Twisted Patterson-Sullivan measures and applications to amenability and coverings

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Contents

Abstract

Let $\Gamma' < \Gamma$ be two discrete groups acting properly by isometries on a Gromov-hyperbolic space X. We prove that their critical exponents coincide if and only if Γ' is co-amenable in Γ, under the assumption that the action of Γ on X is strongly positively recurrent, i.e. has a growth gap at infinity. This generalizes all previously known results on this question, which required either X to be the real hyperbolic space and Γ geometrically finite, or X Gromov hyperbolic and Γ cocompact. This result is optimal: we provide several counterexamples when the action is not strongly positively recurrent.

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1. Introduction

1.1. Main result. Let (X,d) be a proper metric space, $o \in X$ be some fixed origin, and Γ be a discrete group acting properly by isometries on X. The *critical exponent* of Γ (for its action on X) is

$$h_\Gamma = h_\Gamma(X) = \limsup_{r\to+\infty} \frac{1}{r} \ln |\{\gamma \in \Gamma \mid d(o,\gamma o) \leqslant r\}| .$$

Of course, any subgroup $\Gamma' < \Gamma$ satisfies $h_{\Gamma'} \leqslant h_\Gamma$. This paper is devoted to the study of the equality case.

When do we have $h_{\Gamma'} = h_\Gamma$?

We are particularly interested in the case where X is Gromov hyperbolic. The answer to this question is intimately related to the *co-amenability* of Γ' in Γ, as was first independently shown by Grigorchuk [**Gri80**], Cohen [**Coh82**] and Brooks [**Bro81**]. Saying that Γ' is co-amenable in Γ is a natural way to generalise the fact that the quotient Γ/Γ' is amenable when Γ' is not a normal subgroup of Γ, see Definition 6.2. Our main theorem widely extends all previously known results on this question. It holds under the assumption that the action of Γ has a *growth gap at infinity*, i.e. some critical exponent at infinity, representing the growth of Γ far from the orbit of any compact set, is strictly smaller than h_Γ. We also call such actions *strongly positively recurrent*. See below for the rigorous definition. This assumption is much weaker than more usual assumptions such as convex-cocompactness or geometrical finiteness, as shown in [**ST18**].

Theorem 1.1. *Let X be a proper hyperbolic geodesic space. Let Γ be a group acting properly by isometries on X, and Γ' a subgroup of Γ. Assume that the action of Γ is strongly positively recurrent. The following are equivalent.*

(1) $h_{\Gamma'} = h_\Gamma$

(2) The subgroup Γ' is co-amenable in Γ.

Let us emphasize that this result is new even when X is a rank one symmetric manifold. As will be shown in Section 6.3, this result is optimal. None of the assumptions can be weaken without hitting numerous counterexamples. Our main theorem closes the above question for group actions on Gromov hyperbolic spaces.

Beyond this, we believe that our main tool, the *twisted Patterson-Sullivan measure*, is at least as important as the result, and should have various other applications in the future. Indeed, the method is completely original and new, and builds a fruitful bridge between invariant measures and ergodic theory of the geodesic flow on the one hand, and representations of the fundamental group on the other hand.

Theorem 1.1 has also a quantified version (Theorem 5.2) which leads to the following wide generalisation of Corlette's growth rigidity result [**Cor90**], see also [**Dou17, CDS17**].

Theorem 1.2. *Let X be a proper hyperbolic geodesic space. Let Γ be a group with Kazhdan's property (T) acting properly by isometries on X. Assume that the action of Γ is strongly positively recurrent. There exists $\varepsilon \in \mathbb{R}_+^*$ such that for every subgroup Γ' of Γ, either $h_{\Gamma'} \leqslant h_\Gamma - \varepsilon$ or Γ' is a finite index subgroup of Γ.*

We now give a brief historical background on this question, introduce the notion of strongly positively recurrent action, and sketch the proof of Theorem 1.2.

1.2. Historical background. The first relations between critical exponents and amenability appeared independently in the eighties, in the work of Brooks

[**Bro81**, **Bro85**], in the context of hyperbolic manifolds, and Grigorchuk [**Gri80**], Cohen [**Coh82**] in a combinatorial setting.

Let Γ be a finitely generated free group acting on its Cayley graph X, with respect to a free basis. Given any normal subgroup Γ' of Γ, Grigorchuk and Cohen relate by a delicate explicit computation the critical exponent of Γ' (also called *co-growth* of Γ/Γ') to the spectral radius of the random walk on Γ/Γ'. Combined with Kesten's amenability criterion, they obtain the following statement.

Theorem 1.3 (Grigorchuk [**Gri80**], Cohen [**Coh82**]). *Let Γ be a finitely generated free group and X its Cayley graph with respect to a free basis. For every normal subgroup Γ' of Γ, the quotient Γ/Γ' is amenable if and only if $h_{\Gamma'} = h_\Gamma$.*

At the same period, Brooks showed the following statement using the spectral properties of the Laplace-Beltrami operator.

Theorem 1.4 (Brooks, [**Bro85**]). *Let $n \in \mathbb{N}$ and $M = \mathbb{H}^{n+1}/\Gamma$ be a* convex-cocompact hyperbolic *manifold with $h_\Gamma > n/2$. Then for every normal subgroup Γ' of Γ, the quotient Γ/Γ' is amenable if and only if $h_{\Gamma'} = h_\Gamma$.*

Let us discuss briefly the strategy behind this last result. Recall that a negatively curved manifold is convex-cocompact if all closed geodesics are included in a given compact set (or equivalently, if the geodesic flow has a compact nonwandering set). Brooks' approach actually starts in a much larger context. Given a Riemannian manifold M whose Laplacian satisfies a *spectral gap* condition, he showed that for every normal covering M' of M the quotient $\pi_1(M)/\pi_1(M')$ is amenable if and only if the bottom spectra of their respective Laplace-Beltrami operators satisfy $\lambda_0(M) = \lambda_0(M')$. If $M = \mathbb{H}^{n+1}/\Gamma$ is a hyperbolic manifold with $h_\Gamma > n/2$, then Sullivan's formula relates $\lambda_0(M)$ to h_Γ [**Sul87**]. Moreover, Brooks' spectral condition is satisfied for convex-cocompact hyperbolic manifolds with $h_\Gamma > n/2$, which gives Theorem 1.4.

We will not define this spectral gap condition for the Riemannian Laplacian here – see [**Bro85**, Section 1] – but it is exactly the spectral analog to the *growth gap at infinity* (or *strongly positive recurrence*) that we will introduce below for group actions, under which our main theorems are valid.

The assumption $h_\Gamma > n/2$ is specific to this approach and cannot be removed as long as one uses Laplace spectrum.

Sullivan's formula relating the bottom of the spectrum of the Laplacian with critical exponents has been extended by Corlette-Iozzi [**CI99**] to all other locally symmetric hyperbolic manifolds. Therefore Brooks method extends verbatim to these exotic hyperbolic manifolds. Note also that Brooks's result can be extended when Γ' is not normal in Γ. This can be seen following the alternative proof of Brooks' Theorem given in [**RT13**].

Using Patterson-Sullivan theory, Roblin in [**Rob05**] is the first to prove the so-called "easy direction" in a much wider context. Namely, if Γ is a discrete group of isometries acting on a CAT(-1) space X and Γ' is a normal subgroup of Γ such that Γ/Γ' is amenable, then $h_\Gamma = h_{\Gamma'}$. His proof extends easily to actions on Gromov hyperbolic spaces, but requires in a crucial way that Γ' be normal in Γ.

The reciprocal statement was generalised by Stadlbauer in [**Sta13**], using a dynamical argument inspired by Kesten's work on random walks, see also Jaerisch [**Jae14**]. If Γ is an *essentially free* discrete group of isometries of $\mathbb{H}^{n+1}$, then for all normal subgroups Γ' of Γ, the quotient Γ/Γ' is amenable if and only if $h_{\Gamma'} = h_\Gamma$. His method allows to remove the artificial assumption $h_\Gamma > n/2$, and to our knowledge it is the only published work to deal with certain specific non convex-cocompact manifolds (geometrically finite).

Stadlbauer's arguments have been used later on by Dougall-Sharp in [**DS16**] with a symbolic coding in order to extend the result to convex-cocompact manifolds with pinched negative curvature, when Γ' is a normal subgroup of Γ. A generalization by Coulon, Dal'bo and Sambusetti in [**CDS17**] allows to deal with proper cocompact actions of Γ on some Gromov-hyperbolic spaces X, more precisely CAT(-1) spaces or the Cayley graph of Γ. Moreover the subgroup Γ' need not be normal in Γ.

1.3. Strongly positively recurrent actions. The notion of strongly positively recurrent action is crucial in our work. Let us present the definition and its origin. A detailed presentation can be found in Section 3. Let (X, d) be a proper geodesic space and Γ a group acting properly by isometries on X. Given a compact subset K of X, we define Γ_K as the set of elements $\gamma \in \Gamma$ for which there exists two points $x, y \in K$ and a geodesic $c\colon [a, b] \to X$ joining x to γy such that $c \cap \Gamma \cdot K$ is contained in $K \cup \gamma K$. The critical exponent h_{Γ_K} of Γ_K is called the *entropy outside* K. The *entropy at infinity* of Γ is the quantity

$$h_\Gamma^\infty = \inf_K h_{\Gamma_K}$$

The action of Γ on X has a *growth gap at infinity* if $h_\Gamma^\infty < h_\Gamma$. We will say then that the action is *strongly positively recurrent.* This notion which has both a dynamical and a geometric origin has been introduced independently in different contexts.

A dynamical origin. Heuristically, a dynamical system is strongly positively recurrent (with respect to a constant potential) if its entropy at infinity is strictly smaller than its topological entropy, see for instance [**Sar01**]. The terminology *stably positively recurrent* has been first introduced in the context of Markov shifts over a countable alphabet by Gurevič-Savchenko [**GS98**], and became *strongly positively recurrent* later in Sarig [**Sar01**]. This terminology, with the notion of entropy at infinity, has been used later on by several authors considering dynamical systems on a non-compact space, such as Ruette [**Rue03**], Boyle, Buzzi and Gomez [**BBGo14**], or more recently Riquelme and Velozo [**RV18, Vel17**]. We do not define here the entropy at infinity of a dynamical system, however for the geodesic flow of a non-compact negatively curved manifold it coincides with the quantity h_Γ^∞ defined above [**RV18, Vel17, ST18, GSTR20**].

A geometric point of view. Dal'bo, Otal and Peigné in [**DOP00**] introduced the terminology of *parabolic gap* concerning geometrically finite groups Γ of isometries of a negatively curved space X whose parabolic subgroups P all satisfy $h_P < h_\Gamma$. Extending the work of Dal'bo *et al* [**DPPS11**], this was later generalized by Arzhantseva, Cashen and Tao [**ACT15**, Definition 1.6] to the so-called *growth gap* property, which is exactly the *growth gap at infinity* defined above. They showed that if the action of Γ on X has a growth gap at infinity and admits a contracting element, then Γ is *growth tight* (see [**ACT15**] for a definition). This notion has also been studied by Yang [**Yan16**, Definition 1.4] under the name *statistically convex-cocompact action.* His terminology comes from the following intuition. Given $r \in \mathbb{R}_+$, the Γ-orbit of a point $o \in X$ is in general not r-quasi-convex. If K stands for the closed ball $B(o, r)$, then Γ_K is exactly the set of elements $\gamma \in \Gamma$ violating the definition of quasi-convexity. The assumption $h_\Gamma^\infty < h_\Gamma$ states that most elements of Γ behave as in a convex-cocompact setting.

Combining dynamical and geometric approaches. The paper [**ST18**] by Schapira and Tapie introduced strongly positively recurrent actions in order to study the geodesic flow of negatively curved manifolds (independently but identically to Arzhantseva et al. and Yang), and provided several new examples. It was both inspired by the aforementioned dynamical works of Sarig and Buzzi and the geometric approach of Dal'bo, Otal and Peigné.

In the present work, we combine intuitions from dynamical systems – especially many tools used for the ergodic study of the geodesic flow on non-compact negatively curved manifolds – and geometric group theory to get our main result.

Recent developments through Laplace spectrum approach. Since the end of this work, it has been shown by Ballmann, Matthiesen and Polymerakis in [**BMP18**] and [**Pol18**] that for any Riemannian covering $p : M' \to M$, if the bottom of the Laplace spectrum $\lambda_0(M)$ is an isolated eigenvalue, then $\lambda_0(M') \geqslant \lambda_0(M)$ with equality if and only if the covering is amenable. Moreover, it has been shown in [**Tap20**] that if $M = X/\Gamma$, where X is a symmetric space with negative curvature, then $\lambda_0(M)$ is an isolated eigenvalue if and only if the action of Γ on X is strongly positively recurrent and $h_\Gamma > \frac{\kappa}{2}$, where κ is the volume entropy of X.

Recall that on symmetric spaces X, the Patterson-Sullivan-Corlette formula relates h_Γ and $\lambda_0(X/\Gamma)$ (see [**Sul87, Cor90, Tap20**]). This provides an alternative proof of Theorem 1.1 in the special case where X is a symmetric space with negative curvature and $h_\Gamma > \frac{\kappa}{2}$.

1.4. Outline of the proofs. Let us give a brief account of the proofs, and the main novelties of this paper. Theorem 1.1 is the combination of two results:

(1) the so-called *"easy direction"*, i.e. showing that if Γ' is a co-amenable subgroup of Γ, then $h_{\Gamma'} = h_\Gamma$;
(2) conversely, showing that for any subgroup Γ' of Γ, if $h_{\Gamma'} = h_\Gamma$ then Γ' is co-amenable in Γ.

The "easy direction", detailed in Corollary 6.10, is based on an explicit estimation of the spectral radius of some random walks on Γ/Γ', as in [**CDS17**].

The core of this paper is the other direction. In the context of general Gromov hyperbolic spaces instead of negatively curved manifolds or CAT(-1)-spaces, and maybe even more problematic when the action of Γ is not cocompact, all the approaches described above fail. Indeed, the approach via the spectrum of a Laplace-Beltrami operator seems specific to *locally symmetric Riemannian manifolds with negative curvature* and might not be adapted in this more general setting. Moreover, we are not aware of any coding of the geodesic flow which would allow to transpose Stadlbauer's work. We develop therefore a new strategy combining Patterson-Sullivan theory and representation theory.

Assume for simplicity here that X is a proper CAT(-1) space, for example a rank one symmetric manifold. Recall that our results are new and optimal even in the latter case. Let Γ be a discrete group acting properly by isometries on X, Γ' a subgroup of Γ and $\mathcal{H} = \ell^2(\Gamma/\Gamma')$. Then Γ' is co-amenable in Γ if and only if the corresponding unitary representation $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ almost admits invariant vectors. Given $s > 0$, we associate to this representation the following formal twisted Poincaré series

$$A(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}\rho(\gamma),$$

and show that there exists a *critical exponent* h_ρ such that for every $s > h_\rho$, $A(s)$ is a bounded operator of $\mathcal{H}$. Moreover this exponent satisfies

$$h_{\Gamma'} \leqslant h_\rho \leqslant h_\Gamma, \tag{1}$$

see Lemma 6.1. By analogy with the standard Patterson-Sullivan measure, we associate to any $x \in X$, an operator-valued measure

$$\begin{array}{cccc} a^\rho_{x,s}\colon & C\left(\bar{X}\right) & \to & \mathcal{B}(\mathcal{H}) \\ & f & \mapsto & \displaystyle\frac{1}{\|A(s)\|}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma o)} f(\gamma o)\rho(\gamma). \end{array}$$

where $\bar{X}$ is the visual compactification of X and $C(\bar{X})$ the space of continuous functions on $\bar{X}$. When s approaches h_ρ from above, we are able, using an ultrafilter ω (see Section 5.2) to let these measures "converge" to a measure a_x^ρ supported on the boundary ∂X of X and taking its values in the space of bounded operators $\mathcal{B}(\mathcal{H}_\omega)$ on a larger Hilbert space $\mathcal{H}_\omega$. We call it the *twisted Patterson-Sullivan measure*.

In Section 5 we properly define and study this measure. In particular, we show that it satisfies all the properties of the classical Patterson-Sullivan measures: h_ρ-conformality (Lemma 5.15), Γ-invariance twisted by the limit representation $\rho_\omega \colon \Gamma \to \mathcal{U}(\mathcal{H}_\omega)$ induced by ρ (Lemma 5.14), Shadow Lemma (Lemma 5.17), etc.

The existence of a growth gap at infinity is used at a single but crucial place to prove that the measure a_x^ρ gives full mass to the *radial limit set* (Corollary 5.19). This apparently technical result allows to approximate the measure of any Borel set by measures of shadows. Then, the proof of Theorem 1.1 becomes particularly simple. Assume indeed that $h_{\Gamma'} = h_\Gamma$. By (1) the classical and twisted Patterson-Sullivan measures have the same conformal dimension, namely $h_\Gamma = h_\rho$. Using the Shadow Lemma we deduce that a_x^ρ is "absolutely continuous" with respect to the standard Patterson-Sullivan measure (Proposition 5.22). Thanks to the ergodicity of Bowen-Margulis current we prove that the corresponding "Radon-Nikodym derivative" – which takes its values in $\mathcal{B}(\mathcal{H}_\omega)$ – is essentially constant, equal to say $D \in \mathcal{B}(\mathcal{H}_\omega) \setminus \{0\}$. The twisted equivariance of a_x^ρ directly implies that the image of D (which is non trivial) is contained in the subspace of ρ_ω-invariant vectors. It follows then from the construction of ρ_ω that the original representation ρ almost has invariant vectors, i.e. Γ' is co-amenable in Γ (Section 5.6).

When X is a proper Gromov hyperbolic space, the above ideas work exactly in the same way. One just has to be careful that all measures are only quasi-conformal. However, this proof requires an important ergodicity argument. We use the fact that the Bowen-Margulis current is ergodic for the diagonal action of Γ on the double boundary $\partial^2 X$. This is well-known when X is a negatively curved Hadamard manifold, or even a CAT(-1) space and the action of Γ is strongly positively recurrent [**Rob03, ST18**]. Bader and Furman proved that the statement also holds when Γ acts cocompactly on a Gromov hyperbolic space [**BF17**]. Although the result is quite expected, it had not been written yet for a non-cocompact action on a Gromov hyperbolic space, such as strongly positively recurrent actions. As it should be useful to other people, we decided to expose this argument in the fullest possible generality.

More precisely, if Γ is a discrete group acting properly by isometries on a Gromov-hyperbolic space X, using the abstract geodesic flow already studied in [**BF17**], we prove a Hopf-Tsuji-Sullivan dichotomy (Theorem 4.2): the Bowen-Margulis current on the double boundary $\partial^2 X$ is ergodic with respect to the action of Γ if and only if the geodesic flow is ergodic and conservative (for the Bowen-Margulis measure), if and only if the usual Patterson-Sullivan measure gives full measure to the radial limit set. The desired ergodicity for a strongly positively recurrent action then directly follows from Corollary 3.16.

For the sake of completeness, we also included a finiteness criterion for the Bowen-Margulis measure (Theorem 4.16) inspired from [**PS18**], which allows to deduce that the Bowen-Margulis measure is finite in the presence of a growth gap at infinity (Corollary 4.17).

Once again, let us insist on the fact that the key novelty of our argument is the construction of a Patterson-Sullivan measure twisted by a unitary representation.

Outline of the paper. We recall basics on Gromov hyperbolic spaces and the definition of the classical Patterson-Sullivan measure in Section 2. In Section 3 we define and study strongly positively recurrent actions. In Section 4, we develop the ergodic study of Patterson-Sullivan and Bowen-Margulis measures in the context of Gromov-hyperbolic spaces. Section 5 is devoted to the twisted Patterson-Sullivan measures. In Section 6 we introduce the notion of co-amenable subgroup and prove Theorem 1.1 and other applications of our method. We conclude in Section 7 with some questions.

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2. Patterson-Sullivan measures in hyperbolic spaces

2.1. Gromov hyperbolic spaces. We review a few important facts about hyperbolic spaces and their compactifications. For more details we refer the reader to Gromov's original paper [Gro87] or [CDP90, GdlH90].

Let (X, d) be a proper geodesic metric space. We denote by $B(x, r)$ the *closed* ball of radius r centred at x.

The four point inequality. Given three points $x, y, z \in X$, the *Gromov product* is defined by

$$\langle x, y\rangle_z = \frac{1}{2}\left[d(x, z) + d(y, z) - d(x, y)\right].$$

Let $\delta \in \mathbb{R}_+$. The space X is δ-*hyperbolic* if for all $x, y, z, t \in X$, we have

$$\langle x, z\rangle_t \geqslant \min\left\{\langle x, y\rangle_t, \langle y, z\rangle_t\right\} - \delta. \tag{2}$$

It is said to be *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \in \mathbb{R}_+$. Nevertheless, for simplicity we will always assume that $\delta > 0$.

The boundary at infinity. Let o be a base point of X. A sequence (x_n) of points of X *converges to infinity* if $\langle x_n, x_m\rangle_o$ tends to infinity as n and m approach to infinity. The set $\mathcal{S}$ of such sequences is endowed with a binary relation defined as follows. Two sequences (x_n) and (y_n) are related if

$$\lim_{n\to+\infty} \langle x_n, y_n\rangle_o = +\infty.$$

By (2), this relation is an equivalence relation. The *boundary at infinity* of X (also called *Gromov boundary*), denoted by ∂X, is the quotient of $\mathcal{S}$ by this relation. A sequence (x_n) in the class of $\xi \in \partial X$ is said *converging* to ξ. We write

$$\lim_{n\to+\infty} x_n = \xi.$$

The definition of ∂X does not depend on the base point o. As X is proper and geodesic, the Gromov boundary coincides with the visual boundary of X [CDP90, Chapitre 2].

The Gromov product of three points can be extended to the boundary. Let $x \in X$ and $y, z \in X \cup \partial X$. Define $\langle y, z\rangle_x$ as the infimum

$$\liminf_{n\to+\infty} \langle y_n, z_n\rangle_x$$

where (y_n) and (z_n) run over all sequences which converge to y and z respectively. This definition coincides with the original one when $y, z \in X$. By (2), for any two sequences (y_n) and (z_n) converging respectively to $\eta, \xi \in \partial X$ one has

$$\langle \eta, \xi\rangle_x \leqslant \liminf_{n\to\infty} \langle y_n, z_n\rangle_x \leqslant \limsup_{n\to\infty} \langle y_n, z_n\rangle_x \leqslant \langle \eta, \xi\rangle_x + 2\delta.$$

Two points ξ and η of ∂X are equal if and only if $\langle \xi, \eta \rangle_x = +\infty$. Moreover, for every $t \in X$, for every $x, y, z \in X \cup \partial X$, the four point inequality (2) leads to

$$\langle x, z \rangle_t \geqslant \min \left\{ \langle x, y \rangle_t, \langle y, z \rangle_t \right\} - \delta. \tag{3}$$

The Gromov boundary is a metrizable compact space. More precisely there exists a metric on ∂X that we denote $d_{\partial X}$ and two numbers $a_0 \in (0,1)$ and $\varepsilon_0 \in \mathbb{R}_+$ such that for every $\eta, \xi \in \partial X$,

$$\left| \ln d_{\partial X}(\eta, \xi) + a_0 \langle \eta, \xi \rangle_o \right| \leqslant \varepsilon_0. \tag{4}$$

See for instance [**CDP90**, Chapitre 11, Lemme 1.7].

Limit sets. Assume that Γ is a group acting by isometries on X. This action extends to an action by homeomorphisms on ∂X. Given any subset S of Γ, the *limit set of* S, denoted by $\Lambda(S)$, is the intersection $\overline{Sx} \setminus Sx$ of the closure of the orbit Sx with ∂X, for some (hence any) point $x \in X$.

Let K be a compact subset of X. The K*-radial limit set* of Γ, denoted by $\Lambda^K_{\mathrm{rad}}(\Gamma)$, is the set of points $\xi \in \partial X$ for which there exists a geodesic ray $c \colon \mathbb{R}_+ \to X$ ending at ξ whose image intersects infinitely many translates of K by elements of Γ. It is a Γ-invariant subset of $\Lambda(\Gamma)$. The *radial limit set* is the increasing union

$$\Lambda_{\mathrm{rad}}(\Gamma) = \bigcup_{K \subset X} \Lambda^K_{\mathrm{rad}}(\Gamma).$$

If there is no ambiguity we will drop Γ from all the notations.

Horocompactification. We denote by $\mathbf{1}$ the constant function equal to 1. Let $C(X)$ be the set of continuous functions from X to $\mathbb{R}$ endowed with the topology of uniform convergence on every compact subset. We denote by $C_*(X)$ its quotient by the one-dimensional $\mathbb{R}\mathbf{1}$ endowed with the quotient topology. As X is proper, $C_*(X)$ is compact. Alternatively $C_*(X)$ can be seen as the space of continuous cocycles on X, i.e. maps $b \colon X \times X \to \mathbb{R}$ such that $b(x,z) = b(x,y) + b(y,z)$, for every $x, y, z \in X$. These two realisations of $C_*(X)$ are canonically identified via the isomorphism sending a map $f \colon X \to \mathbb{R}$ to the cocycle $b \colon X \times X \to \mathbb{R}$ defined by $b(x,y) = f(x) - f(y)$.

Given $x \in X$, we write $d_x \colon X \to \mathbb{R}$ for the map sending y to $d(x,y)$. The map $x \to d_x$ induces a homeomorphism from X onto its image in $C_*(X)$. The *horocompactification* of X, denoted by $\bar{X}_h$ is the closure of X in $C_*(X)$. The *horoboundary* $\partial_h X$ is defined as $\partial_h X = \bar{X}_h \setminus X$.

We extend the Gromov product to $\bar{X}_h$ as follows. Given $x \in X$ and $b, b' \in \partial_h X$, we set

$$\langle b, b' \rangle_x = \frac{1}{2} \sup_{z \in X} \left[b(x,z) + b'(x,z) \right]. \tag{5}$$

Let Γ be a group acting by isometries on X. This action induces an action of Γ on $C_*(X)$ as follows. For every cocycle $b \in C_*(X)$, for every $\gamma \in \Gamma$, and all $(x,y) \in X^2$,

$$[\gamma \cdot b](x,y) = b(\gamma^{-1}x, \gamma^{-1}y).$$

The horoboundary $\partial_h X$ is invariant under this action. Moreover, the action preserves the Gromov product defined in (5).

Comparison with the Gromov boundary. Given a geodesic ray $\alpha \colon \mathbb{R}_+ \to X$ the *Busemann cocycle along* α is the map $b \colon X \times X \to \mathbb{R}$ defined by

$$b(x,y) = \lim_{t \to \infty} \left[d(x, \alpha(t)) - d(y, \alpha(t)) \right].$$

It is an example of point in the horoboundary $\partial_h X$. Note that there are in general several geodesic rays ending at a given point of the Gromov boundary ∂X, which may induce distinct Busemann cocycles.

Proposition 2.1 (Coornaert-Papadopoulos [**CP01**, Proposition 3.3 and Corollary 3.8]). *There exists a map $\pi\colon \partial_h X \to \partial X$, which is continuous, Γ-invariant and onto.*

Moreover, for every geodesic ray $\alpha\colon \mathbb{R}_+ \to X$ starting at x, the Busemann cocycle along α is a preimage of $\alpha(\infty)$ in $\partial_h X$. In addition, two cocycles $b_1, b_2 \in \partial_h X$ have the same image in ∂X if and only if $\|b_1 - b_2\|_\infty \leqslant 64\delta$.

The following lemma ensures that the extension to the horoboundary of the Gromov product is close to its value in the Gromov boundary.

Lemma 2.2. *Let $b, b' \in \partial_h X$ be two cocycles, and $x \in X$. Let ξ and ξ' be their respective images in ∂X. Then*

$$\langle \xi, \xi' \rangle_x \leqslant \langle b, b' \rangle_x \leqslant \langle \xi, \xi' \rangle_x + 2\delta. \tag{6}$$

Proof. First, if $\xi = \xi'$, then both $\langle \xi, \xi' \rangle_x$ and $\langle b, b' \rangle_x$ are infinite. Indeed the infiniteness of $\langle \xi, \xi \rangle_x$ follows from the definition of the Gromov product on $\bar{X}$. On the other hand, b and b' differ by at most 64δ (Proposition 2.1). Hence

$$\langle b, b' \rangle_x \geqslant \langle b, b \rangle_x - 32\delta \geqslant \infty.$$

Therefore we can assume that $\xi \neq \xi'$. By definition of the horoboundary, there exist two sequences (y_n) and (y'_n) of points of X which respectively converge to b and b' in $\bar{X}_h$. Up to passing to a subsequence we may assume that (y_n) and (y'_n) respectively converge to ξ and ξ' in $\bar{X}$. Let $z \in X$. Triangle inequality gives for all $n \in \mathbb{N}$

$$\frac{1}{2}\left\{[d(y_n, x) - d(y_n, z)] + [d(y'_n, x) - d(y'_n, z)]\right\} \leqslant \langle y_n, y'_n \rangle_x$$

Passing to the limit we get

$$\frac{1}{2}\left\{b(x, z) + b'(x, z)\right\} \leqslant \liminf_{n\to\infty} \langle y_n, y'_n \rangle_x \leqslant \langle \xi, \xi' \rangle_x + 2\delta.$$

This inequality holds for every $x \in X$, hence $\langle b, b' \rangle_x \leqslant \langle \xi, \xi' \rangle_x$, which completes the proof of the right inequality.

For every $n \in \mathbb{N}$, we denote by p_n a projection of x on a geodesic $[y_n, y'_n]$. It follows that

$$d(x, p_n) \leqslant \langle y_n, y'_n \rangle_x + 4\delta,$$

see for instance [**CDP90**, Chapitre 3, Lemme 2.7]. As (y_n) and (y'_n) converges to distinct points in ∂X, the sequence $\langle y_n, y'_n \rangle_x$ is uniformly bounded. Recall that X is proper. Thus, up to passing to a subsequence we can assume that (p_n) converges to a point $p \in X$. Since p_n lies on $[y_n, y'_n]$, for every $n \in \mathbb{N}$, we have

$$\langle y_n, y'_n \rangle_x = \frac{1}{2}\left\{[d(y_n, x) - d(y_n, p_n)] + [d(y'_n, x) - d(y'_n, p_n)]\right\}$$

Passing to the limit we get

$$\langle \xi, \xi' \rangle_x \leqslant \liminf_{n\to\infty} \langle y_n, y'_n \rangle_x \leqslant \frac{1}{2}\left\{b(x, p) + b'(x, p)\right\} \leqslant \langle b, b' \rangle_x ,$$

which corresponds to the left inequality. □

2.2. Patterson-Sullivan measures. The Patterson-Sullivan measure is a well-known very useful object in the study of negatively curved manifolds. It was extended by Coornaert in the context of hyperbolic spaces X [**Coo93**]. His work used the Gromov compactification $\bar{X} = X \cup \partial X$. Nevertheless the measure that he obtains is not exactly conformal but only quasi-conformal. Following [**BM96**], we run the construction in the horocompactification $\bar{X}_h = X \cup \partial_h X$ rather than $\bar{X}$. We obtain thus easily an *exactly* conformal family of measures, and a Γ-invariant measure on the double Gromov boundary $\partial^2 X$, contrarily to the Γ-quasi-invariant construction of [**Coo93**, Corollaire 9.4].

Poincaré series and critical exponent. Let Γ be a group acting properly by isometries on X. We fix a base point $o \in X$. To any subset S of Γ we associate a *Poincaré series* defined by

$$\mathcal{P}_S(s) = \sum_{\gamma \in S} e^{-sd(\gamma o, o)},$$

Its critical exponent h_S is also the *exponential growth rate* of S, i.e.

$$h_S = \limsup_{r\to\infty} \frac{1}{r} \ln \left|\{\gamma \in S \mid d(\gamma o, o) \leqslant r\}\right|.$$

This quantity does not depend on the choice of o. The group Γ is called *convergent* (respectively *divergent*) if the Poincaré series $\mathcal{P}_\Gamma(s)$ converges (respectively diverges) at $s = h_\Gamma$. According to Patterson study of Dirichlet series [Pat76], there exists a map $\theta_0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties.

(1) For every $\varepsilon > 0$, there exists $t_0 \geqslant 0$, such that for every $t \geqslant t_0$ and $u \geqslant 0$, we have $\theta_0(t+u) \leqslant e^{\varepsilon u}\theta_0(t)$.

(2) The weighted series

$$\mathcal{P}'_\Gamma(s) = \sum_{\gamma \in S} \theta_0(d(o, \gamma o))e^{-sd(\gamma o, o)} \tag{7}$$

is divergent whenever $s \leqslant h_\Gamma$, and convergent otherwise.

Measure on the horoboundary. Let us now define the Patterson-Sullivan measure. It is well known that there is a one-to-one correspondence between Radon measures and positive linear forms on the space of continuous functions. We adopt the latter point of view here. It may look overcomplicated, however it emphasizes the analogy with the *twisted Patterson-Sullivan measure* that we are going to define in Section 5.3.

Denote by $C(\bar{X}_h)$ the set of continuous functions from $\bar{X}_h$ to $\mathbb{R}$. Let $x \in X$. For every $s > h_\Gamma$, we define a positive continuous linear form $L \colon C(\bar{X}_h) \to \mathbb{R}$ by

$$L_{x,s}(f) = \frac{1}{\mathcal{P}'_\Gamma(s)} \sum_{\gamma \in \Gamma} \theta_0(d(x, \gamma o))e^{-sd(x,\gamma o)} f(\gamma o).$$

Since $\bar{X}_h$ is compact, the dual of $C(\bar{X}_h)$ endowed with the weak-$*$ topology is compact as well. Thus, there exists a sequence (s_n) converging to h_Γ such that (L_{o,s_n}) converges to a positive continuous linear form $L_o \colon C(\bar{X}_h) \to \mathbb{R}$. By Riesz representation Theorem, there exists a unique Radon measure $\tilde{\nu}_o$ on $\bar{X}_h$ such that for every $f \in C(\bar{X}_h)$

$$L_o(f) = \int f d\tilde{\nu}_o.$$

By construction of θ_0, the series $\mathcal{P}'_\Gamma(s_n)$ diverges when s_n approaches to h_Γ. As a consequence the support of the measure $\tilde{\nu}_o$ is contained in $\partial_h X$. A standard argument shows that for every $x \in X$, (L_{x,s_n}) also converges to a continuous linear form on $C(\bar{X}_h)$ that can be represented by a measure $\tilde{\nu}_x$ on $\bar{X}_h$ which belongs to the same class as $\tilde{\nu}_o$. The resulting family $(\tilde{\nu}_x)_{x\in X}$ is h_Γ-*conformal*, i.e. for $\tilde{\nu}_o$-almost every $b \in \partial_h X$,

$$\frac{d\tilde{\nu}_x}{d\tilde{\nu}_y}(b) = e^{-h_\Gamma b(x,y)}.$$

This family is also Γ-*equivariant* in the sense that for all $\gamma \in \Gamma$ and $x \in X$, we have $\gamma_* \tilde{\nu}_x = \tilde{\nu}_{\gamma x}$.

