{R}^3} |\bar{E}_{1,t}(x) - \bar{E}_{2,t}(x)|^2 \rho_{f_2}(t,x) dx \leq \|\rho_{f_2}(t)\|_{L^\infty(\mathbb{R}^3)} \|\bar{E}_{1,t} - \bar{E}_{2,t}\|_{L^2(\mathbb{R}^3)}^2,$$

and we conclude using Propositions 2.8-2.9 (depending on the model under consideration) and (3.2). $\quad\square$

Lemma 3.5 (Control of I_3). *Let I_3 be defined as in (3.4). Then*

$$I_3(t) \leq CH(D(t)),$$

where the constant $C > 0$ depends only on $\mathcal{E}[f_1(0)]$.

Proof. Note that

$$\Delta \widehat{U}_{1,t} = ge^{\widehat{U}_{1,t}+\bar{U}_{1,t}} \qquad \left(\text{resp. } \Delta \widehat{U}_{1,t} = \frac{ge^{\widehat{U}_{1,t}+\bar{U}_{1,t}}}{\int_{\mathbb{R}^3} ge^{\widehat{U}_{1,t}+\bar{U}_{1,t}} \, \mathrm{d}x} \right).$$

We can thus deduce a log-Lipschitz estimate on $\widehat{U}$ by using Lemma 3.2. To do this we therefore need L^1 and L^∞ estimates on $\Delta \widehat{U}$.

By (2.6) and (2.9)

$$\|\Delta \widehat{U}_{1,t}\|_{L^\infty(\mathbb{R}^3)} \leq C_g e^{\|\bar{U}_{1,t}\|_{L^\infty(\mathbb{R}^3)}}.$$

Then, using the L^∞ estimate on $\bar{U}$ from Lemma 2.1,

$$\|\Delta \widehat{U}_{1,t}\|_{L^\infty(\mathbb{R}^3)} \leq C_g \exp \left[C\|\rho_{f_1}(t)\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}} \|\rho_{f_1}(t)\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}} \right] \leq C,$$

where C depends only on the initial datum $f_1(0)$.

For the L^1 estimates, in the fixed charge case we always have

$$\|\Delta \widehat{U}_{1,t}\|_{L^\infty(\mathbb{R}^3)} = 1.$$

In the variable charge case, by (2.6) we have

$$\|\Delta \widehat{U}_{1,t}\|_{L^1(\mathbb{R}^3)} \leq \|g\|_{L^1(\mathbb{R}^3)} e^{\|\bar{U}_{1,t}\|_{L^\infty(\mathbb{R}^3)}} \leq \|g\|_{L^1(\mathbb{R}^3)} \exp \left[C\|\rho_{f_1}(t)\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}^{\frac{5}{6}} \|\rho_{f_1}(t)\|_{L^1(\mathbb{R}^3)}^{\frac{1}{6}} \right] \leq C,$$

where C depends only on the initial datum $f_1(0)$.

Therefore, by Lemma 3.2,

$$|\nabla \widehat{U}_{1,t}(x) - \nabla \widehat{U}_{1,t}(y)|^2 \leq CH(|x-y|^2) \qquad \text{for all } x, y \in \mathbb{R}^3,$$

for some C depending only on $f_1(0)$.

We then argue as in Lemma 3.3: using the above regularity estimate on $\nabla \widehat{U}_{1,t}$, we have

$$I_3(t) \leq C \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} H\left(|X_t^{(1)} - X_t^{(2)}|^2 \right) \mathrm{d}\pi_0.$$

Since H is concave and non-decreasing,

$$I_3(t) \leq C H\left(\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} |X_t^{(1)} - X_t^{(2)}|^2 \, \mathrm{d}\pi_0 \right) \leq C H(D(t)).$$

which concludes the proof. $\qquad \square$

Lemma 3.6 (Control of I_4). *Let I_4 be defined as in (3.4). Then*

$$I_4(t) \leq C D(t),$$

where D is defined as in (3.1) and $C_{M,d}$ depends on M and d.

