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Existence of geometric ergodic periodic measures of stochastic differential equations

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Abstract

Periodic measures are the time-periodic counterpart to invariant measures for dynamical systems and can be used to characterise the long-term periodic behaviour of stochastic systems. This paper gives sufficient conditions for the existence, uniqueness and geometric convergence of a periodic measure for time-periodic Markovian processes on a locally compact metric space in great generality. In particular, we apply these results in the context of time-periodic weakly dissipative stochastic differential equations, gradient stochastic differential equations as well as Langevin equations. We will establish the Fokker-Planck equation that the density of the periodic measure sufficiently and necessarily satisfies. Applications to physical problems shall be discussed with specific examples.

Keywords: Periodic measures; Markov processes; Existence and uniqueness; Geometric ergodicity; Locally Lipschitz; Stochastic resonance

1. Introduction

In existing literature, there are a vast number of results concerning asymptotic behaviour of both deterministic and stochastic autonomous systems. In particular, there are many powerful

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results on the existence and uniqueness of a limiting invariant measure of time-homogeneous Markovian systems of both finite and infinite dimensions ([8,45,50,43,24,20,47,42,46]). While limiting invariant measure captures the idea that the system "settles" towards an equilibrium, it does not accommodate for systems that are asymptotically periodic. Needless to say, the (asymptotic) periodic solution is natural to study and a central branch within the theory of dynamical systems. However, due to the delicate nature of combining periodicity and randomness, there is still a gap in literature for asymptotic random periodic behaviour of stochastic systems. Filling this gap, in [13], the authors defined rigorously periodic measures which play the role as the time-periodic counterpart of invariant measures. In particular, periodic measures can characterise asymptotic periodic behaviour for stochastic systems. See also [10,12] for some discussions.

As with invariant measures, periodic measures have both theoretical and practical applications to physical sciences. In this paper, we establish explicit criteria for the uniqueness and geometric convergence of a periodic measure for time-periodic Markovian processes applicable to a great general setting. Our results apply to "periodically forced" stochastic systems which have a range of applications. We refer readers to [31,57] and references therein for examples from biology and physics. A notable example includes the overdamped Duffing Oscillator which has been used to model climate dynamics [49,4] to portray the physical phenomena of stochastic resonance. The stochastic resonance model introduced in [4] offered a reasonable physical explanation about the peak observed in the power spectrum of paleoclimatic variations in the last 700,000 years at a periodicity of around 10^5 years. This is in complementary with smaller peaks at periods of 2×10^4 and 4×10^2 years. The major peak represents dramatic climate change to a temperature change of 10 K in Kelvin scale. Except for the dramatic changes, temperature seems to oscillate around fixed values. This phenomenon was suggested to be related to variations in the earth's orbital parameter which also has a similar periodic pattern of changes [48,25]. Such studies were able to reproduce smaller peaks, but failed to explain the 10^5-year cycle major peak. In physics literature, the theory of stochastic resonance for stochastic periodically forced double well potential provides a mathematical model of the transitions between the two equilibria interpreted as climates of the ice age and interglacial period respectively [4,49]. Periodic forcing corresponds to the annual mean variation in insolation due to changes in ellipticity of the earth's orbit, while noise stimulates the global effect of relatively short-term fluctuations in the atmospheric and oceanic circulations on the long-term temperature behaviour. The transition is driven by the noise in the system and happens more likely when one of the well is at or near the highest position and the other one is at the lowest position due to the periodicity. This striking phenomenon is intuitively correct and agrees with the reality. It is noted that stochastic resonance occurs for the right set of parameters in the stochastic periodic double well model, suggested by numerical simulations [18,44,1,6]. The concepts of periodic measures and ergodicity ([13]) provide a rigorous framework and new insight for understanding such physical phenomena.

We will establish the existence and geometric ergodicity of periodic measures for weakly dissipative periodic stochastic differential equations, including the double well problem mentioned above as an example. The periodic measures give a rigorous description of the equilibria observed by physicists and the geometric ergodicity gives the convergence to and the uniqueness of the periodic measures. The uniqueness is significant in explaining the transition between the two wells as otherwise there should be two periodic measures instead of one. However, the current result does not give the estimate of transition time of 10^5 years. We studied this problem in [14] where we derived and analysed the Kramers' equation ([33]) satisfied by the expected exit time**.** Here the Kramers' equation is a parabolic PDE with periodic coefficients while it is an elliptic PDE in the classical case. For autonomous systems with small noise, study by the large deviation theory ([17]) suggests transition would occur at an exponential long time ([16,26]). In our problem, the noise is not necessarily small.

In [56], the authors gave a rigorous definition of random periodic solutions, objects which can be interpreted as the periodic counterpart of stationary solutions. Just as there is an "equivalence" (possibly on an enlarged probability space) between invariant measures and stationary processes ([2,51]), the analogous equivalence between random periodic solution and periodic measures has been proved in [13]. Specifically, by sampling the random periodic solution, one can construct a periodic measure. The existence of a random periodic path was shown for semilinear SDEs in [15] and [12]. Numerical approximations of random periodic paths of SDEs were studied in [10]. In the case of Markovian random dynamical systems, the equivalence of ergodicity of periodic measure with the pure imaginary simple eigenvalues of the infinitesimal generator of the semigroup was established in [13].

In this paper, we establish the existence and uniqueness of a limiting periodic measure for time-periodic Markov processes on locally compact metric spaces. Moreover, we are interested in the geometric convergence. The underlying approach leads to the results for SDEs with weakly dissipative drifts by the means of a Lyapunov function and utilising the coupling method [37,55,47,45,46] of Markov chains. Then, inspired by techniques from [43,8,7,28], we give generally verifiable results in which time-periodic weakly dissipative SDEs, gradient SDEs and Langevin equations possess a unique (geometric) periodic measure. Coefficients of these equations are generally non-Lipschitz.

