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Branes and Non-Invertible Symmetries

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$\mathcal{N} = 4$ supersymmetric Yang-Mills theories with algebra $\mathfrak{so}(4N)$ and appropriate choices of global structure can have non-invertible symmetries. We identify the branes holographically dual to the non-invertible symmetries, and derive the fusion rules for the symmetries from the worldvolume dynamics on the branes.

1. Introduction

The notion of symmetry is undergoing rapid evolution: during the last few years a number of works have convincingly argued that the classical textbook definition of symmetry as a group of transformations acting on local operators can (and should) be extended to include higher form symmetries acting on extended operators,[1] higher groups structures[2–4] and more generally higher categorical structures.

The importance of such higher categorical structures in two dimensions has been realised for a long time, where they often appear from discrete gauging.[5–8] A number of recent works have shown that symmetry operators without inverses (and which are therefore not elements of any group, but should rather be thought of in categorical terms) are also very common in higher dimensional theories.[9–26] In this paper we will focus on one class of theories where such non-invertible symmetries appear: $\mathcal{N} = 4$ theories with gauge group[1] $\mathrm{Pin}^+(4N)$, $Sc(4N)$ and $PO(4N)$.[14] The details are a little different in the three cases, so in this introduction we will focus on the $Sc(4N)$ case for concreteness. This theory has three 2-surface symmetry generators, which we will call $D_2^{\mathrm{c},e}(\Sigma_2)$, $D_2^{\mathrm{s},m}(\Sigma_2)$ and their product $D_2^{\mathrm{c},e}(\Sigma_2)D_2^{\mathrm{s},m}(\Sigma_2)$. There is additionally a three-surface operator $\mathcal{N}(\mathcal{M}^3)$. The terms in the fusion algebra involving $\mathcal{N}(\mathcal{M}^3)$ are

$$\mathcal{N}(\mathcal{M}^3) \times \mathcal{N}(\mathcal{M}^3) = \sum_{\Sigma_2,\Sigma_2' \in H_2(\mathcal{M}^3;\mathbb{Z})} D_2^{\mathrm{c},e}(\Sigma_2)\, D_2^{\mathrm{s},m}(\Sigma_2')\,, \tag{1a}$$

$$\mathcal{N}(\mathcal{M}^3) \times D_2^{\mathrm{c},e}(\Sigma_2) = \mathcal{N}(\mathcal{M}^3)\,, \tag{1b}$$

$$\mathcal{N}(\mathcal{M}^3) \times D_2^{\mathrm{s},m}(\Sigma_2) = \mathcal{N}(\mathcal{M}^3)\,. \tag{1c}$$

The right hand side of (1a) is generically a sum of operators, and therefore $\mathcal{N}(\mathcal{M}^3)$ is not invertible.

All these theories can be obtained from the $\mathcal{N} = 4$ $SO(4N)$ theories by suitable gaugings of discrete symmetries. Whether we have performed the gauging or not is not visible for a local observer measuring processes on a topologically trivial (but arbitrarily large) neighbourhood of a point. This suggests that the holographic dual of all these theories is the same, which is indeed the case: the holographic dual is in all cases IIB on $\mathrm{AdS}_5 \times \mathbb{RP}^5$. The different theories arise from different choices of asymptotic behaviour for discrete gauge fields in the bulk, as discussed in related examples in [29, 30].

Since all these theories share the same bulk description, it should be possible to describe the non-invertible symmetry generators (in the cases where they are present in the field theory) in terms of objects living on the holographic IIB dual. The goal of this note is to identify these objects, and to derive their fusion rules using IIB techniques.[2,3] Surprisingly, given the perhaps unfamiliar fusion relations (1), it will transpire that the symmetry generators are represented holographically by ordinary branes wrapping torsional cycles in the internal $\mathbb{RP}^5$.

In order to explain how this is possible, it is useful to review briefly how the fusion relations (1) are derived in [14] (see also [12]). We start with the $SO(4N)$ theory, which has a $\mathbb{Z}_2$ outer automorphism 0-form symmetry and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry. We will denote the background for the 0-form symmetry A_1, and the backgrounds for the two $\mathbb{Z}_2$ factors B_2^m and C_2^e. There is a cubic 't Hooft anomaly represented by an anomaly theory with action

$$i\pi \int A_1 B_2^e C_2^m\,. \tag{2}$$

We obtain the $Sc(4N)$ theory by gauging both 1-form symmetries simultaneously.[4] (The Pin^+ and PO cases are obtained by gauging other pairs of symmetries involved in the cubic anomaly.)

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[1] In this note we do not aim to analyse fully the mapping from boundary conditions to global structures, so we will ignore the existence of discrete choices of θ angles in some of the theories we discuss.[27] A careful analysis of the mapping from global structures to properties of the holographic duals will be provided in [28].

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[2] The techniques we use in our analysis do not require knowledge of the Lagrangian of the boundary SCFT (although the choice of theories to study is certainly informed by the field theory results in [14], and we will chose our notation to dovetail the field theory analysis), so they apply equally well to the study of non-Lagrangian theories realised either holographically or via geometric engineering. See [21] for a recent study of non-invertible symmetries in non-Lagrangian theories using a different approach.

[3] We refer the reader to [23] for a holographic study of a different class of non-invertible defects.

[4] There is a discrete choice when gauging, related to the precise way in which we sum over B_2^m backgrounds. A slightly different choice (re-

Naively, we would say that the 0-form symmetry is broken due to the cubic anomaly (2). The more precise statement is that due to the anomaly the generator $D_3^{(0)}(\mathcal{M}^3)$ of the 0-form symmetry is not invariant under combined gauge transformations of B_2^m and C_2^e. But as argued in [12, 14] it is possible to "dress" (or stack) $D_3^{(0)}(\mathcal{M}^3)$ with an anomalous TQFT $\mathcal{T}$ depending on B_2^m and C_2^e. The combined topological operator is gauge invariant, and survives as a topological operator of the gauged theory. The price to pay is that the fusion rules for $\mathcal{T}$ are more involved, and lead to non-invertibility of the dressed operator $D_3^{(0)}(\mathcal{M}^3) \times \mathcal{T}$ (the details will be reviewed below).