Measure on the Gromov boundary. Recall that $\pi\colon \partial_h X \to \partial X$ denotes the continuous Γ-invariant map from the horoboundary to the Gromov boundary (Proposition 2.1). For $x \in X$, denote by $\nu_x = \pi_* \tilde{\nu}_x$ the push-forward measure. As π is Γ-equivariant, so is the family (ν_x). Recall that any two cocycles $b, b' \in \partial_h X$ lying in the same fibre of π differ by at most 64δ. It follows that (ν_x) is *h_Γ-quasi-conformal*, i.e. there exists $C \in \mathbb{R}_+^*$, such that for every $x, y \in X$, for ν_0-almost every $\xi \in \partial X$, for every $b \in \pi^{-1}(\xi)$,

$$\frac{1}{C} e^{-h_\Gamma b(x,y)} \leqslant \frac{d\nu_x}{d\nu_y}(\xi) \leqslant C e^{-h_\Gamma b(x,y)}. \tag{8}$$

A key tool is the well known Sullivan Shadow Lemma, due to Coornaert in our context [Coo93]. Recall that o is a fixed base point in X. Let $x \in X$ and $r \in \mathbb{R}_+$. The *shadow of $B(x,r)$ seen from o*, denoted by $\mathcal{O}_o(x,r)$, is the set of points $y \in \bar{X}$ for which there exists a geodesic from o to y intersecting the ball $B(x,r)$.

Lemma 2.3 (Shadow Lemma [Coo93, Proposition 6.1]). *Let $(\alpha_x)_{x\in X}$ be a Γ-invariant a-quasi-conformal family of measures on the Gromov boundary ∂X. There exist $r_0, C \in \mathbb{R}_+^*$ such that for all $r \geqslant r_0$, for all $\gamma \in \Gamma$,*

$$\frac{1}{C} e^{-ad(o,\gamma o)} \leqslant \alpha_o\left(\mathcal{O}_o(\gamma o, r)\right) \leqslant C e^{2ar} e^{-ad(o,\gamma o)}.$$

In terms of shadows, the radial limit set (defined in the previous section) is also the set of points $\xi \in \partial X$ which belong to infinitely many distinct shadows $\mathcal{O}_x(\gamma_n o, r)$ for some $x \in X$ and $r \in \mathbb{R}_+^*$.

Corollary 2.4. *Assume that ν_o gives full measure to the radial limit set. Then it is unique, non-atomic, and is ergodic with respect to the action of Γ on ∂X. Moreover the Poincaré series of Γ diverges at h_Γ*

Proof. The proof is well known and elementary. We recall the arguments, as they will appear later in a more sophisticated manner (see Proposition 5.22). First, it is non-atomic. Indeed, Lemma 2.3 implies that any radial limit point has a sequence of decreasing neighbourhoods whose measure decreases to zero.

As ν_o gives full measure to Λ_{rad}, there exists some compact subset $k \subset X$ large enough so that Λ^k_{rad} has positive measure. In addition, there exists $r > 0$, such that for every finite subset S of Γ the collection

$$\left(\mathcal{O}_o(\gamma o, r)\right)_{\gamma \in \Gamma \setminus S}$$

covers Λ^k_{rad}. By Lemma 2.3, there exists $\varepsilon \in \mathbb{R}_+^*$, independent of S, such that

$$\sum_{\gamma \in \Gamma \setminus S} e^{-h_\Gamma d(o,\gamma o)} \geqslant \varepsilon \sum_{\gamma \in \Gamma \setminus S} \nu_o\left(\mathcal{O}_o(\gamma o, r)\right) \geqslant \varepsilon \nu_o\left(\Lambda^k_{\mathrm{rad}}\right) > 0.$$

Hence the Poincaré series of Γ diverges at h_Γ.

Let us show that ν_o is ergodic. Let $A \subset \partial X$ be a Γ-invariant set with $\nu_o(A) > 0$. Without loss of generality, we can assume that $A \subset \Lambda^k_{\mathrm{rad}}(\Gamma)$, for some compact subset $k \subset X$. Consider the new family of measures

$$\nu'_x = \frac{1}{\nu_o(A)} \mathbf{1}_A \nu_x,$$

which is also Γ-invariant and h_Γ-quasi-conformal. Therefore, it also satisfies the Shadow Lemma. In particular, for all $r \in \mathbb{R}_+$, for all $\gamma \in \Gamma$,

$$\nu_o(\mathcal{O}_o(\gamma o, r)) \leqslant C(r) \nu'_o(\mathcal{O}_o(\gamma o, r)),$$

where $C(r) \in \mathbb{R}_+^*$ is a parameter which only depends on r. By a Vitali type argument, one easily proves that for any compact subset K containing k, in restriction to Λ^K_{rad}, the measure ν_o is absolutely continuous with respect to ν'_o. We deduce that $\nu_o(\Lambda_{\mathrm{rad}}^K \setminus A) = 0$ for all $K \supset k$, so that $\nu_o(\partial X \setminus A) = 0$. Uniqueness directly follows from the ergodicity. $\square$

2.3. The Bowen-Margulis current. Given any two cocycles $b, b' \in \partial_h X$, we define

$$D(b, b') = e^{-\langle b, b' \rangle_o}.$$

It can be thought of as the analogue of the Bourdon distance (cf [**Bou95**]), except that it does not satisfy the triangle inequality. By definition of $\langle b, b' \rangle_o$ we get:

$$D(\gamma^{-1}b, \gamma^{-1}b') = e^{-\frac{1}{2}\left[b(\gamma o, o) + b'(\gamma o, o)\right]} D(b, b'). \tag{9}$$

We follow the standard notations for the double boundary of X and let

$$\partial^2 X = \{(\eta, \xi) \in \partial X \times \partial X \mid \eta \neq \xi\}, \tag{10}$$

$$\partial_h^2 X = \{(b, b') \in \partial_h X \times \partial_h X \mid \pi(b) \neq \pi(b')\}. \tag{11}$$

We still denote by π the continuous Γ-invariant map $\partial_h X \times \partial_h X \to \partial X \times \partial X$ induced by $\pi \colon \partial_h X \to \partial X$.

Definition 2.5. The *Bowen-Margulis current on* $\partial_h^2 X$ is the measure $\tilde{\mu}$ defined by

$$\tilde{\mu} = \frac{1}{D^{2h_\Gamma}} \tilde{\nu}_o \otimes \tilde{\nu}_o.$$

The *Bowen-Margulis current on* $\partial^2 X$ is the push-forward measure $\mu = \pi_* \tilde{\mu}$.

By Lemma 2.2, there exists $C_0 \in \mathbb{R}_+^*$ such that for μ-almost every $(\eta, \xi) \in \partial^2 X$,

$$\frac{1}{C_0} e^{2h_\Gamma \langle \eta, \xi \rangle_o} \leqslant \frac{d\mu}{d\left(\nu_o \otimes \nu_o\right)}(\eta, \xi) \leqslant C_0 e^{2h_\Gamma \langle \eta, \xi \rangle_o}. \tag{12}$$

The above definitions combined with (9) give the following lemma.

Lemma 2.6. *The Bowen-Margulis currents $\tilde{\mu}$ on $\partial_h^2 X$ and μ on $\partial^2 X$ are both Γ-invariant. If the Patterson-Sullivan measure ν_0 on ∂X gives full measure to the radial limit set, then μ gives full measure to $(\Lambda_{\mathrm{rad}} \times \Lambda_{\mathrm{rad}}) \cap \partial^2 X$.*

3. Strongly positively recurrent actions

The presentation is strongly inspired from Schapira-Tapie [**ST18**] but has been slightly modified and simplified to adapt in an easier way to less smooth actions on general hyperbolic spaces.

3.1. Entropy outside a compact set. Let (X, d) be a proper geodesic metric space, and Γ a group acting properly by isometries on X. Given a compact subset K of X, let Γ_K be the set of elements $\gamma \in \Gamma$ for which there exist two points $x, y \in K$ and a geodesic $c \colon [a, b] \to X$ joining x to γy such that $c \cap \Gamma \cdot K$ is contained in $K \cup \gamma K$. We call the critical exponent h_{Γ_K} of the Poincaré series $\mathcal{P}_{\Gamma_K}$ the *entropy outside* K. Given any two compact subsets $k \subset K$ of X, it has been shown in [**ST18**, Prop. 7.9] that Γ_K is contained in a finite union of copies of Γ_k, whence $h_{\Gamma_K} \leqslant h_{\Gamma_k}$.

Definition 3.1. The *entropy at infinity* h_Γ^∞ is the quantity

$$h_\Gamma^\infty = \inf_K h_{\Gamma_K}$$

where the infimum runs over all compact subsets of X.

Definition 3.2. The action of Γ on X is *strongly positively recurrent* if $h_\Gamma^\infty < h_\Gamma$. We also say that the action has a *growth gap at infinity*.

3.2. Examples. We present some examples of strongly positively recurrent actions. Example 3.3 is a trivial one. The simplest non trivial example is a geometrically finite group acting on a negatively curved manifold with a *parabolic gap*, as studied by Dal'bo et al. in [**DOP00**], see Proposition 3.5. We refer to [**ST18**] for more examples in a Riemannian setting such as geometrically finite manifolds, Schottky products, infinite genus Ancona surfaces, etc. If one does not assume that the space X on which Γ acts is hyperbolic, Arzhantseva et al. [**ACT15**] and Yang [**Yan16**] produce other examples, e.g. some rank one actions on CAT(0) spaces and some actions of subgroups of mapping class groups.

EXAMPLE 3.3 (Non elementary hyperbolic groups). Let Γ be a group acting properly cocompactly on a geodesic δ-hyperbolic space X (in particular Γ is a hyperbolic group). If Γ is non elementary, this action is always strongly positively recurrent. Indeed, as the action is cocompact, there exists a compact subset K of X such that ΓK covers X. Thus, Γ_K is contained in

$$\{\gamma \in \Gamma \mid K \cap \gamma K \neq \emptyset\}.$$

Since the action is proper, the latter set is finite, hence $h_{\Gamma_K} = 0$. As Γ is non-elementary $h_\Gamma > 0$. Thus the action is strongly positively recurrent.

For the same reason, if the action of Γ on the Gromov hyperbolic space X is convex co-compact – i.e. some (hence every) orbit of Γ is a quasi-convex subset of X – then it is also strongly positively recurrent.

EXAMPLE 3.4 (Relative hyperbolic groups). There exist many equivalent definitions of relative hyperbolic groups. Let us recall the one that fits to our context, see for instance Bowditch [**Bow12**] or Hruska [**Hru10**, Definition 3.3].

Let Γ be a group and $\mathcal{P}$ a finite collection of finitely generated subgroups of G. Assume that Γ acts properly by isometries on a geodesic hyperbolic space X. We say that the action of $(\Gamma, \mathcal{P})$ on X is *cusp-uniform* if there exists a Γ-invariant family $\mathcal{Z}$ of pairwise disjoint horoballs in X with the following properties.

(1) The action of Γ on $X \setminus U$ is cocompact, where U stands for the union of all horoballs $Z \in \mathcal{Z}$.

(2) For every $Z \in \mathcal{Z}$, the stabilizer of Z is conjugated to some $P \in \mathcal{P}$.

The group Γ is *hyperbolic relative to* $\mathcal{P}$ if $(\Gamma, \mathcal{P})$ admits a cusp-uniform action on a hyperbolic space. The elements of $\mathcal{P}$ and their conjugates are the only maximal *parabolic* subgroups for the action of Γ on X.

The definition of cusp-uniform action mimics the decomposition of finite volume hyperbolic manifolds as the union of a compact part and finitely many cusps. Hence the proof of the next statement works as in Schapira-Tapie [**ST18**, Proposition 7.16]. The details are left to the reader.

PROPOSITION 3.5. *Let Γ be a group and $\mathcal{P}$ a finite collection of finitely generated subgroups of Γ. Let X be a hyperbolic space, endowed with a cusp-uniform action of $(\Gamma, \mathcal{P})$. The critical exponent at infinity for this action is*

$$h_\Gamma^\infty = \max_{P \in \mathcal{P}} h_P.$$

In particular, the action of Γ on X is strongly positively recurrent if $h_P < h_\Gamma$, for every $P \in \mathcal{P}$.

Remark. We recover here the parabolic gap condition, introduced by Dal'bo, Otal and Peigné [DOP00]. It also follows from this statement that if any group Γ (not necessarily a relatively hyperbolic one) admits a strongly positively recurrent action, then it is non-elementary (for this action).

We now focus on a specific cusp-uniform action, following with minor variations the Groves-Manning construction [GM08]. Given a geodesic metric space Y, the *horocone* over Y is the space $Z(Y) = Y \times \mathbb{R}_+$ whose metric is modelled on the standard hyperbolic plane $\mathbb{H}^2$ as follows: if $x = (y, r)$ and $x' = (y', r')$ are two points of $Z(Y)$, then

$$\cosh d(x, x') = \cosh(r - r') + \frac{1}{2} e^{-(r+r')} d(y, y')^2.$$

It is a geodesic hyperbolic space. It comes with a natural 1-Lipschitz embedding $\iota\colon Y \to Z(Y)$ sending y to $(y, 0)$.

Let Γ be a group and $\mathcal{P}$ a finite collection of finitely generated subgroups of G. Let S be a finite generating subset of G such that for every $P \in \mathcal{P}$, the set $S \cap P$ generates P. Let X (respectively Y_P) be the Cayley graph of Γ (respectively P) with respect to S (respectively $S \cap P$). It follows from our assumption that Y_P isometrically embeds in X. The *cone-off* space $\dot{X}$ is the space obtained by attaching for every $P \in \mathcal{P}$ and $\gamma \in \Gamma$, the horocone $Z(\gamma Y_P)$ onto X along γY_P according to the canonical embedding $\gamma Y_P \to Z(\gamma Y_P)$. We endow this space with the largest pseudo-metric such that the maps $X \to \dot{X}$ and $Z(\gamma Y_P) \to \dot{X}$ are 1-Lipschitz. It turns out that this pseudo-metric is actually a distance. Moreover the space $\dot{X}$ is proper and geodesic. In addition, the action of Γ on X extends to a proper action on $\dot{X}$. It is known that Γ is hyperbolic relative to $\mathcal{P}$, if and only if the space $\dot{X}$ is hyperbolic, see [GM08, Theorem 3.25]. In this case, the action of $(\Gamma, \mathcal{P})$ on $\dot{X}$ is cusp-uniform.

Proposition 3.6. *Assume that every $P \in \mathcal{P}$ is virtually nilpotent. If the action of Γ on $\dot{X}$ is non elementary then it is strongly positively recurrent.*

Remark 3.7. Being a strongly positively recurrent is a property of the *action* of Γ and not of the group Γ itself. The proposition states that the action of Γ on the *cone-off space* $\dot{X}$ is strongly positively recurrent. However this is not the case of any cusp-uniform action of $(\Gamma, \mathcal{P})$ on a δ-hyperbolic space. Indeed Dal'bo, Otal and Peigné produced an example of a geometrically finite manifold M with pinched negative curvature whose fundamental group $\Gamma = \pi_1(M)$ contains a parabolic subgroup P (isomorphic to $\mathbb{Z}$) whose critical exponent is the same as the one of Γ [DOP00, Théorème C]. In particular, the action of Γ on the universal cover of M is not strongly positively recurrent. Their construction strongly relies on the fact that the curvature of M is not constant. Indeed, an explicit computation shows that in locally symmetric spaces with negative curvature, all parabolic groups have a divergent Poincaré series (cf [DOP00] for the case of real hyperbolic surfaces). By Remark 3.9 below, this implies that all groups acting on a locally symmetric space with a geometrically finite action have a growth gap at infinity.

In the above construction, the metric on each horocone $Z(Y)$ is modelled on the one of the standard hyperbolic plane $\mathbb{H}^2$. Hence, although there is no appropriate notion of sectional curvature in this context, it is natural to think of $\dot{X}$ as a space with constant curvature equal to -1.

The assumption that the parabolic subgroups in $\mathcal{P}$ are virtually nilpotent is not really restrictive here. Consider indeed a cusp-uniform action of $(\Gamma, \mathcal{P})$ on a Gromov hyperbolic space X. One can prove using hyperbolic geometry that if the critical exponent of any finitely generated parabolic subgroup P is finite, then P

grows at most polynomially with respect to its own word metric. It follows then from Gromov's polynomial growth theorem that P is virtually nilpotent.

A variation of Proposition 3.6 already appears in the course of the proof of [**ACT15**, Theorem 8.1]. However the argument is rather terse. For the convenience of the reader, we expose an alternative approach, which is of independent interest. We start with the following statement.

LEMMA 3.8. *Let $P \in \mathcal{P}$. If P is virtually nilpotent, then the action of P on $\dot{X}$ is divergent.*

PROOF. For simplicity we let $Y = Y_P$. Since $\dot{X}$ is hyperbolic, there exists $r \geqslant 0$ such that the subspace $Z_r(Y) = Y \times [r, \infty)$ of $Z(Y)$ *isometrically* embeds in $\dot{X}$ [**CHK15**, Proposition 3.12]. Hence it suffices to prove that P is divergent for its action on $Z(Y)$. We denote by o the image in $Z(Y)$ of the vertex of Y corresponding to the trivial element in P. For every $\gamma \in P$ we have

$$\cosh d(\gamma o, o) = 1 + \frac{1}{2}\left|\gamma\right|^2,$$

where $|\gamma|$ stands for the length of γ with respect to the word metric on P induced by $S \cap P$. A direct computation shows that

$$e^{-d(\gamma o, o)} = \frac{1}{4}\left(\sqrt{\left|\gamma\right|^2 + 4} - \left|\gamma\right|\right)^2$$

Hence the Poincaré series of P for its action on $Z(Y)$ computed at s is

$$\mathcal{P}_P(s) = \sum_{k \in \mathbb{N}} \left|S(k)\right| a_k \quad \text{where} \quad a_k = \left(\frac{\sqrt{k^2+4} - k}{2}\right)^{2s},$$

and $S(k)$ stands for the sphere of radius k of P with respect to the word metric induced by $S \cap P$. Using Abel's transformation we compute the partial series associated to $\mathcal{P}_P(s)$. More precisely, for every $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} \left|S(k)\right| a_k = \sum_{k=0}^{n-1} \left|B(k)\right| (a_k - a_{k+1}) + \left|B(n)\right| a_n,$$

where $B(k)$ stands for the *ball* of radius k of P with respect to the word metric induced by $S \cap P$. A simple asymptotic expansion yields

$$a_k \underset{k\to\infty}{\sim} \frac{1}{k^{2s}} \quad \text{and} \quad (a_k - a_{k+1}) \underset{k\to\infty}{\sim} \frac{2s}{k^{2s+1}}.$$

Recall that P is virtually nilpotent. According to Bass [**Bas72**] and Guivarc'h [**Gui71**], there exist $A, B \in \mathbb{R}_+^*$ and $d \in \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$Ak^d \leqslant \left|B(k)\right| \leqslant Bk^d.$$

Combining this estimate with the previous asymptotic expansion, we deduce that $\mathcal{P}_P(s)$ converges if and only if $s > d/2$. In particular, the group P is divergent. □

PROOF OF PROPOSITION 3.6. By Proposition 3.5, there exists $P \in \mathcal{P}$ such that $h_\Gamma^\infty = h_P$. Moreover, since the action of Γ on X is non-elementary, the limit set $\Lambda(\Gamma)$ is infinite, whereas $\Lambda(P)$ is a single point. Recall also that P is divergent (Lemma 3.8). By [**DOP00**, Proposition 2], we get $h_\Gamma > h_P = h_\Gamma^\infty$ (this reference is written in the context of negatively curved manifolds, but the proof applies verbatim to our setting). □

REMARK 3.9. The previous proof can be adapted, using a variation of the construction of Abbott-Hume-Osin [AHO17] to get the following combination result. Let $\mathcal{P}$ be a collection of finitely generated subgroups of Γ. Assume that Γ is hyperbolic relative to $\mathcal{P}$ and non-elementary. If each parabolic group $P \in \mathcal{P}$ admits a divergent action on a hyperbolic space X_P, then there exists a hyperbolic space X on which Γ admits a strongly positively recurrent action. The proof is left to the interested reader.

We have seen that the action of a relatively hyperbolic group on its coned-off Cayley graph is strongly positively recurrent. Let us now mention another source of examples of cusp-uniform actions due to Crampon and Marquis [CM14b]. Let Ω be a properly convex subset of the projective space $\mathbb{P}^n$, i.e. there exists an affine chart in which Ω is a relatively compact convex subset. Assume that Ω is strictly convex with C^1 boundary. Using cross-ratio, one defines the Hilbert metric d_Ω on Ω, so that the group $\mathrm{Aut}(\Omega)$ of projective transformations preserving Ω acts by isometries on (Ω, d_Ω). Let Γ be a discrete subgroup of $\mathrm{Aut}(\Omega)$ whose action on Ω is geometrically finite. In this situation the convex hull $C(\Lambda_\Gamma)$ of the limit set $\Lambda_\Gamma \subset \partial\Omega$ endowed with the induced metric is Gromov-hyperbolic [CM14a, Théorème 9.1]. Moreover Γ is hyperbolic relative to the collection of maximal parabolic subgroups of Γ [CM14a, Proposition 9.8] and the action of Γ on $(C(\Lambda_\Gamma), d_\Omega)$ is cusp-uniform. It follows from [CM14b, Lemme 9.8] that every parabolic subgroup of Γ is divergent. Reasoning as in Proposition 3.6, it yields that the action of Γ on $C(\Lambda_\Gamma)$ is strongly positively recurrent.

To complete this section, we recall other examples of strongly positively recurrent actions described by Schapira-Tapie [ST18]. Note that those examples can be *geometrically infinite.*

EXAMPLE 3.10 (Schottky product). Let X be a simply connected Riemannian manifold with pinched negative sectional curvature. Let A and B be two groups of isometries of X is a Schottky position, i.e. there exist two disjoint compact subsets U_A and U_B of $\bar{X} = X \cup \partial X$ such that for every $a \in A \setminus \{1\}$ and $b \in B \setminus \{1\}$ we have

$$a\left(\bar{X} \setminus U_A\right) \subset U_A \quad \text{and} \quad b\left(\bar{X} \setminus U_B\right) \subset U_B$$

Then A and B generate a group Γ isometric to $A * B$. Moreover

$$h_\Gamma^\infty = \max\left\{h_A^\infty, h_B^\infty\right\},$$

see [ST18, Theorem 7.18]. In particular, as we also have

$$\max\left\{h_A^\infty, h_B^\infty\right\} \leq \max\left\{h_A, h_B\right\} \leq h_\Gamma,$$

if the actions of A and B on X are strongly positively recurrent, then so is the one of Γ.

EXAMPLE 3.11 (Ancona type surfaces). Let $N = \mathbb{H}^2/\Gamma$ be a complete hyperbolic surface with $1/2 < h_\Gamma < 1$. Denote by g_0 its Riemmanian metric. For example, N can be build as a non-amenable regular cover of a compact hyperbolic surface N_0. In any pair of pants decomposition of N, choose finitely many pairs of pants $P_1, \dots, P_K$. Change the metric of N to a metric g_ϵ, which is equal to g_0 far from the pants P_i, and modified in the neighborhood of the P_i by shrinking the lengths of the boundary geodesics of the pants P_i to a length ϵ. Let Γ_ϵ be a discrete group such that the new hyperbolic surface (N, g_ϵ) is isometric to $\mathbb{H}^2/\Gamma_\epsilon$. If ϵ is sufficiently small, then the action of Γ_ϵ on $\mathbb{H}^2$ is strongly positively recurrent [ST18, Theorem 7.24].

3.3. Radial limit set. Let Γ be a group with a strongly positively recurrent action on a hyperbolic space X. This assumption has a key consequence: the Patterson-Sullivan measure gives full measure to $\Lambda^r_{\rm rad}$ for some $r \in \mathbb{R}_+$, see Corollary 3.16. As mentioned in the introduction, being strongly positively recurrent is useful but not necessary here, see Corollary 2.4. It will be crucial in Corollary 5.19. Several results in this section have been proven in [**ST18**] in a Riemannian setting. In our Gromov-hyperbolic setting, some arguments need to be slightly adapted.

Let K be a compact subset of X. Denote by $\mathcal{L}_K$ the set of points $\xi \in \partial X$ for which there exists a geodesic ray $c\colon \mathbb{R}_+ \to X$ starting in K, ending at ξ and such that $c \cap \Gamma K$ is contained in K. For every $T \in \mathbb{R}_+$, define U_K^T by

$$U_K^T = \left\{ x \in \bar X \;\middle|\; \exists \xi \in \mathcal{L}_K,\ \langle x, \xi \rangle_o \geqslant T \right\}.$$

Lemma 3.12 (Compare with [**ST18**, Proposition 7.29]). *For every compact subset $K \subset X$, we have*

$$\partial X \setminus \Lambda^K_{\rm rad} \subset \Gamma \mathcal{L}_K.$$

Remark. In comparison with [**ST18**] observe that $\Lambda^K_{\rm rad} \cap \Gamma \mathcal{L}_K$ could be non-empty. This follows from the fact that two points in $X \cup \partial X$ may be joined by several geodesics, one intersecting infinitely many translates of K and the other not.

Proof. Let $\xi \in \partial X \setminus \Lambda^K_{\rm rad}$. Let $c\colon \mathbb{R}_+ \to X$ be any geodesic ray starting in K and ending at ξ. Since ξ does not belong to $\Lambda^K_{\rm rad}$, there exists $t \in \mathbb{R}_+$ and $\gamma \in \Gamma$ such that $c(t)$ belongs to γK and c restricted to (t, ∞) does not intersect ΓK. It follows that $\gamma^{-1}\xi$ belongs to $\mathcal{L}_K$, hence the result. □

Lemma 3.13. *Let $k \subset X$ be a compact subset containing the base point o and K its 6δ-neighbourhood. There exist a finite subset $S \subset \Gamma$ and $r_0 \in \mathbb{R}_+^*$ with the following property. Let $x \in K$, $y \in X \cup \partial X$ and $c\colon I \to X$ a geodesic joining x to y such that $c \cap \Gamma K \subset K$. For every $\gamma \in \Gamma$, there exists $\beta \in S\Gamma_k$ such that*

(1) $\langle x, \gamma o \rangle_{\beta o} \leqslant r_0$ and $\langle y, \gamma o \rangle_{\beta o} \leqslant r_0$,

(2) $d(x, \beta o) \geqslant \langle y, \gamma o \rangle_x - r_0$.

Remark. Working with the Gromov product is very convenient when geodesics are not unique, but sometimes confusing at the first sight. The above statement has the following geometrical meaning. If one approximates the triangle $[x, y, \gamma o]$ by a tripod, then βo lies close to the branch joining γo to the centre of the tripod (see Figure 1).

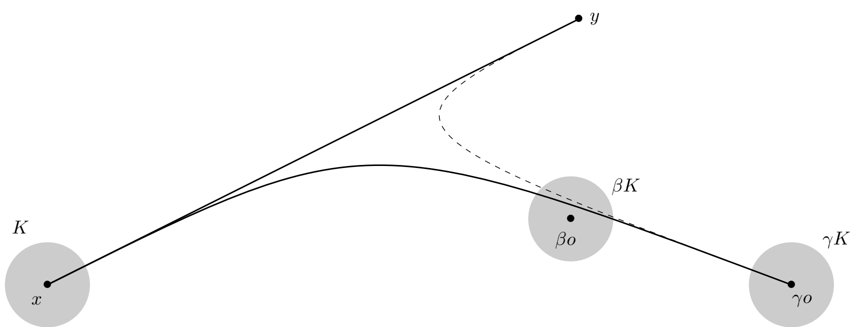

Figure 1. A geodesic tripod

PROOF. Let D be the diameter of K. We let $r_0 = D + 6\delta$ and fix $T > D + 8\delta$. Since the action of Γ on X is proper the set

$$S = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant T + 2D\}$$

is finite. Let $x \in K$, $y \in X \cup \partial X$ and $c \colon I \to X$ be a geodesic joining x to y such that $c \cap \Gamma K \subset K$. We write $z \in k$ for a projection of x on k so that $d(x, z) \leqslant \min\{D, 6\delta\}$.

Let $\gamma \in \Gamma$. Note that the identity belongs to Γ_k. In particular, if γ belongs to S, then we can simply take $\beta = \gamma$ and the conclusion holds. From now on, we assume that γ does not belong to S. It follows from the triangle inequality that $d(z, \gamma o) \geqslant T$. We fix a geodesic $c_\gamma \colon [0, a] \to X$ joining o to γo. According to our previous assumption, $T \in [0, a]$. Recall that the action of Γ on X proper, hence $F = c_\gamma^{-1}(\Gamma k)$ is closed subset containing 0 and a. We let

$$s = \max\left([0, T] \cap F\right) \quad \text{and} \quad t = \min\left([T, a] \cap F\right).$$

By definition of F, there exists $\alpha, \beta \in \Gamma$ such that the points $p = c_\gamma(s)$ and $q = c_\gamma(t)$ belong to αk and βk respectively. It follows from the construction that $\alpha^{-1}\beta$ lies in Γ_k. The points z and o belong to k white p and αo lie in αk. Using the triangle inequality, it yields $d(o, \alpha o) \leqslant 2D + T$. Hence $\alpha \in S$ and $\beta \in S\Gamma_k$.

We now focus on the metric inequalities. From the triangle inequality we get

$$\langle x, \gamma o\rangle_{\beta o} \leqslant \langle z, \gamma o\rangle_q + d(q, \beta o) + d(z, x) \leqslant D + 6\delta \leqslant r_0$$

Applying twice the four point inequality (2) we have

$$\min\left\{\langle z, x\rangle_q, \langle x, y\rangle_q, \langle y, \gamma o\rangle_q\right\} \leqslant \langle z, \gamma o\rangle_q + 2\delta \leqslant 2\delta. \tag{13}$$

Note that

$$\langle z, x\rangle_q \geqslant d(z, q) - d(x, z) \geqslant T - 6\delta > 2\delta.$$

Hence the minimum in (13) cannot be achieved by $\langle z, x\rangle_p$. Suppose now that this minimum is achieved by $\langle x, y\rangle_q$, so that $\langle x, y\rangle_q \leqslant 2\delta$. Hence q is 6δ-close to a point q' on c [**CDP90**, Chapitre 3, Lemme 2.7]. In particular, $q' \in c \cap \beta K$. As $c \cap \Gamma K$ is contained in K, the point q' actually belongs to K. It forces $t \leqslant d(z, q) \leqslant D + 6\delta$, which contradicts the definition of t. It follows from the above discussion that the minimum in (13) is achieved by $\langle y, \gamma o\rangle_q$. Hence

$$\langle y, \gamma o\rangle_{\beta o} \leqslant \langle y, \gamma o\rangle_q + d(q, \beta o) \leqslant D + 2\delta \leqslant r_0,$$

which completes the proof of the first point. It follows from the triangle inequality that

$$\langle y, \gamma o\rangle_x \leqslant \langle y, \gamma o\rangle_q + d(x, \beta o) + d(\beta o, q) \leqslant d(x, \beta o) + D + 2\delta \leqslant d(x, \beta o) + r_0,$$

which corresponds to the second point. ☐

LEMMA 3.14 (Compare with [**ST18**, Equation (27)]). *Let $k \subset X$ be a compact subset containing o and K its 6δ-neighbourhood. There exists a finite subset S of Γ and $r \in \mathbb{R}_+$ with the following properties. For every $T \in \mathbb{R}_+$,*

$$U_K^T \cap \Gamma o \subset \bigcup_{\substack{\beta \in S\Gamma_k, \\ d(o, \beta o) \geqslant T - r}} \mathcal{O}_o(\beta o, r).$$

PROOF. Let $S \subset \Gamma$ and $r_0 \in \mathbb{R}_+$ be given by Lemma 3.13. Set $r = r_0 + 2D + 4\delta$, where D is the diameter of K. Let $T \in \mathbb{R}_+$. Let $\gamma \in \Gamma$ such that γo belongs to U_K^T. In particular, there exists $\xi \in \mathcal{L}_K$ such that $\langle \xi, \gamma o\rangle_o \geqslant T$. By definition of $\mathcal{L}_K$, there exists a geodesic ray $c \colon \mathbb{R}_+ \to X$ starting in K, ending at ξ such that $c \cap \Gamma K$ is contained in K. For simplicity, set $x = c(0)$. By Lemma 3.13, there exists $\beta \in S\Gamma_k$ such that

$$d(x, \beta o) \geqslant \langle \xi, \gamma o\rangle_x - r_0 \quad \text{and} \quad \langle x, \gamma o\rangle_{\beta o} \leqslant r_0.$$

Observe that $d(o,x) \leqslant D$, as o and x both belongs to K. It follows from the triangle inequality that

$$d(o,\beta o) \geqslant \langle \xi, \gamma o\rangle_o - r_0 - 2D \geqslant T - r \quad \text{and} \quad \langle o, \gamma o\rangle_{\beta o} \leqslant r_0 + D < r - 4\delta$$

The latter point implies that γo belongs to $\mathcal{O}_o(\beta o, r)$ [**CDP90**, Chapitre 3, Lemme 2.7], whence the result. $\square$

Proposition 3.15 (Compare with [**ST18**, Proposition 7.31]). *Assume that the action of Γ on X is strongly positively recurrent. There exists a compact subset K of X and numbers $a, C, T_0 \in \mathbb{R}_+^*$ such that for every $T \geqslant T_0$, for every non-negative function $f \in C^+(\bar X)$ whose support is contained in U_K^T,*

$$\int f d\nu_o \leqslant C \left\| f \right\|_\infty e^{-aT}.$$

Proof. Since the action of Γ is strongly positively recurrent, there exists a compact subset k of X such that $h_{\Gamma_k} < h_\Gamma$. Up to enlarging k, we may assume that o belongs to k. Let K be the 6δ-neighbourhood of k. By Lemma 3.14, there exists a finite subset S of Γ and a number $r \in \mathbb{R}_+$ such that for every $T \in \mathbb{R}_+$,

$$U_K^T \cap \Gamma o \subset \bigcup_{\substack{\beta \in S\Gamma_k \\ d(o,\beta o) \geqslant T - r}} \mathcal{O}_o(\beta o, r). \tag{14}$$

Let $\varepsilon > 0$ be such that $h_\Gamma - 2\varepsilon > h_{\Gamma_k}$. Since S is finite, $S\Gamma_k$ and Γ_k have the same critical exponent. In particular, the Poincaré series associated to $S\Gamma_k$ converges at $h_\Gamma - \varepsilon$. More precisely there exists $B \in \mathbb{R}_+$ such that for every $T \geqslant 0$, we have

$$\sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} e^{-(h_\Gamma - \varepsilon) d(o,\beta o)} \leqslant B e^{-(h_\Gamma - h_{\Gamma_k} - 2\varepsilon)T}. \tag{15}$$

Recall that $\theta_0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ is the slowly increasing function used in (7) to define the Patterson-Sullivan measure ν_o. There exists $t_0 \geqslant 0$ such that for every $t \geqslant t_0$ and $u \geqslant 0$, we have $\theta_0(t+u) \leqslant e^{\varepsilon u}\theta_0(t)$. Define

$$F = \{\gamma \in \Gamma \mid d(o,\gamma o) < t_0\}.$$

Now, let $T \geqslant t_0 + r$ and $f \in C^+(\bar X)$ be a map supported in U_T^K. We may assume that $\|f\|_\infty = 1$. Let $s > h_\Gamma$. It follows from (14) that

$$L_{o,s}(f) \leqslant \frac{1}{\mathcal{P}'_\Gamma(s)} \sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} \sum_{\substack{\gamma \in \Gamma \\ \gamma o \in \mathcal{O}_o(\beta o, r)}} \theta_0(d(o,\gamma o)) e^{-s d(o,\gamma o)}. \tag{16}$$

Let $\beta \in S\Gamma_k$ such that $d(o,\beta o) \geqslant T - r$. We are going to estimate the second sum appearing above. For every $y \in \mathcal{O}_o(\beta o, r)$ we have

$$d(o,\beta o) + d(\beta o, y) - 2r \leqslant d(o,y) \leqslant d(o,\beta o) + d(\beta o, y).$$

Moreover if $d(\beta o, y) \geqslant t_0$, then

$$\theta_0(d(o,y)) \leqslant e^{\varepsilon d(o,\beta o)} \theta_0(d(\beta o, y)),$$

otherwise, since $d(o,\beta o) \geqslant t_0$, we get

$$\theta_0(d(o,y)) \leqslant e^{\varepsilon[d(o,\beta o) + d(\beta o, y) - t_0]} \theta_0(t_0) \leqslant e^{\varepsilon d(o,\beta o)} \theta_0(t_0).$$

Consequently

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma o \in \mathcal{O}_o(\beta o, r)}} \theta_0(d(o,\gamma o)) e^{-s d(o,\gamma o)} \leqslant e^{2sr} e^{-(s-\varepsilon) d(o,\beta o)} \left(\Sigma_1 + \Sigma_2\right),$$

where

$$\Sigma_1 = \sum_{\substack{\gamma\in\Gamma \\ \gamma o\in\mathcal{O}_o(\beta o,r),\ d(\beta o,\gamma o)<t_0}} \theta_0(t_0)e^{-sd(\beta o,\gamma o)},$$

$$\Sigma_2 = \sum_{\substack{\gamma\in\Gamma \\ \gamma o\in\mathcal{O}_o(\beta o,r),\ d(\beta o,\gamma o)\geqslant t_0}} \theta_0(d(\beta o,\gamma o))e^{-sd(\beta o,\gamma o)}.$$

The number of terms in Σ_1 is at most $|F|$, so that $\Sigma_1 \leqslant |F|\,\theta(t_0)$. On the other hand Σ_2 is bounded from above by $\mathcal{P}'_\Gamma(s)L_{\beta o,s}(\mathbf{1}) = \mathcal{P}'_\Gamma(s)$. Combining these inequalities, we get

$$L_{o,s}(f) \leqslant C(s) \sum_{\substack{\beta\in S\Gamma_k, \\ d(o,\beta o)\geqslant T-r}} e^{-(s-\varepsilon)d(o,\beta o)},$$

with

$$C(s) = e^{2sr}\left(1+\frac{|F|\,\theta_0(t_0)}{\mathcal{P}'_\Gamma(s)}\right).$$

After passing to the limit, it becomes

$$\int f d\nu_o \leqslant e^{2h_\Gamma r} \sum_{\substack{\beta\in S\Gamma_k, \\ d(o,\beta o)\geqslant T-r}} e^{-(h_\Gamma-\varepsilon)d(o,\beta o)}.$$

Recall that $h_\Gamma - 2\varepsilon > h_{\Gamma_k}$. Thus, we also get from (15)

$$\int f d\nu_o \leqslant Be^{2h_\Gamma r}e^{-(h_\Gamma - h_{\Gamma_k}-2\varepsilon)T}.$$

Recall that B, k, r and ε do not depend on T or f. The result follows. □

Corollary 3.16 (Compare with [**ST18**, Corollary 7.32]). *Assume that the action of Γ on X is strongly positively recurrent. There exists a compact subset K of X such that $\nu_o(\Lambda^K_{\mathrm{rad}}) = 1$.*

Recall that this conclusion is also true under the weaker assumption that ν_o gives full measure to Λ_{rad}, as shown in Corollary 2.4.