Some of the technical ideas in this paper are motivated also by the Lyapunov function and discrete Markov chain method in [27]. They studied periodic stochastic differential equations with Lipschitz coefficients and invariant measures of the grid process on multiple integrals of the period. Since periodic measures were defined some years later after [27], the authors were not aware of periodic measures and their ergodicity in [13], hence were not able to obtain the uniqueness of periodic measures. An invariant measure can be obtained by lifting the periodic measure on a cylinder and considering its average over one period. We would like to remark that in this paper, we require the diffusion coefficient to be non-degenerate. We believe this is a technical requirement which can be relaxed by a locally non-degenerate condition or Hörmander's condition. But the non-degenerate case studied in this paper is already applicable in many physical problems such as the stochastic periodic double well potential problem. We note the objective of this current work is to introduce the main ingredients and techniques to attain the existence and uniqueness of periodic measures rather than being the most general results. We leave the refinement of this paper in this direction to a later publication. Note that the diffusion of Langevin equation we investigate is degenerate, but satisfies Hörmander's condition together with the drift of vector fields.

We expect our approach to apply to SPDEs. Our expectation derives from the existing literature where invariant measures for SPDEs was attained via a coupling method that is similar to ours in spirit. For instance, in [41,9,35,34], the respective authors utilised the coupling method to attain an invariant measure for the 2D SNS (two-dimensional stochastic Navier-Stokes equation). In fact, it is shown that the convergence of invariant measure for the 2D SNS equation is geometric in [22]. Other examples include [20] for a class of degenerate parabolic SPDEs including the complex Ginzburg-Landau equation and [41] for dissipative SPDEs. We refer readers to [42] where key aspects to attain invariant measures via the coupling method in the infinite dimensional setting was discussed.

In the final section, we prove the Fokker-Planck equation for the density of a periodic measure. We give explicitly a formula for this density for periodically forced Ornstein-Uhlenbeck processes.

2. Preliminaries

We recall some basic definitions, notation and standard results of Markovian processes on locally compact separable metric space $(E, \mathcal{B})$ where $\mathcal{B}$ is the natural Borel σ-algebra and time indices $\mathbb{T} = \mathbb{N} := \{0, 1, .., \}$ or $\mathbb{R}^+$. By convention, when $\mathbb{T} = \mathbb{N}$, the Markov process is referred as a Markov chain. The objective of this section is to state important results from time-homogeneous Markov chain that would be crucial in proving vital results for T-periodic Markovian systems.

Let $P : \mathbb{T} \times \mathbb{T} \times E \times \mathcal{B} \to [0, 1]$ be a two-parameter Markov transition kernel. It satisfies

(i) $P(s, t, x, \cdot)$ is a probability measure on $(E, \mathcal{B})$ for all $s \le t$ and all $x \in E$.
(ii) $P(s, t, \cdot, B)$ is a $\mathcal{B}$-measurable function for all $s \le t$ and $\Gamma \in \mathcal{B}$.
(iii) (Chapman-Kolmogorov) For all $s \le r \le t$, one has

$$P(s, t, x, \Gamma) = \int_E P(s, r, x, dy) P(r, t, y, \Gamma), \quad x \in E, \Gamma \in \mathcal{B}.$$

(iv) $P(s, s, x, B) = 1_\Gamma(x)$ for all $s \in \mathbb{T}$, $x \in E$ and $\Gamma \in \mathcal{B}$.

For $s \le t$, define linear operators $P(s, t)$ acting on $\mathcal{B}_b(E)$, the space of bounded measurable functions by

$$P(s, t) f(x) = \int_E f(y) P(s, t, x, dy), \quad f \in \mathcal{B}_b(E), x \in E.$$

We say that $P(\cdot, \cdot)$ is Feller if for all $s \le t$, $P(s, t)f \in C_b(E)$ when $f \in C_b(E)$ and strong Feller if $P(s, t)f \in C_b(E)$ when $f \in \mathcal{B}_b(E)$. For $s \le t$, we define adjoint operator $P^*(s, t)$ acting on $\mathcal{P}(E)$, the space of probability measures on $(E, \mathcal{B})$ by

$$\left(P^*(s, t)\mu\right)(\Gamma) = \int_E P(s, t, x, \Gamma)\mu(dx), \quad \mu \in \mathcal{P}(E), \Gamma \in \mathcal{B}.$$

It is well-known that $P(s, t)$ and $P^*(s, t)$ forms a two-parameter semigroup on $\mathcal{B}_b(E)$ and $\mathcal{P}(E)$ respectively and satisfies $P(s, t) = P(s, r)P(r, t)$ and $P^*(s, t) = P^*(r, t)P^*(s, r)$. On $\mathcal{P}(E)$, we endow the total variation norm defined by

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{\Gamma \in \mathcal{B}} |\mu_1(\Gamma) - \mu_2(\Gamma)|, \quad \mu_1, \mu_2 \in \mathcal{P}(E).$$

It is easy to show that $P^*(s, t) : (\mathcal{P}(E), \|\cdot\|_{TV}) \to (\mathcal{P}(E), \|\cdot\|_{TV})$ has operator norm $\|P^*(s, t)\| = 1$. While many of the convergence results presented here holds in other norms than the total variation norm (such as f-norms). For clarity and simplicity, we shall only consider convergence in the total variation norm. Some results only require weak convergence of measures. Hence we occasionally consider $\mu \in \mathcal{P}(E)$ as a linear functional on $C_b(E)$ by

$$\mu(f) = \int_E f(x)\mu(dx), \quad f \in C_b(E).$$

And we say $\mu, \nu \in \mathcal{P}(E)$ are equal if $\mu(f) = \nu(f)$ for all $f \in C_b(E)$. It is easy to show that $P^*(s,t)\mu(f) = \mu(P(s,t)f)$ for $\mu \in \mathcal{P}(E)$, $f \in C_b(E)$ and $s \leq t$.

We give the definition of a time-periodic Markov transition kernel. We also introduce the stronger definition of minimal time-periodic. Note that time-periodic Markov kernels depend on initial and terminal time.