Coming back to the IIB holographic setup, the main observation of this paper is that $D_3^{(0)}(\mathcal{M}^3) \times \mathcal{T}$ is precisely the IR limit of the theory on branes wrapping suitable torsional cycles in the holographic dual background. For instance, we will see that in the $Sc(4N)$ case the non-invertible operator arises from a D3 brane wrapping $\mathcal{M}^3 \times \mathbb{RP}^1$, where $\mathbb{RP}^1 \subset \mathbb{RP}^5$. Reducing on the $\mathbb{RP}^5$ leaves an effective 3-dimensional brane wrapping $\mathcal{M}^3$ inside the five dimensional bulk, which becomes $D_3^{(0)}(\mathcal{M}^3)$ when pushed to the boundary.

A pleasing consequence of the identification in this note is that anomaly cancellation of the dressed operator follows automatically: the background fields for the symmetries of the theory are given by asymptotic values for the supergravity fields in the IIB dual, and the D3 brane action is necessarily gauge invariant under all possible gauge transformations of these (although the precise way in which this happens is often subtle). Since anomaly cancellation is automatic once we start talking about branes, it is illuminating to understand why non-invertible symmetries appear in the holographic dual without referring to anomalous operators. This is also desirable since the split between the bare $D(\mathcal{M}^3)$ and its "dressing" $\mathcal{T}$ is unnatural in terms of the brane theory, particularly once we try to formulate things in the language of K-theory. We provide such an explanation below in terms of incomplete cancellation of induced brane charges due to quantum effects.

Note Added

I thank the authors of [31] for informing me of their related upcoming work, where they give complementary evidence for the identification of non-invertible symmetries with branes in holographic settings, and for agreeing to coordinate submissions.

2. 4d $\mathcal{N} = 4$ $\mathfrak{spin}(4N)$ SYM and Non-Invertibles

The Spin(4N) SYM theory has a 2-group structure, with one-form symmetry group[5]

$$\Gamma^{(1)} = \mathbb{Z}_2^{\mathsf{s}} \times \mathbb{Z}_2^{\mathsf{c}}\,, \tag{3}$$

and a 0-form symmetry part $\mathbb{Z}_2^{(0)}$ which is an outer automorphism that acts on the 1-form symmetry by exchanging the two factors: $\mathbb{Z}_2^{\mathsf{s}} \leftrightarrow \mathbb{Z}_2^{\mathsf{c}}$. We will now construct the topological defects that generate these symmetries in the holographic dual.

This holographic dual is obtained as the near horizon limit of a stack of D3-branes on top of an O3$^-$ orientifold, and is given by IIB string theory on $\mathrm{AdS}_5 \times \mathbb{RP}^5$.[29] In general we want to put the field theory on some spin[6] manifold $\mathcal{M}^4$ different from S^4, so we will replace AdS_5 by a non-compact manifold X^5 which asymptotically becomes $\mathbb{R} \times \mathcal{M}^4$.[32] There is a non-trivial $SL(2,\mathbb{Z})$ duality fibration over $\mathbb{RP}^5$, which acts with the $-1 \in SL(2,\mathbb{Z})$ element as we go around the non-trivial generator of $\pi_1(\mathbb{RP}^5) = \mathbb{Z}_2$. (This element can be represented alternatively as ΩF_L in worldsheet terms, but with future generalisations in mind we will describe it as an $SL(2,\mathbb{Z})$ bundle instead.) The 2-form supergravity fields B_2 and C_2 get a sign under this action, and project down to $\mathbb{Z}_2$ fields on AdS_5, while C_4 does not get a sign and survives as a continuous field. We will find it useful to work in a democratic formulation, where we also include the B_6 and C_6 fields magnetic dual to B_2 and C_2. $SL(2,\mathbb{Z})$ is a gauge symmetry of the theory on the (orientable) space $\mathrm{AdS}_5 \times \mathbb{RP}^5$, so in order for the action to be well defined we need B_6 and C_6 to also transform with a minus sign under $-1 \in SL(2,\mathbb{Z})$.

What this means is that H_3 and F_3 are elements of the cohomology group with local coefficients $H^3(X^5 \times \mathbb{RP}^5; \widetilde{\mathbb{Z}})$ (we refer the reader to appendix 3.H of [33] for details), and similarly their magnetic duals H_7 and F_7 are elements of $H^7(X^5 \times \mathbb{RP}^5; \widetilde{\mathbb{Z}})$. On the other hand F_5 is classified by $H^5(X^5 \times \mathbb{RP}^5; \mathbb{Z})$. In what follows we will focus on the structure on $\mathbb{RP}^5$, as the $SL(2,\mathbb{Z})$ bundle is trivial on X^5. The untwisted cohomology groups of $\mathbb{RP}^5$ are standard, and the twisted ones can be derived easily from the results in [34]:

$$\begin{aligned} H^*(\mathbb{RP}^5, \mathbb{Z}) &= \{\mathbb{Z}\,,\ 0\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\,,\ \mathbb{Z}\} \\ H^*(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) &= \{0\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\}\,. \end{aligned} \tag{4}$$

Similar considerations hold for homology: (p,q) 1-branes (such as fundamental strings and D1 branes) are elements of $H_2(X^5 \times \mathbb{RP}^5; \widetilde{\mathbb{Z}})$, (p,q) 5-branes are elements of $H_6(X^5 \times \mathbb{RP}^5; \widetilde{\mathbb{Z}})$, and D3 branes are elements of $H_4(X^5 \times \mathbb{RP}^5; \mathbb{Z})$. The relevant homology groups are (by Poincaré duality, which holds since $\mathbb{RP}^5$ is orientable)

$$\begin{aligned} H_*(\mathbb{RP}^5, \mathbb{Z}) &= \{\mathbb{Z}\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}\} \\ H_*(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) &= \{\mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\,,\ 0\,,\ \mathbb{Z}_2\,,\ 0\}\,. \end{aligned} \tag{5}$$

With an understanding of the cycles that the branes can wrap, it is straightforward to identify the charged operators of the Spin(4N) theory[29]: the vector Wilson line W_V is a fundamental string on a point of $\mathbb{RP}^5$, the s-spinor Wilson line W_{s} is a D5-brane on $\mathbb{RP}^4$, and finally the c-spinor Wilson line W_{c} is the

lated by the outer automorphism) gives the $Ss(4N)$ global form instead, which also has non-invertible symmetries. The analysis of both cases is essentially identical, so we will focus on the $Sc(4N)$ case.

5 Our conventions are as follows: Spin(4n) has two spinor irreps unrelated by complex conjugation, which we denote by "s" and "c". $\mathbb{Z}_2^{\mathsf{s}}$ acts on c, and leaves s invariant, while $\mathbb{Z}_2^{\mathsf{c}}$ acts on s and leaves c invariant. This choice of notation is motivated by consistency with the fact that the diagonal $\mathbb{Z}_2$ combination, traditionally denoted $\mathbb{Z}_2^V$, does not act on the vector. We define $Sc(4N) \coloneqq \mathrm{Spin}(4N)/\mathbb{Z}_2^{\mathsf{s}}$.