Proof. Let K be the compact subset of X given by Proposition 3.15. By definition (U^T_K) is a family of neighbourhoods of $\mathcal{L}_K$. Since ν_0 is inner regular, it follows from Proposition 3.15, that $\nu_o(\mathcal{L}_K) = 0$. Therefore $\nu_o(\Gamma.\mathcal{L}_K) = 0$, whence $\nu_o(\Lambda^K_{\mathrm{rad}}) = 1$. □

4. Ergodicity of the Bowen-Margulis current

The aim of this section is to prove the following theorem, which will be of crucial importance in the proof of Theorem 1.1.

Theorem 4.1. *Let Γ be a discrete group with a proper strongly positively recurrent action on a Gromov-hyperbolic space X. Then the diagonal action of Γ on $\partial^2 X$ is ergodic with respect to the Bowen-Margulis current μ.*

The statement is well-known if X is a CAT(-1) space. Indeed the Hopf-Tsuji-Sullivan Theorem states that the action of Γ on $(\partial^2 X, \mu)$ is ergodic if and only if the Patterson-Sullivan measure ν_o gives full measure to the radial limit set Λ_{rad} [**Rob03**, Chapter 1]. The proof goes through the ergodicity of the geodesic flow on the quotient space X/Γ with respect to the Bowen-Margulis measure. A key ingredient is the exponential contraction/expansion of the geodesic flow along stable/unstable manifolds.

The reader may know that when X is Gromov hyperbolic, the definition of a good geodesic flow may be a problem. An option to bypass this difficulty would be to use the construction of either Gromov [Gro87, Cha94] or Mineyev [Min05]. They both define a metric geodesic flow, with the needed exponential contraction/expansion properties. However, the statements available in the literature require some additional assumptions. Although it is likely that the proof would adapt to strongly positively recurrent actions on Gromov hyperbolic spaces, it is probably a long technical work.

Instead, we follow with little variations the strategy developed by Bader and Furman for hyperbolic groups [BF17]. We first define a measurable action of Γ on the abstract space $\partial^2 X \times \mathbb{R}$ and then prove a version of the Hopf-Tsuji-Sullivan theorem, involving the "geodesic flow" on the quotient $(\partial^2 X \times \mathbb{R})/\Gamma$ (Theorem 4.2). In this approach the contraction property of the geodesic flow is replaced by a contraction property for the action of Γ on the boundary ∂X.

4.1. The measurable geodesic flow of Bader-Furman.

The space of the geodesic flow. Recall that $o \in X$ is a fixed base point. The space X does not come with a well behaved geodesic flow. Instead we consider the abstract topological space

$$SX = \partial^2 X \times \mathbb{R},$$

and denote by $\mathcal{B}$ its Borel σ-algebra. As suggested by the notation, it should be thought as the analogue of the unit tangent bundle of X. For this reason we slightly abuse terminology by calling the elements of SX *vectors*.

Measure, flow and action. Recall that (ν_x) is the Γ-invariant h_Γ-quasi-conformal Patterson-Sullivan density on ∂X and μ is the associated Γ-invariant Bowen-Margulis current, which belongs to the same measure class as $\nu_o \otimes \nu_o$. The Bowen-Margulis measure on SX is defined as the product measure

$$m = \mu \otimes dt,$$

where dt is the Lebesgue measure on $\mathbb{R}$. The translation on the $\mathbb{R}$ component defines a flow $(\phi_t)_{t\in\mathbb{R}}$ on SX which preserves the Bowen-Margulis measure m. If $v = (\eta, \xi, t)$ is vector of SX, the points η and ξ are its respective (asymptotic) *past* and *future*. We now endow SX with a measurable Γ-action. To that end, we define a map $\beta \colon \Gamma \times \partial X \to \mathbb{R}$ by

$$\beta(\gamma, \xi) = h_\Gamma^{-1} \ln \left(\frac{d\gamma_*^{-1}\nu_o}{d\nu_o}(\xi) \right).$$

It satisfies the following cocycle relation ν_o-a.s.

$$\beta(\gamma_2\gamma_1, \xi) = \beta(\gamma_2, \gamma_1\xi) + \beta(\gamma_1, \xi), \quad \forall \gamma_1, \gamma_2 \in \Gamma. \tag{17}$$

As (ν_x) is the push-forward by $\pi \colon \partial_h X \to \partial X$ of the h_Γ-conformal Patterson-Sullivan density $(\tilde{\nu}_x)$ on $\partial_h X$, for any cocycle $b \in \pi^{-1}(\xi)$ and $\gamma \in \Gamma$,

$$\left| \beta(\gamma, \xi) - b(\gamma^{-1} o, o) \right| \leqslant 100\delta. \tag{18}$$

For every point $v = (\eta, \xi, t)$ in SX and any element $\gamma \in \Gamma$, define

$$\gamma v = (\gamma\eta, \gamma\xi, t + \kappa_\gamma(\eta, \xi)), \quad \text{where} \quad \kappa_\gamma(\eta, \xi) = \frac{\beta(\gamma, \xi) - \beta(\gamma, \eta)}{2}. \tag{19}$$

It defines a measurable action of Γ on SX which preserves the Bowen-Margulis measure m. Indeed, by (17), as Γ is countable, the set

$$S_0 X = \{ v \in SX \mid \forall \gamma_1, \gamma_2 \in \Gamma,\ \gamma_1(\gamma_2 v) = (\gamma_1\gamma_2) v \} \tag{20}$$

is a Γ-invariant Borel subset of SX, with full m-measure. The action of Γ commutes with the flow (ϕ_t). In particular, the set $S_0 X$ defined above is invariant under the flow (ϕ_t).

Quotient space. By analogy with the Riemannian setting, we wish to study the geodesic flow on the quotient space SX/Γ, viewed as an analogue of the unit tangent bundle on $M = X/\Gamma$. The action of Γ on SX is only a measurable action, but we could work in the quotient space S_0X/Γ where S_0X is the subset defined in (20). We prefer to use a slightly different approach and keep working in SX. Let $\mathcal{B}_\Gamma$, be the sub-σ-algebra of all Borel subsets which are Γ-invariant (up to measure zero). Let D be a Borel fundamental domain for the action of Γ on SX. We endow $(SX, \mathcal{B}_\Gamma)$ with the restriction $\bar{m}$ of the measure m to D. More precisely, for every $B \in \mathcal{B}_\Gamma$, we let

$$\bar{m}(B) = m\left(B \cap D\right).$$

This definition of $\bar{m}$ does not depend on the choice of D. As Γ is countable we observe that $\bar{m}(B) = 0$ if and only if $m(B) = 0$. Since the flow (ϕ_t) commutes with the action of Γ, it induces a measure preserving flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$. We think of this new dynamical system as the geodesic flow on SX/Γ.

The Hopf-Tsuji-Sullivan theorem. Theorem 4.1 is a direct consequence of Corollary 3.16 and the following statement.

THEOREM 4.2 (Hopf-Tsuji-Sullivan theorem on δ-hyperbolic spaces). *Let Γ be a discrete group acting properly by isometries on a Gromov hyperbolic space X. The following assertions are equivalent.*

(1) *The Patterson-Sullivan measure ν_o only charges the radial limit set.*
(2) *The geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative.*
(3) *The geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is ergodic.*
(4) *The diagonal action of Γ on $(\partial^2 X, \mu)$ is ergodic.*

Moreover, if any of these assertions is satisfied, then Γ is divergent.

Remark. If X is CAT(-1), Roblin shows that the above items are *equivalent* to the divergence of the group Γ [**Rob03**, Chapter 1]. His proof would adapt to our setting, but is long and useless for our purpose, so we omit it here.

Note that the equivalence (3) $\Leftrightarrow$ (4) follows immediately from the definition. We will see in Section 4.3 that (1) $\Leftrightarrow$ (2) is also rather easy. As Γ is non-elementary, the Bowen-Margulis measure is not supported on a single orbit, so that (3) $\Rightarrow$ (2), see [**Aar97**, Proposition 1.2.1]. The core of the proof is (2) $\Rightarrow$ (4), which is shown in Section 4.4.

Projection from SX to X. In order to prove the Hopf-Tsuji-Sullivan Theorem, we need to relate the dynamical properties of the abstract space SX to the geometry of the original space X. To that end, we build a "projection" map $\mathrm{proj}\colon SX \to X$ as follows. For every $(\eta, \xi) \in \partial^2 X$ we choose first a bi-infinite geodesic $\sigma_{(\eta,\xi)}\colon \mathbb{R} \to X$ joining η to ξ. Without loss of generality we can assume that $\sigma_{(\xi,\eta)}$ is obtained from $\sigma_{(\eta,\xi)}$ by reversing the orientation. The image $\mathrm{proj}(v)$ of a vector $v = (\eta, \xi, t)$ in SX is now defined as the unique point x on $\sigma_{(\eta,\xi)}$ such that

$$\frac{1}{2}\left[b^+_\xi(o,x) - b^-_\eta(o,x)\right] = t,$$

where b^+_ξ and b^-_η stand for the Busemann cocycle along $\sigma_{(\eta,\xi)}$ and $\sigma_{(\xi,\eta)}$ respectively. This definition of $\mathrm{proj}(v)$ involves many choices. However, any another choice would lead to a point x' such that $d(x,x') \leqslant 100\delta$. It is a standard exercise of hyperbolic geometry to prove that for every vector $v = (\eta, \xi, t)$ in SX,

$$\left|\langle \eta, \xi\rangle_o + |t| - d(o, \mathrm{proj}(v))\right| \leqslant 20\delta. \tag{21}$$

It follows from the construction that for every $v = (\eta, \xi, t)$ in SX the map

$$\begin{array}{ccc} \mathbb{R} & \to & X \\ s & \mapsto & \mathrm{proj} \circ \phi_s(v) \end{array}$$

is (up to changing the origin) the bi-infinite geodesic $\sigma_{(\eta,\xi)}$. The projection $\mathrm{proj}\colon SX \to X$ is not Γ-invariant in general. However, for every $v \in SX$, for every $\gamma \in \Gamma$,

$$d(\gamma\, \mathrm{proj}(v), \mathrm{proj}(\gamma v)) \leqslant 200\delta. \tag{22}$$

Combined with (21) we get the following useful estimate. For every $v = (\eta, \xi, t)$ in SX, for every $\gamma \in \Gamma$,

$$\left| \langle \gamma\eta, \gamma\xi \rangle_o + |t + \kappa_\gamma(\eta, \xi)| - d(o, \gamma\, \mathrm{proj}(v)) \right| \leqslant 220\delta, \tag{23}$$

where $\kappa_\gamma(\eta, \xi)$ has been defined in (19).

4.2. Changing spaces. We use the strategy of Bader-Furman to go back and forth between the spaces $(\partial^2 X, \mu)$, $(SX, \mathcal{B}, m)$ and $(SX, \mathcal{B}_\Gamma, \bar{m})$. We now work at the level of function spaces. We consider first the following operation

$$\begin{array}{ccc} L^1(\partial^2 X, \mu) \times L^1(\mathbb{R}, dt) & \to & L^1(SX, m) \\ (f, \vartheta) & \mapsto & f_\vartheta \end{array} \tag{24}$$

where $f_\vartheta = f \otimes \vartheta$, i.e. for every $(\eta, \xi, t) \in SX$, $f_\vartheta(\eta, \xi, t) = f(\eta, \xi)\vartheta(t)$.

Let $f \in L^1_+(SX, m)$ be a non-negative summable function. We define a Γ-invariant function $\hat{f}\colon SX \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\hat{f}(v) = \sum_{\gamma \in \Gamma} f(\gamma v). \tag{25}$$

Recall that D stands for a Borel fundamental domain for the action of Γ on SX. As m is Γ-invariant, we have

$$\int_{SX} \hat{f} d\bar{m} = \sum_{\gamma \in \Gamma} \int_D (f \circ \gamma) dm = \sum_{\gamma \in \Gamma} \int_{\gamma D} f dm = \int_{SX} f dm. \tag{26}$$

In particular, $\hat{f} \in L^1_+(SX, \bar{m})$. It follows that the map $f \mapsto \hat{f}$ is a well defined isometric embedding of $L^1(SX, \mathcal{B}, m)$ into $L^1(SX, \mathcal{B}_\Gamma, \bar{m})$.

4.3. Conservativity of the flow. This section is devoted to the proof of (1) $\Leftrightarrow$ (2) in Theorem 4.2. For a precise definition of *conservativity* we refer the reader to [**Hop37, Aar97**]. In this article we will only use the following properties. Assume that T is an invertible measure preserving map acting on a Borel space $(Y, \mathcal{B}, m)$. If for m-almost every $y \in Y$ there exists $B \in \mathcal{B}$ with $0 < m(B) < \infty$ such that

$$\sum_{n=1}^{\infty} \mathbf{1}_B \circ T^n(y) = \infty,$$

then T is conservative [**Aar97**, Proposition 1.1.6]. Reciprocally, by Halmos' recurrence theorem [**Aar97**, Theorem 1.1.1], if T is conservative, then for every $B \in \mathcal{B}$, with $m(B) > 0$, for m-almost every $y \in Y$,

$$\sum_{n=1}^{\infty} \mathbf{1}_B \circ T^n(y) = \infty.$$

A measure preserving flow (ϕ_t) on $(Y, \mathcal{B}, m)$ is *conservative* if its time-one map $T = \phi_1$ is conservative.

For every $r \in \mathbb{R}_+$, we define two subsets of $\partial^2 X$ and SX respectively by

$$Z(r) = \left\{ (\eta, \xi) \in \partial^2 X \;\middle|\; \langle \eta, \xi \rangle_o \leqslant r \right\}, \quad \text{and} \quad B(r) = Z(r) \times [0, 1]. \tag{27}$$

Note that $B(r)$ need not be a compact subset of SX (the Gromov product is not necessarily continuous). Still it has positive finite m-measure.

LEMMA 4.3. *For $\bar{m}$-almost every vector $v = (\eta, \xi, t)$ in SX, the future ξ of v belongs to the radial limit set $\Lambda_{\rm rad}$ if and only if there exists $r \in \mathbb{R}_+^*$ such that*

$$\int_0^\infty \mathbf{1}_{\Gamma B(r)} \circ \phi_s(v) ds = \infty. \tag{28}$$

PROOF. Recall that S_0X is the Γ- and flow-invariant subset of SX of full measure given in (20). Let $v = (\eta, \xi, t)$ be a vector in S_0X. As we noticed earlier, the path $\sigma\colon \mathbb{R} \to X$ sending s to $\operatorname{proj} \circ \phi_s(v)$ is a bi-infinite geodesic joining η to ξ. Assume first that ξ belongs to $\Lambda_{\rm rad}$. There exists $r \in \mathbb{R}_+^*$, and an infinite sequence (γ_n) of elements of Γ such that $(\gamma_n o)$ converges to ξ and for every $n \in \mathbb{N}$, we have $\langle \eta, \xi \rangle_{\gamma_n o} \leqslant r$. Set

$$s_n = -t - \kappa_{\gamma_n^{-1}}(\eta, \xi).$$

so that for all $u \in [0,1]$, the vector $\gamma_n^{-1}\phi_{s_n+u}(v)$ belongs to $B(r)$. Since $v \in S_0X$, the vector $\phi_{s_n+u}(v)$ belongs to $\gamma_n B(r)$, hence to $\Gamma B(r)$. By (23), the point $\operatorname{proj} \circ \phi_{s_n}(v)$ is approximatively a projection of $\gamma_n o$ onto σ. As $(\gamma_n o)$ converges to ξ, the sequence (s_n) diverges to infinity. Consequently the positive orbit of v spends an infinite amount of time in $\Gamma B(r)$, whence

$$\int_0^\infty \mathbf{1}_{\Gamma B(r)} \circ \phi_s(v) ds = \infty.$$

Conversely assume that there exists $r \in \mathbb{R}_+^*$ such that the above equality holds. As v belongs to S_0X, it means that there exists a sequence (s_n) diverging to infinity as well as a sequence (γ_n) of elements of Γ such that $\gamma_n^{-1}\phi_{s_n}(v)$ belongs to $B(r)$ for every $n \in \mathbb{N}$. Hence $\langle \eta, \xi \rangle_{\gamma_n o} \leqslant r$. By (23) we also get that $d(\gamma_n o, \operatorname{proj} \circ \phi_{s_n}(v))$ is uniformly bounded. Consequently $(\gamma_n o)$ converges to ξ, hence ξ belongs to $\Lambda_{\rm rad}$. □

PROPOSITION 4.4. *The Patterson-Sullivan measure ν_o gives full support to the radial limit set if and only if the flow (ϕ_t) on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative.*

PROOF. The measure ν_0 gives full measure to the radial limit set if and only if the measures m and thus $\bar{m}$ give full measure to

$$\{(\eta, \xi, t) \in SX \mid (\eta, \xi) \in \Lambda_{\rm rad} \times \Lambda_{\rm rad}\}.$$

In view of the properties of conservative systems recalled above, Lemma 4.3 tells us that ν_0 gives full measure to the radial limit set if and only if the flow (ϕ_t) is conservative. □

4.4. The Hopf argument. This section is devoted to the implication (2) ⇒ (4) of Theorem 4.2, proven in Corollary 4.15.

Preliminaries. We fix once for all a number $a > 2h_\Gamma$ and a map

$$\begin{array}{cccc} \vartheta\colon & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & \frac{1}{2}ae^{-a|t|}. \end{array}$$

For every $T_1, T_2 \in \mathbb{R}_+$, with $T_1 \leqslant T_2$, we write $\Theta_{T_1}^{T_2}\colon \mathbb{R} \to [0,1]$ for the map defined by

$$\Theta_{T_1}^{T_2}(u) = \int_{T_1}^{T_2} \vartheta(u+t)dt = \int_{T_1+u}^{T_2+u} \vartheta(t)dt.$$

This function is "almost constant" on $[-T_2, -T_1]$ and decays exponentially outside this interval. More precisely we have the following useful estimates.

(1) For every $u \in \mathbb{R}$,

$$\Theta_{T_1}^{T_2}(u) \leqslant \frac{1}{2}\min\left\{e^{a(T_2+u)}, e^{-a(T_1+u)}\right\}. \tag{29}$$

(2) For every $u \in [-T_2, -T_1]$,

$$\Theta_{T_1}^{T_2}(u) = 1 - \frac{1}{2}\left[e^{a(T_1+u)} + e^{-a(T_2+u)}\right].$$

Consequently, for every $u, u' \in [-T_2, -T_1]$,

$$\left|\Theta_{T_1}^{T_2}(u) - \Theta_{T_1}^{T_2}(u')\right| \leqslant \left[e^{a\left(T_1+\frac{u+u'}{2}\right)} + e^{-a\left(T_2+\frac{u+u'}{2}\right)}\right] \sinh\left(\frac{a}{2}|u-u'|\right). \tag{30}$$

See Figure 2 for a sketch of the graph of $\Theta_{T_1}^{T_2}$.

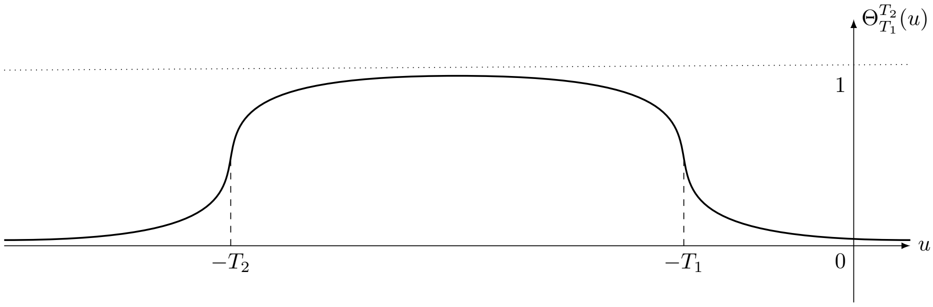

FIGURE 2. Graph of the map $\Theta_{T_1}^{T_2}$.

We say that a function $f\colon \partial^2 X \to \mathbb{R}$ has *exponential decay* if there exists $C \in \mathbb{R}_+$ such that for μ-almost every $(\eta, \xi) \in \partial^2 X$ we have

$$|f(\eta, \xi)| \leqslant Ce^{-a\langle \eta, \xi\rangle_o}. \tag{31}$$

Any such function belongs to $L^1(\mu)$. Indeed, as $a > 2h_\Gamma$, Inequality (12) yields

$$\int |f|\, d\mu \leqslant CC_0 \int e^{(2h_\Gamma - a)\langle \eta, \xi\rangle_o} d\nu_o(\eta) d\nu_o(\xi) \leqslant CC_0.$$

Recall that the boundary ∂X is endowed with a visual metric $d_{\partial X}$ for which there exists $a_0, \varepsilon_0 \in (0,1)$ such that for every $\eta, \xi \in \partial X$, we have

$$|d_{\partial X}(\eta, \xi) + a_0 \langle \eta, \xi\rangle_o| \leqslant \varepsilon_0. \tag{32}$$

The product metric induces a distance on $\partial^2 X$. We write $\mathcal{D}^+(\partial^2 X) \subset L^1_+(\mu)$ for the set of all Lipschitz functions $f\colon \partial^2 X \to \mathbb{R}_+$ with exponential decay.

We complete this preliminary discussion with the following easy but useful statements.

LEMMA 4.5. *There exists $C \in \mathbb{R}_+$ such that for every $x \in X$, for every $r \in \mathbb{R}_+$,*

$$|\{\gamma \in \Gamma \mid d(o, \gamma x) \leqslant r\}| \leqslant Ce^{2h_\Gamma r}.$$

PROOF. If $d(x, \Gamma o) > r$, then $\{\gamma \in \Gamma \mid d(o, \gamma x) \leqslant r\}$ is empty. If $d(x, \Gamma o) \leqslant r$, there exists $\alpha \in \Gamma$ such that $d(x, \alpha o) = d(x, \Gamma o)$. Triangle inequality implies that $\{\gamma \in \Gamma \mid d(o, \gamma x) \leqslant r\}$ is contained in $B_\Gamma(o, 2r)\alpha^{-1}$, where

$$B_\Gamma(o, 2r) = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant 2r\}.$$

It is well-known that $|B_\Gamma(o, 2r)| \leqslant Ce^{2h_\Gamma r}$, for some universal constant C, see for instance [**Coo93**, Corollaire 6.8], which completes the proof. $\square$

LEMMA 4.6. *There exists $C \in \mathbb{R}_+$ such that for every $x \in X$, for every $r \in \mathbb{R}_+$,*

$$\sum_{\substack{\gamma \in \Gamma, \\ d(o, \gamma x) \geqslant r}} e^{-ad(o, \gamma x)} \leqslant Ce^{-(a - 2h_\Gamma)r}.$$

PROOF. Let $x \in X$ and $r \in \mathbb{R}_+$. We split the sum as follows:

$$\sum_{\substack{\gamma \in \Gamma, \\ d(o,\gamma x) \geqslant r}} e^{-ad(o,\gamma x)} \leqslant \sum_{\substack{\ell \in \mathbb{N}, \\ \ell \geqslant r}} |\{\gamma \in \Gamma \mid \ell \leqslant d(o,\gamma x) \leqslant \ell + 1\}| \, e^{-a\ell}.$$

By Lemma 4.5, there exists $C \in \mathbb{R}_+$ (independent of x and r) such that

$$\sum_{\substack{\gamma \in \Gamma, \\ d(o,\gamma x) \geqslant r}} e^{-ad(o,\gamma x)} \leqslant C \sum_{\substack{\ell \in \mathbb{N}, \\ \ell \geqslant r}} e^{-(a-2h_\Gamma)\ell}.$$

As $a > 2h_\Gamma$, we get (up to changing the constant C)

$$\sum_{\substack{\gamma \in \Gamma, \\ d(o,\gamma x) \geqslant r}} e^{-ad(o,\gamma x)} \leqslant Ce^{-(a-2h_\Gamma)r}. \qquad \square$$

LEMMA 4.7. *If $f \in \mathcal{D}^+(\partial^2 X)$, then $\hat{f}_\vartheta$ is bounded, where $\hat{f}_\vartheta$ was defined in* (24) *and* (25).

PROOF. As f has exponential decay, there exists $C \in \mathbb{R}_+$ such that for every $v \in SX$,

$$\hat{f}_\vartheta(v) \leqslant C \sum_{\gamma \in \Gamma} e^{-a\langle \gamma\xi, \gamma\eta\rangle_o} e^{-a|t+\kappa_\gamma(\xi,\eta)|}.$$

Set $x = \operatorname{proj}(v)$. Recall that $\langle \gamma\xi, \gamma\eta\rangle_o + |t + \kappa_\gamma(\xi,\eta)|$ is approximatively the distance between o and γx, see (23). Up to increasing C, we get

$$\hat{f}_\vartheta(v) \leqslant C \sum_{\gamma \in \Gamma} e^{-ad(o,\gamma x)},$$

Recall that $a > 2h_\Gamma$. It follows from Lemma 4.6 that this last sum is bounded independently of v. $\square$

Contraction property. In a CAT(-1) space X, a key fact when running the Hopf argument is that two geodesic rays $\sigma, \sigma' \colon \mathbb{R} \to X$ with the same point at infinity satisfy a contraction property, namely there exists $u \in \mathbb{R}$, such that $t \to d(\sigma(t), \sigma'(t+u))$ converges exponentially fast to zero. As a consequence, if $f : X \to \mathbb{R}$ is a Hölder continuous map, the difference

$$\int_0^T [f(\sigma(t)) - f(\sigma'(t))] \, dt$$

converges when $T \to +\infty$. Exponential decay of the distance along asymptotic geodesics is no longer true when X is Gromov hyperbolic. Indeed two geodesic rays may have the same endpoint at infinity, but only stay at bounded distance one from the other. In this setting, the contraction of geodesics is replaced by the following fact.

PROPOSITION 4.8 (Contraction lemma). *Let $f \in \mathcal{D}^+(\partial^2 X)$. Let $v = (\eta, \xi, 0)$ and $v' = (\eta', \xi, 0)$ be two vectors of SX with the same future. The map*

$$\begin{array}{ccc} \mathbb{R}_+ & \to & \mathbb{R}_+ \\ T & \to & \displaystyle\int_0^T \left(\hat{f}_\vartheta \circ \phi_s(v) - \hat{f}_\vartheta \circ \phi_s(v')\right) ds \end{array}$$

is bounded.

PROOF. Note that since f is continuous, the function $\hat{f}_\vartheta$ is defined everywhere (and not just $\bar{m}$-almost everywhere). Recall that the map $\sigma \colon \mathbb{R} \to X$ sending s to $\operatorname{proj} \circ \phi_s(v)$ is a bi-infinite geodesic joining η to ξ. Similarly, using the vector v', we define a geodesic $\sigma' \colon \mathbb{R} \to X$ from η' to ξ. We start by defining a time shift, to make sure that σ and σ' fellow travel.

By hyperbolicity of X, there exists $u, T_0 \in \mathbb{R}$, such that for every $s \geqslant T_0$, we have $d(\sigma(s), \sigma'(s+u)) \leqslant 16\delta$ [GdlH90, Chaptitre 7, Proposition 2]. To have enough flexibility, we let $T_1 = T_0 + 10^{10}\delta$ (this estimate is very generous). As we already observed from (21) the point $\sigma(0) = \operatorname{proj}(v)$ is approximately a projection of o on σ. Similarly, by (23) for every $\gamma \in \Gamma$, $\sigma(-\kappa_\gamma(\eta,\xi))$ is approximately a projection of $\gamma^{-1}o$ on σ. The same interpretation holds for σ'. It follows that for every $\gamma \in \Gamma$ such that $\kappa_\gamma(\eta,\xi) \leqslant -T_1 + 5000\delta$ or $u + \kappa_\gamma(\eta',\xi') \leqslant -T_1 + 5000\delta$, the following holds

(1) $|\langle \eta', \xi \rangle_{\gamma^{-1}o} - \langle \eta, \xi \rangle_{\gamma^{-1}o}| \leqslant 500\delta$,
(2) $|\kappa_\gamma(\eta', \xi) + u - \kappa_\gamma(\eta,\xi)| \leqslant 2000\delta$,
(3) $\langle \eta, \eta' \rangle_{\gamma^{-1}o} \geqslant -\kappa_\gamma(\eta,\xi) - T_1$.

This general configuration is sketched on Figure 3.

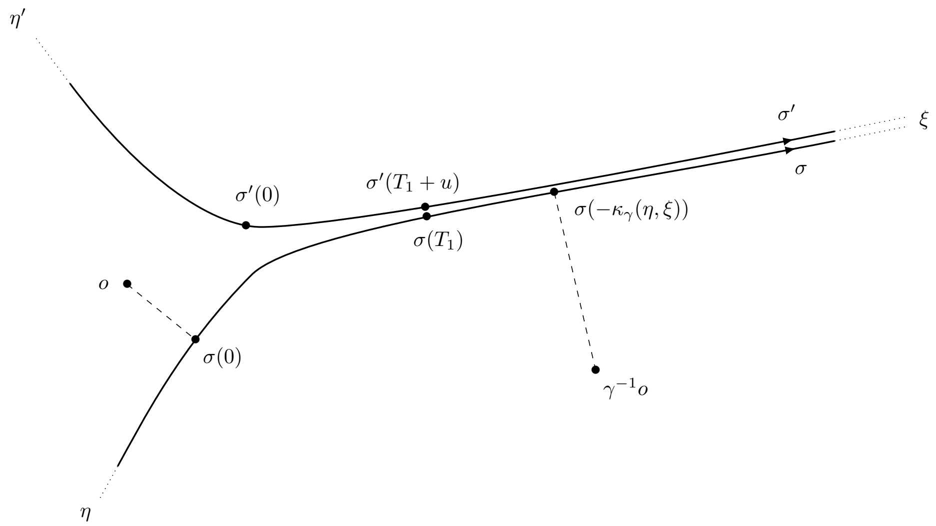

FIGURE 3. General configuration of σ and σ'.

Since the map $\hat{f}_\vartheta$ is bounded (Lemma 4.7) there exists $C_0 \in \mathbb{R}_+$, such that for every $T \geqslant T_0$,

$$\left| \int_0^T \left(\hat{f}_\vartheta \circ \phi_s(v) - \hat{f}_\vartheta \circ \phi_s(v') \right) ds - \int_{T_1}^T \left(\hat{f}_\vartheta \circ \phi_s(v) - \hat{f}_\vartheta \circ \phi_{s+u}(v') \right) ds \right| \leqslant C_0.$$

Set $v'_u = \phi_u(v')$ and define F_T by

$$\begin{array}{rccc} F_T\colon & SX & \to & \mathbb{R} \\ & w & \mapsto & \displaystyle\int_{T_1}^T \hat{f}_\vartheta \circ \phi_s(v) ds, \end{array}$$

To get Proposition 4.8, it suffices to show that the map $T \to F_T(v) - F_T(v'_u)$ is bounded. A Fubini argument gives

$$F_T(w) = \sum_{\gamma \in \Gamma} f \otimes \Theta_{T_1}^T (\gamma w) = \widehat{f \otimes \Theta_{T_1}^T}(w). \tag{33}$$

Figure 4 represents the value of F_T.

We are now going to decompose the sum in (33) according to the value of $\kappa_\gamma(\eta,\xi)$. For every $t \in \mathbb{R}_+$, we define a subset $S(t)$ of Γ as follows.

$$S(t) = \{\gamma \in \Gamma \mid -\delta < t + \kappa_\gamma(\eta,\xi) \leqslant 0\}.$$

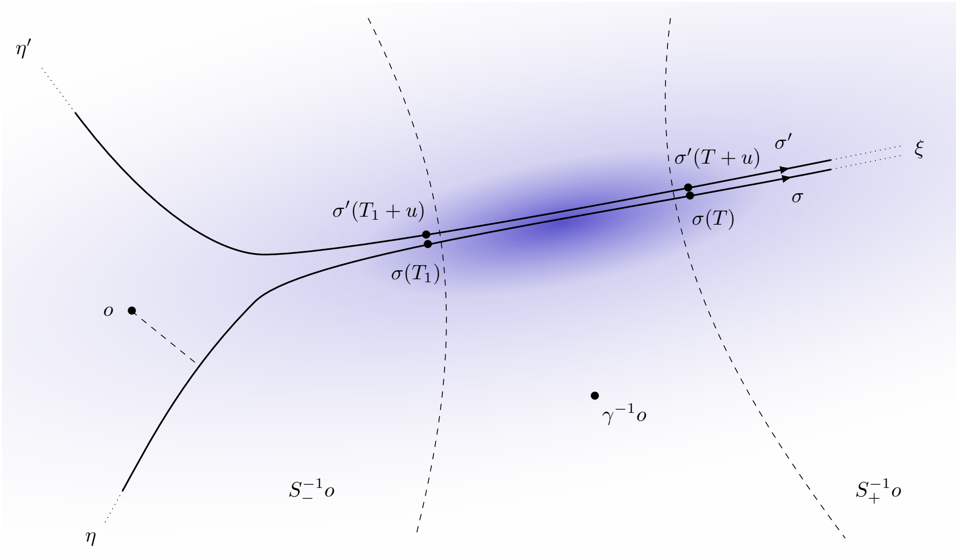

FIGURE 4. The function F_T. The shade represents the magnitude of F_T. The dark (respectively light) area corresponds to vectors $w \in SX$ for which $|F_T(w)|$ is large (respectively small). The dashed lines split the orbit $\{\gamma^{-1}o \mid \gamma \in \Gamma\}$ in three parts according to whether γ belongs to S_-, S_+ or $\Gamma \setminus (S_- \cup S_+)$.