Definition 2.1. The two-parameter Markov transition kernel $P(\cdot,\cdot,\cdot,\cdot)$ is said to be T-periodic for some $T > 0$ if

$$P(s,t,x,\cdot) = P(s+T, t+T, x, \cdot), \quad \text{for all } x \in E, s \leq t. \tag{2.1}$$

Moreover, we say $P(\cdot,\cdot,\cdot,\cdot)$ is minimal T-periodic if for every $\delta \in (0,T) \cap \mathbb{T}$

$$P(s,t,x,\cdot) \neq P(s+\delta, t+\delta, x, \cdot), \quad \text{for all } x \in E, s \leq t. \tag{2.2}$$

And we say $P(\cdot,\cdot,\cdot,\cdot)$ is time-homogeneous if

$$P(s,t,x,\cdot) = P(0, t-s, x, \cdot), \quad \text{for all } x \in E, s \leq t.$$

The definition of T-periodic should be clear and intuitive. Observe that minimal T-periodic assumption is stronger. It rules out the possibility of being time-homogeneous and enforces non-trivial period for every state. Equation (2.1) on the other hand allows states to have trivial period. This implies results of this paper assuming T-periodic P recovers results for the usual time-homogeneous case.

As a convention, we denote by $P(t)$ for the time-homogeneous Markov semigroup and $P^*(t)$ for its adjoint depending only on the elapsed time $0 \leq t \in \mathbb{T}$. Specifically for $\mathbb{T} = \mathbb{N}$, we denote $P := P(1)$ and $P^* := P^*(1)$ for the "one-step" semigroup and adjoint semigroup respectively. We now define our central objects of study characterising stationary and periodic behaviour.

Definition 2.2. A probability measure $\pi \in \mathcal{P}(E)$ is called an invariant (probability) measure with respect to $P(s,t)$ if

$$P^*(s,t)\pi = \pi \quad \text{for all } s \leq t.$$

When P is time-homogeneous, π satisfies $P^*(t)\pi = \pi$ for all $t \geq 0$. In particular, when $\mathbb{T} = \mathbb{N}$, π needs only to satisfy the one-step relation $P^*\pi = \pi$.

Invariant measures have been well-studied for many decades in many general settings. For example time-homogeneous Markov chains on finite dimensional state space [45,50,47], and Markov processes on finite state space [54,50], on infinite dimensional state spaces [8]. On the other hand, the formulation of periodic measure below is new and was first formally defined [13].

Definition 2.3. A measure-valued function $\rho : \mathbb{T} \to \mathcal{P}(E)$ is called a T-periodic (probability) measure with respect to $P(\cdot, \cdot)$ if for all $0 \le s \le t$

$$\rho_{s+T} = \rho_s, \quad \rho_t = P^*(s,t)\rho_s.$$

Note that periodic measures are invariant measures when the period is trivial. We shall give sufficient conditions to ensure the periodic measure has a minimal positive period. In classic literature, see [8,24,45,47,54,50] for instance, appropriate assumptions yield asymptotic convergence of the Markov kernel towards a unique invariant measure. However, these classical asymptotic results seem to have neglected the possibility of asymptotically periodic behaviour. While conceptually simple, it seems that asymptotic periodic behaviour was first formally pointed by Feng and Zhao in [13] and formalised under the definition of periodic measures. Nonetheless, these limiting invariant measures results can still be utilised for time-periodic Markovian system. We end this section by quoting without proof two now-classical results for time-homogeneous Markov chain result taken as special cases from [45,47]. To state the results, we require the following definitions.

Definition 2.4. Let P be a one-step time-homogeneous Markov transition kernel. We say that P satisfies the "minorisation" or "local Doeblin" condition if there exists a non-empty measurable set $K \in \mathcal{B}$, constant $\eta \in (0, 1]$ and a probability measure φ such that

$$P(x, \cdot) \ge \eta\varphi(\cdot), \quad x \in K. \tag{2.3}$$

Definition 2.5. A function $V : \mathbb{T} \times E \to \mathbb{R}^+$ is norm-like (or coercive) if $V(s, x) \to \infty$ as $\|x\| \to \infty$ for every fixed $s \in \mathbb{T}$ i.e. the level-sets $\{x \in E|V(s, x) \le r\}$ are pre-compact for each $r > 0$.

Lemma 2.6. *(Theorem 4.6 [45]) Let P be a one-step time-homogeneous Markov transition kernel and assume there exists a norm-like function $U : E \to \mathbb{R}^+$, a compact set $K \in \mathcal{B}$ and $\epsilon > 0$ such that*

$$PU - U \le -\epsilon \quad on\ K^c, \tag{2.4}$$

$$PU < \infty \quad on\ K. \tag{2.5}$$

Then there exists a unique invariant measure π with respect to P. Moreover if P satisfies the local Doeblin condition (2.3) then the invariant measure is limiting i.e. for any $x \in E$,

$$\|P^n(x, \cdot) - \pi\|_{TV} \to 0, \quad as\ n \to \infty.$$

In literature, conditions (2.4) and (2.5) are typically referred as the Foster-Lyapunov drift criteria and have the interpretation that the process moves inwards on average when outside the compact set. And U is referred as (Foster-)Lyapunov function. Lemma 2.6 is a qualitative result and does not give any rate of convergence. The following result from [45,47] gives sufficient condition for a time-homogeneous Markov chain to possess a unique invariant measure that converges geometrically.