6 We will assume for simplicity that neither $\mathcal{M}^4$ nor any of the submanifolds where we will wrap defects contains torsion in homology. This is not physically required, but it simplifies some of the formulas below.

combination of both previous lines: a D5-brane/F1 bound state, again wrapped on $\mathbb{RP}^4$. In all cases the branes wrap a surface on X^5 extending to the boundary, where they end on a line.

We can go to the $SO(4N)$ theory by gauging the diagonal factor $\mathbb{Z}_2^V \subset \mathbb{Z}_2^{\rm s} \times \mathbb{Z}_2^{\rm c}$. The vector line W_V is unaffected by the gauging, so it survives, but the $W_{\rm s}$ and $W_{\rm c}$ lines are no longer gauge invariant, and become non-genuine (that is, boundaries of surface operators). A non-genuine line H_V of the Spin$(4N)$ theory, with $w_2^{\rm s} = w_2^{\rm c}$ flux around it, now becomes a genuine line operator in the $SO(4N)$ theory. Holographically this operator corresponds to a D1 brane wrapping a point in $H_0(\mathbb{RP}^5; \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$.[7] We denote the generators for these two symmetries $D_2^{B,e}(\mathcal{M}^2)$ (acting on fundamental strings) and $D_2^{C,m}(\mathcal{M})$ (acting on D1 branes), and the corresponding background fields B_2^e and C_2^m.

Starting from the $SO(4N)$ theory we can gauge various pairs of global symmetries, an operation that, due to the cubic anomaly (2), results in theories with non-invertible symmetries [12, 14]:

$$\begin{aligned} \mathrm{Pin}^+(4N)&: \quad \text{gauge } D_3^{(0)} \text{ and } D_2^{C,m} \\ Sc(4N)&: \quad \text{gauge } D_2^{B,e} \text{ and } D_2^{C,m} \\ PO(4N)&: \quad \text{gauge } D_3^{(0)} \text{ and } D_2^{B,e}. \end{aligned} \tag{6}$$

In these expressions $D_3^{(0)}$, or more precisely $D_3^{(0)}(\mathcal{M}^3)$ is the generator for the outer automorphism 0-form symmetry of the $SO(4N)$ theory.

We are thus led to the crucial question in this paper: having identified the *charged* operators in the field theory in terms of the holographic dual, what is the holographic description of the *charge* operators implementing the global symmetries in the $SO(4N)$ theory?

For concreteness, let us specialise to the holographic dual of the symmetry generator $D_2^{C,m}(\mathcal{M}^2)$ of the $SO(4N)$ theory, measuring how many 't Hooft lines (mod 2) H_V are linked by $\mathcal{M}^2$, without taking into account the Wilson lines W_V. Given our identification of lines above, a natural guess would be

$$D_2^{C,m}(\mathcal{M}^2) \stackrel{?}{\rightarrow} e^{i\pi \int_{\mathcal{M}_2 \times \mathbb{RP}^4} C_6}, \tag{7}$$

where $\mathcal{M}^2$ lives on $\mathcal{M}^4$, and becomes the symmetry operator when pushed to the boundary. This holonomy certainly measures the number of D1 branes linked by $\mathcal{M}^2$ (the basic argument is given below in case of the outer automorphism 0-form symmetry), but it cannot be the right answer for a number of reasons. First, we know that in IIB string theory fluxes are not measured by cohomology, but rather K-theory.[35–37] A way of capturing the right K-theoretic formula is to phrase the answer in terms of the Wess-Zumino coupling in the D5 brane action:

$$D_2^{C,m}(\mathcal{M}^2) \stackrel{?}{\rightarrow} e^{\mathrm{WZ}(\mathcal{M}_2 \times \mathbb{RP}^4)}. \tag{8}$$

where [37–39]

$$\mathrm{WZ}(X) = 2\pi i \int_X e^{F_2 - B_2} \sqrt{\frac{\hat{A}(TX)}{\hat{A}(NX)}} (C_0 + C_2 + \cdots) \tag{9}$$

A second reason why we expect neither (7) nor (8) to be the full answer is that in string theory there are no local operators, only dynamical objects. So we should aim to represent the charge generator by a dynamical object, and not simply a defect. The dynamical objects that are electrically charged under C_6, and would arise when fixing the insertion of the defect as a boundary condition, are D5 branes.

While neither argument is conclusive, they both suggest that the holographic description of the symmetry generator is a full D5, pushed to the boundary:[8]

$$D_2^{C,m}(\mathcal{M}^2) \rightarrow \mathrm{D5}(\mathcal{M}_2 \times \mathbb{RP}^4). \tag{10}$$

This ansatz has the additional virtue of restoring the common origin between lines and charge generators, familiar from the formulation of symmetries in terms of relative field theories.[40]

An objection one might raise about (10) is that branes are not topological, while charge operators should be. As we will see in a moment, the worldvolume theory on the branes, when reduced to $\mathcal{M}^2 \subset X^5$, is a discrete $\mathbb{Z}_2$ gauge theory. Therefore the potential lack of deformation-invariance coming from the gauge fields on the brane is not an issue. There is still an overall factor of the volume, but it does not couple to the dynamical fields of the field theory on the boundary, so it can be absorbed into a counterterm.

A subtle feature of (10) is that the worldvolume theories on the brane are quantum field theories, so we should sum over them. As we will argue, the sum over worldvolume degrees of freedom provides precisely the minimal anomalous TQFT "dressing" the bare symmetry generator identified in [14]. This is a very non-trivial test of the identification (10).

Clearly, if the ansatz (10) is correct, the holographic dual of the operator counting 't Hooft lines H_V is the S-dual of (10):

$$D_2^{B,e}(\mathcal{M}^2) \rightarrow \mathrm{NS5}(\mathcal{M}_2 \times \mathbb{RP}^4). \tag{11}$$

Additionally, the $SO(4M)$ theory has the 0-form parity symmetry discussed above. The point operator charged under this symmetry is known as the Pfaffian operator. As discussed in [29] the Pfaffian operator is represented holographically by a D3 brane wrapping the $\mathbb{RP}^3$ cycle inside $\mathbb{RP}^5$, and extending to a point on the boundary. We will refer to this brane as the "Pfaffian brane".