Roughly speaking $S(t)$ corresponds to the set of all elements $\gamma \in \Gamma$ such that the projection of $\gamma^{-1}o$ on σ is approximatively $\sigma(t) = \operatorname{proj} \circ \phi_t(v)$. The sets $(S(n\delta))_{n\in\mathbb{Z}}$ form a partition of Γ. In particular

$$F_T(w) = \sum_{n\in\mathbb{Z}} \sum_{\gamma\in S(n\delta)} f \otimes \Theta_{T_1}^T(\gamma w)\,. \tag{34}$$

The first lemma handles the tails of this sum.

LEMMA 4.9. *There exists $C_1 \in \mathbb{R}_+$ such that for every $T \geqslant T_1$, for every $w \in \{v, v'_u\}$, we have*

$$\max\left\{ \sum_{n\delta\leqslant T_1+2000\delta} \sum_{\gamma\in S(n\delta)} f\otimes\Theta_{T_1}^T(\gamma w)\,, \sum_{n\delta\geqslant T-2001\delta} \sum_{\gamma\in S(n\delta)} f\otimes\Theta_{T_1}^T(\gamma w) \right\} \leqslant C_1.$$

PROOF. Let $T \geqslant T_1$. Observe that

$$\bigcup_{n\delta\geqslant T-2001\delta} S(n\delta) = \{\gamma\in\Gamma \mid \kappa_\gamma(\eta,\xi) \leqslant -T+2001\delta\}\,.$$

For simplicity we denote this set by S_+ (see Figure 4). Similarly, set

$$S_- = \bigcup_{n\delta\leqslant T_1+2000\delta} S(n\delta) = \{\gamma\in\Gamma \mid \kappa_\gamma(\eta,\xi) \geqslant -T_1-2001\delta\}\,.$$

We focus now on the right tail of $F_T(v)$. Recall that f has exponential decay, whereas the tails of $\Theta_{T_1}^T$ decay exponentially – see (29). It follows that

$$\begin{aligned}\sum_{n\delta\geqslant T-2001\delta} \sum_{\gamma\in S(n\delta)} f\otimes\Theta_{T_1}^T(\gamma v) &\leqslant \sum_{\gamma\in S_+} f(\gamma\eta,\gamma\xi)\Theta_{T_1}^T(\kappa_\gamma(\eta,\xi)) \\ &\leqslant \frac{1}{2}\sum_{\gamma\in S_+} e^{-a\langle\gamma\eta,\gamma\xi\rangle_o} e^{a[T+\kappa_\gamma(\eta,\xi)]}.\end{aligned}$$

However $(T - 2001\delta) + \kappa_\gamma(\eta, \xi) \leqslant 0$, for every $\gamma \in \Gamma$. Consequently

$$\langle \gamma\eta, \gamma\xi\rangle_o - [(T - 2001\delta) + \kappa_\gamma(\eta, \xi)] = \langle \gamma\eta, \gamma\xi\rangle_o + |(T - 2001\delta) + \kappa_\gamma(\eta, \xi)|,$$

which, according to (23), differs from $d(o, \gamma\sigma(T - 2001\delta))$ by at most 220δ. Hence there exists a constant C (which does not depends on T) such that

$$\sum_{n\delta \geqslant T-\delta} \sum_{\gamma \in S(n\delta)} f \otimes \Theta_{T_1}^T(\gamma v) \leqslant C \sum_{\gamma \in S_+} e^{-ad(o,\gamma\sigma(T-2001\delta))}$$

It follows from Lemma 4.6 that the latter sum is bounded from above independently of T. The upper bound for the left tail of $F_T(v)$ follows the exact same strategy.

For the tails of $F_T(v'_u)$ we have to be slightly more careful. Indeed the sets $S(t)$ were defined according to v (the definition involves its past η) and not v'_u. Nevertheless, as we observed at the beginning of the proof if either $\kappa_\gamma(\eta, \xi) \leqslant -T_1 + 5000\delta$ or $u + \kappa_\gamma(\eta', \xi') \leqslant -T_1 + 5000\delta$, then these two quantities differ by as most 2000δ. Consequently

$$\begin{aligned} S_- &\subset \{\gamma \in \Gamma \mid u + \kappa_\gamma(\eta', \xi') \geqslant -T_1 - 4001\delta\}, \\ S_+ &\subset \{\gamma \in \Gamma \mid u + \kappa_\gamma(\eta', \xi') \leqslant -T + 4001\delta\}. \end{aligned}$$

The estimation of the tails of $F_T(v'_u)$ now works as for the one of $F_T(v)$. □

The next step is to estimate in (34) each sum over $S(n\delta)$ whenever $n\delta$ belongs to $[T_1 + 2000\delta, T - 2001\delta]$.

LEMMA 4.10. *There exists $C_2 \in \mathbb{R}_+$ such that for every $T \geqslant T_1$, for every $n \in \mathbb{Z}$ such that $T_1 + 2000\delta \leqslant n\delta \leqslant T - 2001\delta$, we have*

$$\begin{aligned} \Delta(n) := \sum_{\gamma \in S(n\delta)} &\left| f \otimes \Theta_{T_1}^T(\gamma v) - f \otimes \Theta_{T_1}^T(\gamma v'_u)\right| \\ &\leqslant C_2 \left[\left(e^{a(T_1 - n\delta)} + e^{-a(T - n\delta)}\right) + \frac{1}{n^2} + n^q e^{-a_0 n\delta}\right], \end{aligned}$$

where a_0 is given in (32) and

$$q = \frac{4h_\Gamma}{a - 2h_\Gamma}.$$

PROOF. Let $n \in \mathbb{Z}$, be such that $T_1 + 2000\delta \leqslant n\delta \leqslant T - 2001\delta$. Observe that $\Delta(n) \leqslant \Delta_f(n) + \Delta_\Theta(n)$ where

$$\begin{aligned} \Delta_f(n) &= \sum_{\gamma \in S(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)|\, \Theta_{T_1}^T(u + \kappa_\gamma(\eta', \xi)), \\ \Delta_\Theta(n) &= \sum_{\gamma \in S(n\delta)} f(\gamma\eta, \gamma\xi) \left|\Theta_{T_1}^T(\kappa_\gamma(\eta, \xi)) - \Theta_{T_1}^T(u + \kappa_\gamma(\eta', \xi))\right|. \end{aligned}$$

We start with the term $\Delta_\Theta(n)$. Let $\gamma \in S(n\delta)$. As observed at the beginning of the proof, since $\kappa_\gamma(\eta, \xi) \leqslant -T_1$, the quantity $\kappa_\gamma(\eta, \xi)$ differs from $u + \kappa_\gamma(\eta', \xi)$ by at most 2000δ. In particular, $u + \kappa_\gamma(\eta', \xi')$ belongs to $[-n\delta - 2001\delta, -n\delta + 2000\delta]$, hence to $[-T, -T_1]$. On this interval the function $\Theta_{T_1}^T$ is almost constant. More precisely, using (30) we get

$$\left|\Theta_{T_1}^T(\kappa_\gamma(\eta, \xi)) - \Theta_{T_1}^T(u + \kappa_\gamma(\eta', \xi'))\right| \leqslant C\left(e^{a(T_1 - n\delta)} + e^{-a(T - n\delta)}\right),$$

for some parameter C, which does not depends on n or T. Consequently

$$\Delta_\Theta(n) \leqslant C\left(e^{a(T_1 - n\delta)} + e^{-a(T - n\delta)}\right) \sum_{\gamma \in S(n\delta)} f(\gamma\eta, \gamma\xi).$$

Recall that $-\delta \leqslant n\delta + \kappa_\gamma(\eta, \xi) \leqslant 0$, for every $\gamma \in S(n\delta)$. Consequently the latter sum can be bounded above as follows

$$\sum_{\gamma \in S(n\delta)} f(\gamma\eta, \gamma\xi) \leqslant e^{a\delta} \sum_{\gamma \in S(n\delta)} f(\gamma\eta, \gamma\xi) e^{-a|n\delta + \kappa_\gamma(\eta,\xi)|}.$$

Since f decays exponentially, following the same proof as in Lemma 4.9 we get that this sum is bounded from above independently of n and T. To summarize, we have proved that there exists $C_\Theta \in \mathbb{R}_+$ (which does not depend on n or T) such that

$$\Delta_\Theta(n) \leqslant C_\Theta \left(e^{a(T_1 - n\delta)} + e^{-a(T - n\delta)} \right). \tag{35}$$

Let us now focus on $\Delta_f(n)$. First, as $\Theta_{T_1}^T \leqslant 1$, we have

$$\Delta_f(n) \leqslant \sum_{\gamma \in S(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)|.$$

We split again this sum in two parts according to the value of $\langle \eta, \xi \rangle_{\gamma^{-1}o}$. More precisely, we set

$$p = \frac{2}{a - 2h_\Gamma},$$

and

$$S_0(n\delta) = \left\{ \gamma \in S(n\delta) \;\middle|\; \langle \eta, \xi \rangle_{\gamma^{-1}o} \leqslant p\ln(n\delta) \right\},$$
$$S_\infty(n\delta) = \left\{ \gamma \in S(n\delta) \;\middle|\; \langle \eta, \xi \rangle_{\gamma^{-1}o} > p\ln(n\delta) \right\}.$$

Roughly speaking, $S_0(n\delta)$ is the set of all $\gamma \in S(n\delta)$ such that $\gamma^{-1}o$ stay close to σ. We will bound the corresponding sum using the regularity of f. On the other hand $S_\infty(n\delta)$ is the set of all elements $\gamma \in S(n\delta)$ such that $\gamma^{-1}o$ is far from σ. The corresponding sum will be controlled using the exponential decay of f. We split the details in three claims.

Claim 4.11. *There exists $C \in \mathbb{R}_+$ (which does not depend on n or T) such that $|S_0(n\delta)| \leqslant Cn^q$.*

Let $\gamma \in S_0(n\delta)$. Using (23) we observe that, up to 220δ the distance between o and $\gamma\sigma(n\delta)$ is at most

$$\langle \gamma\eta, \gamma\xi \rangle_o + |n\delta + \kappa_\gamma(\eta, \xi)| \leqslant p\ln(n\delta) + \delta.$$

Consequently $S_0(n\delta)$ is contained in

$$U = \{\gamma \in \Gamma \mid d(o, \gamma\sigma(n\delta)) \leqslant r\}, \quad \text{where} \quad r = p\ln(n\delta) + 221\delta.$$

By Lemma 4.5, there exists $C \in \mathbb{R}_+$ (independent of n or T) such that

$$|U| \leqslant Ce^{2h_\Gamma r} \leqslant Ce^{442h_\Gamma\delta}(n\delta)^{2ph_\Gamma},$$

which completes the proof of the first claim.

Recall that a_0 denotes the parameter which allows to approximate the visual metric on ∂X by Gromov products (32).

Claim 4.12. *There exists $C \in \mathbb{R}_+$ (which does not depend on n or T) such that*

$$\sum_{\gamma \in S_0(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)| \leqslant Cn^q e^{-a_0 n\delta}.$$

According to our assumption f is Lipschitz with respect to the product metric on $\partial^2 X$. Moreover, v and v' have the same future, namely ξ. These observations together with (32) imply that there exists $M \in \mathbb{R}_+$ such that for every $\gamma \in S_0(n\delta)$.

$$|f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)| \leqslant M e^{-a_0 \langle \eta, \eta' \rangle_{\gamma^{-1} o}}.$$

However, as $\kappa_\gamma(\eta, \xi) \leqslant -T_1 - \delta$, we observed at the beginning of the proof that

$$\langle \eta, \eta' \rangle_{\gamma^{-1} o} \geqslant -\kappa_\gamma(\eta, \xi) - T_1 \geqslant n\delta - T_1.$$

Consequently

$$\sum_{\gamma \in S_0(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)| \leqslant M e^{a_0 T_1} |S_0(n\delta)| \, e^{-a_0 n\delta}.$$

Claim 4.12 now follows from the estimate of $|S_0(n\delta)|$ given by Claim 4.11.

Claim 4.13. *There exists $C \in \mathbb{R}_+$ (independent of n or T) such that*

$$\sum_{\gamma \in S_\infty(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)| \leqslant \frac{C}{n^2}.$$

We split this sum in two parts as follows.

$$\sum_{\gamma \in S_\infty(n\delta)} |f(\gamma\eta, \gamma\xi) - f(\gamma\eta', \gamma\xi)| \leqslant \sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) + \sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta', \gamma\xi).$$

Proceeding as for Δ_Θ, we observe that

$$\sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) \leqslant e^{a\delta} \sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) e^{-a|n\delta + \kappa_\gamma(\eta, \xi)|}$$

We now argue as in Lemma 4.9 and prove that there exists a constant $C \in \mathbb{R}_+$ (which does not depends on n or T) such that

$$\sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) \leqslant C \sum_{\gamma \in S_\infty(n\delta)} e^{-ad(o, \gamma\sigma(n\delta))}.$$

As usual the distance $d(o, \gamma\sigma(n\delta))$ can be approximated by

$$\langle \eta, \xi \rangle_{\gamma^{-1} o} + |n\delta + \kappa_\gamma(\eta, \xi)|$$

It follows from the very definition of $S_\infty(n\delta)$ that $d(o, \gamma\sigma(n\delta)) > p \ln(n\delta) - 220\delta$, for every $\gamma \in S_\infty(n\delta)$. Hence

$$\sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) \leqslant C \sum_{\substack{\gamma \in \Gamma, \\ d(o, \gamma\sigma(n\delta)) > p \ln(n\delta) - 220\delta}} e^{-ad(o, \gamma\sigma(n\delta))}.$$

An upper bound of the last sum is given by Lemma 4.6. More precisely, up to replacing C by a larger constant (which still does not depend on n or T) we get

$$\sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta, \gamma\xi) \leqslant C e^{-(a - 2h_\Gamma) p \ln(n\delta)} \leqslant \frac{C}{(n\delta)^2}.$$

The last inequality follows by definition of p. Recall that whenever $\kappa_\gamma(\eta, \xi) \leqslant -T_1$, then $\kappa_\gamma(\eta, \xi)$ and $u + \kappa_\gamma(\eta', \xi)$ differ by at most 2000δ. Following the exact same argument we get a similar upper bound for

$$\sum_{\gamma \in S_\infty(n\delta)} f(\gamma\eta', \gamma\xi'),$$

which completes the proof of Claim 4.13. To summarize, the last two claims tell us that there exists C_f (which does not depend on n or T) such that

$$\Delta_f(n) \leqslant C_f \left(\frac{1}{n^2} + n^q e^{-a_0 n\delta} \right). \tag{36}$$

Lemma 4.10 is the combination of (35) and (36). □

Recall that we need to estimate $F_T(v) - F_T(v'_u)$. According to Lemma 4.9 there exists $C_1 \in \mathbb{R}_+$ such that for every $T \geqslant 0$,

$$|F_T(v) - F_T(v'_u)| \leqslant C_1 + \sum_{T_1+\delta \leqslant n\delta \leqslant T-\delta} \sum_{\gamma \in S(n\delta)} \left| f \otimes \Theta^T_{T_1}(\gamma v) - f \otimes \Theta^T_{T_1}(\gamma v'_u) \right|$$

Combined with Lemma 4.10, we see that there exists $C_2 \in \mathbb{R}_+$ such that

$$\begin{aligned} &|F_T(v) - F_T(v'_u)| \\ &\quad \leqslant C_1 + C_2 \sum_{T_1+\delta \leqslant n\delta \leqslant T-\delta} \left[\left(e^{a(T_1 - n\delta)} + e^{-a(T-n\delta)} \right) + \frac{1}{n^2} + n^q e^{-a_0 n\delta} \right]. \end{aligned}$$

Observe that for every integer n indexing the sum $T_1 - n\delta$ is negative, whereas $T - n\delta$ is positive. Consequently, the latter sum is bounded from above independently of T, which completes the proof of the proposition. □

Running the Hopf argument. We fix until the end of this section a *bounded positive* function $g \in \mathcal{D}^+(\partial^2 X)$, i.e. g is Lipschitz with exponential decay. For instance one can chose $g(\eta,\xi) = d_{\partial X}(\eta,\xi)^p$ for a sufficiently large $p \in \mathbb{R}_+$. Recall that g belongs to $L^1(\mu)$. Up to rescaling g we can assume that

$$\int \hat{g}_\vartheta d\bar{m} = \int g_\vartheta dm = \int g d\mu = 1.$$

In addition we define an auxiliary map

$$\begin{array}{cccc} g'\colon & \partial^2 X & \to & \mathbb{R}^*_+ \\ & (\eta,\xi) & \to & \displaystyle\int_{\mathbb{R}} \hat{g}_\vartheta(\eta,\xi,t)\vartheta(t)dt. \end{array}$$

Note that as $\hat{g}_\vartheta$ is bounded (Lemma 4.7), g' is a bounded positive map.

Proposition 4.14. *Assume that the geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative. If $f \in L^1(\mu)$, then for $\bar{m}$-almost every $v \in SX$,*

$$\lim_{T\to\pm\infty} \frac{\int_0^T \hat{f}_\vartheta \circ \phi_t(v)dt}{\int_0^T \hat{g}_\vartheta \circ \phi_t(v)dt} = \int_{\partial^2 X} f g' d\mu.$$

The same conclusion holds with $v = (\eta,\xi,0)$, for μ-almost all $(\eta,\xi) \in \partial^2 X$.

Proof. Recall that the map $\hat{f}_\vartheta \colon SX \to \mathbb{R}$ defined as in (24) and (25) is Γ-invariant and $\bar{m}$-integrable. Since the geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative, the Hopf ergodic theorem [Hop37] tells us that for $\bar{m}$-almost every $v \in SX$,

$$\lim_{T\to\pm\infty} \frac{\int_0^T \hat{f}_\vartheta \circ \phi_t(v)dt}{\int_0^T \hat{g}_\vartheta \circ \phi_t(v)dt} = f_\infty(v), \quad \text{where} \quad f_\infty(v) = \mathbb{E}_{\hat{g}_\vartheta \bar{m}}\left(\hat{f}_\vartheta \,\middle|\, \mathcal{I} \right)(v) \tag{37}$$

is the conditional expectation of $\hat{f}_\vartheta$ with respect to the sub-σ-algebra $\mathcal{I}$ of $\mathcal{B}_\Gamma$ of all (ϕ_t)-invariant Borel subsets.

Assume that f belongs to $\mathcal{D}^+(\partial^2 X)$. As the geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative, both the numerator and the denominator in (37) diverge to infinity. Since $\hat{f}_\vartheta$ and $\hat{g}_\vartheta$ are bounded (Lemma 4.7), the map $f_\infty(v)$ does not depend on the time coordinate of $v = (\eta,\xi,t)$, hence we write $f_\infty(v) = f_\infty(\eta,\xi)$. The crucial ingredient is Proposition 4.8, which implies that the map f_∞ only depends on the future, $\bar{m}$- or m- or μ-almost surely. As the flow is flip invariant, the map f_∞ depends also only on the past, μ-almost surely. Since μ is equivalent to a product measure, the standard Hopf argument (based on Fubini Theorem) shows that f_∞ is constant $\bar{m}$- or m- or μ-almost surely.

By construction $\hat{g}_\vartheta$ is bounded (see Lemma 4.7) so that $f_\vartheta \hat{g}_\vartheta \in L^1(m)$. As $\hat{g}_\vartheta$ is Γ-invariant, (26) yields

$$\int f g' d\mu = \int f_\vartheta \hat{g}_\vartheta dm = \int \widehat{f_\vartheta \hat{g}_\vartheta} d\bar{m} = \int \hat{f}_\vartheta \hat{g}_\vartheta d\bar{m}.$$

By definition of conditional expectation, we deduce that the almost sure value of f_∞, say $M \in \mathbb{R}$, satisfies

$$M = \int f_\infty \hat{g}_\vartheta d\bar{m} = \int \hat{f}_\vartheta \hat{g}_\vartheta d\bar{m} = \int f g' d\mu.$$

As g' is bounded, both maps

$$f \mapsto \mathbb{E}_{\hat{g}_\vartheta \bar{m}} \left(\hat{f}_\vartheta \,\middle|\, \mathcal{I} \right) \quad \text{and} \quad f \mapsto \int_{\partial^2 X} f g' d\mu$$

are bounded linear functionals, which coincide on $\mathcal{D}^+(\partial^2 X) \subset L^1_+(\mu)$. As it is a dense subset of $L^1_+(\mu)$, they coincide everywhere. It completes the proof of the main statement. The proof of the last statement is a direct corollary of the previous argument. We omit it. □

We have not quite proved yet that the measure $\bar{m}$ is ergodic for the flow (ϕ_t). Indeed Proposition 4.14 does not a priori apply for any function in $L^1(\bar{m})$. Nevertheless it is sufficient to deduce that μ is ergodic for the diagonal action of Γ on $\partial^2 X$. The next statement completes the proof of Theorem 4.2.

Corollary 4.15. *Assume that the geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative. The action of Γ on $(\partial^2 X, \mu)$ is ergodic.*

Proof. Let B be a Γ-invariant subset of $\partial^2 X$ such that $\mu(B) > 0$. We want to prove that $\mu(\partial^2 X \setminus B) = 0$. Let $K \subset B$ be a compact set with $\mu(K) > 0$. By Proposition 4.14 applied to $f = \mathbf{1}_K$, for μ-almost every (η, ξ), for every sufficiently large $T \in \mathbb{R}_+$,

$$\int_0^T \hat{f}_\vartheta \circ \phi_t(v) dt > 0, \quad \text{where} \quad v = (\eta, \xi, 0).$$

It implies that for μ-almost every (η, ξ), some $(\gamma\xi, \gamma\eta)$ lies in K, and therefore B. As B is Γ-invariant, it means that μ-almost every (η, ξ) belongs to B, i.e. B has full measure. □

4.5. Finiteness of the Bowen-Margulis measure. As a by-product of our technique, we will show that, when the action of Γ has a growth gap at infinity, the Bowen-Margulis measure $\bar{m}$ on $(SX, \mathcal{B}_\Gamma)$ is finite. This statement is not needed for the proof of Theorem 1.1. We include it because it is an important dynamical result, which follows easily from the previous material. In fact, we prove the following more general statement, inspired from the work of Pit and Schapira [**PS18**, Section 5].

Theorem 4.16. *Let Γ be a discrete group acting properly by isometries on a Gromov-hyperbolic space X. The Bowen-Margulis measure $\bar{m}$ on $(SX, \mathcal{B}_\Gamma)$ is finite if and only if the Patterson-Sullivan measure ν_0 gives full measure to the radial limit set $\Lambda_{\mathrm{rad}}(\Gamma)$ and there exists a compact subset K of X such that the series*

$$\sum_{\gamma \in \Gamma_K} d(o, \gamma o)\, e^{-h_\Gamma d(o, \gamma o)}$$

converges.

Recall that if the action of Γ on X is strongly positively recurrent, then ν_0 gives full measure to the radial limit set (Corollary 3.16). Moreover there exists a compact subset K of $\bar{X}$ such that $h_{\Gamma_K} < h_\Gamma$. Therefore Theorem 4.16 has the following immediate corollary.

COROLLARY 4.17. *Let Γ be a discrete group acting properly by isometries on a Gromov-hyperbolic space X. If the action of Γ on X is strongly positively recurrent, then the Bowen-Margulis measure $\bar{m}$ on $(SX, \mathcal{B}_\Gamma)$ is finite.*

If the Bowen-Margulis measure $\bar{m}$ on $(SX, \mathcal{B}_\Gamma)$ is finite, it follows from Poincaré's recurrence theorem that the geodesic flow on $(SX, \mathcal{B}_\Gamma, \bar{m})$ is conservative. Therefore, by Theorem 4.2, the Patterson-Sullivan measures gives full measure to the radial limit set. It is therefore enough to show Theorem 4.16 when ν_0 gives full measure to the radial limit set.

By definition, $\Lambda_{\rm rad}$ is the increasing union of all $\Lambda^K_{\rm rad}$ where K runs over all compact subsets of X. As already noticed before, there exists a compact subset $k \subset X$, such that $\nu_o(\Lambda^k_{\rm rad}) = 1$ (Corollary 2.4). Up to enlarging k we may assume that o belongs to k. We now fix a parameter $r \geqslant \operatorname{diam}(k) + 1000\delta$. For the moment r is fixed, it will vary only at the very end of the proof. For simplicity let

$$Z = Z(r) = \left\{ (\eta, \xi) \in \partial^2 X \;\middle|\; \langle \eta, \xi \rangle_o \leqslant r \right\},$$

and define

$$\Sigma = \Sigma(r) = \{ (\eta, \xi, 0) \in SX \mid (\eta, \xi) \in Z(r) \},$$

which we think of as a "compact" subset of a section of the flow. As in the preceding section, we work in SX modulo Γ. This motivates the next definition. Given a vector $v = (\eta, \xi, 0)$ in Σ, the *first return time* of v in Σ (modulo Γ), denoted by $\tau(v)$, is defined by

$$\tau(v) = \inf \left\{ t > 2r + 500\delta \;\middle|\; \exists \gamma \in \Gamma,\ \gamma^{-1}\phi_t(v) \in \Sigma \right\}.$$

Remark. As X is Gromov hyperbolic, we only control its large scale geometry, which causes some edge effects. For this reason, it is convenient to require the first return time to be larger that $2r + 500\delta$ (see for instance the proof of Lemma 4.26).

Define now

$$\Sigma' = \{ v \in \Sigma, \tau(v) < +\infty \}.$$

Finally, the *first return core* is defined by

$$W = \{ \phi_t(v) \mid v \in \Sigma',\ 0 \leqslant t \leqslant \tau(v) \}.$$

We are going to prove that ΓW has full $\bar{m}$-measure (Proposition 4.19) and that its measure $\bar{m}(\Gamma W)$ is finite if and only if a certain series converges (Propositions 4.23 and 4.27). We start with the following lemma which provides a useful criterion *in the space* X to determine when a vector $v \in SX$ belongs to a translate of Σ.

LEMMA 4.18. *Let $v \in SX$ and $\gamma \in \Gamma$. If $d(\gamma o, \operatorname{proj}(v)) \leqslant r - 220\delta$, then there exists $s \in \mathbb{R}$, with $|s| \leqslant r$ such that $\gamma^{-1}\phi_s(v) \in \Sigma$.*

PROOF. We write $v = (\eta, \xi, t)$. Combining our assumption with (23) we get

$$\left\langle \gamma^{-1}\eta, \gamma^{-1}\xi \right\rangle_o + \left| t + \kappa_{\gamma^{-1}}(\eta, \xi) \right| \leqslant d(\gamma o, \operatorname{proj}(v)) + 220\delta \leqslant r.$$

It follows first that $\langle \gamma^{-1}\eta, \gamma^{-1}\xi \rangle_o \leqslant r$, i.e. the pair $(\gamma^{-1}\eta, \gamma^{-1}\xi)$ belongs to Z. Moreover $s = -\kappa_{\gamma^{-1}}(\eta, \xi) - t$ satisfies $|s| \leqslant r$. One easily checks that $\gamma^{-1}\phi_s(v) = \left(\gamma^{-1}\eta, \gamma^{-1}\xi, 0\right)$, which, according to our previous observation, belongs to Σ. $\square$

PROPOSITION 4.19. *The set ΓW is a Γ-invariant set of full $\bar{m}$-measure. In particular, $\bar{m}(SX) \leqslant m(W)$.*

PROOF. By assumption, $\nu_0(\Lambda^k_{\rm rad}) = 1$. Since μ belongs to the same measure class as $\nu_o \otimes \nu_o$, it gives full measure to the set $(\Lambda^k_{\rm rad} \times \Lambda^k_{\rm rad}) \cap \partial^2 X$. It follows from Lemma 4.3, that for m-almost every $v \in SX$, for every $T \geqslant 0$, there exists $t \geqslant T$ and $\gamma \in \Gamma$ such that $\gamma^{-1}\phi_t(v) \in \Sigma$. The same holds for negative times. Hence W contains a Borel fundamental domain for the action of Γ on SX. Consequently

ΓW has full m-measure and thus full $\bar{m}$-measure. The inequality $\bar{m}(SX) \leqslant m(W)$ directly follows from the definition of $\bar{m}$. $\square$

In order to estimate the measure of W it will be convenient to decompose it according to which translates of Σ the first return map falls in. This motivates the next definitions. For all $\gamma \in \Gamma$, we define

$$\begin{aligned} \Sigma'_\gamma &= \left\{ v \in \Sigma' \;\middle|\; \exists s,\ \tau(v) \leqslant s \leqslant \tau(v) + 2r + 500\delta \text{ and } \gamma^{-1}\phi_s(v) \in \Sigma \right\} \\ Z'_\gamma &= \left\{ (\eta, \xi) \in \partial^2 X \;\middle|\; (\eta, \xi, 0) \in \Sigma'_\gamma \right\} \\ W_\gamma &= \left\{ \phi_t(v) \;\middle|\; v \in \Sigma'_\gamma,\ 0 \leqslant t \leqslant \tau(v) \right\}. \end{aligned}$$

Finally we denote by $\Gamma(\Sigma')$ the set of all elements $\gamma \in \Gamma$ for which Σ'_γ is non-empty. It follows from these definitions that

$$W \subset \bigcup_{\gamma \in \Gamma(\Sigma')} W_\gamma. \tag{38}$$

Let us study the properties of these sets. We start with a series of lemmas that will provide an *upper* bound of $\bar{m}(SX)$.

Lemma 4.20. *For every $\gamma \in \Gamma(\Sigma')$, for every $v \in \Sigma'_\gamma$, the vectors v and $v' = \phi_{\tau(v)}(v)$ satisfy*

$$d(o, \operatorname{proj}(v)) \leqslant r + 20\delta \quad \text{and} \quad d(\gamma o, \operatorname{proj}(v')) \leqslant 3r + 720\delta$$

Moreover $|d(o, \gamma o) - \tau(v)| \leqslant 4r + 740\delta$.

Proof. Note that the proof would be rather obvious if the projection $SX \to X$ were Γ-equivariant. Let $v = (\eta, \xi, 0)$ in Σ'_γ. As observed in (21), the quantity $\langle \eta, \xi \rangle_o$ roughly measures the distance between o and $\operatorname{proj}(v)$. Since $v \in \Sigma$, we get

$$d(o, \operatorname{proj}(v)) \leqslant \langle \eta, \xi \rangle_o + 20\delta \leqslant r + 20\delta.$$

By definition of Σ'_γ, there exists $t \in [\tau(v), \tau(v) + 2r + 500\delta]$ such that $\gamma^{-1}\phi_t(v)$ belongs to Σ. As before, we get from (23)

$$d(\gamma o, \operatorname{proj}(\phi_t(v))) \leqslant \left\langle \gamma^{-1}\eta, \gamma^{-1}\xi \right\rangle_o + 220\delta \leqslant r + 220\delta.$$

The map $\operatorname{proj} \circ \phi_s : \mathbb{R} \to X$ is a bi-infinite geodesic, so that

$$d(\operatorname{proj}(v'), \operatorname{proj}(\phi_t(v))) \leqslant 2r + 500\delta \quad \text{and} \quad d(\operatorname{proj}(v), \operatorname{proj}(v')) = \tau(v).$$

It yields $d(\gamma o, \operatorname{proj}(v')) \leqslant 3r + 720\delta$, which completes the first part of the lemma. The second part follows from the triangle inequality. $\square$

Lemma 4.21. *For every $\gamma \in \Gamma(\Sigma')$, the set Z'_γ is contained in the product $\mathcal{O}_{\gamma o}(o, r + 30\delta) \times \mathcal{O}_o(\gamma o, r + 30\delta)$.*

Proof. Let $(\eta, \xi) \in \partial^2 X$ and $v = (\eta, \xi, 0)$. As usual we write $\sigma \colon \mathbb{R} \to X$ for the bi-infinite geodesic sending s to $\phi_s(v)$. Assume first that $(\eta, \xi) \in Z'_\gamma$, i.e. the vector v belongs to Σ'_γ. It follows that $\langle \eta, \xi \rangle_o \leqslant r$ and $\langle \eta, \xi \rangle_{\gamma o} \leqslant r$. In particular, o and γo are $(r + 6\delta)$-close to σ. As the ideal geodesic triangles in X are 24δ-thin, γo (respectively o) is $(r + 30\delta)$-close to any geodesic joining o to ξ (respectively γo to η). Whence the result. $\square$

Lemma 4.22. *Assume that K is a compact subset contained in $B(r - 300\delta)$. There exist two finite subsets S_1 and S_2 of Γ such that $\Gamma(\Sigma') \setminus S_1$ is contained in $S_2 \Gamma_K S_2$.*

PROOF. Set

$$S_1 = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant 8r + 2000\delta\},$$
$$S_2 = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant 5r + 1000\delta\}.$$

Let $\gamma \in \Gamma(\Sigma') \setminus S_1$ and choose an arbitrary $v \in \Sigma'_\gamma$. For simplicity, set $\tau = \tau(v)$ for the first return time of v in Σ. Recall that the map $\sigma \colon \mathbb{R} \to X$ sending t to $\mathrm{proj} \circ \phi_t(v)$ is a bi-infinite geodesic of X joining η to ξ. According to Lemma 4.20

$$d(o, \sigma(0)) \leqslant r + 20\delta \quad \text{and} \quad d(\gamma o, \sigma(\tau)) \leqslant 3r + 720\delta$$

Fix a geodesic $c \colon [0, \ell] \to X$ joining o to γo. Let $s_1, s_2 \in [0, 4r + 1000\delta]$ be the largest times such that $c(s_1)$ (respectively $c(\ell - s_2)$) belongs to $\alpha_1 K$ for some $\alpha_1 \in \Gamma$ (respectively $\gamma\alpha_2 K$ for some $\alpha_2 \in \Gamma$). It follows from the previous claim combined with the triangle inequality that $d(o, \alpha_i o) \leqslant 5r + 1000\delta$. In other words γ can be written $\gamma = \alpha_1(\alpha_1^{-1}\gamma\alpha_2)\alpha_2^{-1}$ where α_1 and α_2^{-1} belong to S_2. Thus we are left to prove that $\alpha_1^{-1}\gamma\alpha_2$ belongs to Γ_K. As γ does not belong to S_1, the points x, $c(s_1)$, $c(\ell - s_2)$ and γy are aligned in this order along c. Hence it suffices to prove that c restricted to $(s_1, \ell - s_2)$ does not intersect ΓK. Assume on the contrary that there exists $s \in (s_1, \ell - s_2)$ such that $y = c(s)$ belongs to βK for some $\beta \in \Gamma$. By construction $d(o, c(s)) > d(o, \sigma(0)) + 3r + 510\delta$ and $d(\gamma o, c(s)) > d(\gamma o, \sigma(\tau)) + r + 10\delta$. It is a standard exercise in hyperbolic geometry to observe that y is 6δ-close to a point $x = \sigma(t)$ with $t \in (3r + 500\delta, \tau - r)$. In particular, $d(\beta o, x) \leqslant r - 220\delta$. It follows then from Lemma 4.18 that there exists $t' \in (2r + 500\delta, \tau(v))$ such that $\beta^{-1}\phi_{t'}(v)$ belongs to Σ. This contradicts the definition of the first return time and completes the proof of the claim. □

PROPOSITION 4.23. *Assume that K is a compact subset contained in $B(r - 300\delta)$. There exists $C \in \mathbb{R}_+$ such that*

$$\bar{m}(SX) \leqslant C \sum_{\gamma \in \Gamma_K} d(o, \gamma o)\, e^{-h_\Gamma d(o, \gamma o)}$$

PROOF. As we observed earlier $\bar{m}(SX) \leqslant m(W)$ (Proposition 4.19). For every $(\eta, \xi) \in Z$ define $\tau(\eta, \xi) = \tau(v)$ where $v = (\eta, \xi)$. Recall that $m = \mu \otimes dt$. Thus the decomposition of the first return core W given in (38) yields

$$\bar{m}(SX) \leqslant \sum_{\gamma \in \Gamma(\Sigma')} m(W_\gamma) \leqslant \sum_{\gamma \in \Gamma(\Sigma')} \int \mathbf{1}_{Z'_\gamma}(\eta, \xi)\tau(\eta, \xi) d\mu(\eta, \xi),$$

By Lemma 4.20, the first return time τ is approximatively $d(o, \gamma o)$ when restricted to Z'_γ. Moreover by (12) μ restricted to Z' is comparable to $\nu_o \otimes \nu_o$. Hence there exists $C \in \mathbb{R}_+$ such that

$$\bar{m}(SX) \leqslant C \sum_{\gamma \in \Gamma(\Sigma')} d(o, \gamma o)\, (\nu_o \otimes \nu_o)(Z'_\gamma).$$

According to Lemma 4.21, Z'_γ is contained in $\partial X \times \mathcal{O}_o(\gamma o, r + 30\delta)$. Hence (up to increasing C) the Shadow Lemma (Lemma 2.3) gives

$$\bar{m}(SX) \leqslant C \sum_{\gamma \in \Gamma(\Sigma')} d(o, \gamma o)\, e^{-h_\Gamma d(o, \gamma o)}.$$

The conclusion now follows from Lemma 4.22. □

Let us now provide a *lower* bound of $\bar{m}(SX)$. To that end we define

$$W^0 = \{\phi_s(v) \mid v \in \Sigma',\ 0 \leqslant s < \tau(v)\}$$

The first step is to estimate the multiplicity of certain families.