Lemma 2.7. *(Theorem 6.3 [45], Theorem 15.0.1 [47]) Let P be a one-step time-homogeneous Markovian transition kernel satisfying (2.3). Assume there exists a norm-like function $U : E \to \mathbb{R}^+$, $\alpha \in (0,1)$ and $\beta > 0$ such that*

$$PU \leq \alpha U + \beta \quad on\ E. \tag{2.6}$$

Then there exists a unique geometric invariant measure $\pi \in \mathcal{P}(E)$ i.e. there exist constants $0 < R < \infty$ and $r \in (0,1)$ such that

$$\left\| P^n(x,\cdot) - \pi \right\|_{TV} \leq R(U(x)+1)r^n, \quad x \in E, n \in \mathbb{N}.$$

3. Periodic measures for time-periodic Markovian systems

The aim of this section is to give new results to establish the existence, uniqueness and convergence of a periodic measure in the general setting of time-periodic Markovian systems on a locally compact metric space E. We will give sufficient conditions in which a periodic measure to have a minimal positive period (hence not an invariant measure) for T-periodic Markov processes on Euclidean space. We start with the following basic existence and uniqueness lemma.

Lemma 3.1. *Let $P(\cdot,\cdot,\cdot,\cdot)$ be a two-parameter T-periodic Markov transition kernel. Assume for some fixed $s_* \in \mathbb{T}$ there exists invariant measure ρ_{s_*} with respect to the one-step Markov transition kernel $P(s_*, s_*+T)$. Then there exists a T-periodic measure ρ with respect to $P(\cdot,\cdot)$. If ρ_{s_*} is unique then ρ is also unique.*

Proof. Given ρ_{s_*}, define the following measures

$$\rho_s := P^*(s_*, s)\rho_{s_*}, \quad s \geq s_*. \tag{3.1}$$

Extend ρ_s by periodicity for $s \leq s_*$. Then it is clear that $\rho : \mathbb{T} \to \mathcal{P}(E)$ and is easy to show that ρ is a periodic measure with respect to $P(\cdot,\cdot)$. Now suppose that ρ_{s_*} is the unique invariant measure with respect to $P(s_*, s_*+T)$, we prove ρ is also unique. Suppose there are two T-periodic measures $\rho^i = \left(\rho^i_s\right)_{s\in\mathbb{T}}$ for $i = 1,2$ with respect to $P(\cdot,\cdot)$. By definition of periodic measures, ρ^i_s satisfies $\rho^i_s = P^*(s_*, s)\rho^i_{s_*}$ for $s \geq s_*$, hence by the linearity of P^*

$$\|\rho^1_s - \rho^2_s\|_{TV} = \|P^*(s_*, s)(\rho^1_{s_*} - \rho^2_{s_*})\|_{TV} \leq \|P^*(s_*, s)\| \|\rho^1_{s_*} - \rho^2_{s_*}\|_{TV}$$

The result follows by the assumption of uniqueness i.e. $\rho^1_{s_*} = \rho^2_{s_*}$. □

Note that, by definition, an invariant measure is always a periodic measure with a trivial period. For applications, we expect it is important to distinguish periodic measures of minimal positive period and those of a trivial period. However, it is not immediate whether the periodic measure constructed in Lemma 3.1 has a trivial period. The distinction can be subtle as SDE with periodic coefficients does not immediately yield a periodic measure with a non-trivial period. For example, let W_t be a one-dimensional Brownian motion and S is a continuously differentiable T-periodic function and consider the following SDE

$$dX_t = (-\alpha X_t^3 + S(t)X_t)dt + \sigma dW_t, \quad \alpha > 0, \sigma \neq 0.$$

The results from Section 4 of this paper yield the existence and uniqueness of a periodic measure with a minimal positive period. On the other hand, the same SDE with multiplicative linear noise,

$$dX_t = (-\alpha X_t^3 + S(t)X_t)dt + X_t dW_t, \quad \alpha > 0,$$

has δ_0 (Dirac mass at the origin) as an invariant measure which is also a periodic measure with a trivial period of the system. In fact, for time-homogeneous Markovian systems in general, the time-average of a periodic measure always yields an invariant measure. This is known for Markovian cocycles in the general framework of random dynamical systems (RDS); formal statements and proof can be found in [13]. This means that T-periodic measures and invariant measures coexist for time-homogeneous Markovian systems. However, for time-inhomogeneous and specifically time-periodic Markovian systems, invariant measures and periodic measures can be mutually exclusive. Should the measures be mutually exclusive, this has the important implication that the long term behaviour is characterised by strictly periodic behaviour.

We make the following trivial but important observation. If $(X_t)_{t\in\mathbb{T}}$ is a T-periodic Markov process, then $(Z_n^s)_{n\in\mathbb{N}} := (X_{s+nT})_{n\in\mathbb{N}}$ is a time-homogeneous Markov chain. This enables the usage of classical time-homogeneous Markov chain theory that is already well-established. Beyond the theoretical advantage, this observation is practically important in applications.

We can now discuss ergodicity of time-periodic Markovian systems. Classically, an ergodic (time-homogeneous) Markov process has the property that the Markov transition kernel converges to an invariant measure as time tends to infinity. In this sense, the invariant measure characterises the long-time behaviour of the system. On the other hand, periodic measure (with a minimal positive period) cannot be limiting in the same way because the periodic measure evolves over time. However, it is possible that the Markov transition kernel can converge along integral multiples for T-periodic Markovian processes. This captures the idea that the periodic measure describes long-time periodic behaviour of the system. This shall be apparent and rigorously written in the forthcoming theorem. We remark also that the forthcoming theorem can be regarded as the time-periodic generalisation of Lemma 2.6.

Theorem 3.2. *Let P be a T-periodic Markov transition kernel and Feller. Assume there exists $s_* \in \mathbb{T}$, norm-like function $U_{s_*} : E \to \mathbb{R}^+$, a non-empty compact set $K \in \mathcal{B}$, $\epsilon > 0$, $\eta_{s_*} \in (0, 1]$, $\varphi_{s_*} \in \mathcal{P}(E)$ such that*

$$P(s_*, s_* + T)U_{s_*} - U_{s_*} \le -\epsilon \quad \text{on } K^c, \tag{3.2}$$

$$P(s_*, s_* + T)U_{s_*} < \infty \quad \text{on } K, \tag{3.3}$$

$$P(s_*, s_* + T, x, \cdot) \ge \eta_{s_*}\varphi_{s_*}(\cdot), \quad x \in K, \tag{3.4}$$

i.e. (2.3), (2.4) and (2.5) are satisfied for $P(s_*, s_* + T)$. Then there exists a unique T-periodic measure ρ that satisfies all the convergences below.