We now argue that the holographic dual of the generator of this symmetry is

$$D_3^{(0)}(\mathcal{M}^3) \rightarrow \mathrm{D3}(\mathcal{M}^3 \times \mathbb{RP}^1). \tag{12}$$

More precisely, we will show that the Pfaffian brane is charged under this D3 in the Hamiltonian formalism, so we take the boundary to be of the form $\mathcal{M}^3 \times \mathbb{R}_t$, with the last component the time direction along the boundary. We choose coordinates so

[7] The simplest derivation of this fact follows from recalling that the $SO(4N)$ field theory is invariant under $SL(2,\mathbb{Z})$, which maps to an $SL(2,\mathbb{Z})$ action on the holographic dual. We refer the reader to [28] for a systematic analysis.

[8] The authors of [31] provide complementary evidence for the same proposal.

that the endpoint of the Pfaffian operator is at $t = 0$. Now we wrap the putative symmetry generator D3 on $\mathcal{M}^3 \times \mathbb{RP}^1$, where $\mathcal{M}^3$ is at the boundary at $t = 0$. Because the F_5 RR flux is self-dual, the two D3 branes that we have introduced do not commute[41–44]:

$$\begin{aligned} \mathrm{D3}(\mathcal{M}^3 \times \mathbb{RP}^1)\,\mathrm{Pf}(\mathrm{pt}) &= e^{2\pi i\,\mathsf{L}(\mathbb{RP}^1,\mathbb{RP}^3)}\,\mathrm{Pf}(\mathrm{pt})\mathrm{D3}(\mathcal{M}^3 \times \mathbb{RP}^1) \\ &= -\,\mathrm{Pf}(\mathrm{pt})\mathrm{D3}(\mathcal{M}^3 \times \mathbb{RP}^1) \end{aligned} \tag{13}$$

where Pf(pt) is the D3 brane representing the Pfaffian operator, $\mathsf{L}(\mathbb{RP}^1, \mathbb{RP}^3) = \frac{1}{2}$ is the linking pairing between the given cycles of $\mathbb{RP}^5$, and we have used that the given branes intersect at a point on the $t = 0$ spatial slice on X^5. Equation (13) is the Hamiltonian version of the statement that the Pfaffian operator is charged under $\mathrm{D3}(\mathcal{M}^3 \times \mathbb{RP}^1)$, as claimed. This discussion can be generalised straightforwardly to show that the branes (10) and (11) do indeed give the expected charges to the W_V and H_V lines of the $SO(4N)$ theory, as claimed.

3. TQFT Stacking from Wess-Zumino Couplings

Our task in this section will be to deduce the non-invertibility of the symmetry generators of the theories in (6) from our assumption that symmetry generators are represented holographically by branes.

3.1. Fluxes and Twisted Differential Cohomology

Our basic tool will be differential cohomology. We refer the reader to [45] for a review of the basic techniques and notation that we use. The analysis in this paper has some novelties with respect to the discussion in [45], which we now discuss.

The main difference is that we will be working with *twisted* differential cohomology. The twisted and untwisted cohomology groups of $\mathbb{RP}^5$ were given in (4) above. The ring structure induced by the cup product for $\mathbb{RP}^5$ can be obtained by adapting the discussion in Lemma 1 of [46] (see also [34]). It is most easily described by adjoining the twisted and untwisted cohomology groups

$$\begin{aligned} \mathsf{H}^*(\mathbb{RP}^5) &= H^*(\mathbb{RP}^5;\mathbb{Z}) \oplus H^*(\mathbb{RP}^5;\widetilde{\mathbb{Z}}) \\ &= \mathbb{Z}[t_1, u_5]/(2t_1, t_1^6, u_5^2)\,. \end{aligned} \tag{14}$$

That is, we have free components of degree 0 and 5, and $\mathbb{Z}_2$ torsional components of degrees 1 to 5, generated by t_1^n. In particular, taking an even number of powers of t_1 gives an untwisted class, while taking an odd number of powers gives a twisted one. In what follows we will use the notation $t_{2n+1} = t_1^{2n+1}$ for twisted classes and $u_{2n} = t_1^{2n}$ together with u_5 for untwisted ones.

We denote by $\breve{t}_k$ a flat differential cohomology class with characteristic class t_k, which we denote by $I(\breve{t}_k) = t_k$, and similarly for $\breve{u}_k$. We note that $I(\breve{t}_1^2) = u_2$, and similarly $I(\breve{t}_1^4) = u_4$, so perfectness of the linking pairing on $H^2(\mathbb{RP}^5;\mathbb{Z}) \times H^4(\mathbb{RP}^5;\mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ implies that

$$\int_{\mathbb{RP}^5} \breve{t}_1^6 = \int_{\mathbb{RP}^5} \breve{u}_2 \star \breve{u}_4 = \frac{1}{2} \mod 1\,. \tag{15}$$

This equation together with the ring structure (14) will be our workhorses in what follows.

Finally, before moving on to the examples, we need to know how to represent background fluxes in terms of differential cohomology. To lighten notation, in this section we introduce $b_2 := B_2^e$ and $c_2 := C_2^m$. Recall that the objects charged under these backgrounds are F1 and D1 branes, respectively, so the holographic fluxes encoding these backgrounds are $\breve{H}_3$ and $\breve{F}_3$, which are asymptotically of the form $\breve{H}_3 = b_2 \star \breve{t}_1$ and $\breve{F}_3 = c_2 \star \breve{t}_1$. By imposing this asymptotic form we ensure that the charged lines in the field theory acquire the right holonomies, see [45, 47] for analysis of similar examples. We could also include terms proportional to $\breve{t}_3$ in these expansions, but they would correspond to a change of the gauge algebra to $\mathfrak{so}(4N+1)$ (for $\breve{F}_3$) of $\mathfrak{usp}(4N)$ (for $\breve{H}_3$)[29] so we will not consider these terms further.[9] Finally, a field theory 0-form symmetry background a_1 for $D_2^{(0)}$ is represented by $\breve{F}_5 = a_1 \star \breve{u}_4$. There are additional terms possible in the expansion for $\breve{F}_5$, we will discuss these below.

3.2. Non-Invertibles in $Sc(4N)$ from D3 Branes

We start with the case of the $Sc(4N)$ theory, where following the analysis in [12, 14], we expect to get the topological defect that is non-invertible from the generator of the 0-form symmetry in the $SO(4N)$ theory. We identified this generator above with the D3-brane wrapped on $\mathcal{M}^3 \times \mathbb{RP}^1 = \mathcal{M}^3 \times S^1$. The worldvolume $U(1)$ field on the D3 brane is odd under $-1 \in SL(2,\mathbb{Z})$, because it is a trivialisation of B_2, which is odd. Therefore it takes values in the twisted cohomology group $H^2(\mathcal{M}^3 \times \mathbb{RP}^1; \widetilde{\mathbb{Z}})$. We have $H^*(\mathbb{RP}^1; \widetilde{\mathbb{Z}}) = \{0, \mathbb{Z}_2\}$.