Proof. Using (3.5), we deduce that

$$I_4(t) = \int_{\mathbb{R}^3} |\widehat{E}_{1,t}(x) - \widehat{E}_{2,t}(x)|^2 \rho_{f_2}(t,x) \, \mathrm{d}x \leq \|\rho_{f_2}(t)\|_{L^\infty(\mathbb{R}^3)} \|\widehat{E}_{1,t} - \widehat{E}_{2,t}\|_{L^2(\mathbb{R}^3)}^2,$$

and we conclude by Propositions 2.8-2.9 and (3.2). $\qquad \square$

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4 Moment Estimates

In this section, we turn to the existence of strong solutions. We adapt the method of construction of solutions developed by Lions and Perthame [10] for the Vlasov-Poisson system for electrons. The key step is to prove an a priori estimate on the velocity moments of a solution. This is the content of the following proposition.

Proposition 4.1. *Let $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, $f_0 \geq 0$, $\|f_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} = 1$. Assume that f_0 also satisfies, for some $m_0 > 3$,*

$$M_{m_0}(0) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{m_0} f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v < +\infty.$$

Let f be a solution of (1.3) (resp. (1.4)) such that for all t,

$$\mathcal{E}_V[f](t) \leq C \qquad (\text{resp. } \mathcal{E}_F[f](t) \leq C),$$

and satisfying

$$\|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leq \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}, \qquad \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v = 1.$$

Then, for all $k < m_0$,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \leq \exp\left[C\big(1 + \log(1 + M_k(0))\big) \exp(Ct)\right],$$

for some constant C depending only on $\|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}$ and $\mathcal{E}_V[f_0]$ (resp. $\mathcal{E}_F[f_0]$).

The proof of this proposition follows by the classical argument by Lions and Perthame [10], simplified with the use of Lorentz spaces. Indeed, since our electric field can be split as the sum of the classical Vlasov-Poisson field plus a term uniformly bounded in $L^{\frac{3}{2},\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, the proof in [10] can be adapted with some minor modifications. For convenience of the interested reader, we detail the argument in Appendix A.

A Proof of Proposition 4.1.

The aim is to control the velocity moments M_k by use of a Gronwall estimate, where

$$M_k(t) := \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(s, x, v) \, \mathrm{d}x \, \mathrm{d}v.$$

From [10], by using the equation we can deduce the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} M_k(t) \leq C \|E_t\|_{L^{k+3}(\mathbb{R}^3)} M_k(t)^{\frac{k+2}{k+3}}. \tag{A.1}$$

It therefore remains to control $\|E_t\|_{L^{k+3}(\mathbb{R}^3)}$. We assume from now on, without loss of generality, that $k > 3$.

First, we note that the conservation of energy gives us uniform in time bounds on ρ_f and therefore E. By Lemma 1.6 and conservation of mass, we have the uniform bounds

$$\|\rho_f(t, \cdot)\|_{L^1(\mathbb{R}^3)} \equiv 1, \qquad \sup_t \|\rho_f(t, \cdot)\|_{\frac{5}{3}(\mathbb{R}^3)} \leq C.$$

From the regularity estimates above, we deduce that we have uniform bounds on the electric field:

$$\sup_t \|\bar{E}_t\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq C, \qquad \sup_t \|\bar{E}_t\|_{L^{\frac{15}{4}}(\mathbb{R}^3)} \leq C$$

$$\sup_t \|\widehat{E}_t\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \leq C, \qquad \sup_t \|\widehat{E}_t\|_{L^\infty(\mathbb{R}^3)} \leq C.$$

If $k + 3 > \frac{15}{4}$, we require further estimates on $\|\bar{E}_t\|_{L^{k+3}(\mathbb{R}^3)}$. To do this, we will follow the strategy of [10]. We first note some preliminary estimates relating the $L^p(\mathbb{R}^3)$ norms of ρ_f and $\bar{E}$ and similar quantities to moments of f.