LEMMA 4.24. *There exists $N \in \mathbb{N}$ such that for $\bar{m}$-almost every $v \in SX$ the set $\{\gamma \in \Gamma \mid v \in \gamma W^0\}$ contains at most N elements.*

PROOF. Recall that S_0X is the full measure, Γ and flow invariant subset defined in (20). Let $v \in S_0X$. Let $\alpha, \beta \in \Gamma$ be such that v belongs to $\alpha W^0 \cap \beta W^0$. We can write $\alpha\phi_s(u) = v = \beta\phi_t(w)$, where $u, w \in \Sigma'$,

$$0 \leqslant s < \tau(u), \quad \text{and} \quad 0 \leqslant t < \tau(w).$$

In particular, $\alpha^{-1}\beta\phi_{t-s}(w) = u$ and $\beta^{-1}\alpha\phi_{s-t}(u) = w$ both belong to Σ. By construction either $0 \leqslant t - s < \tau(w)$ or $0 \leqslant s - t < \tau(u)$. It follows from our definition of first return time that $|t - s| \leqslant 2r + 500\delta$. Since $\mathrm{proj}\colon SX \to X$ is almost Γ-equivariant (22) and maps orbits of the flow to geodesics we get

$$d(\alpha \,\mathrm{proj}(u), \beta \,\mathrm{proj}(w)) \leqslant 2r + 700\delta.$$

On the other hand, since w belongs to Σ, we have $d(o, \mathrm{proj}(w)) \leqslant r+20\delta$ (Lemma 4.20). Consequently

$$\left\{\gamma \in \Gamma \;\middle|\; v \in \gamma W^0\right\} \subset \{\gamma \in \Gamma \mid d(x, \gamma o) \leqslant 3r + 800\delta\}$$

where $x = \alpha \,\mathrm{proj}(u)$. The conclusion follows from Lemma 4.5. □

LEMMA 4.25. *There exists $N \in \mathbb{N}$ such that for every $v \in SX$ the set $\{\gamma \in \Gamma \mid v \in W_\gamma\}$ contains at most N elements.*

PROOF. Set $v' = \phi_{\tau(v)}(v)$. It follows from Lemma 4.20 that

$$\{\gamma \in \Gamma \mid v \in W_\gamma\} \subset \{\gamma \in \Gamma \mid d(\mathrm{proj}(v'), \gamma o) \leqslant 3r + 720\delta\}\,.$$

Hence the result follows from Lemma 4.5. □

LEMMA 4.26. *There exists a compact subset $K \subset X$ and a finite subset $S \subset \Gamma$ such that for every $\gamma \in \Gamma_K \setminus S$, the product $\mathcal{O}_{\gamma o}(o, r-\delta) \times \mathcal{O}_o(\gamma o, r-\delta)$ is contained in Z'_γ. In particular, $\Gamma_K \setminus S \subset \Gamma(\Sigma')$*

PROOF. Let K be the closed ball $K = \bar{B}(o, r + 250\delta)$ and set

$$S = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant 6r + 1000\delta\}\,.$$

Let $\gamma \in \Gamma_K \setminus S$ and $(\eta, \xi) \in \mathcal{O}_{\gamma o}(o, r - \delta) \times \mathcal{O}_o(\gamma o, r - \delta)$. Since η belongs to $\mathcal{O}_{\gamma o}(o, r - \delta)$, it follows from the four point inequality (3) that

$$\min\left\{\langle \eta, \xi\rangle_o, \langle \gamma o, \xi\rangle_o\right\} \leqslant \langle \gamma o, \eta\rangle_o + \delta \leqslant r.$$

As ξ belongs to $\mathcal{O}_{\gamma o}(o, r - \delta)$ and $\gamma \notin S$, we have

$$\langle \gamma o, \xi\rangle_o \geqslant d(o, \gamma o) - r - \delta > r,$$

thus the minimum cannot be achieved by $\langle \gamma o, \xi\rangle_o$. Hence $\langle \eta, \xi\rangle_o \leqslant r$, which means that $v = (\eta, \xi, 0)$ lies in Σ. Similarly we prove that $\langle \eta, \xi\rangle_{\gamma o} \leqslant r$, thus there exists $t \in \mathbb{R}$ such that $\gamma^{-1}\phi_t(v)$ belongs to Σ. Since $d(o, \gamma o) > 6r + 1000\delta$, we can assume that $t > 0$. In particular, $\tau(v) \leqslant t$. We now need to prove that $t \leqslant \tau(v) + r + \delta$. Assume on the contrary that is its not the case. In particular, there exists $s \in [\tau(v), t - r - \delta)$ such that $\alpha^{-1}\phi_s(v) \in \Sigma$, for some $\alpha \in \Gamma$. For simplicity we let $z_0 = \mathrm{proj}(v)$, $z_s = \mathrm{proj} \circ \phi_s(v)$ and $z_t = \mathrm{proj} \circ \phi_t(v)$. By (23) we have

$$\max\left\{d(o, z_0)\,, d(\alpha o, z_s)\,, d(\gamma o, z_t)\right\} \leqslant r + 220\delta.$$

Since γ belongs to Γ_K, there exists $x, y \in K$ and a geodesic $c\colon [0, \ell] \to X$ joining x to γy such that $c \cap \Gamma K \subset K \cup \gamma K$. It follows then from the triangle inequality that $d(x, z_0) \leqslant 2r + 470\delta$ and $d(\gamma y, z_t) \leqslant 2r + 470\delta$. On the other hand since $\mathrm{proj}\colon SX \to X$ maps orbits of the flow to geodesics, hence $|s - t| \leqslant r + \delta$, we have

$$d(z_0, z_s) \geqslant \tau(v) > 2r + 500\delta \quad \text{and} \quad d(z_t, z_s) \geqslant t - s > 2r + 500\delta. \tag{39}$$

A standard exercise of hyperbolic geometry show that z_s it 6δ-close to a point $c(s)$ on c. In particular, $d(\alpha o, c(s)) \leqslant r + 250\delta$, i.e. $c(s) \in \alpha K$. It follows from the definition of c that $c(s)$ belongs to $K \cup \gamma K$. Consequently either $d(z_0, z_s) \leqslant 2r+500\delta$ or $d(z_t, z_s) \leqslant 2r + 500\delta$, which violates (39). ☐

PROPOSITION 4.27. *There exist $C \in \mathbb{R}_+^*$ and a compact subset $K \subset X$ such that*

$$C \sum_{\gamma \in \Gamma_K} d(o, \gamma o)\, e^{-h_\Gamma d(o,\gamma o)} \leqslant \bar{m}(SX)$$

PROOF. We write K for the compact subset of X given by Lemma 4.26. Obviously $\bar{m}(\Gamma W^0) \leqslant \bar{m}(SX)$. Note that the collection (γW^0) may not be pairwise disjoint, nevertheless thanks to Lemma 4.24 we control its multiplicity. Thus there exists $C \in \mathbb{R}_+^*$ such that

$$Cm(W^0) \leqslant \bar{m}(\Gamma W^0) \leqslant \bar{m}(SX).$$

Similarly (up to decreasing C) we get by Lemma 4.24

$$C \sum_{\gamma \in \Gamma(\Sigma')} m\left(W^0 \cap W_\gamma\right) \leqslant \bar{m}(SX).$$

Reasoning as in Proposition 4.23, we get

$$C \sum_{\gamma \in \Gamma(\Sigma')} d(o, \gamma o)\, (\nu_o \otimes \nu_o)(Z'_\gamma) \leqslant \bar{m}(SX).$$

By Lemma 4.26, $\mathcal{O}_{\gamma o}(o, r-\delta) \times \mathcal{O}_o(\gamma o, r-\delta)$ is contained in Z'_γ, for all but finitely many $\gamma \in \Gamma_K$. Combined with the Shadow Lemma (Lemma 2.3) it yields

$$C \sum_{\gamma \in \Gamma_K} d(o, \gamma o)\, e^{-h_\Gamma d(o,\gamma o)} \leqslant \bar{m}(SX). \qquad ☐.$$

We complete this section with the proof of Theorem 4.16.

PROOF OF THEOREM 4.16. Assume first that the Bowen-Margulis measure $\bar{m}$ is finite. It follows form Proposition 4.27 that there exists a compact subset $K \subset X$ such that the series

$$\sum_{\gamma \in \Gamma_K} d(o, \gamma o)\, e^{-h_\Gamma d(o,\gamma o)}$$

converges. Assume on the contrary that there exists a compact subset K for which the above series converges. Up to enlarging the value of r, we can always assume that K is contained in $B(o, r - 300\delta)$. It follows from Proposition 4.23 that $\bar{m}$ is finite. ☐

5. A twisted Patterson-Sullivan measure

5.1. Main theorem.

Setting. Let (X, d) be a proper geodesic δ-hyperbolic space. We fix once and for all a base point $o \in X$. Let Γ be a group acting properly by isometries on X. Recall that h_Γ stands for the critical exponent of the Poincaré series of Γ.

Let $(\mathcal{H}, \prec)$ be a *Hilbert lattice*, i.e. a Hilbert space endowed with a partial order $\prec$, compatible with the vector space structure as well as the norm, which induces a lattice structure on $\mathcal{H}$. We refer the reader to Section 2 for a precise definition. All properties of Hilbert lattices that we will use are also recalled in this appendix. Denote by $\mathcal{H}^+$ its positive cone, i.e. the set of elements $\phi \in \mathcal{H}$ such that $0 \prec \phi$. Let $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a *positive* unitary representation, i.e. $\rho(\gamma)\phi \in \mathcal{H}^+$, for every $\gamma \in \Gamma$ and every $\phi \in \mathcal{H}^+$.

Twisted Poincaré series. For every $s \in \mathbb{R}_+$ we consider the formal series $A(s)$ defined as follows

$$A(s) = \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)} \rho(\gamma).$$

We say that this series is *bounded* if there exists $M \in \mathbb{R}_+$ such that for every finite subset S of Γ,

$$\left\| \sum_{\gamma \in S} e^{-sd(\gamma o, o)} \rho(\gamma) \right\| \leqslant M.$$

The *critical exponent* of the representation ρ is defined as

$$h_\rho = \inf \left\{ s \in \mathbb{R}_+ \mid A(s) \text{ is bounded} \right\}.$$

According to Proposition 2.3, for every $s > h_\rho$, the series pointwise converges to a bounded operator of $\mathcal{H}$. The following lemma is straightforward.

Lemma 5.1. *For every $s > h_\Gamma$, the series $A(s)$ is bounded and $\|A(s)\| \leqslant \mathcal{P}_\Gamma(s)$. In particular, $h_\rho \leqslant h_\Gamma$.*

Almost invariant vectors. Let S be a finite subset of Γ and $\varepsilon \in \mathbb{R}_+^*$. A vector $\phi \in \mathcal{H}$ is (S, ε)*-invariant* (with respect to ρ) if

$$\sup_{\gamma \in S} \|\rho(\gamma)\phi - \phi\| < \varepsilon \|\phi\|.$$

The representation $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ *almost has invariant vectors* if for every finite subset S of Γ, for every $\varepsilon \in \mathbb{R}_+^*$, there exists an (S, ε)-invariant vector. The goal of this section is to prove the following statement.

Theorem 5.2. *Let Γ be a discrete group acting properly by isometries on a hyperbolic space (X, d). Assume that the action of Γ on X is strongly positively recurrent. For every finite subset S of Γ, for every $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$ with the following property. Let $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary positive representation of Γ into a Hilbert lattice. If $h_\rho \geqslant (1 - \eta) h_\Gamma$, then ρ has an (S, ε)-invariant vector.*

The proof of this result is given in Sections 5.3 - 5.6. For the moment let us mention a first consequence of this statement.

Corollary 5.3. *Let $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary positive representation of Γ into a Hilbert lattice. The representation ρ almost has invariant vectors if and only if $h_\rho = h_\Gamma$. In this case, $\|A(s)\| = \mathcal{P}_\Gamma(s)$, for every $s > h_\Gamma$.*

Proof. Assume first that the representation almost has invariant vectors. Let $s > h_\rho$, and S be a finite subset of Γ and $\varepsilon \in \mathbb{R}_+^*$. There exists a vector $\phi \in \mathcal{H} \setminus \{0\}$ such that for every $\gamma \in S$, we have $\|\rho(\gamma)\phi - \phi\| < \varepsilon \|\phi\|$. It yields

$$\left\| \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \rho(\gamma)\phi - \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \phi \right\| \leqslant \varepsilon \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \|\phi\|.$$

As

$$\left\| \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \phi \right\| = \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \|\phi\|$$

we obtain

$$(1 - \varepsilon) \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \|\phi\| \leqslant \left\| \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \rho(\gamma)\phi \right\| \leqslant \left\| \sum_{\gamma \in S} e^{-sd(o, \gamma o)} \rho(\gamma) \right\| \|\phi\|.$$

Since ϕ is a non-zero vector we get

$$(1-\varepsilon)\sum_{\gamma\in S} e^{-sd(o,\gamma o)} \leqslant \left\| \sum_{\gamma\in S} e^{-sd(o,\gamma o)}\rho(\gamma)\right\|.$$

This inequality holds for all $\varepsilon > 0$. Hence, for any finite subset S of Γ, we have

$$\sum_{\gamma\in S} e^{-sd(o,\gamma o)} \leqslant \left\| \sum_{\gamma\in S} e^{-sd(o,\gamma o)}\rho(\gamma)\right\|.$$

We deduce that for all $s > h_\rho$, $\mathcal{P}_\Gamma(s) \leqslant \|A(s)\|$. It follows that $h_\Gamma \leqslant h_\rho$. By Lemma 5.1, we get $h_\Gamma = h_\rho$ and $\|A(s)\| = \mathcal{P}_\Gamma(s)$. The converse implication follows from Theorem 5.2. □

5.2. Ultra-limit of Hilbert spaces. Inspired by the standard Patterson-Sullivan construction we are going to build in the next section a linear map on $C(\bar{X}_h)$ which we think of as an operator valued measure on $\bar{X}_h$. Since $\bar{X}_h$ is compact, the set of *probability* measures on $\bar{X}_h$ is compact for the weak-$*$ topology. This is no more the case for general vector-valued measures event with the appropriate normalization, see Remarks 5.12 and 5.28. To bypass this difficulty we let our measures converge in a bigger space obtained as the ultra-limit of a sequence of Banach spaces. This section reviews the main properties of ultra-limits of Banach spaces. For more details see Druţu-Kapovich [**DK18**, Chapter 19].

A *non-principal ultra-filter* is a finitely additive map $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ such that $\omega(\mathbb{N}) = 1$ and which vanishes on every finite subset of $\mathbb{N}$. A property P_n is true ω-*almost surely* (ω-as) if

$$\omega\left(\{n\in\mathbb{N} \mid P_n \text{ is true}\}\right) = 1.$$

A real sequence (u_n) is ω-*essentially bounded* (ω-eb) if there exists M such that $|u_n| \leqslant M$, ω-as. Given $\ell \in \mathbb{R}$, we say that the ω-limit of (u_n) is ℓ and write $\lim_\omega u_n = \ell$ if for all $\varepsilon > 0$, we have $|u_n - \ell| \leqslant \varepsilon$, ω-as. Any sequence which is ω-eb admits a ω-limit [**Bou71**].

Let (E_n) be a sequence of Banach spaces. We define a restricted product by

$$\prod_\omega E_n = \left\{ (\phi_n) \in \prod_{n\in\mathbb{N}} E_n \;\middle|\; \|\phi_n\| \text{ is } \omega\text{-eb}\right\}$$

Pointwise addition and scalar multiplication define a vector space structure on this set. We define a pseudonorm by

$$\|(\phi_n)\| = \lim_\omega \|\phi_n\|.$$

Definition 5.4. The ω-*limit* of (E_n), denoted by $\lim_\omega E_n$ or simply E_ω, is the quotient of $\prod_\omega E_n$ by the equivalence relation which identifies two sequences (ϕ_n) and (ϕ'_n) whenever $\|(\phi_n) - (\phi'_n)\| = 0$.

The vector space structure on $\prod_\omega E_n$ passes to the quotient and turns E_ω into a vector space. Similarly the pseudonorm on $\prod_\omega E_n$ defines a norm on $\lim_\omega E_n$ for which E_ω is complete [**Pap96**, Preliminaries]. Hence E_ω is a Banach space. In addition, if for every $n \in \mathbb{N}$, the space E_n is a Hilbert space, then so is E_ω. Indeed, the parallelogram law only involves four points, thus it passes to the limit [**DK18**, Corollary 19.3].

Notation. If (ϕ_n) is a sequence in $\prod_\omega E_n$ we denote its image in $\lim_\omega E_n$ by $\lim_\omega \phi_n$.

Let (E_n) and (F_n) be two sequences of Banach spaces. Let

$$E_\omega = \lim_\omega E_n \quad \text{and} \quad F_\omega = \lim_\omega F_n.$$

For every $n \in \mathbb{N}$, the space $\mathcal{B}(E_n, F_n)$ of bounded linear operators from E_n to F_n is a Banach space. In particular, we can consider the limit space $\lim_\omega \mathcal{B}(E_n, F_n)$. Given an element $A = \lim_\omega A_n$ in $\lim_\omega \mathcal{B}(E_n, F_n)$, one defines an operator $\iota(A)$ in $\mathcal{B}(E_\omega, F_\omega)$ as follows. For every $\phi = \lim_\omega \phi_n$ in E_ω, we let

$$\iota(A)\phi = \lim_\omega [A_n \phi_n].$$

One checks easily that $\iota(A)$ is well-defined. In particular, it does not depend on the choice of the sequences (A_n) or (ϕ_n). The resulting map

$$\iota\colon \lim_\omega \mathcal{B}(E_n, F_n) \to \mathcal{B}(E_\omega, F_\omega)$$

is both a linear map and an isometric embedding. As $\lim_\omega \mathcal{B}(E_n, F_n)$ is complete, its image is closed. In this article, we will omit the map ι and see $\lim_\omega \mathcal{B}(E_n, F_n)$ as a closed linear subspace of $\mathcal{B}(E_\omega, F_\omega)$. Similarly $\lim_\omega \mathcal{B}(E_n)$ embeds as a closed *subalgebra* of $\mathcal{B}(E_\omega)$. This leads to the following statement.

Proposition 5.5. *Let Γ be a group. Let (ρ_n) be a sequence of unitary representations of Γ into a Hilbert space $\mathcal{H}_n$. There exists a unique unitary representation $\rho_\omega\colon \Gamma \to \mathcal{U}(\mathcal{H}_\omega)$ such that for every $\gamma \in \Gamma$, for every element $\phi = \lim_\omega \phi_n$ of $\mathcal{H}_\omega$ we have*

$$\rho_\omega(\gamma)\phi = \lim_\omega [\rho_n(\gamma)\phi_n].$$

It is denoted by $\rho_\omega = \lim_\omega \rho_n$, and called the (ultra-)limit representation.

Lattice structure. Assume now that each space E_n comes with a partial order $\prec$ that turns E_n into a Banach lattice. We define E_ω^+ as the set

$$E_\omega^+ = \left\{ \lim_\omega \phi_n \in E_\omega \;\middle|\; \phi_n \in E_n^+,\ \omega\text{-as} \right\}.$$

It is a positive convex cone. Hence one can define a partial order on E_ω by declaring that $\phi \prec \phi'$ if $\phi' - \phi \in E_\omega^+$.

Lemma 5.6 (Druţu-Kapovich [DK18, Proposition 19.12]). *The ordered vector space $(E_\omega, \|\,.\,\|, \prec)$ is a Banach lattice.*

Lemma 5.7. *Let Γ be a group. Let (ρ_n) be a sequence of unitary representations of $\rho_n\colon \Gamma \to \mathcal{U}(\mathcal{H}_n)$ into a Hilbert lattice $\mathcal{H}_n$. If ρ_n is positive ω-as, then so is the limit representation $\rho_\omega = \lim_\omega \rho_n$.*

Proof. It directly follows from the definition of ρ_ω. $\square$

5.3. Conformal family of operator valued measures. The next sections are dedicated to the proof of Theorem 5.2.

Setting. Let (X, d) be a proper geodesic hyperbolic space. We fix once and for all a base point $o \in X$. Recall that $\bar{X}$ stands for the Gromov compactification of X whereas $\bar{X}_h$ is its horocompactification. Let Γ be a group acting properly by isometries on X. We assume that this action is strongly positively recurrent. Let ω be a non-principal ultra-filter. For every $n \in \mathbb{N}$, we fix a Hilbert lattice $\mathcal{H}_n$, as well as a unitary positive representation $\rho_n\colon \Gamma \to \mathcal{U}(\mathcal{H}_n)$. We denote by $A_n(s)$ the formal series

$$A_n(s) = \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)} \rho_n(\gamma).$$

We set
$$h_\omega = \lim_\omega h_{\rho_n}.$$
We are going to prove that if $h_\omega = h_\Gamma$, then ρ_ω has a non-zero invariant vector (Proposition 5.27). This will imply Theorem 5.2.

Remark. To prove Theorem 1.1, we can assume ρ_n is constantly equal to the Koopman representation associated to the right action of Γ on $\Gamma'\backslash\Gamma$. In this case $h_\omega = h_\rho$. Nevertheless, the quantified version of our main theorem as stated in Theorem 5.2 requires this level of full generality. Note that even if (ρ_n) is a constant sequence, we cannot avoid using ultra-filters. Indeed, as our Hilbert spaces are not locally compact, ultra-limit of Hilbert spaces provides a convenient tool to make bounded sequences converge.

Weighted Poincaré series. Since the action of Γ is strongly positively recurrent, the standard Poincaré series $\mathcal{P}_\Gamma(s)$ is divergent at the critical exponent $s = h_\Gamma$, see Corollary 3.16. However there is no reason that the sequence $\|A_n(s)\|$ should diverge, at $s = h_{\rho_n}$. We bypass this difficulty by adapting the usual Patterson argument [**Pat76**, Lemma 3.1].

LEMMA 5.8. *Let (s_n) be a sequence converging to h_ω. There exists a non decreasing map $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties.*

(1) For every $\varepsilon \in \mathbb{R}_+^*$, there exists $t_0 \in \mathbb{R}_+$ such that for every $u \in \mathbb{R}_+$ and $t \geqslant t_0$, one has $\theta(t+u) \leqslant e^{\varepsilon u}\theta(t)$.

(2) The operator series
$$A'_n(s) = \sum_{\gamma\in\Gamma} \theta(d(\gamma o, o)) e^{-sd(\gamma o,o)}\rho_n(\gamma)$$
is bounded whenever $s > h_{\rho_n}$ and unbounded whenever $s < h_{\rho_n}$.

(3) The sequence $\|A'_n(s_n)\|$ diverges as n approaches infinity.

PROOF. Recall that both (h_{ρ_n}) and (s_n) converge to h_ω. Hence we can find a decreasing sequence (ε_n) of positive numbers converging to zero, such that $s_n - \varepsilon_n < h_{\rho_n}$ for every $n \in \mathbb{N}$. We are going to build by induction an increasing sequence (t_n) diverging to infinity and a map $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ whose restriction to $[t_n, t_{n+1}]$ is logarithmically affine with slope ε_n. We start by letting $t_0 = 0$ and $\theta(t_0) = 1$. Let $n \in \mathbb{N}$. Assume now that t_n and θ restricted to $[t_0, t_n]$ have already be defined. By assumption, the series $A_n(s)$ is divergent at $s = s_n - \varepsilon_n$. Consequently there exists $t_{n+1} > t_n + 1$ such that

$$\left\| \sum_{\gamma\in S_n} e^{-(s_n-\varepsilon_n)d(\gamma o,o)}\rho_n(\gamma) \right\| \geqslant n, \tag{40}$$

where
$$S_n = \{\gamma \in \Gamma \mid t_n < d(\gamma o, o) \leqslant t_{n+1}\}.$$
We define θ on $]t_n, t_{n+1}]$ by $\theta(t) = e^{\varepsilon_n(t-t_n)}\theta(t_n)$. This completes the induction step. Points (1) and (2) are proved exactly as for regular Patterson-Sullivan measures. By construction,
$$\left\| \sum_{\gamma\in S_n} \theta(d(\gamma o,o)) e^{-s_n d(\gamma o,o)}\rho_n(\gamma) \right\| = \left\| \sum_{\gamma\in S_n} \theta(t_n) e^{-(s_n-\varepsilon_n)d(\gamma o,o)}\rho_n(\gamma) \right\|.$$
Consequently, (40) yields
$$\|A'_n(s_n)\| \geqslant \left\| \sum_{\gamma\in S_n} \theta(d(\gamma o,o)) e^{-s_n d(\gamma o,o)}\rho_n(\gamma) \right\| \geqslant n.$$

Hence the sequence $\|A'_n(s_n)\|$ diverges as n approaches infinity, whence (3). $\square$

A limit of bounded operators. We fix once for all a sequence (s_n) converging to h_ω as well as a slowly increasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ as in Lemma 5.8. Following the exposition of Section 5.2, let $\mathcal{H}_\omega = \lim_\omega \mathcal{H}_n$ be the limit Hilbert space (Definition 5.4) and $\rho_\omega = \lim_\omega \rho_n$ the limit representation (Proposition 5.5).

Let $x \in X$. For every $n \in \mathbb{N}$, we define a linear map

$$a^\rho_{x,n}\colon C(\bar{X}_h) \to \mathcal{B}(\mathcal{H}_n)$$

as follows. For every $f \in C(\bar{X}_h)$, define

$$a^\rho_{x,n}(f) = \frac{1}{\|A'_n(s_n)\|}\sum_{\gamma\in\Gamma}\theta(d(x,\gamma o))e^{-s_n d(x,\gamma o)}f(\gamma o)\rho(\gamma). \tag{41}$$

By Lemma 5.8, there exists a parameter $C(x)$ (which does not depends on n) such that for every $\gamma \in \Gamma$,

$$\theta(d(x,\gamma o)) \leqslant C(x)\theta(d(o,\gamma o)).$$

Let $f \in C(\bar{X}_h)$. Using the previous inequality, we observe that the series defining $a^\rho_{x,n}(f)$ is bounded for every $n \in \mathbb{N}$. Moreover its norm is bounded above by

$$\left\|a^\rho_{x,n}(f)\right\| \leqslant C(x)e^{s_n d(x,o)}\left\|f\right\|_\infty. \tag{42}$$

We define a bounded operator of $\mathcal{H}_\omega$ by

$$a^\rho_x(f) = \lim_\omega a^\rho_{x,n}(f).$$

This provides a positive continuous linear map

$$a^\rho_x\colon C(\bar{X}_h) \to \mathcal{B}(\mathcal{H}_\omega). \tag{43}$$

Remark. This map can be interpreted as an *operator-valued* measure on $\bar{X}_h$. Indeed by construction for every continuous function $f \in C(\bar{X}_h)$ with $f \geqslant 0$, the associated operator $a^\rho_x(f)$ is positive. It follows that a^ρ_x takes its values in the set $\mathcal{L}_r(\mathcal{H}_\omega)$ of regular operators on $\mathcal{H}_\omega$, which is an order-complete vector lattice (Proposition 2.2). By Wright [**Wri71**, Theorem 1], there exists a unique quasi-regular $\mathcal{L}_r(\mathcal{H}_\omega)$-valued Borel measure on $\bar{X}_h$ such that for every $f \in C(\bar{X}_h)$, the operator $a^\rho_x(f)$ equals the integral of f against this measure. We refer the reader to [**Wri71**] and the references therein for the theory of lattice-valued measures. We can hence see a^ρ_x as an operator-valued measure. However, it is simpler to express all the properties of this measure in terms of the map a^ρ_x. Still, it justifies the following terminology.

Definition 5.9. We call the family $(a^\rho_x)_{x\in X}$ the *twisted Patterson-Sullivan measure* associated to Γ and $\rho = (\rho_n)_{n\in\mathbb{N}}$.

The core of the proof of Theorem 5.2 consists in understanding the properties of this twisted Patterson-Sullivan measure. This new powerful definition and the study below are the main novelty of our paper. It has been inspired from the weighted Patterson-Sullivan measures of the thermodynamical formalism on the one hand, see [**BL98**, **Bab96**, **PPS15**] but also several papers of Sambarino as [**Sam14**] or all his later works, and from weighted Ruelle operators on the other hand, as for example [**Bow08**] and particularly the twisted Ruelle operators used in [**CDS17**].

First properties. The following statement translates the well-known properties of usual Patterson-Sullivan measures in this context.

THEOREM 5.10. *Let Γ be a discrete group acting properly by isometries on a proper geodesic hyperbolic space X. With the previous notations, the family $(a^\rho_x)_{x\in X}$ is ρ_ω-equivariant, h_ω-conformal, gives full support to $\partial_h X$ and is normalized at o.*

The proof of this theorem, as well as its precise meaning, is detailed in Lemmas 5.11 to 5.15.

LEMMA 5.11 (Normalization). *The operator $a^\rho_o(\mathbf{1})$ has norm 1.*

PROOF. By construction, for every $n \in \mathbb{N}$, if $f = \mathbf{1} \in C(\bar{X}_h)$, the operator

$$a^\rho_{o,n}(\mathbf{1}) = \frac{1}{\|A'_n(s_n)\|} A'_n(s_n)$$

has norm 1. We get the result by passing to the limit. □

REMARK 5.12. The previous lemma justifies our use of ultra-filters. Indeed, there are several topologies on the set of operators, and therefore on the space of linear maps $C(\bar{X}_h) \to \mathcal{B}(\mathcal{H})$, that we will denote here by $\mathcal{M}$. A natural choice would have been to consider the weak-* topology on $\mathcal{M}$: a sequence (L_n) converges to $L \in \mathcal{M}$ if for every $f \in C(\bar{X}_h)$ and every $\phi, \psi \in \mathcal{H}$ we have

$$\lim_{n\to\infty} (L_n(f)\phi, \psi) = (L(f)\phi, \psi)\,.$$

One checks that every bounded subset of $\mathcal{M}$ is pre-compact for this topology. In particular, the sequence $(a^\rho_{o,n})$ has an accumulation point in $\mathcal{M}$, say a^ρ_o. However there is no guarantee, that $a^\rho_o(\mathbf{1})$ is not the zero map. This is actually the case in many situations, see Remark 5.28.

LEMMA 5.13 (Support). *Let $x \in X$. Let $f \in C(\bar{X}_h)$. If its support is contained in X, then $a^\rho_x(f) = 0$.*

PROOF. As the support of f is a compact subset of X, there exists a finite subset S of Γ such that $\mathrm{Supp}(f)\cap\Gamma o \subset So$. Consequently, for every $n \in \mathbb{N}$, we have

$$a^\rho_{x,n}(f) = \frac{1}{\|A'_n(s_n)\|} \sum_{\gamma\in S} \theta(d(x,\gamma o))e^{-s_n d(x,\gamma o)} f(\gamma o)\rho(\gamma).$$

This finite sum is uniformly bounded whereas $\|A'_n(s_n)\|$ diverges to infinity (Lemma 5.8). Passing to the limit, we get $a^\rho_x(f) = 0$. □

LEMMA 5.14 (ρ_ω-equivariance). *Let $x \in X$ and $\gamma \in \Gamma$. For every $f \in C(\bar{X}_h)$ we have*

$$a^\rho_{\gamma x}(f) = \rho_\omega(\gamma)a^\rho_x(f \circ \gamma).$$

PROOF. Let $f \in C(\bar{X}_h)$. A direct computation shows that for every $n \in \mathbb{N}$, we have

$$a^\rho_{\gamma x,n}(f) = \rho_n(\gamma)a^\rho_{x,n}(f \circ \gamma).$$

The result follows by taking the ω-limit. □

Let $x, y \in X$. Recall that a point in the horoboundary $\partial_h X$ of X can be seen as a cocycle b. With this in mind, we define for $s \in \mathbb{R}_+$, a map $\chi^s_{x,y} \in C(\bar{X}_h)$. If $z \in X$, then

$$\chi^s_{x,y}(z) = \frac{\theta(d(x,z))}{\theta(d(y,z))} e^{-s[d(x,z)-d(y,z)]}.$$

If $b \in \partial_h X$, then

$$\chi^s_{x,y}(b) = e^{-sb(x,y)}.$$

The sequence $(\chi^{s_n}_{x,y})$ uniformly converges to $\chi^{h_\omega}_{x,y}$.