(i) For any fixed $x \in E$ and $s \in \mathbb{T}$

$$\|P(s, s + nT, x, \cdot) - \rho_s\|_{TV} \to 0, \quad \text{as } n \to \infty. \tag{3.5}$$

(ii) For any fixed $x \in E$ and $s \in \mathbb{T}$, the following "moving" convergence holds,

$$\|P(s,t,x,\cdot) - \rho_t\|_{TV} = 0, \quad \text{as } t \to \infty. \tag{3.6}$$

(iii) Allowing for negative initial time, for any fixed $x \in E$, $s,t \in \mathbb{T}$, the following pullback convergence holds

$$\|P(s-nT,t,x,\cdot) - \rho_t\|_{TV} = 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Proof. Since $P(s,s+T,\cdot,\cdot)$ is a one-step time-homogeneous Markov kernel for all $s \in \mathbb{T}$, by Lemma 2.6, there exists a unique $\rho_{s_*} \in \mathcal{P}(E)$ with respect to $P(s_*, s_*+T)$. Moreover by Lemma 3.1, there exists a unique periodic measure ρ. To show the convergences, we need to prove that $P(s,s+T)$ satisfies (2.3), (2.4) and (2.5) for all $s \in \mathbb{T}$. By T-periodicity of P and the semigroup properties of P, observe that

$$P(s,s_*)P(s_*,s_*+T) = P(s,s_*)P(s_*,s+T)P(s+T,s_*+T) = P(s,s+T)P(s,s_*), \quad s \le s_*.$$

Hence applying $P(s,s_*)$ to both sides of (3.2) yields

$$P(s,s+T)P(s,s_*)U_{s_*} - P(s,s_*)U_{s_*} < -\epsilon, \quad \text{on } K^c.$$

i.e. $U_s := P(s,s_*)U_{s_*}$ satisfies (2.4) with respect to $P(s,s+T)$. Analogously, U_s satisfies (2.5). It is easy to verify that $U_s \ge 0$. We extend U_s for all $s \in \mathbb{T}$ by periodicity. We claim that for any $s \ge s_*$, $\eta_s := \eta_{s_*} \in (0,1]$ and $\varphi_s := P^*(s_*,s)\varphi_{s_*} \in \mathcal{P}(E)$ satisfies

$$P(s,s+T,x,\cdot) \ge \eta_s \varphi_s(\cdot), \quad x \in K. \tag{3.8}$$

i.e. $P(s,s+T)$ satisfies (2.3). Should this not be the case, then there exists some $x \in K$ and $\Gamma \in \mathcal{B}$ such that $P(s,s+T,x,\Gamma) < \eta_s\varphi_s(\Gamma)$. Then

$$P(s_*,s)P(s,s+T,x,\Gamma) = P(s_*,s+T,x,\Gamma) < \eta_s\varphi_s(\Gamma),$$

by applying $P(s_*,s)$ to both sides and Chapman-Kolmogorov equation. However by assumption (3.4),

$$\begin{aligned}\eta_s\varphi_s(\Gamma) &> P(s_*,s+T,x,\Gamma)\\ &= P^*(s_*+T,s+T)P(s_*,s_*+T,x,\Gamma)\\ &= \eta_{s_*}P^*(s_*,s)\varphi_{s_*}(\Gamma),\end{aligned}$$

which is a contradiction. We again extend by periodicity for all $s \in \mathbb{T}$. Thus, the assumptions of Lemma 2.6 are satisfied to deduce (3.5) for all $s \in \mathbb{T}$. Observe that for $t \ge s+nT$,

$$\begin{aligned}\|P(s,t,x,\cdot) - \rho_t\|_{TV} &= \|P^*(s+nT,t)P(s,s+nT,x,\cdot) - P^*(s+nT,t)\rho_{s+nT}\|_{TV}\\ &= \|P^*(s+nT,t)P(s,s+nT,x,\cdot) - P^*(s+nT,t)\rho_s\|_{TV}\\ &\le \|P(s,s+nT,x,\cdot) - \rho_s\|_{TV}.\end{aligned}$$

Hence (3.6) follows by (3.5), by taking $t \to \infty$ followed by $n \to \infty$. Using (3.5), convergence (3.7) holds due to

$$\begin{aligned} P(s-nT,t,x,\cdot) &= P(s,t+nT,x,\cdot) = P^*(s+nT,t+nT)P(s,s+nT,x,\cdot) \\ &= P^*(s,t)P(s,s+nT,x,\cdot). \quad \square \end{aligned}$$

We elaborate on the convergences given in Theorem 3.2. The first convergence (3.5) is clear where the convergence is along integral multiples of the period towards a fixed measure. That is, ergodicity of the grid chain. Convergence (3.6) extends (3.5) by allowing the convergence to be taken continuously in time. Observe that equation (3.6) captures the idea that long-term behaviour is characterised by the periodic measure. Note that this convergence is towards a "moving target" as the periodic measure evolves over time. It is typical in the theory of non-autonomous dynamical systems [32] and RDS (random dynamical systems) [5] to study "pullback" convergence. This is convergence where one takes initial time further and further back in time rather than the forward time. The advantage is that the convergence will be to a fixed target rather than a moving one. This is the content of convergence (3.7). In general, (forward) convergence and pullback convergence do not coincide (see [32,5] for examples). In this T-periodic case, we see that the convergences coincide.

Assuming we have a stochastic Lyapunov function for a T-periodic Markovian kernel, Theorem 3.2 gives a limiting periodic measure provided the local Doeblin condition (3.4). The following two results give sufficient conditions in which (3.4) holds. We denote for convenience $\mathcal{M}(E)$ to be the space of measures on $(E,\mathcal{B})$.