Lemma A.1. *For any $s, t \geq 0$ and $k \geq 0$,*

$$\left\| \int_{\mathbb{R}^3} f(s, \cdot - vt, v) \, dv \right\|_{L^{\frac{k+3}{3}}(\mathbb{R}^3)} \leq M_k(s)^{\frac{3}{k+3}}.$$

Proof. This is a standard interpolation argument. For any $R > 0$,

$$\int_{\mathbb{R}^3} f(s, x - vt, v) \, dv \leq R^{-k} \int_{|v| > R} |v|^k f(s, x - vt, v) \, dv + \|f(s, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} R^3.$$

Optimising over R gives

$$\int_{\mathbb{R}^3} f(s, x - vt, v) \, dv \leq C \left(\int_{|v| > R} |v|^k f(s, x - vt, v) \, dv \right)^{\frac{3}{k+3}}.$$

Then

$$\left\| \int_{\mathbb{R}^3} f(s, \cdot - vt, v) \, dv \right\|_{L^{\frac{k+3}{3}}(\mathbb{R}^3)} \leq \left(\int_{\mathbb{R}^3} |v|^k f(s, x - vt, v) \, dx \, dv \right)^{\frac{3}{k+3}} \leq M_k(s)^{\frac{3}{k+3}}.$$

$\square$

Using Lemma 2.1, we deduce that control of moments implies integrability of $\bar{E}$.

Lemma A.2. *Let $n \in (0, 6)$ and $q \in (\frac{3}{2}, +\infty)$ satisfy*

$$q = \frac{3}{6 - n} \cdot (n + 3).$$

Then there exists a constant $C_q > 0$ such that, for all $t \geq 0$,

$$\|\bar{E}_t\|_{L^q(\mathbb{R}^3)} \leq C_q M_n^{\frac{3}{n+3}}(t).$$

The resulting estimate on $\|\bar{E}_t\|_{L^{k+3}}$ is not sufficient to allow us to obtain a long term estimate from the differential inequality (A.1). The next step is to obtain an improved estimate on $\bar{E}$. We start by obtaining a formula for ρ_f by solving the equation along characteristics with $-E \cdot \nabla_v f$ as a source term. From a Duhamel representation of f, we deduce as in [10] that

$$\rho_f(t, x) = -\operatorname{div}_x \int_0^t s \int_{\mathbb{R}^3} [Ef(t - s, x - vs, v)] \, dv \, ds + \int_{\mathbb{R}^3} f_0(x - vt, v) \, dv.$$

Since $\bar{E} = \nabla \Delta^{-1} \rho_f$, by using Sobolev inequality and Calderon-Zygmund theory we deduce that

$$\|\bar{E}_t\|_{L^{k+3}(\mathbb{R}^3)} \leq \left\| \int_0^t s \int_{\mathbb{R}^3} [Ef(t - s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} + \left\| \int_{\mathbb{R}^3} f_0(\cdot - vt, v) \, dv \right\|_{L^{\frac{3(k+3)}{k+6}}(\mathbb{R}^3)}.$$

To estimate the term involving f_0, we use Lemma A.1 to deduce that

$$\left\| \int_{\mathbb{R}^3} f_0(\cdot - vt, v) \, dv \right\|_{L^{\frac{3(k+3)}{k+6}}(\mathbb{R}^3)} \leq C M_l(0)^{\frac{3}{l+3}},$$

where l is chosen such that

$$\frac{l+3}{3} = \frac{3(k+3)}{k+6} = \frac{9}{k+6} \cdot \frac{k+3}{3}.$$

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Since we have assumed that $k > 3$, then $l < k$ and so $M_l(0)$ is controlled by $M_k(0)$.

To estimate the term involving Ef, we proceed as in [10], and we split the time integral into a short time and a long time part:

$$\left\| \int_0^t s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \leq \left\| \int_0^{t_0} s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)}$$
$$+ \left\| \int_{t_0}^t s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)}.$$

We complete the estimates on these terms in the following two subsections.

A.1 Long Time Estimate

In this subsection, we prove that

$$\left\| \int_{t_0}^t s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \leq C |\log t_0| \, M_k(t)^{\frac{1}{k+3}}. \tag{A.2}$$

We will require the following Hölder-like inequalities for the Lorentz spaces $L^{p,q}$ - see O'Neil [14] and Tartar [17].

Lemma A.3. *(i) Let $0 < p_1, p_2, p < \infty$, $0 < q_1, q_2, q \leq \infty$ satisfy $p^{-1} = p_1^{-1} + p_2^{-1}$, $q^{-1} = q_1^{-1} + q_2^{-1}$. Then*

$$\|fg\|_{L^{p,q}} \leq C_{p_1,p_2,q_1,q_2} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}},$$

whenever the right hand side is finite.