Lemma 5.15 (h_ω-conformality). *Let $x, y \in X$. For every $f \in C(\bar X_h)$, we have*

$$a_x^\rho(f) = a_y^\rho\left(\chi_{x,y}^{h_\omega} f\right).$$

Proof. A standard computation shows that for every $n \in \mathbb{N}$,

$$a_{x,n}^\rho(f) = a_{y,n}^\rho\left(\chi_{x,y}^{s_n} f\right).$$

Consequently

$$a_x^\rho f = \lim_\omega a_{y,n}^\rho\left(\chi_{x,y}^{s_n} f\right).$$

On the other hand, by the very definition of a_y^ρ we have

$$a_y^\rho\left(\chi_{x,y}^{h_\omega} f\right) = \lim_\omega a_{y,n}^\rho\left(\chi_{x,y}^{h_\omega} f\right).$$

Hence it suffices to prove that

$$\lim_\omega a_{y,n}^\rho\left(\left[\chi_{x,y}^{h_\omega} - \chi_{x,y}^{s_n}\right] f\right) = 0.$$

The norm of $a_{y,n}^\rho$, as a linear map form $C(\bar X_h)$ to $\mathcal{B}(\mathcal{H}_n)$, is uniformly bounded – see (42). In particular, there exists $M \in \mathbb{R}_+$ such that for every $n \in \mathbb{N}$,

$$\left\| a_{y,n}^\rho\left(\left[\chi_{x,y}^{h_\omega} - \chi_{x,y}^{s_n}\right] f\right)\right\| \leqslant M \left\| \left[\chi_{x,y}^{h_\omega} - \chi_{x,y}^{s_n}\right] f\right\|_\infty.$$

The result follows from the fact that $(\chi_{x,y}^{s_n})$ uniformly converges to $\chi_{x,y}^{h_\omega}$. □

As a corollary of the above lemmas, we get the following useful formula, for every $\gamma \in \Gamma$ and $f \in C(\bar X_h)$:

$$a_{\gamma o}^\rho(f) = \rho_\omega(\gamma) a_o^\rho(f \circ \gamma) = a_o^\rho(\chi_{\gamma o, o}^{h_\omega} f). \tag{44}$$

5.4. Twisted measure on the Gromov boundary. We study now how the family (a_x^ρ) – thought as a family of measures on $\partial_h X$ – behaves compared to usual Patterson-Sullivan measures ν_x. From a dynamical point of view, it is more appropriate to work in the Gromov boundary ∂X rather than in the horoboundary $\partial_h X$. Therefore, we push forward the family (a_x^ρ) by the natural Γ-equivariant continuous map $\pi\colon \bar X_h \to \bar X$. For every $x \in X$, we set

$$\begin{array}{cccc} \pi_* a_x^\rho\colon & C\left(\bar X\right) & \to & \mathcal{B}\left(\mathcal{H}_\omega\right) \\ & f & \mapsto & a_x^\rho(f \circ \pi). \end{array}$$

It follows from the previous study that $\pi_* a_o^\rho(\mathbf{1})$ has norm 1 (Lemma 5.11) and the support of $\pi_* a_x^\rho$ is contained in ∂X for every $x \in X$ (Lemma 5.13). Since $\pi\colon \bar X_h \to \bar X$ is Γ-equivariant, the family $(\pi_* a_x^\rho)$ is ρ_ω-equivariant (Lemma 5.14). Let us now focus on the conformality of $\pi_* a_x^\rho$ which is slightly more technical.

Lemma 5.16 (h_ω-quasi-conformality). *There exists $C \in \mathbb{R}_+^*$ with the following property. Let $x, y \in X$ and $\xi \in \partial X$. There is a neighbourhood $V_\xi \subset \bar X$ of ξ such that for every cocycle $b \in \pi^{-1}(\xi)$ and for every $f \in C(\bar X)$ whose support is contained in V_ξ we have*

$$\frac{1}{C}\pi_* a_x^\rho(f) \prec e^{-h_\omega b(x,y)}\pi_* a_y^\rho(f) \prec C \pi_* a_x^\rho(f).$$

Proof. Let $x, y \in X$. Let $\xi \in \partial X$. Using the hyperbolicity of X, we observe that there exists a neighbourhood $V_\xi \subset \bar X$ such that for every $b \in \pi^{-1}(\xi)$ the following holds: if z in a point in $V_\xi \cap X$ then

$$|[d(z,x) - d(z,y)] - b(x,y)| \leqslant 100\delta;$$

moreover if b' is a cocycle in $\pi^{-1}(V_\xi) \cap \partial_h X$, then $|b'(x,y) - b(x,y)| \leqslant 100\delta$. We now fix $b \in \pi^{-1}(\xi)$ and $f \in C(\bar X)$ whose support is contained in V_ξ. Let $\varepsilon > 0$. Since θ is a slowly increasing function, there exists $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$ and $u \geqslant 0$, we have $\theta(t+u) \leqslant e^{\varepsilon u}\theta(t)$. We fix a map $g\colon \bar X \to [0,1]$ whose support is contained in X and whose restriction to $B(x, t_0)$ and $B(y, t_0)$ is constant equal to 1. It follows from Lemma 5.13 that both $\pi_* a_x^\rho(gf)$ and $\pi_* a_y^\rho(gf)$ vanish. Consequently it suffices to compare $\pi_* a_x^\rho(f')$ and $\pi_* a_y^\rho(f')$ where $f' = (1-g)f$. Using the conformality of (a_x^ρ) we get

$$\pi_* a_x^\rho(f') = a_x^\rho(f' \circ \pi) = a_y^\rho\left(\chi_{x,y}^{h_\omega} f' \circ \pi\right). \tag{45}$$

It follows from our choice of t_0 and V_ξ that for every $z \in \bar{X}_h$ lying in the support of f' we have

$$\frac{1}{C(\varepsilon)} e^{-h_\omega b(x,y)} \leqslant \chi_{x,y}^{h_\omega}(z) \leqslant C(\varepsilon) e^{-h_\omega b(x,y)},$$

where $C(\varepsilon) = e^{100 h_\omega \delta} e^{\varepsilon d(x,y)}$. Since a_y^ρ is a positive linear map, (45) becomes

$$\frac{1}{C(\varepsilon)} \pi_* a_x^\rho(f') \prec e^{-h_\omega b(x,y)} a_y^\rho(f' \circ \pi) \prec C(\varepsilon) \pi_* a_x^\rho(f'),$$

hence

$$\frac{1}{C(\varepsilon)} \pi_* a_x^\rho(f) \prec e^{-h_\omega b(x,y)} \pi_* a_y^\rho(f) \prec C(\varepsilon) \pi_* a_x^\rho(f).$$

This inequality holds for every $\varepsilon \in \mathbb{R}_+^*$, consequently

$$\frac{1}{C} \pi_* a_x^\rho(f) \prec e^{-h_\omega b(x,y)} \pi_* a_y^\rho(f) \prec C \pi_* a_x^\rho(f),$$

where $C = e^{100\delta h_\omega}$ is a universal parameter. □

Remark. Note that if $h_\omega < h_\Gamma$, then the operator valued measures $(\pi_* a_x^\rho)$ cannot have bounded variation – see [**DUJ77**, Chapter 1] for a definition. Indeed otherwise their variations would be a Γ-invariant, h_ω-quasi-conformal family of measures on ∂X. Such measures do not exist unless $h_\omega \geqslant h_\Gamma$ [**Coo93**, Corollaire 6.6]. Later we will use a Radon-Nikodym derivative theorem for $\pi_* a_o^\rho$. This observation somehow tells us that all the theory exposed in [**DUJ77**] does not apply here unless $h_\omega = h_\Gamma$.

Shadow lemma.

LEMMA 5.17 (Half shadow lemma). *For every $r \in \mathbb{R}_+$, there exists $C \in \mathbb{R}_+$, with the following property. Let $\gamma \in \Gamma$ and $f \in C^+(\bar{X})$. If the support of f is contained in $\mathcal{O}_o(\gamma o, r)$, then*

$$\|\pi_* a_o^\rho(f)\| \leqslant C e^{-h_\omega d(o,\gamma o)} \|f\|_\infty.$$

PROOF. Combining Lemmas 5.14 and 5.15 we observe that

$$\rho_\omega(\gamma^{-1}) \pi_* a_o^\rho(f) = a_{\gamma^{-1}o}^\rho(f \circ \pi \circ \gamma) = a_o^\rho\left(\chi_{\gamma^{-1}o,o}^{h_\omega} f \circ \pi \circ \gamma\right).$$

For simplicity we set

$$f_\gamma = \chi_{\gamma^{-1}o,o}^{h_\omega} f \circ \pi \circ \gamma.$$

Let $\varepsilon > 0$. By Lemma 5.8, there exists $t_0 \in \mathbb{R}_+$ such that for $t \geqslant t_0$ and $u \geqslant 0$,

$$\theta(t+u) \leqslant e^{\varepsilon u} \theta(t).$$

We fix a continuous map $g\colon X \to [0,1]$, with compact support whose restriction to $B(o, t_0)$ is constant equal to 1. It allows to decompose f_γ as $f_\gamma = g f_\gamma + (1-g) f_\gamma$. Since the support of $g f_\gamma$ is contained in X we have $a_o^\rho(g f_\gamma) = 0$ (Lemma 5.13). Consequently

$$\rho_\omega(\gamma^{-1}) a_o^\rho(f) = a_o^\rho((1-g) f_\gamma).$$

Let us now consider a point $z \in \bar{X}_h$ in the support of $(1-g) f_\gamma$. By construction z belongs to $\pi^{-1}(\mathcal{O}_{\gamma^{-1}o}(o,r)) \setminus B(o, t_0)$. If $z = b \in \partial_h X$ is a cocycle, then

$$b(\gamma^{-1}o, o) \geqslant d(o, \gamma o) - 2r.$$

On the other hand, if $z \in X$, then

$$d\big(\gamma^{-1}o, z\big) - d(o,z) \geqslant d(o,\gamma o) - 2r.$$

In addition $d(o,\gamma o) \geqslant t_0$, thus according to our choice of t_0,

$$\theta\left(d\big(\gamma^{-1}o, z\big)\right) \leqslant \theta(d(o,\gamma o) + d(o,z)) \leqslant e^{\varepsilon d(o,\gamma o)}\theta(d(o,z)).$$

In both cases, we get

$$\chi^{h_\omega}_{\gamma^{-1}o,o}(z) \leqslant e^{2h_\omega r} e^{-(h_\omega - \varepsilon)d(o,\gamma o)}.$$

Hence

$$0 \leqslant (1-g)f_\gamma \leqslant e^{2h_\omega r} e^{-(h_\omega - \varepsilon)d(o,\gamma o)} \left\| f \right\|_\infty \mathbf{1}.$$

Since a^ρ_o is a positive map, we get

$$\rho_\omega(\gamma^{-1})\pi_* a^\rho_o(f) \prec e^{2h_\omega r} e^{-(h_\omega - \varepsilon)d(o,\gamma o)} \left\| f \right\|_\infty a^\rho_o(\mathbf{1}).$$

Recall that ρ_ω is a unitary representation. Taking the norm, we get

$$\|\pi_* a^\rho_o(f)\| \leqslant C e^{-(h_\omega - \varepsilon)d(o,\gamma o)} \left\| f \right\|_\infty,$$

where $C = e^{2h_\omega r}\|a^\rho_o(\mathbf{1})\|$. As it holds for all $\varepsilon > 0$, the result follows. □

5.5. Absolute continuity.

Radial limit set. Let K be a compact subset of X. Recall that the K-radial limit set Λ^K_{rad} is the set of all points $\xi \in \partial X$ for which there exists a geodesic ray $c\colon \mathbb{R}_+ \to X$ ending at ξ whose image $c(\mathbb{R}_+)$ intersects infinitely many copies γK of K. As explained before, we think of $\pi_* a^\rho_o$ as an operator valued measure on $\bar{X}$. The next step consists in proving that this "measure" gives full mass to Λ^K_{rad} for some compact subset K (Corollary 5.19). This is probably the *most crucial* point in the proof. Indeed, Shadow Lemmas Lemma 2.3 and Lemma 5.17 tell us that when $h_\omega = h_\Gamma$, the measures $\pi_* a^\rho_o$ and ν_o can be compared on shadows. As both measures give full measure to Λ^K_{rad} for closed ball $K = \bar{B}(o,r)$ with fixed $r > 0$, a Vitali type argument, approximating any Borel set by a union of shadows, allows to deduce that $\pi_* a^\rho_o$ is absolutely continuous with respect to ν_o (Proposition 5.22). Corollary 5.19 is the only place where we use in an essential way the fact that the action of Γ on X is strongly positively recurrent. All other arguments in the article work under a weaker assumption (e.g. if the geodesic flow is conservative).

The proof of the next statements follows exactly the same steps as the one of Corollary 3.16. It relies on the same auxiliary sets $\mathcal{L}_K$ and U^T_K defined in Section 3.3. However since it is the only place where we use (in a crucial way!) the existence of a growth gap at infinity to get our main theorem, we decided to detail it here.

Proposition 5.18. *Assume that $h^\infty_\Gamma < h_\omega$. There exists a compact subset K of X and numbers $\alpha, C, T_0 \in \mathbb{R}^*_+$ such that for every $T \geqslant T_0$, for every $f \in C^+(\bar{X})$ whose support is contained in U^T_K, we have*

$$\|\pi_* a^\rho_o(f)\| \leqslant C e^{-\alpha T} \left\| f \right\|_\infty.$$

Proof. By assumption, there exists a compact subset k of X containing o such that $h_{\Gamma_k} < h_\omega$. Let K be the 7δ-neighbourhood of k. By Lemma 3.14, there exists a finite subset S of Γ and a number $r \in \mathbb{R}_+$ such that for every $T \in \mathbb{R}_+$,

$$U^T_K \cap \Gamma o \subset \bigcup_{\substack{\beta \in S\Gamma_k \\ d(o,\beta o) \geqslant T - r}} \mathcal{O}_o(\beta o, r). \tag{46}$$

We fix $\varepsilon > 0$ such that $h_\omega - 2\varepsilon > h_{\Gamma_k}$. Let us fix $t_0 > 0$ such that for all $T \geqslant t_0$ and all $u \geqslant 0$, we have

$$\theta(t+u) \leqslant e^{\epsilon u}\theta(t).$$

Define F as

$$F = \{\gamma \in \Gamma \mid d(o, \gamma o) \leqslant t_0\}.$$

Let $T \geqslant t_0 + r$, and $f \in C^+(\bar{X}_h)$ be a non-negative function whose support is contained in U_K^T. Up to rescaling f, we can assume that $\|f\|_\infty = 1$. Let $n \in \mathbb{N}$. By (46),

$$a^\rho_{o,n}(f) \prec \frac{1}{\|A'_n(s_n)\|} \sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} \sum_{\substack{\gamma \in \Gamma \\ \gamma o \in \mathcal{O}_o(\beta o, r)}} \theta(d(o,\gamma o)) e^{-s_n d(o,\gamma o)} \rho_n(\gamma). \tag{47}$$

Let $\beta \in S\Gamma_k$ be such that $d(o, \beta o) \geqslant T - r$. As in the proof of Proposition 3.15, when $y \in \mathcal{O}_o(\beta o, r)$,

— if $d(\beta o, y) \geqslant t_0$, then $\theta(d(o,y)) \leqslant e^{\varepsilon d(o,\beta o)} \theta(d(\beta o, y))$, whereas
— if $d(o, \beta o) \geqslant t_0$, then $\theta(d(o,y)) \leqslant e^{\varepsilon d(o,\beta o)} \theta(t_0)$.

Consequently,

$$a^\rho_{o,n}(f) \prec \frac{1}{\|A'_n(s_n)\|} \sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} e^{2s_n r} e^{-(s_n - \varepsilon) d(o,\beta o)} \left(\Sigma_1 + \Sigma_2\right),$$

where

$$\Sigma_1 = \sum_{\substack{\gamma \in \Gamma \\ \gamma o \in \mathcal{O}_o(\beta o, r),\ d(\beta o, \gamma o) < t_0}} \theta(t_0) e^{-s_n d(\beta o, \gamma o)} \rho_n(\gamma),$$

$$\Sigma_2 = \sum_{\substack{\gamma \in \Gamma \\ \gamma o \in \mathcal{O}_o(\beta o, r),\ d(\beta o, \gamma o) \geqslant t_0}} \theta(d(\beta o, \gamma o)) e^{-s_n d(\beta o, \gamma o)} \rho_n(\gamma).$$

The number of terms in Σ_1 is at most $|F|$, so that $\|\Sigma_1\| \leq |F|\, \theta(t_0)$, whereas $\|\Sigma_2\|$ is bounded above by $\|A'_n(s_n)\|$. Combining all these inequalities we get

$$\left\| a^\rho_{o,n}(f) \right\| \leqslant e^{2s_n r} \left(\frac{|F|\, \theta(t_0)}{\|A'_n(s_n)\|} + 1 \right) \sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} e^{-(s_n - \varepsilon) d(o,\beta o)}.$$

After passing to the limit, it becomes

$$\|\pi_* a^\rho_o(f)\| \leqslant e^{2h_\omega r} \sum_{\substack{\beta \in S\Gamma_k, \\ d(o,\beta o) \geqslant T - r}} e^{-(h_\omega - \varepsilon) d(o,\beta o)}.$$

Since $h_\omega - 2\varepsilon > h_{\Gamma_k}$, we obtain as in (15)

$$\|\pi_* a^\rho_o(f)\| \leqslant B e^{2h_\omega r} e^{-(h_\omega - h_{\Gamma_k} - 2\varepsilon) T}.$$

Recall that B, k, r and ε do not depend on T or f, whence the result. □

Corollary 5.19. *Assume that $h^\infty_\Gamma < h_\omega$. There exists a compact subset K of X such that for every $\varepsilon > 0$, there is an open subset $V \subset \bar{X}$ containing $\partial X \setminus \Lambda^K_{\mathrm{rad}}$ with the following property. For every $f \in C^+(\bar{X})$ whose support is contained in V we have*

$$\|\pi_* a^\rho_o(f)\| \leqslant \varepsilon \|f\|_\infty.$$

Proof. According to Proposition 5.18 there exists a compact subset K of X as well as numbers $C, \alpha, T_0 \in \mathbb{R}^*_+$ such that for every $T \geqslant T_0$, for every $f \in C^+(\bar{X})$ whose support is contained in U^T_K,

$$\|\pi_* a^\rho_o(f)\| \leqslant C e^{-\alpha T} \|f\|_\infty.$$

We fix a summable function $w\colon \Gamma \to \mathbb{R}_+^*$ whose sum is 1. Let $\varepsilon > 0$. For every $\gamma \in \Gamma$, we fix $T_\gamma \geqslant T_0$ such that

$$Ce^{-\alpha T_\gamma} \leqslant \varepsilon w(\gamma)$$

and an open subset V_γ of $\bar{X}$ such that

$$\mathcal{L}_K \subset V_\gamma \subset U_K^{T_\gamma}.$$

According to Lemma 3.12, the set $\partial X \setminus \Lambda^K_{\mathrm{rad}}$ is contained in $\Gamma\mathcal{L}_K$. Hence the set

$$V = \bigcup_{\gamma\in\Gamma} V_\gamma$$

is an open neighbourhood of $\partial X \setminus \Lambda^K_{\mathrm{rad}}$. Let $f \in C^+(\bar{X})$ be a map whose support is contained in V. Without loss of generality we can assume that $\|f\|_\infty = 1$. As this support is compact, it is actually contained in

$$\bigcup_{\gamma\in S} V_\gamma,$$

where S is a finite subset of Γ. We fix a partition of unity, i.e. a family $(g_\gamma)_{\gamma\in S}$ of elements of $C^+(\bar{X})$ such that the support of g_γ is contained in V_γ, for every $\gamma \in S$ and

$$\sum_{\gamma\in S} g_\gamma$$

is constant equal to 1, when restricted to the support of f. Combining Proposition 5.18 with our choice of T_γ, we get

$$\|\pi_* a_o^\rho(f)\| \leqslant \sum_{\gamma\in S} \|\pi_* a_o^\rho(fg_\gamma)\| \leqslant \sum_{\gamma\in S} Ce^{-\alpha T_\gamma} \leqslant \varepsilon \sum_{\gamma\in\Gamma} w(\gamma) \leqslant \varepsilon. \qquad \square$$

A Vitali type argument. We now exploit the previous result to prove that whenever $h_\omega = h_\Gamma$ the "measure" $\pi_* a_o^\rho$ is absolutely continuous with respect to the usual Patterson-Sullivan measure ν_o. The first lemma is an easy exercise of hyperbolic geometry. Its proof is left to the reader.

LEMMA 5.20. *Let $r \in \mathbb{R}_+$. Let $x, y \in X$ such that $d(o,x) \leqslant d(o,y)$. If $\mathcal{O}_o(x,r)$ and $\mathcal{O}_o(y,r)$ have a non-empty intersection, then $\mathcal{O}_o(y,r)$ is contained in $\mathcal{O}_o(x, 3r+4\delta)$.*

The second lemma is a Vitali like Lemma.

LEMMA 5.21 (Vitali's Lemma). *Let K be a compact subset of X. There exists $r_1 \in \mathbb{R}_+^*$ such that for every $r \geqslant r_1$, for every $R \in \mathbb{R}_+$, there exists a subset S of Γ with the following properties.*

(1) For all $\alpha \in S$, $d(o,\alpha o) \geqslant R$.

(2) The union $\bigcup_{\alpha\in S} \mathcal{O}_o(\alpha o, 4r)$ covers Λ^K_{rad}.

(3) The shadows $(\mathcal{O}_o(\alpha o, r))_{\alpha\in S}$ are pairwise disjoint.

PROOF. Let $r_1 = \max\{\mathrm{diam}(K \cup \{o\}), 4\delta\}$. Let $r \geqslant r_1$ and $R \in \mathbb{R}_+$. For simplicity we set

$$U_R = \{\gamma \in \Gamma \mid d(o,\gamma o) \geqslant R\}.$$

We build the set S by induction, adding one element at each step. We start with $S_0 = \emptyset$. For every $n \in \mathbb{N}$, we define the set S_{n+1} by adding to S_n an element $\gamma \in U_R \setminus S_n$ such that $\mathcal{O}_o(\gamma o, r)$ is disjoint from all the previous shadows $(\mathcal{O}_o(\alpha o, r))_{\alpha\in S_n}$ and which minimizes $d(o,\gamma o)$. Standard elementary arguments using Lemma 5.20 show that the increasing union $S = \bigcup_n S_n$ satisfies the above statement. $\square$

PROPOSITION 5.22. *Assume that $h_\omega = h_\Gamma$. There exists $C \in \mathbb{R}_+^*$ such that for every $f \in C(\bar{X})$,*

$$\|\pi_* a_o^\rho(f)\| \leqslant C \int_{\partial X} |f| \, d\nu_o.$$

As already mentioned, this proposition is a direct consequence of Shadow lemmas. Indeed, the key Corollary 5.19 allows to approximate every Borel set by unions of shadows of fixed radius, through a Vitali type argument.

PROOF. Let K be the compact subset of $\bar{X}$ given by Corollary 5.19. Fix $r \geqslant \max\{r_0, r_1\}$ where r_0 and r_1 are respectively given by Lemmas 2.3 and 5.20. By Shadow Lemmas 2.3 and 5.17, there exists $C_0 \in \mathbb{R}_+^*$ such that for every $\gamma \in \Gamma$,

— $\nu_o\left(\mathcal{O}_o(\gamma o, r)\right) \geqslant \frac{1}{C_0} e^{-h_\Gamma d(o,\gamma o)}$

— for every $f \in C^+(\bar{X})$ whose support is contained in $\mathcal{O}_o(\gamma o, 4r)$ we have

$$\|\pi_* a_o^\rho(f)\| \leqslant C_0 e^{-h_\omega d(o,\gamma o)} \|f\|_\infty .$$

Let $f \in C(\bar{X})$. We first assume that f is non-negative. Let $\varepsilon > 0$. We fix some auxiliary subsets of X to decompose the map f into a sum of functions supported on appropriate small shadows. Since the action of Γ is strongly positively recurrent, $h_\Gamma^\infty < h_\Gamma = h_\omega$. According to Corollary 5.19 there exists an open set V containing $\partial X \setminus \Lambda_{\mathrm{rad}}^K$ such that for every $g \in C(\bar{X})$ whose support is contained in V, we have

$$\|\pi_* a_o^\rho(g)\| \leqslant \varepsilon \|g\|_\infty .$$

Since f is continuous, for all $\varepsilon > 0$, there exists $R > 0$ such that on any shadow $\mathcal{O}_o(y, 4r)$, with $d(o, y) \geq R$, the variations of f are bounded by ε. Let S be the collection of elements of Γ given by Vitali's Lemma 5.21. Since f is continuous, there exists a finite subset S_0 of S such that the support of f is contained in

$$\left(\bigcup_{\gamma \in S_0} \mathcal{O}_o(\gamma o, 2r) \right) \cup V.$$

We now fix a partition of unity, i.e. a collection $\{g\} \cup \{g_\gamma\}_{\gamma \in S_0}$ of continuous functions from $\bar{X}$ to $[0, 1]$ such that the support of g_γ (respectively g) is contained in $\mathcal{O}_o(\gamma o, 4r)$ (respectively V) and

$$g + \sum_{\gamma \in S_0} g_\gamma$$

is constant equal to 1 when restricted to the support of f. We now first estimate $\|\pi_* a_o^\rho(f)\|$ from above. The triangle inequality yields

$$\|\pi_* a_o^\rho(f)\| \leqslant \|\pi_* a_o^\rho(gf)\| + \sum_{\gamma \in S_0} \|\pi_* a_o^\rho(g_\gamma f)\| .$$

By Corollary 5.19, $\|\pi_* a_o^\rho(gf)\| \leqslant \varepsilon \|f\|_\infty$. For every $\gamma \in S_0$ we let

$$f_\gamma = \sup_{x \in \mathcal{O}_o(\gamma o, 2r)} f(x).$$

so that $\|g_\gamma f\|_\infty \leqslant f_\gamma$. It follows from the Half-Shadow Lemma 5.17 that

$$\sum_{\gamma \in S_0} \|\pi_* a_o^\rho(g_\gamma f)\| \leqslant C_0 \sum_{\gamma \in S_0} e^{-h_\omega d(o,\gamma o)} f_\gamma.$$

Consequently

$$\|\pi_* a_o^\rho(f)\| \leqslant \varepsilon \|f\|_\infty + C_0 \sum_{\gamma \in S_0} e^{-h_\omega d(o,\gamma o)} f_\gamma. \tag{48}$$

Let us now estimate $\nu_o(f)$ from below. Let $\gamma \in S_0$. Since $d(\gamma o, o) \geqslant R$, the map f restricted to $\mathcal{O}_o(\gamma o, 4r)$ varies by at most ε. On the other hand the shadows $(\mathcal{O}_o(\gamma o, r))_{\gamma \in S}$ are pairwise disjoint. We get from the standard Shadow Lemma

$$\begin{aligned}\int f d\nu_0 &\geqslant \sum_{\gamma \in S_0} \int_{O_o(\gamma o, r)} f d\nu_0 \geqslant \sum_{\gamma \in S_0} (f_\gamma - \varepsilon)\nu_o\left(\mathcal{O}_o(\gamma o, r)\right) \\ &\geqslant \frac{1}{C_0} \sum_{\gamma \in S_0} f_\gamma e^{-h_\Gamma d(o,\gamma o)} - \varepsilon.\end{aligned} \tag{49}$$

Recall that $h_\omega = h_\Gamma$. Hence combining (48) and (49) yields

$$\|\pi_* a_o^\rho(f)\| \leqslant \varepsilon \|f\|_\infty + C_0^2 \left(\int f d\nu_o + \varepsilon \right).$$

This inequality holds for every $\varepsilon > 0$, hence

$$\|\pi_* a_o^\rho(f)\| \leqslant C_0^2 \int f d\nu_o.$$

If f is not nonnegative anymore, decomposing f into its positive and negative part leads immediately to the result. $\square$

Corollary 5.23. *Assume that $h_\omega = h_\Gamma$. There exists a unique continuous linear map $D\colon \mathcal{H}_\omega \to L^\infty((\partial X, \nu_0), \mathcal{H}_\omega)$ such that for every $\phi \in \mathcal{H}_\omega$, for every $f \in C(\bar{X})$, we have*

$$\pi_* a_o^\rho(f)\phi = \int f D(\phi)\; d\nu_o.$$

Remark. The integral in the statement is an integral in the sense of Bochner (Section 1.1). The map D can be thought as a kind of Radon-Nikodym derivative of $\pi_* a_o^\rho$ with respect to ν_o.

Proof. Let C be the constant given by Proposition 5.22. Let $\phi \in \Phi$. It follows from Proposition 5.22 that for every $f \in C(\bar{X})$ we have

$$\|\pi_* a_o^\rho(f)\phi\| \leqslant C \int |f|\, d\nu_0\, \|\phi\|.$$

Thus the map sending $f \in C(\bar{X})$ to $\pi_* a_o^\rho(f)\phi$ extends to a continuous map $L^1(\partial X, \nu_o) \to \mathcal{H}_\omega$, whose norm is at most $C\|\phi\|$. As a Hilbert space, $\mathcal{H}_\omega$ is reflexive, hence satisfies the Radon-Nikodym property (Theorem 1.4). Consequently there exists a vector $D(\phi) \in L^\infty((\partial X, \nu_0), \mathcal{H}_\omega)$, whose norm is at most $C\|\phi\|$ such that for every $f \in C(\bar{X})$ we have

$$\pi_* a_o^\rho(f)\phi = \int f D(\phi)\; d\nu_o.$$

This construction defines a map $D\colon \mathcal{H}_\omega \to L^\infty((\partial X, \nu_o), \mathcal{H}_\omega)$. Uniqueness and linearity of D follow from Proposition 1.1. By construction, $\|D(\phi)\|_\infty \leqslant C\|\phi\|$, for every $\phi \in \mathcal{H}_\omega$. Hence D is continuous. $\square$

5.6. Invariant vectors. From now on we assume that $h_\omega = h_\Gamma$. The goal is now to study the map $D\colon \mathcal{H}_\omega \to L^\infty((\partial X, \nu_0), \mathcal{H}_\omega)$ given by Corollary 5.23.

Heuristically the idea is the following. Using the ergodicity of the action of Γ on $(\partial^2 X, \mu)$ we are going to prove that $D(\phi)$ is almost surely constant, so that viewed as a measure with values in $\mathcal{B}(\mathcal{H}_\omega)$, the twisted Patterson-Sullivan measure $\pi_* a_o^\rho$ satisfies

$$\pi_* a_o^\rho(f)\phi = D(\phi) \int f d\nu_o, \quad \forall f \in C(\bar{X}).$$

Comparing the invariance of ν_o and $\pi_* a^\rho_o$, we will observe that $D(\phi)$ is a ρ_ω-invariant vector, that is a limit of ρ_n almost-invariant vectors. Below is a rigorous exposition of this strategy.

Fix $\phi \in \mathcal{H}^+_\omega$. For simplicity, set $\Psi = D(\phi)$. Recall that Ψ is a bounded map from $\bar{X}$ to $\mathcal{H}_\omega$. Actually it directly follows from Lemma 5.13 that the support of Ψ is contained in ∂X. Since ϕ is positive, Ψ takes its values in $\mathcal{H}^+_\omega$ (Lemma 2.7).

LEMMA 5.24. *There exists $C \in \mathbb{R}^*_+$, which does not depend on ϕ, such that for every $\gamma \in \Gamma$, we have*

$$\frac{1}{C}\Psi \prec \rho_\omega(\gamma)\Psi \circ \gamma^{-1} \prec C\Psi.$$

Remark. Comparing pointwise two functions defines an order which endows $L^\infty((\partial X, \nu_o), \mathcal{H}_\omega)$ with a lattice structure (Lemma 2.6). The inequalities in the lemma are meant in $L^\infty((\partial X, \nu_o), \mathcal{H}_\omega)$.

PROOF. Recall that $\pi\colon \partial_h X \to \partial X$ is a surjective continuous map between compact sets. Hence, we can fix a measurable section of π

$$\begin{array}{cccc} \sigma\colon & \partial X & \to & \partial_h X \\ & \xi & \mapsto & b_\xi. \end{array}$$

see for instance [**Bou74**, Chapitre IX, §6.9, Corollaire 1]. Since (ν_x) is h_Γ-quasi-conformal, there exists $C_0 \in \mathbb{R}^*_+$ such that for every $\gamma \in \Gamma$, for ν_o-almost every $\xi \in \partial X$, we have

$$\frac{1}{C_0} e^{-h_\Gamma b_\xi(\gamma o, o)} \leqslant \frac{d\gamma_*\nu_o}{d\nu_o}(\xi) \leqslant C_0 e^{-h_\Gamma b_\xi(\gamma o, o)} \tag{50}$$

We denote by $C_1 \in \mathbb{R}^*_+$ the universal constant given by the h_ω-quasi-conformality of $(\pi_* a^\rho_x)$ (Lemma 5.16). Let $\gamma \in \Gamma$. We are going to work with the points $x = \gamma o$ and $y = o$. For every $\xi \in \partial X$, we write V_ξ for the neighbourhood of ξ given by Lemma 5.16. Up to decreasing V_ξ we can always assume that for any $b, b' \in \pi^{-1}(V_\xi) \cap \partial_h X$,

$$|b(x,y) - b'(x,y)| \leqslant 100\delta.$$

Let $f \in C(\bar{X})$. Since the support of f is compact, there exists a finite subset F of ∂X such that this support is contained in

$$\left(\bigcup_{\eta \in F} V_\eta \right) \cup X.$$

We now fix a partition of unity, i.e. a collection of continuous maps $g\colon \bar{X} \to [0,1]$ and $g_\eta\colon \bar{X} \to [0,1]$ (one for each $\eta \in F$) such that the support of g (respectively g_η) is contained in X (respectively V_η) and the sum

$$g + \sum_{\eta \in F} g_\eta$$

equals 1 when restricted to the support of f. Since the support of gf is contained in X, we have $\pi_* a^\rho_{\gamma o}(gf) = 0$. Hence the ρ_ω-equivariance (Lemma 5.14) of $(\pi_* a^\rho_x)$ yields

$$\rho_\omega(\gamma) \int (f \circ \gamma)\Psi d\nu_o = \rho_\omega(\gamma)\pi_* a^\rho_o(f \circ \gamma)\phi = \pi_* a^\rho_{\gamma o}(f)\phi = \sum_{\eta \in F} \pi_* a^\rho_{\gamma o}(g_\eta f)\phi$$

Combined with the h_ω-quasi-conformality (Lemma 5.16) of $(\pi_* a^\rho_x)$,we get

$$\rho_\omega(\gamma) \int (f \circ \gamma)\Psi d\nu_o \prec C_1 \sum_{\eta \in F} e^{-h_\omega b_\eta(\gamma o, o)} \pi_* a^\rho_o(g_\eta f)\phi.$$

This inequality can be written using the definition of Ψ as

$$\rho_\omega(\gamma)\int (f\circ\gamma)\Psi d\nu_o \prec C_1\left(\sum_{\eta\in F} e^{-h_\omega b_\eta(o,\gamma o)}\int g_\eta f\Psi d\nu_o\right)$$

Recall now first that the support of ν_o is contained in ∂X, second that for every ξ in the support of g_η the quantities $b_\xi(\gamma o, o)$ and $b_\eta(\gamma o, o)$ differ by at most 100δ. Consequently Lemma 2.7 gives

$$\begin{aligned}\rho_\omega(\gamma)\int (f\circ\gamma)\Psi d\nu_o &\prec C_1 e^{100h_\omega\delta}\left(\sum_{\eta\in F}\int g_\eta(\xi)f(\xi)\Psi(\xi)e^{-h_\omega b_\xi(\gamma o,o)}d\nu_o(\xi)\right)\\ &\prec C_1 e^{100h_\omega\delta}\int f(\xi)\Psi(\xi)e^{-h_\omega b_\xi(\gamma o,o)}d\nu_o(\xi)\end{aligned}$$

Recall that $h_\omega = h_\Gamma$. Hence the invariance and quasi-conformality of (ν_x) yields

$$\rho_\omega(\gamma)\int (f\circ\gamma)\Psi d\nu_o \prec C_0C_1e^{100h_\omega\delta}\int (f\circ\gamma)(\Psi\circ\gamma)d\nu_o$$

Note that this inequality holds for every $f\in C(\bar X)$, hence $\rho_\omega(\gamma)\Psi \prec C(\Psi\circ\gamma)$ where $C = C_0C_1e^{100h_\omega\delta}$ is a universal constant (Proposition 2.8). The other inequality follows by symmetry. □

If ∂X and $\partial_h X$ coincide, all the Patterson-Sullivan measures are Γ-equivariant and conformal (not just quasi-conformal). Hence our argument proves that for every $\gamma\in\Gamma$, we have

$$\rho_\omega(\gamma)\Psi\circ\gamma^{-1} = \Psi.$$

When the two boundaries differ we do not have quite equality. To deal with this problem, we proceed as follows. Since ν_o is a finite measure, Ψ is also an element of $L^2((\partial X,\nu_o),\mathcal{H}_\omega)$ which has a natural structure of Hilbert space. We endow this space with an action of Γ defined as follows:

$$\gamma\cdot\Phi = \rho_\omega(\gamma)\Phi\circ\gamma^{-1},\quad \forall\gamma\in\Gamma,\ \forall\Phi\in L^2((\partial X,\nu_o),\mathcal{H}_\omega).$$

We denote by Ψ' the projection of the zero function on the convex hull of $\Gamma\cdot\Psi$ in $L^2((\partial X,\nu_o),\mathcal{H}_\omega)$. The projection on a closed convex subset of a Hilbert space is unique, hence

$$\rho_\omega(\gamma)\Psi'\circ\gamma^{-1} = \gamma\cdot\Psi' = \Psi',\quad \forall\gamma\in\Gamma. \tag{51}$$

Note that the set $\Gamma\cdot\Psi$ is uniformly bounded $L^\infty((\partial X,\nu_0),\mathcal{H}_\omega)$. Hence Ψ' is essentially bounded as well, that is $\Psi'\in L^\infty((\partial X,\nu_0),\mathcal{H}_\omega)$. According to Lemma 5.24, there exists $C>0$ such that for every $\gamma\in\Gamma$, we have

$$\frac{1}{C}\Psi \prec \gamma\cdot\Psi \prec C\Psi.$$

Consequently

$$\frac{1}{C}\Psi \prec \Psi' \prec C\Psi.$$

In particular, Ψ' is non-zero.