Proposition 3.3. *Let P be a T-periodic Markov transition kernel and assume there exists some $s_* \in \mathbb{T}$, a non-empty set $K \in \mathcal{B}$, $\epsilon > 0$ and $\Lambda \in \mathcal{M}(E)$ such that $\Lambda(K) > 0$, $P(s,t,x,\cdot)$ possesses a density $p(s,t,x,y)$ with respect to Λ and*

$$\inf_{x,y\in K} p(s_*,s_*+T,x,y) > 0. \tag{3.9}$$

Then the local Doeblin condition (3.4) of Theorem 3.2 holds.

Proof. By assumption that $\Lambda(K) > 0$,

$$\begin{aligned} \eta := &\int_E \inf_{x\in K} p(s_*,s_*+T,x,y)\Lambda(dy) \\ \geq &\int_K \inf_{x\in K} p(s_*,s_*+T,x,y)\Lambda(dy) \\ \geq &\inf_{x,y\in K} p(s_*,s_*+T,x,y)\Lambda(K) \\ > &\,0. \end{aligned}$$

Clearly, $\eta \in (0,1]$. Define for any $\Gamma \in \mathcal{B}$,

$$\varphi(\Gamma) := \frac{1}{\eta} \int\limits_{\Gamma} \inf_{x \in K} p(s_*, s_* + T, x, y)\Lambda(dy).$$

It is easy to verify that $\varphi \in \mathcal{P}(E)$ and for any $x \in K$ and any $\Gamma \in \mathcal{B}$

$$\begin{aligned} P(s_*, s_* + T, x, \Gamma) &= \int\limits_{\Gamma} p(s_*, s_* + T, x, y)\Lambda(dy) \\ &\geq \int\limits_{\Gamma} \inf_{x \in K} p(s_*, s_* + T, x, y)\Lambda(dy) \\ &= \eta\varphi(\Gamma). \end{aligned}$$

Thereby (3.4) holds with constant η and probability measure φ. $\square$

In practice, assumption (3.9) in Proposition 3.3 can be difficult to verify as well as being stronger than required. By assuming the Markov transition kernel possesses a continuous density, we can relax (3.9). For the forthcoming theorem, we define $\mathcal{M}^+(E) = \{\mu \in \mathcal{M}(E) | \mu(\Gamma) > 0, \quad \text{non-empty open } \Gamma \in \mathcal{B}\}$. We will make explicit use of the metric d on $(E, \mathcal{B})$ and define $B_r(x) := \{y \in E | d(x, y) < r\}$ to be the open ball of radius $r > 0$ centred at $x \in E$.

Theorem 3.4. *Let P be a T-periodic Markov transition kernel and assume there exist some $s_* \in \mathbb{T}$, a non-empty compact set $K \in \mathcal{B}$, $0 \leq r \leq T$ and $\Lambda \in \mathcal{M}^+(E)$ such that $P(s, t, x, \cdot)$ possesses a (local) density $p(s, t, x, y)$ with respect to Λ and is jointly continuous on $K \times K$. Assume further that for any non-empty open set $\Gamma_1, \Gamma_2 \subset K$ and $x \in K$*

$$P(s_*, s_* + r, x, \Gamma_1) > 0, \quad P(s_* + r, s_* + T, x, \Gamma_2) > 0. \tag{3.10}$$

Then the local Doeblin condition (3.4) of Theorem 3.2 holds.

Proof. Fix any $y' \in K$, by (3.10), then for any non-empty open set $\Gamma \subset K$,

$$P(s_* + r, s_* + T, y', \Gamma) > 0.$$

By the existence of a density, there exists $z' \in \Gamma$ such that

$$p(s_* + r, s_* + T, y', z') \geq 2\epsilon,$$

for some $\epsilon > 0$. Joint continuity assumption implies there exist $r_1, r_2 > 0$ such that

$$p(s_* + r, s_* + T, y, z) \geq \epsilon, \quad \text{for all } y \in B_{r_1}(y') \subset K, z \in B_{r_2}(z') \subset K.$$

Hence for any $\Gamma \in \mathcal{B}$ and $y \in B_{r_1}(y')$,

$$\begin{aligned}P(s_*+r, s_*+T, y, \Gamma) &= \int_{\Gamma} p(s_*+r, s_*+T, y, z)\Lambda(dz)\\ &\geq \int_{\Gamma\cap B_{r_2}(z')} p(s_*+r, s_*+T, y, z)\Lambda(dz)\\ &\geq \epsilon\Lambda(\Gamma\cap B_{r_2}(z')).\end{aligned}$$

By (3.10), we have

$$P(s_*, s_*+r, x, B_{r_1}(y')) > 0, \quad \text{for all } x\in K.$$

As $p(s_*, s_*+r, x, y)$ is a continuous function of x, by dominated convergence theorem, $P(s_*, s_*+r, x, \Gamma)$ is also continuous function of x. Hence, the compactness of K yields

$$\inf_{x\in K} P(s_*, s_*+r, x, B_{r_1}(y')) \geq \gamma',$$

for some $\gamma' > 0$. In particular,

$$\inf_{x\in K} P(s_*, s_*+r, x, B_{r_1}(y')) \geq \gamma := \min\left\{\gamma', \frac{1}{\epsilon\Lambda(B_{r_2}(z'))}\right\}.$$

Putting them together via Chapman-Kolmogorov equation, we have for any $x\in K$ and $\Gamma\in\mathcal{B}$,

$$\begin{aligned}P(s_*, s_*+T, x, \Gamma) &= \int_E P(s_*+r, s_*+T, y, \Gamma)p(s_*, s_*+r, x, y)\Lambda(dy)\\ &\geq \int_{B_{r_1}(y')} P(s_*+r, s_*+T, y, \Gamma)p(s_*, s_*+r, x, y)\Lambda(dy)\\ &\geq \epsilon\Lambda(\Gamma\cap B_{r_2}(z'))\int_{B_{r_1}(y')} p(s_*, s_*+r, x, y)\Lambda(dy)\\ &= \epsilon\Lambda(\Gamma\cap B_{r_2}(z'))P(s_*, s_*+r, x, B_{r_1}(y'))\\ &\geq \epsilon\gamma\Lambda(\Gamma\cap B_{r_2}(z')).\end{aligned}$$