(ii) Let $f \in L^1 \cap L^\infty$. Then, for any $p \in [1, \infty)$,

$$\|f\|_{L^{p,1}} \leq C_p \|f\|_{L^1}^{\frac{1}{p}} \|f\|_{L^\infty}^{1-\frac{1}{p}}.$$

Consequently, we have the following estimate. Let $p \in (1, \infty]$. Let $f \in L^{p,\infty}$ and $g \in L^1 \cap L^\infty$. Then

$$\|fg\|_{L^1} \leq C_p \|f\|_{L^{p,\infty}} \|g\|_{L^1}^{1-\frac{1}{p}} \|g\|_{L^\infty}^{\frac{1}{p}}.$$

Using this, we prove (A.2). By Minkowski's inequality,

$$\left\| \int_{t_0}^t s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \leq \int_{t_0}^t s \left\| \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \right\|_{L^{k+3}(\mathbb{R}^3)} ds.$$

By Lemma A.3, we have the following estimate:

$$\int_{\mathbb{R}^3} [Ef(t-s, x-vs, v)] \, dv \leq \|E(t-s, x-s\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbb{R}^3)} \|f\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}} \left| \int_{\mathbb{R}^3} f(t-s, x-vs, v) \, dv \right|^{\frac{1}{3}}.$$

By Lemma 2.1, and Propositions 2.5, 2.7, E is bounded in $L^{\frac{3}{2},\infty}$, uniformly in time. Thus

$$\left\| \int_{t_0}^t s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \leq C \int_{t_0}^t s^{-1} \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(t-s, x-vs, v)|^{\frac{k+3}{3}} \, dx \, dv \right)^{\frac{1}{k+3}} ds,$$

where $C > 0$ depends only on the initial datum.

By Lemma A.1,

$$\left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(t-s, x-vs, v)|^{\frac{k+3}{3}} \, dx \, dv \right)^{\frac{1}{k+3}} \leq M_k(t-s)^{\frac{1}{k+3}} \leq M_k(t)^{\frac{1}{k+3}},$$

since $s > 0$. Therefore (A.2) follows.

A.2 Short Time Estimate

In this subsection we show that

$$\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} [Ef(t-s, x-vs, v)] \, dv \, ds \right\|_{L^{k+3}} \le C t_0^{2-\frac{3}{r}} \left[M_m(0)^{\frac{1}{k+3}} + \left(1 + M_k(t)^\delta \right) \right],$$

where

$$\delta = \frac{3(m+3)}{(k+3)^2}.$$

By Minkowski's inequality,

$$\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \le \int_0^{t_0} s \left\| \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \right\|_{L^{k+3}(\mathbb{R}^3)} ds.$$

By Hölder's inequality, for any $r > \frac{3}{2}$ we obtain

$$\int_{\mathbb{R}^3} [Ef(t-s, x-vs, v)] \, dv \le \left(\int_{\mathbb{R}^3} |E(t-s, x-vs)|^r \, dv \right)^{1/r} \|f\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{r}} \left| \int_{\mathbb{R}^3} f(t-s, x-vs, v) \, dv \right|^{1-\frac{1}{r}}$$

$$\le s^{-\frac{3}{r}} \|E_{t-s}\|_{L^r(\mathbb{R}^3)} \|f\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{r}} \left| \int_{\mathbb{R}^3} f(t-s, x-vs, v) \, dv \right|^{1-\frac{1}{r}}.$$

Thanks to Lemma 2.1, and Propositions 2.5, 2.7 we have

$$\bar{E}_t \in L^{\frac{3}{2},\infty} \cap L^{\frac{15}{4}}(\mathbb{R}^3), \qquad \widehat{E}_t \in L^{\frac{3}{2},\infty} \cap L^\infty(\mathbb{R}^3),$$

with uniform in time estimates depending only on $M_2(0)$. We therefore choose $r \in (\frac{3}{2}, \frac{15}{4})$ and obtain

$$\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} [Ef(t-s, \cdot - vs, v)] \, dv \, ds \right\|_{L^{k+3}(\mathbb{R}^3)} \le C \int_0^{t_0} s^{1-\frac{3}{r}} \left\| \int_{\mathbb{R}^3} f(t-s, \cdot - vs, v) \, dv \right\|_{L^{\frac{k+3}{r'}}(\mathbb{R}^3)}^{\frac{1}{r'}} ds,$$

where r' satisfies $1/r + 1/r' = 1$, and the constant $C > 0$ depends only on the initial datum.