When computing the path integral on the D3, the field strength F_2 on the brane will induce D1 charge due to the Wess-Zumino term (9), while the magnetic field strength F_2^D will induce F1 charge. The Wess-Zumino action written in terms of the electric variable F_2 is

$$S_{\mathrm{D3},e} = 2\pi i \int_{\mathcal{M}^3\times\mathbb{RP}^1} \breve{F}_5 + \breve{F}_3 \star (\breve{\mathscr{F}}_2) + \breve{F}_1 \star \left(\frac{1}{2}\breve{\mathscr{F}}_2 \star \breve{\mathscr{F}}_2 + \frac{1}{24}\breve{e}\right), \tag{16}$$

with $\breve{\mathscr{F}}_2 = \breve{F}_2 - \breve{B}_2$, $\breve{F}_1$ a differential cohomology uplift of C_0, and $\breve{e}$ is a differential cohomology uplift of the Euler class of $\mathcal{M}^3 \times \mathbb{RP}^1$. The term proportional to $\breve{F}_1$ will be relevant only for analysis of anomalies in the space of coupling constants, which we do not analyse in this note (although this is certainly an interesting direction to explore further).

We also need to consider couplings of the form $\breve{F}_3 \star \breve{B}_2$. As argued in [37] (elaborating on results of [48, 49]), these couplings are not present when measuring the actual K-theory charges, which is what we are ultimately interested in, so we will simply set $\breve{B}_2$ to 0. A more careful treatment of this issue would be desirable, but given that inclusion of these background fields would

[9] When doing this sort of expansion there is an additional subtlety involving topologically trivial differential characters that is discussed at length in [45]. It will not affect our considerations, so we will ignore such terms.

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not change our conclusions (since they would provide overall invertible prefactors on the brane action in any case, even if we included them), we will postpone a more careful treatment of this point to future work.

With these simplifications taken into account, the relevant part of the Wess-Zumino action for the D3 becomes

$$S_{\mathrm{D3},e} = 2\pi i \int_{\mathcal{M}^3\times\mathbb{RP}^1} \breve{F}_5 + \breve{F}_3 \star \breve{F}_2\,. \tag{17}$$

The first term is the differential cohomology avatar of the naive guess $\int_{\mathcal{M}^3\times\mathbb{RP}^1} C_4$ for the flux operator in the field theory. $\breve{F}_5$ is even under the $-1\in SL(2,\mathbb{Z})$ action, so its general decomposition is of the form $\breve{F}_5 = N\star\breve{u}_5 + a_1\star\breve{u}_4 + a_3\star\breve{u}_2 + N\star\breve{1}$. Here N are the number of units of RR 5-form flux on the $\mathbb{RP}^5$, and we have used that F_5 is self-dual to relate the components of degrees 5 and 0.

In terms of this decomposition we have an effective operator in AdS_5 of the form

$$\begin{aligned} D(\mathcal{M}^3) &= \exp\left(2\pi i\int_{\mathcal{M}^3\times\mathbb{RP}^1}\breve{F}_5\right)\\ &= \exp\left(2\pi i\int_{\mathcal{M}^3\times\mathbb{RP}^5}\breve{F}_5\star\breve{u}_4\right)\\ &= \exp\left(\pi i\int_{\mathcal{M}^3} a_3\right), \end{aligned} \tag{18}$$

where in the second equality we have used Poincaré duality on $\mathbb{RP}^5$ to relate $\mathbb{RP}^1$ to u_4, and in the third used that the only non-trivial pairing in $\mathbb{RP}^5$ appearing after the expansion of $\breve{F}_5$ is (15). This is the expected formula for the operator measuring discrete electric flux for the outer automorphism symmetry in the $SO(4N)$ theory.

The second term is the more interesting one for our purposes. As explained above, field theory backgrounds for the symmetry $D_2^{C,m}$ are described holographically by fluxes with asymptotic form $\breve{F}_3 = c_2\star\breve{t}_1$. Similarly, we can expand $\breve{F}_2 = \gamma_1\star\breve{t}_1$. We then have (using the formulas for integration on products reviewed in [45])

$$2\pi i\int_{\mathcal{M}^3\times\mathbb{RP}^1}\breve{F}_3\star\breve{F}_2 = 2\pi i\int_{\mathcal{M}^3} c_2\gamma_1\int_{\mathbb{RP}^1}\breve{t}_1\star\breve{t}_1 = \pi i\int_{\mathcal{M}^3} c_2\gamma_1 \tag{19}$$

where in the last step we have again used the fact that the linking pairing is perfect, so

$$\int_{\mathbb{RP}^1}\breve{t}_1\star\breve{t}_1 = \frac{1}{2} \mod 1. \tag{20}$$

So far we have considered the charge induced on a D3 due to the gauge field strength F_2. The computation above shows that it induces an effective coupling on $\mathcal{M}^3$ to the background for $D_2^{C,m}$. By IIB S-duality, this implies that a dual field strength $F_2^D = \phi_1\star\breve{t}_1$ induces a coupling of the form

$$\pi i\int_{\mathcal{M}^3} b_2\phi_1 \tag{21}$$

on the effective operator on AdS_5. The same result can be obtained from the effective action presented in the magnetic variables obtained in [50].

In elementary terms, the two couplings (19) and (21) that we have just derived can be understood as encoding the well known facts that worldvolume flux on the D3 induces D1 charge, and magnetic worldvolume flux F1 charge. Recall that the D1 and F1 are the charged objects in the $SO(4N)$ theory before gauging their corresponding symmetries. After gauging, they will become the symmetry generators for the dual magnetic symmetries in the $Sc(4N)$ theory (at least if our general philosophy of identifying branes with symmetries is correct). So what we have just shown, is that when doing the path integral on the D3 we will have to sum over insertions of the symmetry generators for the 1-forms of the theory. This is certainly suggestive that condensations[9,13,16] are going to enter the picture after gauging.