Thus, the probability measure

$$\varphi(\cdot) = \frac{\Lambda(\cdot\cap B_{r_2}(z'))}{\Lambda(B_{r_2}(z'))},$$

and the constant $\eta = \epsilon\gamma\Lambda(B_{r_2}(z')) \in (0, 1]$ collectively satisfy the local Doeblin condition (3.4). □

Remark 3.5. Note that in Theorem 3.4, if E is a locally compact metrisable topological group, then any Haar measure Λ (for which a local density exists and is jointly continuous) will suffice.

Similar to Theorem 3.2, we end this section with a theorem for the existence and uniqueness of a geometric periodic measure. Observe in the theorem that the geometric convergence intrinsically depends on the initial time and state. This is akin to the autonomous case where the convergence depends on initial state.

Theorem 3.6. *Let P be a T-periodic Markov transition kernel and assume there exists $s_* \in \mathbb{T}$, a norm-like function $U_{s_*} : E \to \mathbb{R}^+$, a non-empty compact set $K \in \mathcal{B}$, $\epsilon > 0$ such that $P(s_*, s_* + T)$ satisfies the local Doeblin condition (3.4) and there exist constants $\alpha \in (0, 1)$ and $\beta > 0$ satisfying*

$$P(s_*, s_* + T)U_{s_*} \leq \alpha U_{s_*} + \beta, \quad \text{on } E. \tag{3.11}$$

Then there exists a unique geometric periodic measure ρ. Specifically, there exists a norm-like function $V : \mathbb{T} \times E \to \mathbb{R}^+$ constants $R_s < \infty$ and $r_s \in (0, 1)$ such that the following all holds:

(i) For any $s \in \mathbb{T}$ and $x \in E$, we have

$$\|P(s, s + nT, x, \cdot) - \rho_s\|_{TV} \leq R_s(V(s, x) + 1)r_s^n, \quad n \in \mathbb{N}. \tag{3.12}$$

(ii) For any $s \leq t$, $x \in E$, we have

$$\|P(s, t, x, \cdot) - \rho_t\|_{TV} \leq R_s(V(s, x) + 1)r_s^n, \quad \mathbb{N} \ni n \leq \lfloor \frac{t-s}{T} \rfloor.$$

(iii) Allowing for negative initial time, for any $s \leq t$, $x \in E$, we have

$$\|P(s - nT, t, x, \cdot) - \rho_t\|_{TV} \leq R_s(V(s, x) + 1)r_s^n, \quad \mathbb{N} \ni n \leq \lfloor \frac{t-s}{T} \rfloor.$$

(iv) The periodic measure is uniformly geometric convergence over initial time i.e. there exist constants $R > 0, r \in (0, 1)$ and a norm-like function $V : E \to \mathbb{R}^+$ such that

$$\|P(s, s + nT, x, \cdot) - \rho_s\|_{TV} \leq R(V(x) + 1)r^n, \quad \text{for all } x \in \mathbb{R}^d, s \in \mathbb{T}, n \in \mathbb{N}. \tag{3.13}$$

Proof. Define $V(s, x) := P(s, s_*)U_{s_*}(x)$ for all $s \leq s_*$ and extend by periodicity for all $s \in \mathbb{T}$. Then analogous to Theorem 3.2, the function $V(s, \cdot)$ satisfies (2.6) with respect to $P(s, s + T)$. Likewise from Theorem 3.2, the local Doeblin condition holds. Then (3.12) holds immediately by Lemma 2.7. To see the uniform convergence (3.13), first consider (3.12) when $s = 0$, i.e. there exists $R_0 > 0, r_0 \in (0, 1)$,

$$\|P(0, nT, x, \cdot) - \rho_0(\cdot)\|_{TV} \leq R_0(1 + V(x))r_0^n.$$

For any $s \in [0, T]$, note that

$$\begin{aligned} &\|P(s, nT, x, \cdot) - \rho_0(\cdot)\|_{TV} \\ &= \left\| P^*(T, nT)P(s, T, x, \cdot) - \rho_0(\cdot) \right\|_{TV} \end{aligned}$$

$$
\begin{aligned}
&= \left\| \int_E (P(T, nT, y, \cdot) - \rho_0(\cdot)) P(s, T, x, dy) \right\|_{TV} \\
&\leq \int_E \|(P(0, (n-1)T, y, \cdot) - \rho_0(\cdot))\|_{TV} \, P(s, T, x, dy) \\
&\leq \frac{R_0}{r_0} \int_E (1 + V(y)) P(s, T, x, dy) r_0^n. \qquad (3.14)
\end{aligned}
$$

Note $\rho_s = P^*(0, s)\rho_0$ and

$$
P(s, s+nT, x, \cdot) = P^*(nT, s+nT) P(s, nT, x, \cdot) = P^*(0, s) P(s, nT, x, \cdot).
$$

Thus,

$$
\begin{aligned}
&\|P(s, s+nT, x, \cdot) - \rho_s(\cdot)\|_{TV} \\
&= \left\| P^*(0, s) P(s, nT, x, \cdot) - P^*(0, s)\rho_0 \right\|_{TV} \\
&\leq \left\| P^*(0, s) \right\| \|P(s, nT, x, \cdot) - \rho_0\|_{TV} \qquad (3.15)
\end{aligned}
$$

Then (3.13) follows from (3.14), (3.15) and (3.11) for $s \in [0, T]$. The case for other $s \in \mathbb{R}$ follows from periodicity. □

4. Time-periodic stochastic differential equations

4.1. Limiting periodic measures

Using the developed theory from Section 3, we apply the results specifically in the context of T-periodic SDEs evolving on Euclidean state space. In this subsection, we are particularly interested in results that can be verified to possess a limiting periodic measure. We will study the T-periodic SDEs with white noises as its source of randomness. We note however that Theorem 3.2 can accommodate other types of noise. For instance, Höpfner and Löcherbach [27] studied at a periodically forced Ornstein-Uhlenbeck process under the influence of Lévy noise. Generalising to Lévy noise would be an area of future works and foresee applications to time-periodic financial models with jumps.