Lemma 5.25. *The function $\Psi'\in L^\infty((\partial X,\nu_o),\mathcal{H}_\omega)$ is constant ν_o-almost everywhere.*

Proof. According to (51) for every $\gamma\in\Gamma$, for ν_o-almost every $\eta,\xi\in\partial X$, we have

$$(\Psi'(\gamma\eta),\Psi'(\gamma\xi)) = (\rho_\omega(\gamma)\Psi'(\eta),\rho_\omega(\gamma)\Psi'(\xi)) = (\Psi'(\eta),\Psi'(\xi)).$$

It exactly means that the map

$$\begin{array}{rccc} Q\colon & (\partial X \times \partial X, \nu_o \otimes \nu_o) & \to & \mathbb{R}_+ \\ & (\eta, \xi) & \to & (\Psi'(\eta), \Psi'(\xi)) \end{array}$$

is Γ-invariant. Recall now that by Theorem 4.1, the action of Γ on the space $(\partial X \times \partial X, \nu_o \otimes \nu_o)$ is ergodic. The map Q is hence constant $\nu_o \otimes \nu_o$-almost everywhere. We write $m \in \mathbb{R}$ for this value. Observe now that for every $f_1, f_2 \in L^1(\nu_o)$ we have

$$\left(\int f_1 \Psi' d\nu_o, \int f_2 \Psi' d\nu_o\right) = m \left(\int f_1 d\nu_o\right)\left(\int f_2 d\nu_o\right). \tag{52}$$

A standard argument using the equality case of the Cauchy-Schwarz inequality shows that there exists $\psi'_0 \in \mathcal{H}_\omega$ such that for every $f \in L^1_+(\nu_o)$ we have

$$\int_{\bar{X}} f \Psi' d\nu_o = \sqrt{m}\left[\int_{\bar{X}} f d\nu_o\right] \psi'_0.$$

Consequently Ψ' is ν_0-almost surely constant, equal to $\sqrt{m}\psi'_0$ (Proposition 1.1). □

Lemma 5.26. *The unique essential value of Ψ' is a ρ_ω-invariant vector of $\mathcal{H}_\omega$.*

Proof. As we proved in Lemma 5.25, Ψ' is constant ν_o-almost surely. To avoid ambiguity we write $\psi' \in \mathcal{H}_\omega$ for its value. Recall that for every $\gamma \in \Gamma$ we have $\rho_\omega(\gamma)\Psi' \circ \gamma^{-1} = \Psi'$, see (51). Replacing Ψ' by its value exactly says that ψ' is ρ_ω-invariant. □

Remark. If the horoboundary $\partial_h X$ coincides with the Gromov boundary ∂X, our arguments prove that for every vector $\phi \in \mathcal{H}_\omega$, there exists a ρ_ω-invariant vector $\psi \in \mathcal{H}^+_\omega$ such that for every $f \in C(\bar{X})$, we have

$$a^\rho_o(f)\phi = \left(\int f d\nu_o\right)\psi.$$

Next proposition summarizes the results of this section.

Proposition 5.27. *If $h_\omega = h_\Gamma$, then the representation ρ_ω has non-zero invariant vectors.*

Proof. The operator $\pi_* a^\rho_o(\mathbf{1})$ has norm 1 (Lemma 5.11). Hence there exists a vector $\phi \in \mathcal{H}^+_\omega$ such that $\pi_* a^\rho_o(\mathbf{1})\phi$ is not zero. To such a vector we associate a bounded function $\Psi\colon \partial X \to \mathcal{H}^+_\omega$ such that for every $f \in C(\bar{X})$

$$\pi_* a^\rho_o(f)\phi = \int f \Psi d\nu_o.$$

In particular, Ψ is a non-zero function. We proved that the map $\Psi'\colon \partial X \to \mathcal{H}^+_\omega$ defined before Lemma 5.25 is constant and its value ψ' is ρ_ω-invariant (Lemma 5.25). Moreover there exists $C > 0$, which does not depend on ϕ, such that $(1/C)\Psi \prec \Psi' \prec C\Psi$ (Lemma 5.24). It follows from this inequality that ψ' is non-zero. Indeed otherwise Ψ' and thus Ψ would be zero as well. □

We complete this section with the proof of Theorem 5.2.

Proof of Theorem 5.2. The proof proceeds by contradiction. Let S be a finite subset of Γ and $\varepsilon \in \mathbb{R}^*_+$. Assume that the theorem is false. For each $n \in \mathbb{N}$, we can find a Hilbert lattice $\mathcal{H}_n$ and a positive representation $\rho_n\colon \Gamma \to \mathcal{U}(\mathcal{H}_n)$ with the following properties.

(1) (h_{ρ_n}) converges to h_Γ.
(2) For every $n \in \mathbb{N}$, the representation does not have any (S, ε)-invariant vector.

Let ω be a non-principal ultra-filter. We let $\mathcal{H}_\omega = \lim_\omega \mathcal{H}_n$ and denote by $\rho_\omega \colon \Gamma \to \mathcal{U}(\mathcal{H}_\omega)$ the limit representation induced by (ρ_n). Observe that we are exactly in the setting of Section 5.3. Moreover

$$h_\omega = \lim_\omega h_{\rho_n} = h_\Gamma.$$

It follows from Proposition 5.27 that ρ_ω admits an invariant unit vector ψ that we can write $\psi = \lim_\omega \psi_n$, where ψ_n is a unit vector in $\mathcal{H}_n$. By definition of the representation ρ_ω, for all $\gamma \in \Gamma$, we have

$$\lim_\omega \|\rho_n(\gamma)\psi_n - \psi_n\| = 0.$$

Since S is finite, it forces

$$\sup_{\gamma \in S} \|\rho_n(\gamma)\psi_n - \psi_n\| < \varepsilon, \ \omega\text{-as.}$$

Hence ψ_n is an (S,ε)-invariant vector of ρ_n ω-as, which contradicts the definition of ρ_n. □

Remark 5.28. Let us complete Remark 5.12 regarding the topology of the space of "operator valued measures". We assume here that Γ is a group whose abelianization is infinite. Consider the derived subgroup $\Gamma' = [\Gamma, \Gamma]$ and let $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ be the unitary representation in $\mathcal{H} = \ell^2(\Gamma/\Gamma')$ associated to the action of Γ on its abelianization. Take for (ρ_n) a constant sequence equal to ρ. Since Γ/Γ' is amenable, ρ almost has invariant vectors and thus $h_\rho = h_\Gamma$ (see Corollary 5.3).

We claim that if we had worked in the space of linear maps $C(\bar{X}_h) \to \mathcal{B}(\mathcal{H})$ *endowed with the weak-* topology*, then any accumulation point of $a^\rho_{o,n}$ is such that $a^\rho_o(\mathbf{1}) = 0$. Indeed, besides Lemma 5.11 all the rest of our proof should work verbatim. In particular, we would get that the image of $\pi_* a^\rho_o(\mathbf{1})$ is contained in the subspace of ρ-invariant vectors in $\mathcal{H}$. Hence if $a^\rho_o(\mathbf{1})$ is not the zero operator, then ρ admits an non-zero invariant vector, and thus Γ/Γ' is finite. A contradiction.

6. Applications to group theory

Let X be a hyperbolic proper geodesic space. Let Γ be a group acting by isometries on X. Let $\mathcal{H}$ be a Hilbert space and $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Let S be a finite subset of Γ and $\varepsilon > 0$. Recall that an (S,ε)-invariant vector is a vector $\phi \in \mathcal{H}$ such that

$$\sup_{\gamma \in S} \|\rho(\gamma)\phi - \phi\| < \varepsilon \|\phi\|.$$

Moreover, the representation ρ almost admits invariant vectors if for every finite subset S of Γ for every $\varepsilon > 0$, it has an (S,ε)-invariant vector. We now investigate the consequences of Theorem 5.2 by varying the representations of Γ.

Our main source of applications deals with the growth of subgroups of Γ. Let Γ' be a subgroup of Γ. We denote by Y the space of left cosets $Y = \Gamma' \backslash \Gamma$ on which Γ acts on the right. Let $\mathcal{H} = \ell^2(Y)$ be the space of square summable maps $Y \to \mathbb{R}$ endowed with its usual Hilbert structure and order (Section 2.2.1) We denote by $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$ the corresponding Koopman representation. Recall that h_ρ is the critical exponent of the operator series

$$A(s) = \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)} \rho(\gamma),$$

whereas h_Γ is the exponential growth rate of Γ (for its action on X).

Lemma 6.1. *The critical exponents h_ρ and $h_{\Gamma'}$ satisfy $h_{\Gamma'} \leqslant h_\rho$.*

PROOF. Let $s > h_\rho$. We write y_0 for the point of Y corresponding to the coset Γ' and $\psi \in \ell^2(Y)$ for the Dirac mass at y_0. Note that $\rho(\gamma)\psi = \psi$, for every $\gamma \in \Gamma'$. Hence

$$\mathcal{P}_{\Gamma'}(s)\psi = \sum_{\gamma \in \Gamma'} e^{-sd(\gamma o, o)} \rho(\gamma)\psi \prec \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)} \rho(\gamma)\psi = A(s)\psi.$$

Consequently $\mathcal{P}_{\Gamma'}(s)$ converges. This statement holds for every $s > h_\rho$, hence the result $h_\rho \geqslant h_{\Gamma'}$. □

Remark. In the next sections we explore various properties of groups defined in terms of unitary representations. These properties make sense for locally compact groups. However we restrict ourselves to discrete groups as they are the only ones that we consider in this article.

6.1. Amenability.

Amenability. There are numerous equivalent definitions of amenability. The most suitable for our purpose can be formulated in terms of the regular representation.

DEFINITION 6.2. The action of a discrete group Γ on a set Y is *amenable* if and only if the Koopman representation $\rho\colon \Gamma \to \mathcal{U}(\ell^2(Y))$ associated to the action of Γ on Y almost admits invariant vectors. A subgroup Γ' of Γ is *co-amenable in* Γ if the action of Γ on $Y = \Gamma' \backslash \Gamma$ is amenable.

The action of Γ on Y is amenable if and only if one of the following equivalent facts holds.

- (Invariant mean) There exists a Γ-invariant positive mean on the set $\ell^\infty(Y)$.
- (Følner sets) For every finite subset S of Γ, for every $\varepsilon > 0$, there exists a finite subset Y_0 of Y such that

$$\sup_{\gamma \in S} \frac{|\gamma Y_0 \Delta Y_0|}{|Y_0|} \leqslant \varepsilon.$$

- (Reiter's criterion) For every finite subset S of Γ, for every $\varepsilon > 0$ there exists a non-zero map $L^1_+(Y)$ such that

$$\sup_{\gamma \in S} \|f \circ \gamma - f\| \leqslant \varepsilon \|f\|$$

The proof for amenable actions works verbatim as for amenable groups, see for instance [**BdlHV08**, Appendix G] or [**Jus15**]. Another reference for amenable action is [**Eym72**]. We can now prove our main theorem, that we recall.

THEOREM 6.3. *Let (X, d) be a hyperbolic proper geodesic space. Let Γ be a group acting properly by isometries on X and Γ' a subgroup of Γ. Assume that the action of Γ is strongly positively recurrent. The following are equivalent.*

(1) $h_{\Gamma'} = h_\Gamma$

(2) The subgroup Γ' is co-amenable in Γ.

From critical exponent to amenability. We start with the proof of the implication (1) ⇒ (2). Assume that $h_{\Gamma'} = h_\Gamma$. Since $h_{\Gamma'} \leqslant h_\rho \leqslant h_\Gamma$ (Lemmas 5.1 and 6.1) we have $h_\rho = h_\Gamma$. It follows from Corollary 5.3 that ρ almost has invariant vectors, which exactly means that Γ' is co-amenable in Γ.

From amenability to critical exponents. We now focus on the so called "easy direction", i.e. (2) $\Rightarrow$ (1). As explained in the introduction, if Γ' is a *normal* subgroup of Γ, then Roblin's proof for CAT(-1) spaces [**Rob05**] directly extends to our setting. However if Γ' is no more a normal subgroup, we are not aware of any existing proof in the literature that would work in the general context of Gromov hyperbolic spaces. We expose here a strategy based on the approach of Coulon-Dal'bo-Sambusetti [**CDS17**] revisited through ideas of Roblin-Tapie [**RT13**].

Let Γ' be a subgroup of Γ. We denote by $Y = \Gamma' \backslash \Gamma$ the space of left cosets of Γ'. The strategy is to estimate in terms of $h_{\Gamma'}$ the spectral radius of a certain random walk on the space Y. When Γ' is co-amenable in Γ, Kesten's amenability criterion tells us that any random walk on Y has spectral radius 1, which leads to the expected relation between $h_{\Gamma'}$ and h_Γ.

We begin with general considerations on random walks. Let $\mathcal{F}(Y, \mathbb{C})$ be the set of *all* maps from Y to $\mathbb{C}$ and $\mathcal{H} = \ell^2(Y)$ the subset consisting of all square summable functions with its canonical Hilbert space structure. The group Γ acts on the right on Y inducing a left action of Γ on $\mathcal{F}(Y, \mathbb{C})$ as follows. For every $\phi \in \mathcal{F}(Y, \mathbb{C})$, for every $\gamma \in \Gamma$,

$$[\gamma \cdot \phi](y) = \phi(y\gamma), \quad \forall y \in Y.$$

Restricted to $\mathcal{H}$, this action defines a unitary representation $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$. Let $p\colon \Gamma \to [0,1]$ be a *symmetric* probability measure on Γ with *finite* support. The convolution by p defines an operator M on $\mathcal{F}(Y, \mathbb{C})$ given by

$$M\phi = \phi * p = \sum_{\gamma \in \Gamma} p(\gamma) \left[\gamma^{-1} \cdot \phi\right]. \tag{53}$$

Its restriction to $\mathcal{H}$, still denoted by M, is the *Markov operator* of the random walk on Y associated to p. Seen as an operator of $\mathcal{H}$, the spectral radius $\tau(M)$ of M is at most 1. The "easy direction" of Kesten's amenability criterion tells us that if Γ' is co-amenable in Γ then $\tau(M) = 1$. Our goal is to relate $\tau(M)$ to the critical exponents of Γ' and Γ. To that end we use a discrete version of Barta's inequality [**Bar37**] exposed in the next two statements.

Lemma 6.4. *Let $u, \phi\colon \Gamma \to \mathbb{R}_+$ be two non negative maps. Then*

$$(M(u\phi), u\phi) \leqslant \left(u^2, \phi M \phi\right).$$

Remark. We do not assume that u or ϕ are square summable. In particular, we allow the above scalar products to be infinite.

Proof. Assume first that both u and ϕ have finite support, so that all objects in the following computations are well-defined. Observe that

$$(M(u\phi), u\phi) - \left(u^2, \phi M\phi\right) = \sum_{y \in Y} \sum_{\gamma \in \Gamma} \left[u\left(y\gamma^{-1}\right) - u(y)\right] u(y)\phi\left(y\gamma^{-1}\right)\phi(y)p(\gamma).$$

Recall that p is symmetric. Reindexing the double sum provides another way to write this difference, namely

$$\begin{aligned}(M(u\phi), u\phi) &- \left(u^2, \phi M\phi\right) \\ &= \sum_{y \in Y} \sum_{\gamma \in \Gamma} \left[u(y) - u\left(y\gamma^{-1}\right)\right] u\left(y\gamma^{-1}\right)\phi\left(y\gamma^{-1}\right)\phi(y)p(\gamma).\end{aligned}$$

Averaging these two expressions yields

$$(M(u\phi), u\phi) - \left(u^2, \phi M\phi\right) = -\frac{1}{2} \sum_{y \in Y} \sum_{\gamma \in \Gamma} \left[u\left(y\gamma^{-1}\right) - u(y)\right]^2 \phi\left(y\gamma^{-1}\right)\phi(y)p(\gamma).$$

Hence

$$\left(M(u\phi), u\phi\right) \leqslant \left(u^2, \phi M\phi\right).$$

If u and ϕ are any non-negative maps, we approximate them by functions supported on larger and larger finite subsets of Y. The conclusion then follows from the monotone convergence theorem. □

PROPOSITION 6.5 (Barta's inequality). *Let $\lambda \in [0,1]$. If there exists a positive function $\phi\colon Y \to \mathbb{R}_+^*$ such that $M\phi \leqslant \lambda\phi$, then $\tau(M) \leqslant \lambda$, where $\tau(M)$ is the spectral radius of M seen as an operator on $\mathcal{H}$.*

Remark. We think of ϕ as a kind of λ super-harmonic function for M. The strength of this statement is that it provides an estimate of $\tau(M)$ without assuming that ϕ is square summable.

PROOF. Recall that $\mathcal{H}^+$ stands for the functions in $\mathcal{H}$ taking values in $\mathbb{R}_+$ Since p is symmetric, M is a self-adjoint positive operator of $\mathcal{H}$. Hence its spectral radius can be computed as follows

$$\tau(M) = \sup_{\psi \in \mathcal{H}^+ \setminus \{0\}} \frac{(M\psi, \psi)}{\|\psi\|^2}.$$

Let $\psi \in \mathcal{H}^+$. Since ϕ is positive we can always write $\psi = u\phi$ where $u\colon Y \to \mathbb{R}_+$ is a non-negative function. It follows from Lemma 6.4 that

$$(M\psi, \psi) = (M(u\phi), u\phi) \leqslant \left(u^2, \phi M\phi\right) \leqslant \lambda \left(u^2, \phi^2\right) = \lambda \left\|\psi\right\|^2$$

This inequality holds for every $\psi \in \mathcal{H}^+$, hence the result. □

We now exploit the previous proposition to estimate the spectral radius of M. To that end we fix a base point $o \in X$ and a Γ'-invariant, $h_{\Gamma'}$-quasi-conformal family of measures (ν'_x) on ∂X. In addition we choose a measurable section $\partial X \to \partial_h X$, sending ξ to b_ξ, see [Bou74, Chapitre IX, §6.9, Corollaire 1]. We define a function $\phi\colon \Gamma \to \mathbb{R}_+^*$ sending γ to the total mass of $\nu'_{\gamma o}$, i.e.

$$\phi(\gamma) = \int \mathbf{1} d\nu'_{\gamma o}.$$

Since the family (ν'_x) is Γ'-invariant, ϕ induces a map $Y \to \mathbb{R}_+^*$ that we still denote by ϕ. This function will play the role of the function ϕ in Proposition 6.5. To that end we need to compute $M\phi$. Since (ν'_x) is $h_{\Gamma'}$-quasi-conformal, there exists a constant $C_1 \in \mathbb{R}_+$ such that for every point $y = \Gamma'\beta$ of Y, we have

$$[M\phi](y) \leqslant C_1 \int B(\beta^{-1}\xi) d\nu'_{\beta o}(\xi) \tag{54}$$

where $B\colon \partial X \to \mathbb{R}_+$ is defined by

$$B(\xi) = \sum_{\gamma \in \Gamma} e^{-h_{\Gamma'} b_\xi(\gamma o, o)} p(\gamma). \tag{55}$$

Consequently, to estimate $M\phi$ and thus $\tau(M)$, it suffices to bound $B(\xi)$ uniformly from above.

Until now we worked with an arbitrary symmetric probability measure p. In order to estimate the map $B\colon \partial X \to \mathbb{R}_+$ defined above we now specialize to a specific measure. Basically we are going to consider measures supported by "spheres" of large radius. Before doing so we make a small digression in order to study the growth of spheres. Let $r, a \in \mathbb{R}_+$ and $x \in X$. We denote by

$$S_\Gamma(x, r, a) = \{\gamma \in \Gamma \mid r - a < d(\gamma o, x) \leqslant r\}$$

the "sphere" of radius r (and thickness a) centred at x. Similarly we define the "ball" of radius r centred at x by

$$B_\Gamma(x,r) = \{\gamma \in \Gamma \mid d(\gamma o, x) \leqslant r\}.$$

Since the action of Γ on X is proper, these sets are finite. Since the usual Patterson-Sullivan measure associated to the ambient group Γ gives full measure to the radial limit set (Corollary 3.16), there exists $C_2 \in \mathbb{R}_+$, such that for every $r \in \mathbb{R}_+$

$$|B_\Gamma(o,r)| \leqslant C_2 e^{rh_\Gamma}, \tag{56}$$

see for instance [**Coo93**, Corollaire 6.8]. The next statement precise these estimates in the presence of a growth gap at infinity.

Lemma 6.6 (Yang [**Yan16**, Theorem 5.3]). *Let Γ be a discrete group acting properly by isometries on a Gromov-hyperbolic space X. Assume that the action is strongly positively recurrent, i.e. there exists a growth gap at infinity. There exists $a, C_3 \in \mathbb{R}_+$, such that for every $r \in \mathbb{R}_+$, we have*

$$\frac{1}{C_3} e^{rh_\Gamma} \leqslant |S_\Gamma(o,r,a)| \leqslant C_3 e^{rh_\Gamma}.$$

The previous lemma provides an estimate for the cardinality of any ball centred at a point in the Γ-orbit of o. The goal of the next proposition is to provide a similar estimate for balls centred at any point $x \in X$.

Proposition 6.7. *Let Γ be a discrete group acting properly by isometries on a Gromov-hyperbolic space. Assume that the action is strongly positively recurrent. For all $\varepsilon \in \mathbb{R}_+^*$, there exists $C_4(\varepsilon) \in \mathbb{R}_+^*$, such that for all $x \in X$, we have*

$$|B_\Gamma(x,r)| \leqslant C_4(\varepsilon) e^{(2h_\Gamma^\infty + \varepsilon - h_\Gamma) d(x,\Gamma o)} e^{rh_\Gamma}.$$

Remark. This estimate is reminiscent from [**Sch04**, Theorem 3.2]. Following the same proof, it is likely that in geometric situations where the growth of Γ_K is purely exponential, this estimate should admit a similar lower bound.

Proof. In the course of this proof, many parameters will appear. Those parameters only depend on ε (and not on x). We denote them all by C, or $C(\varepsilon)$ if we want to emphasize the dependence in ε. We choose $0 < \varepsilon < h_\Gamma - h_\Gamma^\infty$. There exists a compact subset $k \subset X$ such that $h_{\Gamma_k} < h_\Gamma^\infty + \varepsilon/4$. Up to enlarging k, we assume that o belongs to k. Let K be the 6δ-neighbourhood of k and D its diameter. Let $S \subset \Gamma$ and $r_0 \in \mathbb{R}_+$ be given by Lemma 3.13 applied to k and K. By definition of the exponential growth rate, there exists $C(\varepsilon) \in \mathbb{R}_+$ such that for every $r \in \mathbb{R}_+$, we have

$$|S\Gamma_k \cap B_\Gamma(o,r)| \leqslant C(\varepsilon) e^{r(h_{\Gamma_k} + \varepsilon/4)}.$$

Let $x \in X$. For simplicity, set $d = d(x, \Gamma K)$. We fix $\alpha \in \Gamma$ and $q \in \alpha K$ such that q is a projection of x on ΓK. Given any geodesic $[q,x]$ from q to x, the intersection $\alpha^{-1}[q,x] \cap \Gamma K$ is contained in K. By Lemma 3.13, for every $\gamma \in \Gamma$, there exists $\beta \in S\Gamma_k$ such that $\langle \alpha^{-1}x, \alpha^{-1}\gamma o\rangle_{\beta o} \leqslant r_0$. In particular,

$$d\big(\beta o, \alpha^{-1}\gamma o\big) \leqslant d(x, \gamma o) - d\big(\alpha^{-1}x, \beta o\big) + 2r_0.$$

Consequently

$$\alpha^{-1} B_\Gamma(x,r) \subset \bigcup_{\beta \in S\Gamma_k} B_\Gamma\left(\beta o, r - d\big(\alpha^{-1}x, \beta o\big) + 2r_0\right)$$

Combined with (56) it yields

$$|B_\Gamma(x,r)| \leqslant C e^{rh_\Gamma} \sum_{\beta \in S\Gamma_k} e^{-h_\Gamma d(\alpha^{-1}x, \beta o)}. \tag{57}$$

Let us now estimate the latter sum. Recall that D is the diameter of K, which contains both o and $\alpha^{-1}q$. Hence for every $\beta \in S\Gamma_k$, we have $d(\alpha^{-1}x, \beta o) \geqslant d - D$ and

$$d(o, \beta o) \leqslant d\big(o, \alpha^{-1}x\big) + d\big(\alpha^{-1}x, \beta o\big) \leqslant D + d + d\big(\alpha^{-1}x, \beta o\big).$$

Consequently (57) becomes

$$\begin{aligned}|B_\Gamma(x,r)| &\leqslant e^{rh_\Gamma} \sum_{\substack{\ell \in \mathbb{N} \\ \ell \geqslant d-D}} \sum_{\substack{\beta \in S\Gamma_k \\ \ell \leqslant d(\alpha^{-1}x, \beta o) \leqslant \ell+1}} e^{-\ell h_\Gamma} \\ &\leqslant e^{rh_\Gamma} \sum_{\substack{\ell \in \mathbb{N} \\ \ell \geqslant d-D}} |S\Gamma_k \cap B_\Gamma(o, \ell+1+d+D)|\, e^{-\ell h_\Gamma} \\ &\leqslant C(\varepsilon) e^{rh_\Gamma} \sum_{\substack{\ell \in \mathbb{N} \\ \ell \geqslant d-D}} e^{(h_{\Gamma_k}+\varepsilon/4)(\ell+d)} e^{-\ell h_\Gamma}.\end{aligned}$$

Recall that $h_{\Gamma_k} + \varepsilon/4 < h_\Gamma$. Up to increasing $C(\varepsilon)$, we get

$$|B_\Gamma(x,r)| \leqslant C(\varepsilon) e^{(2h_{\Gamma_k}+\varepsilon/2-h_\Gamma)d} e^{rh_\Gamma}.$$

As o belongs to K, we have $d \leqslant d(x, \Gamma o)$. Moreover $h_{\Gamma_k} \leqslant h_\Gamma^\infty + \varepsilon/4$, whence the result. □

We now come back to the study of random walks in Y. Let a and C_3 be the parameters given by Lemma 6.6. Without loss of generality we can assume that $a > 1$. For every $n \in \mathbb{N}$, we denote by p_n the uniform probability measure on $S_\Gamma(o,n,a)$, M_n the associated Markov operator (53) and $B_n\colon \partial X \to \mathbb{R}_+$ the auxiliary map associated to p_n in (55) . By Lemma 6.6, we have $p_n(\gamma) \leqslant C_3 e^{ah_\Gamma} e^{-nh_\Gamma}$ if $\gamma \in S_\Gamma(o,n,a)$, and $p_n(\gamma) = 0$ otherwise.

Proposition 6.8. *For every $\varepsilon > 0$, there exists $C_5(\varepsilon) \in \mathbb{R}_+$, such that for every $n \in \mathbb{N}$, for every $\xi \in \partial X$, we have*

$$B_n(\xi) \leqslant C_5(\varepsilon) \max\left\{ e^{-nh_{\Gamma'}}, e^{n(h_\Gamma^\infty+\varepsilon-h_\Gamma)}, e^{n(h_{\Gamma'}-h_\Gamma)} \right\}.$$

Proof. As above, the proof involves many parameters which only depend on ε (and not on n or ξ). We still denote them all by C, or $C(\varepsilon)$. Choose $\varepsilon > 0$ such that $h_\Gamma^\infty + \varepsilon < h_\Gamma$ and define

$$h_{\mathrm{aux}} = \max\{\varepsilon, 2h_\Gamma^\infty + \varepsilon - h_\Gamma\}.$$

Up to decreasing ε, we can assume that $h_{\Gamma'} \neq (h_\Gamma \pm h_{\mathrm{aux}})/2$. Note that $0 < h_{\mathrm{aux}} \leqslant h_\Gamma^\infty + \varepsilon$. Let $n \in \mathbb{N}$ and $\xi \in \partial X$. We fix a geodesic $[o,\xi)$ joining o to ξ. For every $\ell \in \mathbb{N}$ we denote by x_ℓ the point on $[o,\xi)$ at distance ℓ from o. We now split the sum defining $B_n(\xi)$ according to the value of the Gromov product $\langle \gamma o, \xi\rangle_o$.

$$B_n(\xi) = \sum_{\ell \in \mathbb{N}} \sum_{\substack{\gamma \in \Gamma \\ \ell \leqslant \langle \gamma o, \xi\rangle_o \leqslant \ell+1}} e^{-h_{\Gamma'} b_\xi(\gamma o, o)} p_n(\gamma).$$

Note that the first sum is actually a finite sum. Indeed for every $\gamma \in S_\Gamma(o,n,a)$ the Gromov product $\langle \gamma o, \xi\rangle_o$ is at most n. Let $\ell \in \mathbb{N}$ and $\gamma \in S_\Gamma(o,n,a)$ such that

$$\ell \leqslant \langle \gamma o, \xi\rangle_o \leqslant \ell + 1.$$

A standard exercise of hyperbolic geometry shows that γ belongs to $B_\Gamma(x_\ell, n-\ell+\delta)$ and $b_\xi(\gamma o, o) \geqslant n - 2\ell - (a+\delta)$. On the other hand, as we noticed before

$$p_n(\gamma) \leqslant Ce^{-nh_\Gamma}$$

Consequently

$$B_n(\xi) \leqslant Ce^{-n(h_\Gamma + h_{\Gamma'})} \sum_{\ell \leqslant n} \sum_{\gamma \in B_\Gamma(x_\ell, n-\ell+\delta)} e^{2\ell h_{\Gamma'}}.$$

Note that if $B_\Gamma(x_\ell, n-\ell+\delta)$ is non-empty, then

$$d(x_\ell, \Gamma o) \leqslant \min\{\ell, n-\ell\} + \delta.$$

Using Proposition 6.7 we get

$$B_n(\xi) \leqslant C(\varepsilon) e^{-nh_{\Gamma'}} \sum_{\ell \leqslant n} e^{(2h_{\Gamma'} - h_\Gamma)\ell} e^{h_{\mathrm{aux}} \min\{\ell, n-\ell\}}.$$

We now split the sum according to the value of $\min\{\ell, n-\ell\}$. We get
(58)

$$B_n(\xi) \leqslant C(\varepsilon) e^{-nh_{\Gamma'}} \left[\sum_{\ell \leqslant n/2} e^{(2h_{\Gamma'} - h_\Gamma + h_{\mathrm{aux}})\ell} + e^{nh_{\mathrm{aux}}} \sum_{n/2 < \ell \leqslant n} e^{(2h_{\Gamma'} - h_\Gamma - h_{\mathrm{aux}})\ell} \right].$$

We now distinguish several cases depending on the value of $h_{\Gamma'}$ compared to $(h_\Gamma \pm h_{\mathrm{aux}})/2$. Recall that we chose ε in such a way that $h_{\Gamma'} \neq (h_\Gamma \pm h_{\mathrm{aux}})/2$.

Case 1. Assume that $h_{\Gamma'} < (h_\Gamma - h_{aux})/2$. Then both terms within the bracket in (58) are bounded. We get

$$B_n(\xi) \leqslant C(\varepsilon) e^{-nh_{\Gamma'}}.$$

Case 2. Assume that $(h_\Gamma - h_{aux})/2 < h_{\Gamma'} < (h_\Gamma + h_{aux})/2$. In this case the two terms within the brackets in (58) have exactly the same asymptotic behaviour. More precisely, the computation yields

$$B_n(\xi) \leqslant C(\varepsilon) e^{(h_{\mathrm{aux}} - h_\Gamma)n/2} \leqslant C(\varepsilon) e^{(h_\Gamma^\infty + \varepsilon - h_\Gamma)n}$$

Case 3. Assume that $h_{\Gamma'} > (h_\Gamma + h_{aux})/2$. Both sums in (58) diverge exponentially, however the second term dominates the first one. Hence

$$B_n(\xi) \leqslant C(\varepsilon) e^{(h_{\Gamma'} - h_\Gamma)n}.$$

The result is the combination of these three cases. □

Corollary 6.9. *The asymptotic behaviour of the spectral radius $\tau(M_n)$ of M_n is asymptotically controlled as follows*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \tau(M_n) \leqslant \max\{-h_{\Gamma'}, h_\Gamma^\infty - h_\Gamma, h_{\Gamma'} - h_\Gamma\}.$$

Proof. Let $\varepsilon > 0$. Recall that $\phi\colon Y \to \mathbb{R}_+^*$ is the map sending $y = \Gamma'\beta$ to the total mass of the measure $\nu'_{\beta o}$. Let $n \in \mathbb{N}$. Injecting in (54) the estimate given by Proposition 6.8, we get

$$M_n\phi \leqslant C(\varepsilon)\lambda_n\phi, \quad \text{where} \quad \lambda_n = \max\left\{ e^{-nh_{\Gamma'}}, e^{n(h_\Gamma^\infty + \varepsilon - h_\Gamma)}, e^{n(h_{\Gamma'} - h_\Gamma)} \right\}.$$

By Barta's inequality (Proposition 6.5), we deduce $\tau(M_n) \leqslant C(\varepsilon)\lambda_n$. Observe that $C(\varepsilon)$ does not depend on n. Passing to the limit we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \ln \tau(M_n) \leqslant \max\{-h_{\Gamma'}, h_\Gamma^\infty + \varepsilon - h_\Gamma, h_{\Gamma'} - h_\Gamma\}.$$

This inequality holds for every $\varepsilon \in \mathbb{R}_+^*$, whence the result. □

The next corollary completes the proof of the "easy direction" in Theorem 6.3.