The precise details are nevertheless somewhat subtle. In general, when performing the path integral the standard prescription is that we choose whether we formulate the theory in terms of electric or magnetic variables, and then sum over the specified variables only. From this point of view the two couplings (19) and (21) seem somewhat at odds, and it is not clear which one we should choose. What saves the day is that this standard prescription has to be subtly modified whenever the cohomology groups where the electric and magnetic fluxes live contain torsional components. In this case, as originally pointed out by [42–44], the electric and magnetic flux operators do not commute. As shown in [45] (see also [51] for a different derivation of the same result) this flux non-commutativity leads to the existence of a discrete gauge theory when the theory is compactified on the space with torsion. The argument, adapted to the system at hand, goes as follows.

Our initial theory is four dimensional $U(1)$ Maxwell theory on the D3, compactified on $\mathcal{M}^3\times\mathbb{RP}^1$. We will present a Hamiltonian quantisation analysis, so we assume that $\mathcal{M}^3 = \mathcal{N}^2\times\mathbb{R}$, and we identify the last component with the time direction.[10] The spatial slice is of the form $\mathcal{N}^2\times\mathbb{RP}^1$. There is a non-trivial $SL(2,\mathbb{Z})$ duality bundle along the $\mathbb{RP}^1 = S^1$ direction inherited from the $\mathbb{RP}^5$ background, with holonomy -1, which induces a $(F_2, F_2^D)\to(-F_2,-F_2^D)$ transformation of the worldvolume gauge field. Therefore, just as in the IIB background itself, the worldvolume gauge fields on the D3 are valued in twisted cohomology. In particular $H^1(\mathbb{RP}^1;\widetilde{\mathbb{Z}}) = \mathbb{Z}_2$, which justifies the statement above that there is torsion in this problem.

Consider the operators $\Phi_e(a\otimes t_1)$, $\Phi_m(b\otimes t_1)$ that measure electric and magnetic fluxes on the torsional sector. They are associated with flat, topologically non-trivial elements of $\mathrm{Tor}\,H^2(\mathcal{N}^2\times\mathbb{RP}^1;\widetilde{\mathbb{Z}})$,[43,44] which in our case are all of the form $a\otimes t_1$, where $a\in H^1(\mathcal{N}^2;\mathbb{Z})$ and t_1 is the generator of $H^1(\mathbb{RP}^1;\widetilde{\mathbb{Z}})$. Alternatively, using Poincaré duality, we can view these operators as the holonomy of the twisted fluxes $\breve{F}_2$ and $\breve{F}_2^D$ on cycles $\alpha\times\widetilde{\mathrm{pt}}$ and $\beta\times\widetilde{\mathrm{pt}}$, where α and β are Poincaré dual to a and b in $\mathcal{N}^2$, and $\widetilde{\mathrm{pt}}$

[10] A Lagrangian derivation will appear in [52].

the (twisted) point in $H_0(\mathcal{N}^2;\widetilde{\mathbb{Z}})$, which is Poincaré dual on $\mathbb{RP}^1$ to t_1. So we have

$$\begin{aligned}\Phi_e(a\otimes t_1) &= \exp\left(2\pi i\int_\alpha \gamma_1\int_{\widetilde{\text{pt}}}\check{t}_1\right)\\ &= \exp\left(2\pi i\int_\alpha \gamma_1\int_{\mathbb{RP}^1}\check{t}_1^2\right)\\ &= \exp\left(\pi i\int_\alpha \gamma_1\right)\end{aligned} \tag{22}$$

and similarly

$$\Phi_m(b\otimes t_1) = \exp\left(\pi i\int_\beta \phi_1\right). \tag{23}$$

Now, it follows from the general analysis of [42–44] that

$$\Phi_e(a\otimes t_1)\Phi_m(b\otimes t_1) = (-1)^{\int_{\mathcal{N}^2} ab}\Phi_m(b\otimes t_1)\Phi_e(a\otimes t_1), \tag{24}$$

or equivalently, formulating everything in terms of homology on $\mathcal{N}^2$ (and abusing notation slightly):

$$\Phi_e(\alpha)\Phi_m(\beta) = (-1)^{\alpha\cdot\beta}\Phi_m(\beta)\Phi_e(\alpha). \tag{25}$$

These commutation relations are precisely those of a $\mathbb{Z}_2$ theory. We can represent this theory by a gauge theory on the two fields γ_1, ϕ_1 with action[53]

$$S_{\mathbb{Z}_2} = \pi i\int_{\mathcal{M}^3}\gamma_1\delta\phi_1. \tag{26}$$

We have identified the fields appearing in the Lagrangian with γ_1 and ϕ_1 since these are precisely the fields whole holonomies are measured by the operators in the theory, by construction.

Assembling all the pieces together, we find that the effective partition function on the D3, seen as an 3-surface dynamical object on X^5, is (up to an overall normalisation)

$$\begin{aligned}\mathcal{N}(\mathcal{M}^3) = D_3^{(0)}(\mathcal{M}^3)\cdot&\\ \int\mathcal{D}\gamma_1\mathcal{D}\phi_1\exp&\left(\pi i\int_{\mathcal{M}^3}\gamma_1\delta\phi_1 + c_2\gamma_1 + b_2\phi_1\right).\end{aligned} \tag{27}$$

The path integral over γ_1,ϕ_1 is the remnant of the $U(1)$ YM path integral in this torsional setting. This is precisely the non-invertible operator found in [14].

Fusion Rules

Now that we have a full description of the symmetry defect, including its TQFT sector, we can derive the fusion rules for the extended operators in the $Sc(4N)$ theory, in particular showing that $\mathcal{N}(\mathcal{M}^3)$ is a non-invertible operator of the $Sc(4N)$ theory. Since the TQFT that comes out of the brane dynamics is identical to the one conjectured in [14], the rest of our derivation of the fusion rules can proceed exactly as in that paper (and the similar analysis in [12]). We include the details of the argument for completeness and convenience for the reader, and then offer some comments reinterpreting some of the features of the computation from a brane perspective.