To control the density term, we use Lemma A.1 with a moment of higher order than k. Choose $m \in (k, m_0)$ such that

$$\frac{m+3}{3} = \frac{k+3}{r'}.$$

Then

$$\int_0^{t_0} s^{1-\frac{3}{r}} \left\| \int_{\mathbb{R}^3} f(t-s, \cdot - vs, v) \, dv \right\|_{L^{\frac{k+3}{r'}}(\mathbb{R}^3)}^{\frac{1}{r'}} ds \le C \int_0^{t_0} s^{1-\frac{3}{r}} \, ds \, M_m(t_0)^{\frac{1}{k+3}} \le C t_0^{2-\frac{3}{r}} M_m(t_0)^{\frac{1}{k+3}}.$$

We control M_m by using (A.1), which implies that for all $t \ge 0$,

$$M_m(t) \le C \left(M_m(0) + \left(t \sup_{s \le t} \|E_s\|_{L^{m+3}(\mathbb{R}^3)} \right)^{m+3} \right).$$

$\|\widehat{E}_s\|_{L^{m+3}(\mathbb{R}^3)}$ is uniformly bounded by Lemma 2.6. For $\bar{E}_s$, we use Lemma A.2 to obtain

$$\|\bar{E}_s\|_{L^{m+3}(\mathbb{R}^3)} \le M_n^{\frac{3}{n+3}}(s),$$

where $n = n_m \in (0, 6)$ is related to m via the formula

$$m + 3 = \frac{3}{6-n} \cdot (n+3).$$

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We now aim to control M_n by M_k. Note that if $n > 3$ then $n < m$. Recall that m depends on r and k, and that $m \searrow k$ as $r \searrow 3/2$. As $m \searrow k > 3$ by assumption, $n_m \searrow n_k < k$. Therefore, by choosing r sufficiently close to $3/2$, we can ensure that $n_m \le k < m$. Then, since $M_n \le M_k^{\frac{n}{k}}$ (by Hölder inequality), for $s \le t$ we have

$$\|\bar{E}_s\|_{L^{m+3}(\mathbb{R}^3)}^{m+3} \le M_n(t)^{\frac{3(m+3)}{n+3}} \le M_k(t)^{\frac{3n(m+3)}{k(n+3)}} \le (1+M_k(t))^{\frac{3n(m+3)}{k(n+3)}} \le (1+M_k(t))^{\frac{3(m+3)}{k+3}}.$$

Thus

$$M_m(t_0) \le C\left[M_m(0) + \left[t_0\left(1 + M_k(t)^{\frac{3}{k+3}}\right)\right]^{m+3}\right].$$

Then, for $t_0 \le 1$,

$$\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} [Ef(t-s, x-vs, v)]\, dv\, ds\right\|_{L^{k+3}} \le C t_0^{2-\frac{3}{r}} \left[M_m(0)^{\frac{1}{k+3}} + \left(1 + M_k(t)^{\delta}\right)\right],$$

where

$$\delta = \frac{3(m+3)}{(k+3)^2}.$$

A.3 Full Estimate

Closing the estimate is identical to [10]. Choosing $t_0 = (1+M_k(t))^{-\frac{\delta r}{2r-3}}$, and combining all the previous estimates, gives a bound of the form

$$\|E_t\|_{L^{k+3}(\mathbb{R}^3)} \le C(1 + \log(1 + M_k(t)))(1 + M_k(t))^{\frac{1}{k+3}}.$$

Thus, recalling (A.1), one obtains

$$\frac{d}{dt} M_k(t) \le C\big(1 + |\log 1 + M_k(t)|\big)(1 + M_k(t)),$$

which completes the proof of Proposition 4.1.

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