Corollary 6.10. *If Γ' is co-amenable in Γ, then $h_{\Gamma'} = h_\Gamma$.*

PROOF. It follows from Kesten's amenability criterion that the spectral radius of any random walk on $Y = \Gamma'\backslash\Gamma$ is 1 [Kes59, Day64], see also [CDS17] for the case where Γ' is not a normal subgroup of Γ. Consequently Corollary 6.9 yields

$$\max\{-h_{\Gamma'}, h_\Gamma^\infty - h_\Gamma, h_{\Gamma'} - h_\Gamma\} \geqslant 0.$$

Since $h_\Gamma^\infty < h_\Gamma$, the only options are $h_{\Gamma'} = 0$ or $h_{\Gamma'} = h_\Gamma$. It remains to rule out the first case. Assume that $h_{\Gamma'} = 0$. We claim that Γ' is amenable. Since Γ' is countable, it can be written as an increasing union of finitely generated subgroups. Hence, it suffices to prove that every finitely generated subgroup of Γ' is amenable. Let S be a finite subset of Γ' and Γ'_S the subgroup of Γ generated by S. Obviously $h_{\Gamma'_S} = 0$. However the word metric on Γ'_S (with respect to S) dominates the metric induced by the action on X. It follows that Γ'_S has sub-exponential growth with respect to the word metric, hence Γ'_S is amenable, which completes the proof of our claim. By assumption the action of Γ on Y is amenable. Moreover the stabiliser of any point $y \in Y$ is conjugated to Γ', hence amenable. It follows that Γ is amenable [JM13, Lemma 3.2] or [GM07, Lemma 4.5], which contradicts the fact that Γ is non-elementary. □

6.2. Rigidity and growth gap. We now exploit rigidity properties to exhibit the existence of growth gaps for subgroups of Γ. We first recall the definition of the famous Kazhdan property (T). For more details we refer to [BdlHV08].

DEFINITION 6.11 (Kazhdan property). A discrete group Γ has *Kazhdan property* (T), if any unitary representation of Γ with almost invariant vectors admits a non-zero invariant vector.

For our purpose this property is too strong. Indeed we only consider unitary representations induced by an action on a countable set. In this context the appropriate rigidity property is Property (FM) studied by Monod and Glasner [GM07] or de Cornulier [dC15]. Similar properties have also been considered by Bekka and Olivier [BO14].

DEFINITION 6.12. A discrete group Γ has *Property* (FM) if every amenable action of Γ on a discrete countable set has a finite orbit.

Let Y be a countable discrete set endowed with an action of Γ. The induced representation $\rho\colon \Gamma \to \mathcal{U}(\ell^2(Y))$ has a non-zero invariant vector if and only if Γ has a finite orbit. In view of this remark, Property (FM) can be reformulated as follows.

PROPOSITION 6.13. *A discrete group Γ has property (FM) if and only if for every action of Γ on a discrete countable set Y, if the induced representation $\rho\colon \Gamma \to \mathcal{U}(\ell^2(Y))$ almost admits invariant vectors, then it has a non-zero invariant vector.*

Obviously, Property (T) implies Property (FM). However the converse is not true. For instance the free product of two infinite simple groups with Property (T) has property (FM) [GM07, Lemma 3.2] but cannot have property (T) as it acts on the corresponding Bass-Serre tree without global fixed point [BdlHV08, Theorem 2.3.6]. The next statement is an analogue of the existence of Kazhdan pairs, which quantifies Property (FM). The proof works verbatim as in [BdlHV08, Proposition 1.2.1] and is left to the reader.

LEMMA 6.14. *A discrete group Γ has Property (FM) if and only if there exists a finite subset S of Γ and $\varepsilon \in \mathbb{R}_+^*$ with the following property: for every action of Γ on a discrete countable set Y, if the induced representation $\rho\colon \Gamma \to \mathcal{U}(\ell^2(Y))$ has an (S, ε)-invariant vector, then it has a non-zero invariant vector.*

Theorem 6.15. *Let X be a hyperbolic proper geodesic space. Let Γ be a group with Property (FM) acting properly by isometries on X. We assume that the action of Γ is strongly positively recurrent. There exists $\eta > 0$ such that for every subgroup Γ' of Γ, if $h_{\Gamma'} \geqslant (1-\eta)h_\Gamma$, then Γ' is a finite index subgroup of Γ.*

Proof. Since Γ has Property (FM), there exists a finite subset S of Γ and $\varepsilon \in \mathbb{R}_+^*$ such that for every action of Γ on a discrete countable set Y, if the induced representation $\Gamma \to \mathcal{U}(\ell^2(Y))$ has an (S,ε)-invariant vector, then it admits a non-zero invariant vector (Lemma 6.14). According to Theorem 5.2 there exists $\eta \in \mathbb{R}_+^*$ with the following property: assume that $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation in a Hilbert lattice; if $h_\rho \geqslant (1-\eta)h_\Gamma$ then $\mathcal{H}$ admits (S,ε)-invariant vectors. Let Γ' be a subgroup of Γ such that $h_{\Gamma'} \geqslant (1-\eta)h_\Gamma$. We write $Y = \Gamma'\backslash\Gamma$ for the space of left cosets. Let $\mathcal{H} = \ell^2(Y)$ the Hilbert lattice of square summable functions and $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ the corresponding Koopman representation. It follows from Lemma 6.1 that $h_\rho \geqslant (1-\eta)h_\Gamma$. According to our choice of η, the representation ρ admits an (S,ε)-invariant vector, hence a non-zero invariant vector. This exactly means that the action of Γ on Y has a finite orbit. However this action being transitive, Y is finite. In other words Γ' has finite index in Γ. □

6.3. Counterexamples.

Counterexample without negative curvature. If the space X is not hyperbolic, the "easy direction" of our main theorem fails. Indeed there exists finitely generated amenable groups Γ whose action on their Cayley graph X has exponential growth, for instance Baumslag Solitar groups $\mathrm{BS}(1,n)$, lamplighter groups, etc. More generally, any solvable group which is not virtually nilpotent is so. For such a group Γ the trivial subgroup $\Gamma' = \{1\}$ obviously satisfies $h_{\Gamma'} < h_\Gamma$ although the quotient Γ/Γ' is amenable. Note that the action of a group on its Cayley graph is cocompact, hence strongly positively recurrent.

This problem cannot be "fixed" by strengthening the assumption on the quotient Γ/Γ', e.g. by asking that Γ/Γ' has polynomial growth. Consider indeed the lamplighter group L defined by

$$L = V \rtimes \mathbb{Z}, \quad \text{where} \quad V = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2.$$

An element $v = (v_n)$ of V is a sequence of elements of the finite groups $\mathbb{Z}_2$ which are trivial for all but finitely many $n \in \mathbb{Z}$. In particular, we write $a = (a_n)$ for the sequence which is trivial everywhere except at $n = 0$. The generator t of $\mathbb{Z}$ acts on V by the usual shift. The set $\{a,t\}$ generates L. Let X be the Cayley graph of Γ with respect to this set (on which L acts properly cocompactly). Parry [Par92] computed the associated growth series of L. One can extract from his result that

$$h_L(X) = \frac{1+\sqrt{5}}{2} \approx 1.618,$$

see for instance [BT17]. Actually Parry provides an explicit formula for the length of an element in L with respect to $\{a,t\}$ [Par92, Theorem 1.2]. In particular, the length $|v|$ of an element $v = (v_n)$ in V is the sum of two contributions:

(1) the *length* of the shortest loop in $\mathbb{Z}$, based at the identity, that visits all indices n for which $v_n \neq 1$.

(2) the *number* of indices $n \in \mathbb{Z}$ such that $v_n \neq 1$.

This can be used to compute the growth series $\zeta_V(z)$ of V for its action on X. All computations done we get

$$\zeta_V(z) = \sum_{v\in V} z^{|v|} = 1 + z + \frac{z^2(1+z)(1-z)\left(2+3z+2z^2\right)}{\left[1-z^2(z+1)\right]^2}.$$

Hence $h_V(X)$ is the root of $X^3 - X - 1 = 0$ which approximatively equals 1.3247. In particular, $h_V(X) < h_L(X)$ while the quotient L/V is isomorphic to $\mathbb{Z}$.

Counterexample without a growth gap at infinity. We now provide a few counterexamples acting on Gromov hyperbolic spaces where the "hard direction" of our main theorem fails when we drop the strongly positively recurrent assumption.

Parabolic discrete groups acting of $\mathbb{H}^n$ act by isometries on horospheres, which are Euclidean for their induced metric. Therefore, by Bieberbach theorem they are virtually abelian, hence amenable. Still they have non-zero critical exponent, hence our main theorem cannot apply to such groups. One can elementarily show, using convexity of Busemann functions, that such parabolic groups do not have a growth gap at infinity. Let us now construct non-elementary examples.

For fundamental groups of negatively curved surfaces, having a strongly positively recurrent action is an optimal assumption to get Theorem 6.3, as shown in the next proposition.

Proposition 6.16. *Let S be a locally $\mathrm{CAT}(-1)$ surface, Γ its fundamental group and X its universal cover. It the action of Γ on X does not have a growth gap at infinity, then it admits normal subgroups $\Gamma' \lhd \Gamma$ with $h_\Gamma = h_{\Gamma'}$ and such that Γ/Γ' contains a free group.*

Proof. Choose two disjoint closed non-separating geodesics c_1 and c_2 on S. Such disjoint closed curve exist up to taking a finite covering of S. Cut S along these curves; using the surface with boundary thus obtained, it is elementary to build a surface S' which is a regular cover of S with a covering group isomorphic to $\mathbb{F}_2$. If K is a compact set containing c_1 and c_2 in S, this surface S' contains many copies of $S \setminus K$ so that $\Gamma' = \pi_1(S')$ satisfies $h_{\Gamma'} \geqslant h_\Gamma^\infty = h_\Gamma$. The proposition follows. □

This proposition is really due to the fact that Γ is a surface group. It follows from [DOP00] that there exists such surfaces with finitely generated fundamental group and pinched negative curvature. Negatively curved finite volume surfaces without growth gap at infinity were constructed in [DPPS17]. Note that some of these examples even have a finite Bowen-Margulis measure. Constant curvature surfaces with finitely generated fundamental group always have a growth gap at infinity. A $\mathbb{Z}$-cover of a compact hyperbolic surface is typically a constant curvature surface which does not have a critical gap, and hence satisfies the above proposition.

Let us give a three dimensional constant curvature example.

Proposition 6.17. *Let $M = \mathbb{H}^3/\Gamma_1$ where Γ_1 is the image of a* simply degenerated *representation of a surface group in $\mathbb{H}^3$. Then there exists a hyperbolic isometry $h \in Isom^+(\mathbb{H}^3)$ satisfying the following. Let $\Gamma = \langle \Gamma_1, h\rangle$. Then Γ_1 is not co-amenable in Γ, and $h_\Gamma = h_{\Gamma_1} = 2$.*

Sketch of proof. A simply degenerated representation of a surface group Γ_1 is the geometric limit of a sequence of *quasi-fuchsian representations* $\rho_n(\Gamma_0)$ of a fixed surface group Γ_0 such that one end of $\mathbb{H}^3/\Gamma_1$ remains convex-cocompact, whereas the other end becomes geometrically infinite. We refer to [Mar07, Chapters 4 and 5] for a precise definition of this terminology.

It follows from [BJ97] that $h_{\Gamma_1} = 2$. Now, since Γ_1 is simply degenerated, its *discontinuity set* $\partial\mathbb{H}^3 \backslash \Lambda(\Gamma_1)$ is non-empty. It is therefore possible to find a hyperbolic isometry $h \in \mathrm{Isom}^+(\mathbb{H}^3)$ whose axis has end points in a ball contained in this discontinuity set. The groups Γ_1 and $\langle h\rangle$ are said to be in *Schottky position*: an easy application of Klein's ping pong lemma shows then that

$$\Gamma = \langle \Gamma_1, h\rangle = \Gamma_1 * \langle h\rangle.$$

In particular, Γ_1 is not co-amenable in Γ. Moreover, $2 = h_{\Gamma_1} \leqslant h_\Gamma \leqslant 2$ since any kleinian group in dimension 3 has critical exponent at most 2. □

We complete this section with a last example coming from geometric group theory.

PROPOSITION 6.18. *Let Γ be a group and $\mathcal{P}$ a finite collection of residually finite subgroups of Γ such that Γ is hyperbolic relative to $\mathcal{P}$. Let X be a metric space endowed with proper cusp-uniform action of $(\Gamma, \mathcal{P})$. If $\mathcal{P}$ contains a subgroup P such that $h_P = h_\Gamma$, then there exists a normal subgroup Γ' of Γ such that*

(1) $h_{\Gamma'} = h_\Gamma$;

(2) Γ/Γ' is non-elementary hyperbolic, hence non-amenable.

PROOF. Using the group theoretic Dehn filling [**GM08**, **Osi07**], there exists a finite index subgroup P_0 of P such that the quotient of Γ by $\Gamma' = \langle\langle P_0 \rangle\rangle$ is non-elementary hyperbolic. Since P_0 is a finite index subgroup of P, it has the same growth rate as P, i.e. h_Γ. As Γ' contains P_0, its growth rate is h_Γ. □

7. Comments and questions

Let us present some natural opening directions of this work.

7.1. Generalizations of Theorem 1.1 and its variations.

Beyond hyperbolicity. The approach presented in this paper is most likely applicable to various context beyond groups acting on a δ-hyperbolic space. Let Γ be a discrete group acting by isometries on a general proper geodesic metric space (X, d). As already noticed by Arzhantseva et al. [**ACT15**] and Yang [**Yan16**], the existence of a growth gap at infinity provides many interesting results as soon as this action admits *contracting elements* – see for instance [**Yan16**] for a definition. This setting includes for instance CAT(0) groups with rank one elements or all convex-cocompact subgroups of the mapping class groups acting on Teichmüller space (including the mapping class group itself). We currently work on the extension of our strategy to this more general context.

Locally compact groups. Instead of considering a *discrete* group Γ acting on a metric space, we could also work with locally compact groups. Let X be a Gromov hyperbolic space such that $G = \mathrm{Isom}(X)$ is a locally compact group containing a lattice. Define its critical exponent h_G to be the infimum of $s > 0$ such that

$$\mathcal{P}_G(s) = \int_G e^{-sd(o,go)} dg < \infty,$$

where dg is the Haar measure on G. Still replacing Poincaré series by Haar integrals, we can then define analogously the entropy at infinity of G, Patterson-Sullivan theory on the horoboundary of X, etc. It seems likely that all the theory would extend in this larger setting. In particular, it should lead to the following wide generalization of Corlette's rigidity result [**Cor90**]. Assume that $\mathrm{Isom}(X)$ has Kazhdan's Property (T) and its action on X is strongly positively recurrent. Then there exists $\varepsilon \in \mathbb{R}_+^*$ such that for every discrete group Γ of isometries of X either Γ is a lattice or $h_\Gamma \leqslant \dim_{\mathrm{vis}}(\partial X) - \varepsilon$, where $\dim_{\mathrm{vis}}(\partial X)$ stands for the visual dimension of ∂X.

7.2. Twisted Patterson-Sullivan measures. Let Γ be a discrete group acting on a δ-hyperbolic space, and let ρ be a positive unitary representation of Γ on some Hilbert lattice. The twisted Patterson-Sullivan density $a^\rho = (a^\rho_x)_{x\in X}$ which we introduced in Section 5 is a powerful tool whose exploration should be fruitful. Let us mention some natural problems raised by our study.

(1) If $h_\rho = h_\Gamma$, understand the relation between the operator $a_o^\rho(\mathbf{1})$ and the orthogonal projection on the subspace of invariant vectors of the limit representation ρ_ω.

(2) If $h_\rho < h_\Gamma$, what can be said about the operator $a_o^\rho(\mathbf{1})$?

(3) Let Γ' be a subgroup of Γ and $\mathcal{H} = \ell^2(\Gamma/\Gamma')$. The Patterson-Sullivan density twisted by the induced representation $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ can be seen as a Γ/Γ'-*extension* of the classical Patterson-Sullivan density. Many recent works deal with group extensions of Markov shifts over a finite alphabet, in particular when studying covers of negatively curved convex-cocompact manifolds or Schottky manifolds (see for instance [**CG13**, **Jae16**, **Sta13**, **DS16**]). It seems plausible that, using twisted Patterson-Sullivan measure, many ergodic results which have been obtained for group extensions of Markov shifts could be carried to the geodesic flow.

1. Integration of vector-valued functions

1.1. Bochner spaces. We start by recalling the notion of Bochner integral and Bochner spaces. The goal is to give a rigorous definition for the integral of a Hilbert valued map. For our purpose, everything works verbatim as for the usual Lebesgue integral. We refer the reader to the original article of Bochner [**Boc33**] or [**Din67**, **DUJ77**].

Let $(X, \mathcal{B}, \nu)$ be a finite measure space and $(E, \|\,.\,\|)$ a Banach space.

Measurable functions. A map $\Phi\colon X \to E$ is *simple* if it can be written $\Phi = \mathbf{1}_{B_1}\phi_1 + \cdots + \mathbf{1}_{B_n}\phi_n$ where $B_i \in \mathcal{B}$ and $\phi_i \in E$. A function $\Phi\colon X \to E$ is ν-*measurable* if there exists a sequence (Φ_n) of simple functions from X to E which converges ν-almost everywhere to Φ.

Bochner spaces. Let $p \in [1, \infty)$. Observe that if $\Phi\colon X \to E$ is a ν-measurable map, then the function $X \to \mathbb{R}_+$ mapping x to $\|\Phi(x)\|$ is measurable (in the usual sense). Hence we can define the *p-norm* of Φ by

$$\|\Phi\|_p = \left(\int \|\Phi(x)\|^p d\nu(x)\right)^{1/p}.$$

The *Bochner space* $L^p(\nu, E)$ is the set of ν-measurable maps $\Phi\colon X \to E$ such that $\|\Phi\|_p < \infty$, up to the standard equivalence relation which identifies two maps that coincide ν-almost everywhere. The norm $\|\,.\,\|_p$ gives to $L^p(\nu, E)$ a structure of Banach space. Similarly we define a uniform norm by

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{x \in X} \|f(x)\|.$$

The *Bochner space* $L^\infty(\nu, E)$ consists of all ν-measurable maps $\Phi\colon X \to E$ that are essentially bounded. Again this definition is meant up to equality ν-almost everywhere. It is a Banach space.

Since ν is a finite measure, a standard argument shows that $L^q(\nu, E)$ embeds in $L^p(\nu, E)$ provided $1 \leqslant p \leqslant q \leqslant \infty$. For every $p \in [1, \infty)$ the set of simple functions is dense in $L^p(\nu, E)$.

If $E = \mathbb{R}$, these spaces coincide with the usual function spaces $L^p(\nu)$. If $\mathcal{H}$ is a Hilbert space, the Bochner space $L^2(\nu, \mathcal{H})$ has a structure of Hilbert space, where the scalar product is given by

$$(\Phi_1, \Phi_2) = \int (\Phi_1(x), \Phi_2(x))\, d\nu(x), \quad \forall \Phi_1, \Phi_2 \in L^2(\nu, \mathcal{H}).$$

Bochner integral. The definition of the Bochner integral follows exactly the same steps as the one of the Lebesgue integral. More precisely one starts by defining the integral of a simple function. Given a simple function $\Phi = \mathbf{1}_{B_1}\phi_1 + \dots + \mathbf{1}_{B_n}\phi_n$, its integral is the vector of E defined by

$$\int \Phi d\nu = \sum_{i \in I} \nu(B_i)\phi_i.$$

A ν-measurable function $\Phi\colon X \to E$ is *Bochner integrable* if there exists a sequence (Φ_n) of simple functions from X to E such that

$$\lim_{n\to\infty} \int \|\Phi - \Phi_n\| \, d\nu = 0,$$

in which case we define the integral of Φ as

$$\int \Phi d\nu = \lim_{n\to\infty} \int \Phi_n d\nu.$$

One checks easily that this integral is well defined and does not depend on the choice of (Φ_n). A function Φ is Bochner integrable if and only if it belongs to $L^1(\nu, E)$ [**DUJ77**, Chapter II, Theorem 2]. The Bochner integral defines a 1-Lipschitz linear map $L^1(\nu, E) \to E$ satisfying the following useful properties.

Proposition 1.1 ([**DUJ77**, Chapter II, Corollary 5]). *Let E be a Banach space. Let $\Phi, \Phi' \in L^1(\nu, E)$. If*

$$\int \mathbf{1}_B \Phi d\nu = \int \mathbf{1}_B \Phi' d\nu, \quad \forall B \in \mathcal{B},$$

then $\Phi = \Phi'$ ν-almost everywhere.

Proposition 1.2 ([**DUJ77**, Chapter II, Theorem 6]). *Let E and F be two Banach space. Let $T\colon E \to F$ be a continuous linear operator. For every $\Phi \in L^1(\nu, E)$, the function $T(\Phi)$ belongs to $L^1(\nu, F)$. Moreover*

$$T\left(\int \Phi d\nu\right) = \int T(\Phi) d\nu.$$

1.2. The Radon-Nikodym property. Let $(X, \mathcal{B}, \nu)$ be a measure space. The standard Radon-Nikodym theorem states that $L^\infty(\nu)$ is the dual of $L^1(\nu)$. In general if E is an arbitrary Banach space and E' its dual, the space $L^\infty(\nu, E')$ is not necessarily the dual of $L^1(\nu, E)$. The Radon-Nikodym property defined below is precisely designed to prevent this kind of pathology. See [**DUJ77**, Chapter III, Definition 3 and Theorem 5].

Definition 1.3. A Banach space E has the *Radon-Nikodym property* if for every finite measure space $(X, \mathcal{B}, \nu)$ the following holds: for every continuous linear map $T\colon L^1(\nu) \to E$ there exists a function $\Phi \in L^\infty(\nu, E)$ such that

$$T(f) = \int f\Phi \, d\nu, \quad \forall f \in L^1(\nu).$$

In this definition the integral is a Bochner integral as defined previously. Note that the function Φ given by the definition is necessarily unique (Proposition 1.1). Moreover one checks that $\|\Phi\|_\infty = \|T\|$ [**DUJ77**, Chapter III, Lemma 4].

Recall that a Banach space $(E, \| \, . \, \|)$ is *reflexive* if the evaluation map $E \to E''$ from E to its bidual space E'' is an isomorphism. For instance every Hilbert space is reflexive. The following important result is due to Phillips [**Phi43**].

Theorem 1.4 ([**DUJ77**, Chapter III, Corollary 13]). *Reflexive Banach spaces have the Radon-Nikodym property.*

2. Banach lattices

In this section we review the basic properties of Banach spaces endowed with a lattice structure. For an in-depth study of Banach lattices we refer to [**Sch74**] or [**AB06**].

2.1. Definitions and main properties.

Vocabulary and notations. A *vector lattice* $(E, \prec)$ (also called Riesz space) is a vector space E equiped with a partial order $\prec$, compatible with the vector space structure, which provides E with a *lattice* structure, i.e. such that for all $\phi, \psi \in E$, the set $\{\phi, \psi\}$ has a *least upper bound* usually denoted by $\phi \vee \psi \in E$ and a *greater lower bound*, usually denoted by $\phi \wedge \psi \in E$. Given $\phi \in E$, its *absolute value*, is the vector

$$|\phi| = \phi \vee (-\phi) = \phi_+ + \phi_-$$

where $\phi_+ = \phi \vee 0$ and $\phi_- = (-\phi) \vee 0$ are respectively the positive and negative part of ϕ. The *positive cone* of E, denoted by E^+, is the set of vectors $\phi \in E$ such that $0 \prec \phi$. An *ideal* of E is a vector subspace of F of E satisfying the following property: for every $\phi \in E$ and $\psi \in F$, if $|\phi| \prec |\psi|$, then ϕ belongs to F. The vector lattice $(E, \prec)$ is *(countably) order complete* if every non-empty (countable) subset of E which is bounded from above admits a least upper bound. A norm $\| \, . \, \|$ on E is *monotone* if we have $\|\phi_1\| \leqslant \|\phi_2\|$ whenever $\phi_1, \phi_2 \in E$ satisfy $|\phi_1| \prec |\phi_2|$. If E is (topologically) complete for such a norm, it is called a *Banach lattice.*

Monotone convergence. Recall that a *directed set* $(A, \prec)$ is a set A endowed with a partial order $\prec$ such that for every $a, a' \in A$, there exists $b \in A$ with $a \prec b$ and $a' \prec b$. If I is a countable set, the collection of all finite subsets of I endowed with the inclusion is an example of directed set. A *net* is a map $f \colon A \to E$ from a directed set $(A, \prec)$ to $(E, \prec)$. Such a net

- is *non-decreasing* if $f(a) \prec f(a')$ whenever $a \prec a'$;
- is *norm-bounded* if there exists $M \in \mathbb{R}_+$ such that for every $a \in A$, we have $\|f(a)\| \leqslant M$;
- *converges to* $b \in E$ if for every $\varepsilon \in \mathbb{R}_+^*$, there exists $a_0 \in A$, such that for every $a \in A$, with $a_0 \prec a$, we have $\|f(a) - b\| \leqslant \varepsilon$. In this case we write $b = \lim f$.

PROPOSITION 2.1 (Schaefer [**Sch74**, Chapter II, Theorem 5.11]). *Assume that E is a reflexive Banach lattice. Then E is order complete. Moreover, every non-decreasing norm-bounded net $f \colon A \to E$ converges.*

Operator between lattices. Let E and F be two vector lattices. A linear operator $U \in \mathcal{L}(E, F)$ is *positive* if it maps E^+ into F^+. This defines a partial order on $\mathcal{L}(E, F)$: given $U_1, U_2 \in \mathcal{L}(E, F)$ we say that $U_1 \prec U_2$ if $U_2 - U_1$ is positive. However $\mathcal{L}(E, F)$ endowed with the order is in general not a vector lattice. To bypass this difficult, we consider a smaller subspace of $\mathcal{L}(E, F)$. A linear operator $U \colon E \to F$ is *regular* if is can be written as $U = U_+ - U_-$ where U_+ and U_- are two positive linear operators from E to F. The set of all regular operators from E to F, that we denote by $\mathcal{L}_r(E, F)$, is a vector subspace of $\mathcal{L}(E, F)$.

PROPOSITION 2.2 (Schaefer [**Sch74**, Chapter IV, Propositions 1.3]). *If E and F are two vector lattices and F is order complete, then $\mathcal{L}_r(E, F)$ is an order complete vector lattice.*

Suppose now that E and F are two Banach lattices and F is order complete. We write $\mathcal{B}_r(E, F)$ for the set of *bounded regular operators*, i.e. the elements $U \in \mathcal{L}_r(E, F)$ such that $|U|$ is a bounded operator. This space is endowed with a *regular norm* defined by $\|U\|_r = \||U|\|$ which turns $\mathcal{B}_r(E, F)$ into a Banach lattice [**Sch74**, Chapter IV, Propositions 1.4]. Note that both norms $\| . \|_r$ and $\| . \|$ coincide on positive operators.

Although $\mathcal{B}_r(E, F)$ is a Banach lattice, we cannot expect as in Proposition 2.1 that every non-decreasing norm-bounded net of regular operator converges for the norm $\| . \|_r$. However for our purpose, pointwise convergence will be enough.

PROPOSITION 2.3. *Assume that E and F are two Banach lattices and F is reflexive. Let $f \colon A \to \mathcal{B}_r(E, F)$ be a non-decreasing norm-bounded net. For every $\phi \in E$, the net $f_\phi \colon A \to E$ mapping a to $f(a)\phi$ converges. Moreover the map $V \colon E \to F$ defined by $V\phi = \lim f_\phi$ is a bounded regular operator.*

Remarks. If $f(a)$ is positive, for every $a \in A$, one easily checks that

$$\|V\| = \sup_{a \in A} \|f(a)\| .$$

PROOF. Let $\phi \in E$. We write ϕ_+ and ϕ_- for its positive and negative part respectively. Observe that the nets f_{ϕ_+} and f_{ϕ_-} are non-decreasing and norm-bounded, hence they converge (Proposition 2.1). Thus f_ϕ converges as well. One checks easily that the map $V \colon E \to F$ sending ϕ to $\lim f_\phi$ satisfies the announced properties. ☐

DEFINITION 2.4. Let Γ be a group. We say that a unitary representation $\rho \colon \Gamma \to \mathcal{B}(E)$ is *positive* if $\rho(\gamma)$ is positive for every $\gamma \in \Gamma$.

Dual space. Suppose that E is a Banach lattice. Its (topological) dual space E' endowed with the order inherited from $\mathcal{L}(E, \mathbb{R})$ is an order complete Banach lattice [**Sch74**, Chapter II, Proposition 5.5]. Actually it is isomorphic to $\mathcal{B}_r(E, \mathbb{R})$ [**Sch74**, Chapter IV, Theorem 1.5]. Recall that a subspace F of E' *separates points* if for every distinct $\phi, \phi' \in E$, there exists $\lambda \in F$ such that $\lambda(\phi) \neq \lambda(\phi')$.

PROPOSITION 2.5 ([**AB06**, Corollary 8.35]). *Assume that E is a Banach lattice. Let F be an ideal of E' which separates points. A vector $\phi \in E$ belongs to E^+ if and only if for every $\lambda \in F$ such that $0 \prec \lambda$, we have $\lambda(\phi) \geqslant 0$.*

2.2. Examples. We review here the main examples of Banach lattices that are used in the article.

2.2.1. *Koopman representations.* In this article we are mostly interested with the following situation. Let Y be a set endowed with the counting measure. The space $\mathcal{H} = \ell^2(Y)$ of square summable maps $\phi \colon Y \to \mathbb{R}$, endowed with the scalar product defined as

$$(\phi_1, \phi_2) = \sum_{y \in Y} \phi_1(y)\phi_2(y),$$

is a Hilbert space, hence a reflexive Banach space. We endow this space with a partial order defined as follows. Given $\phi, \phi' \in \mathcal{H}$ we say that $\phi \prec \phi'$ if $\phi(y) \leqslant \phi'(y)$ for every $y \in Y$. It turns $\mathcal{H}$ into a Banach lattice.

Let Γ be a discrete group acting on Y. This action induces a positive unitary representation $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$, called the *Koopman representation.*

2.2.2. *Bochner spaces.* Let $(E, \prec, \| . \|)$ be a Banach lattice and $(X, \mathcal{B}, \nu)$ a be a finite measure space. Let $p \in [1, \infty) \cup \{\infty\}$. We define a binary relation on the Bochner space $L^p(\nu, E)$ as follows. Given $\Phi, \Phi' \in L^p(\nu, E)$, we say that $\Phi \prec \Phi'$ if $\Phi(x) \prec \Phi'(x)$ ν-almost everywhere. It is obvious that this defines indeed a partial order on $L^p(\nu, E)$.

LEMMA 2.6. *The Bochner space $L^p(\nu, E)$ endowed with $\prec$ is a Banach lattice.*

PROOF. It is obvious that the order $\prec$ is compatible with the vector space structure on $L^p(\nu, E)$. Let $\Phi, \Phi' \in L^p(\nu, E)$. We define a map $\Psi\colon X \to E$ by $\Psi(x) = \Phi(x) \vee \Phi'(x)$, for all $x \in X$. We are going to prove that Ψ is the least upper bound of Φ and Φ'. Let us first prove that Ψ is ν-measurable and belongs to $L^p(\nu, E)$. By definition there exists two sequences (Φ_n) and (Φ'_n) of simple functions converging ν-almost everywhere to Φ and Φ' respectively. One checks easily that the function $\Psi_n\colon X \to E$ sending x to $\Phi_n(x) \vee \Phi'_n(x)$ is also a simple function. On the other hand the operation $\vee$ is uniformly continuous [**Sch74**, Chapter II, Proposition 5.1]. It follows that (Ψ_n) converges ν-almost everywhere to Ψ, hence Ψ is ν-measurable. For every $x \in X$, we have

$$\|\Phi(x) \vee \Phi'(x)\| \leqslant \|\Phi(x)\| + \|\Phi'(x)\| .$$

See for instance [**Sch74**, Chapter II, Proposition 1.4(6)]. Since Φ and Φ' belongs to $L^p(\nu, E)$ so does Ψ. It is now obvious to check that Ψ is the least upper bound of Φ and Φ'. We check in the same manner that the greatest lower bound of Φ and Φ' is the function $X \to E$ sending x to $\Phi(x) \wedge \Phi'(x)$. Thus $L^p(\nu, E)$ is a vector lattice. Let us prove now that the norm is monotone. Let $\Phi, \Phi' \in L^p(\nu, E)$ such that $|\Phi| \prec |\Phi'|$. It follows from the previous discussion that $|\Phi|\colon X \to E$ is exactly the function sending x to $|\Phi(x)|$. The same holds for Φ', hence $|\Phi(x)| \prec |\Phi'(x)|$ ν-almost surely. Since the norm of E is monotone, $\|\Phi(x)\| \leqslant \|\Phi'(x)\|$ ν-almost surely, hence $\|\Phi\|_p \leqslant \|\Phi'\|_p$. Consequently $L^p(\nu, E)$ is a Banach lattice. □

2.2.3. *Positivity of the Bochner integral.* We now focus on the case where $p = 1$ and study the behaviour of the Bochner integral with respect the partial order on $L^1(\nu, E)$.

LEMMA 2.7 (Positivity). *Let $\Phi, \Phi' \in L^1(\nu, E)$. If $\Phi \prec \Phi'$, then*

$$\int \Phi d\nu \prec \int \Phi' d\nu.$$

PROOF. Since the Bochner integral is linear it suffices to prove that the

$$0 \prec \int \Phi d\nu.$$

whenever $0 \prec \Phi$. Note that the statement is obvious if Φ is a simple function. Hence we are left to prove that every positive function Φ is the limit of a sequence (Φ_n) of *positive* simple functions. Let $\Phi \in L^1(\nu, E)$ be such a positive function. There exists a sequence (Φ_n) of simple functions converging to Φ in $L^1(\nu, E)$. One checks that $(\Phi_n \vee 0)$ is a sequence of positive simple functions. As $L^1(\nu, E)$ is a Banach lattice, the operation $\vee$ on $L^1(\nu, E)$ is uniformly continuous, hence $(\Phi_n \vee 0)$ converges to $\Phi \vee 0$, i.e Φ. □

PROPOSITION 2.8. *Let E be a Banach lattice. Let $\Phi \in L^1(\nu, E)$. If for every $B \in \mathcal{B}$, we have*

$$0 \prec \int \mathbf{1}_B \Phi d\nu,$$

then $0 \prec \Phi$.

PROOF. Let E' be the dual of E. We consider the bilinear map

$$\begin{array}{ccc} L^\infty(\nu, E') \times L^1(\nu, E) & \to & \mathbb{R} \\ (\Lambda, \Phi) & \to & \displaystyle\int \Lambda(x) \left[\Phi(x)\right] d\nu(x) \end{array}$$

that we denote by (Λ, Φ). This duality product induces an isometric embedding from $L^\infty(\nu, E')$ into the dual D of $L^1(\nu, E)$ [**DUJ77**, Chapter IV, §1]. Moreover, seen as a subspace of D, the space $L^\infty(\nu, E')$ is an ideal that separates the points.

Let $\lambda \in E'$ such that $0 \prec \lambda$ and B be a Borel subspace of X. It follows from our assumption and Proposition 1.2 that the quantity

$$(\mathbf{1}_B\lambda, \Phi) = \int \mathbf{1}_B\lambda \circ \Phi d\nu = \lambda\left(\int \mathbf{1}_B \Phi d\nu\right)$$

is non negative. By linearity, for every positive simple function $\Lambda \in L^\infty(\nu, E')$, we have $(\Lambda, \Phi) \geqslant 0$. Let $\Lambda \in L^\infty(\nu, E')$ be an arbitrary positive function. By definition of ν-measurability, there exists a sequence (Λ_n) of simple functions of $L^\infty(\nu, E')$ which converge to Λ ν-almost everywhere. Up to replacing Λ_n by $\Lambda_n \vee 0$ we can assume that each Λ_n is positive. According to the dominated convergence theorem (for Lebesgue integrals) (Λ_n, Φ) converges to (Λ, Φ) which is thus non-negative. It follows then from Proposition 2.5 that $0 \prec \Phi$. $\square$

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