Consider first the fusion of two copies of $\mathcal{N}(\mathcal{M}^3)$. Each defect comes with its own $\mathbb{Z}_2$ TQFT, so we have two sets of dynamical fields:

$$\begin{aligned}\mathcal{N}(\mathcal{M}^3)\times\mathcal{N}(\mathcal{M}^3) = \int\mathcal{D}\gamma_1\mathcal{D}\phi_1\mathcal{D}\gamma_1'\mathcal{D}\phi_1'&\\ \exp\left(\pi i\int_{\mathcal{M}^3}\gamma_1\delta\phi_1 + \gamma_1'\delta\phi_1' + c_2(\gamma_1+\gamma_1') + b_2(\phi_1+\phi_1')\right).&\end{aligned} \tag{28}$$

Switching to new variables $\gamma_1,\hat{\gamma}_1 := \gamma_1+\gamma_1'$, $\phi_1,\hat{\phi}_1 := \phi_1+\phi_1'$, the action becomes

$$\begin{aligned}\mathcal{N}(\mathcal{M}^3)\times\mathcal{N}(\mathcal{M}^3) = \int\mathcal{D}\gamma_1\mathcal{D}\phi_1\mathcal{D}\hat{\gamma}_1\mathcal{D}\hat{\phi}_1&\\ \exp\left(\pi i\int_{\mathcal{M}^3}\hat{\gamma}_1\delta\hat{\phi}_1 + \hat{\gamma}_1\delta\phi_1 + \gamma_1\delta\hat{\phi}_1 + c_2\hat{\gamma}_1 + b_2\hat{\phi}_1\right).&\end{aligned} \tag{29}$$

We can integrate ϕ_1 and γ_1 out, which imposes $\delta\hat{\gamma}_1 = \delta\hat{\phi}_1 = 0$, so $\hat{\gamma}_1\delta\hat{\phi}_1 = 0$. We then have

$$\mathcal{N}(\mathcal{M}^3)\times\mathcal{N}(\mathcal{M}^3) = \int\mathcal{D}\hat{\gamma}_1\mathcal{D}\hat{\phi}_1\exp\left(\pi i\int_{\mathcal{M}^3}c_2\hat{\gamma}_1 + b_2\hat{\phi}_1\right). \tag{30}$$

Poincaré dualising $\hat{\gamma}_1$ and $\hat{\phi}_1$ to $\Gamma,\Phi\in H_2(\mathcal{M}^3;\mathbb{Z})$, this can be rewritten as

$$\mathcal{N}(\mathcal{M}^3)\times\mathcal{N}(\mathcal{M}^3) = \sum_{\Gamma,\Phi\in H_2(\mathcal{M}^3;\mathbb{Z})} D_2^{\text{c},e}(\Gamma)D_2^{\text{s},m}(\Phi) \tag{31}$$

where $D_2^{\text{c},e}$ and $D_2^{\text{s},m}$ are the 1-form symmetry generators of the $Sc(4N)$ theory. (The notation is explained below.) So $\mathcal{N}$ is indeed a non-invertible defect in the $Sc(4N)$ theory, since the right hand side is a sum of operators.

This was the derivation in [14]. Holographically, the physical meaning of the computation can be understood as follows. We have argued that the defect $\mathcal{N}(\mathcal{M}^3)$ corresponds to a D3 wrapping $\mathcal{M}^3\times\mathbb{RP}^1$, including its quantum dynamics. The effect of the quantum dynamics is to sum over induced charges, which in this case means summing over D3/F1 and D3/F1 bound states. (The precise way in which this sum happens involves, as shown above, a $\mathbb{Z}_2$ gauge theory.) If there was no sum, but only a fixed induced charge (the trivial one, say), then taking the square would lead to a complete annihilation of the $\mathbb{Z}_2$ charges, and therefore a trivial operator. Since there is a sum involved some of the cross-terms in the square of the sum will lead to incomplete annihilations, leaving a sum over F1 and D1 insertions along the worldvolume of the D3. The D3 charge is always there no matter the induced charge, and disappears, so only the sum over D1 and F1 insertions remains.

In order to show that this physical process does indeed produce (31), all we need to verify is that the symmetry generators of the $Sc(4N)$ theory are the F1 and D1. This is immediate, since they are the genuine lines in the $SO(4N)$ theory, and we are gauging the symmetry they are charged under, so they become the magnetic symmetry generators in the gauged theory. It is also instructive to derive it from the Spin$(4N)$ starting point. The s-spinor Wilson line $W_{\rm s}$, given by a D5 brane wrapped on $\mathbb{RP}^4$, is neutral under $\mathbb{Z}_2^{\rm s}$ (recall our conventions from footnote), which is the symmetry that we gauge to go to $Sc(4N)$. So the corresponding charge operator, the D1 on $\widetilde{\rm pt}$, survives as a charge operator on the $Sc(4N)$ theory. We have denoted it above by $D_2^{{\rm c},e}$. On the other hand the c-spinor and vector Wilson lines are not invariant, due to the presence of fundamental strings in them, which are not invariant under $\mathbb{Z}_2^{\rm s}$. So after gauging $\mathbb{Z}_2^{\rm s}$ the fundamental string on a twisted point in $\mathbb{RP}^5$ becomes the second (magnetic) symmetry generator in the $Sc(4N)$ theory, which we have denoted above by $D_2^{{\rm s},m}$.

We are finally left with the task of determining the fusion of $\mathcal{N}(\mathcal{M}^3)$ with the one-form symmetry generators $D_2^{{\rm c},e}(\Gamma)$ and $D_2^{{\rm s},m}(\Phi)$. Consider for example $D_2^{{\rm c},e}(\Gamma)$. We have just argued that it corresponds to a D1 brane on $\Gamma\times\widetilde{\rm pt}$. Fusing it with $\mathcal{N}(\mathcal{M}^3)$, which involves a sum over induced D1 branes wrapping the Poincaré dual $\mathrm{PD}[\gamma_1]\times\widetilde{\rm pt}$ to γ_1 amounts to shifting $\gamma_1\to\gamma_1+\mathrm{PD}[\Gamma]$ in (27). But this can clearly be reabsorbed in a change of variables, giving back $\mathcal{N}(\mathcal{M}^3)$. So

$$\mathcal{N}(\mathcal{M}^3)\times D_2^{{\rm c},e}(\Gamma)=\mathcal{N}(\mathcal{M}^3)\,. \tag{32}$$

An identical argument shows

$$\mathcal{N}(\mathcal{M}^3)\times D_2^{{\rm s},m}(\Phi)=\mathcal{N}(\mathcal{M}^3)\,. \tag{33}$$

We have shown that the $\mathcal{N}(\mathcal{M}^3)$ operators of the $Sc(4N)$ theory are non-invertible, and are represented holographically by D3 branes. A small puzzle remains: our starting point was that the bulk of the holographic dual was the same for all global forms, so the same D3 brane appears in the bulk of all theories with the same local dynamics, including theories that are not expected to have non-invertible symmetries. The reason that the D3 does not lead to non-invertible symmetries in some cases has to do with boundary conditions (as it should, as this is the only thing that is different in the various cases). Consider for instance the $SO(4N)$ theory, where the D3 on $\mathbb{RP}^1$ is also a symmetry operator, implementing the outer automorphism. As we push to the boundary, we obtain an operator of the form (27), but with a crucial difference: the IIB B_2 and C_2 fields have a Dirichlet boundary condition in this case, so they are not dynamical but instead they provide backgrounds for the global 1-form symmetries for the $SO(4N)$ theory. So the term

$$\int \mathcal{D}\gamma_1\mathcal{D}\phi_1\exp\left(\pi i\int_{\mathcal{M}^3}\gamma_1\delta\phi_1+C_2\gamma_1+B_2\phi_1\right) \tag{34}$$

in (27) (where we have capitalised B_2 and C_2 to indicate that now they are fixed background fields) does not depend on any dynamical field in the $SO(4N)$ theory, so it is essentially trivial as an operator of the $SO(4N)$ theory (it can be taken out of the path integral). In this case it is consistent to split it off from the invertible part $D_3^{(0)}(\mathcal{M}^3)$, which can meaningfully be considered in isolation.

3.3. 4d $PO(4N)$ and Pin$^+$ Non-Invertibles

The other two theories with non-invertible symmetries in (6) can be analysed in a very similar way.

Let us start with the $PO(4N)$ case. Here we gauge $D_3^{(0)}$ and $D_2^{B,e}$, so we expect the non-invertible 2-surface operator to be associated with $D_2^{C,m}$, which we argued above is given by a D5 brane wrapping $\mathbb{RP}^4\subset\mathbb{RP}^5$. We will need the twisted and untwisted cohomology groups of $\mathbb{RP}^4$, these are

$$\begin{aligned} &H^*(\mathbb{RP}^4;\mathbb{Z})=\{\mathbb{Z},0,\mathbb{Z}_2,0,\mathbb{Z}_2\}\\ &H^*(\mathbb{RP}^4;\widetilde{\mathbb{Z}})=\{0,\mathbb{Z}_2,0,\mathbb{Z}_2,\mathbb{Z}\}\,. \end{aligned} \tag{35}$$

(The second line follows from analysing the twisted Gysin sequence in [34].) As above, $\breve{F}_2$ is in the twisted sector, so it expands as $\gamma_1\otimes\breve{t}_1$, but its magnetic dual $\breve{F}_4^D$ is now *untwisted*: this is needed to be able to write a kinetic term on the twisted $\mathbb{RP}^4$. It therefore has an expansion of the form $\breve{F}_4^D=\phi_4\star 1+\phi_2\star\breve{u}_2+\phi_0\star\breve{u}_4$.

In the electric frame the action on the D5 is of the form

$$S=\int_{\mathcal{M}^2\times\mathbb{RP}^4}\breve{F}_7+\breve{F}_2\star\breve{F}_5+\left(\frac{1}{2}\breve{F}_2^2+\frac{1}{24}\breve{e}\right)\star\breve{F}_3+\cdots \tag{36}$$

where the missing terms are proportional to $\breve{F}_1$, so we will ignore them. The term proportional to $\breve{F}_2^2\star\breve{F}_3$ does not contribute for degree reasons, as it goes as $\breve{t}_1^3$. The curvature term $\breve{e}\star\breve{F}_3$ could in principle contribute, but it does not depend on the electric field so it will not enter our considerations. We are left with the first two terms. The first one does clearly contribute, and leads to the expected "naive" 2-surface holonomy operator on $\mathcal{M}^4$, entirely analogously to the discussion around (18). The second term is also interesting. Given our expansion of $\breve{F}_5$ above, there is a single non-vanishing contribution of the form

$$\begin{aligned} &2\pi i\int_{\mathcal{M}^2\times\mathbb{RP}^4}(\gamma_1\star\breve{t}_1)\star(N\star\breve{u}_5+a_1\star\breve{u}_4+a_3\star\breve{u}_2+N\star\breve{1})\\ &=\pi i\int_{\mathcal{M}^2}\gamma_1 a_1\,, \end{aligned} \tag{37}$$

where we have used that $\mathbb{RP}^4$ is Poincaré dual to t_1 in $\mathbb{RP}^5$. This is the statement that worldvolume flux F_2 induces D3 charge. The magnetic flux F_4^D will induce F1 charge (by a generalisation of the analysis in [50]), via a coupling of the form

$$2\pi i\int_{\mathcal{M}^2\times\mathbb{RP}^4}\breve{F}_4^D\star\breve{H}_3=\pi i\int_{\mathcal{M}^2}\phi_0 b_2\,, \tag{38}$$

where we have used the expansion $\breve{H}_3=b_2\star\breve{t}_1$ as above.

All that remains is to obtain the prescription for how to sum over electric and magnetic fluxes. As above, flux non-commutativity can be used to argue that there is an effective $\mathbb{Z}_2$ gauge theory on $\mathcal{M}^2$ with action

$$i\pi\int_{\mathcal{M}^2}\gamma_1\delta\phi_0\,. \tag{39}$$

The only new subtlety in this derivation comes from the fact that on $\mathbb{RP}^4$, being non-orientable, the perfect torsional pairing is between a twisted class $\breve{t}_1$ and an untwisted one $\breve{u}_4$. An easy way to verify the existence of such a coupling is to use Poincaré duality on $\mathbb{RP}^5$:

$$\int_{\mathbb{RP}^4} \breve{t}_1 \breve{u}_4 = \int_{\mathbb{RP}^5} \breve{t}_1^6 = \frac{1}{2} \mod 1\,. \tag{40}$$

Putting all these terms together we obtain the topological action

$$S_{TFT}^{PO(4N)} = i\pi \int_{\mathcal{M}^2} \gamma_1 \delta\phi_0 + \phi_0 b_2 + \gamma_1 a_1 \tag{41}$$

which is precisely the action proposed in [14]. The fusion algebra can be derived as above.

Finally, in the 4d $\mathrm{Pin}^+(4N)$ SYM theory we gauge $D_3^{(0)}$ and $D_2^{C,m}$, so the non-invertible surface defects are realised as NS5-branes on $\mathbb{RP}^4 \times \mathcal{M}^2$. The worldvolume theory on the NS5 is just as on the D5, but the gauge fields couple to the S-dual supergravity fields. We can therefore write down the answer immediately from (41):

$$S_{TFT}^{\mathrm{Pin}^+(4N)} = i\pi \int_{\mathcal{M}^2} \gamma_1 \delta\phi_0 + \phi_0 c_2 + \gamma_1 a_1\,. \tag{42}$$

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Conflict of Interest

The authors declare no conflict of interest.

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