We fix some nomenclature and notation. Non-autonomous refers to SDEs with coefficients depending explicitly on time. We always denote by $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ the Euclidean space where $\mathcal{B}(\mathbb{R}^d)$ denote the standard Borel σ-algebra on $\mathbb{R}^d$ and let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the standard inner-product and norm on $\mathbb{R}^d$. Then we can define $B_r(y) := \{x \in \mathbb{R}^d \,|\, \|x - y\| < r\}$ for the open ball of radius $r > 0$ centred at y. And denote for convenience $B_r := B_r(0)$. On $\mathbb{R}^d$, we re-use Λ as the Lebesgue measure. We let $GL(\mathbb{R}^d)$ denote the space of invertible $d \times d$ matrices and let $L_2(\mathbb{R}^{d\times d}) := \{\sigma \in \mathbb{R}^{d\times d} | \|\sigma\|_2 < \infty\}$ where $\|\sigma\|_2 = \sqrt{\mathrm{Tr}(\sigma\sigma^T)} = \sqrt{\sum_{i,j=1}^d \sigma_{ij}^2}$ as the standard Frobenius norm.

We let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ denote the space of functions which are continuously differentiable in the first variable and twice differentiable in the spatial variables and $C_b^\infty(B_n)$ denote the space of bounded infinitely differentiable real-valued functions on B_n. Functions $b:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ are said to be locally Lipschitz if for any compact set $K\subset\mathcal{B}(\mathbb{R}^d)$ there exist constants $L=L(K)$ and $M=M(K)$ such that $\|b(t,x)-b(t,y)\|\le L\|x-y\|$ and $\|\sigma(t,x)-\sigma(t,y)\|_2\le M\|x-y\|_2$ for $x,y\in K$. They are (globally) Lipschitz if $K=E$. We say that σ has linear growth if there exists a constant $C>0$ such that

$$\|\sigma(t,x)\|_2^2\le C(1+\|x\|^2),\quad t\in\mathbb{R}^+,x\in\mathbb{R}^d. \tag{4.1}$$

We say σ has bounded inverse if

$$\|\sigma^{-1}\|_\infty:=\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d}\|\sigma^{-1}(t,x)\|_2<\infty. \tag{4.2}$$

For tuple $\alpha=(\alpha_0,\alpha_1,...,\alpha_d)\in\mathbb{N}^{d+1}$, define the partial derivatives $\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}$ where $|\alpha|=\sum_{i=0}^d\alpha_i$. We say that the functions $b:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ are locally smooth and bounded if for all $n\in\mathbb{N}$

$$\sigma_{ij}\in C_b^\infty(B_n),\quad 1\le i,j\le d, \tag{4.3}$$

and

$$b(t,x)+\partial^\alpha b(t,x)\quad\text{bounded on }\mathbb{R}^+\times B_n,\alpha\in\mathbb{N}^{d+1},|\alpha|=d. \tag{4.4}$$

Note that (4.3) and (4.4) imply the respective functions are locally Lipschitz. Whenever we assume (4.3), we always demand that σ is a function of spatial variables only.

We study Markov processes $X_t=X_t^{s,x}$ satisfying T-periodic SDEs of the form

$$\begin{cases}dX_t=b(t,X_t)dt+\sigma(t,X_t)dW_t,\\ X_s=x.\end{cases} \tag{4.5}$$

Here $x\in\mathbb{R}^d$, $T>0$ and functions $b\in C(\mathbb{R}^+\times\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma\in C(\mathbb{R}^+\times\mathbb{R}^d,GL(\mathbb{R}^d))$ are both T-periodic i.e.

$$b(t,\cdot)=b(t+T,\cdot),\quad\text{and}\quad\sigma(t,\cdot)=\sigma(t+T,\cdot),$$

and W_t is a d-dimensional Brownian motion on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. The infinitesimal generator of (4.5), $\tilde{\mathcal{L}}$ given by

$$\begin{aligned}\tilde{\mathcal{L}}f(t,x)=&\ \partial_t f(t,x)+\sum_{i=1}^d b_i(t,x)\partial_i f(t,x)+\frac{1}{2}\sum_{i,j=1}^d\left(\sigma\sigma^T\right)_{ij}(t,x)\partial_{ij}^2 f(t,x)\\ =:&\ \partial_t f(t,x)+\mathcal{L}(t)f(t,x),\quad f\in C^{1,2}(\mathbb{R}^+\times\mathbb{R}^d).\end{aligned} \tag{4.6}$$

We used the short hand notation $\mathbb{P}^{s,x}$ and $\mathbb{E}^{s,x}$ for the associated probability measure and expectation respectively for the process starting at $(s,x)\in\mathbb{R}^+\times\mathbb{R}^d$. When a unique solution exists, one can define the Markov transition kernel

$$P(s,t,x,\Gamma) := \mathbb{P}^{s,x}(X_t \in \Gamma), \quad s < t, \ \Gamma \in \mathcal{B}. \tag{4.7}$$

Here we say that a unique solution exists implies that the Markov process X_t is regular i.e. for any $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$,

$$\mathbb{P}^{s,x}\{\tau = \infty\} = 1, \tag{4.8}$$

where

$$\tau := \lim_{n\to\infty} \tau_n, \quad \tau_n := \inf_{t\geq s}\{\|X_t\| \geq n\}, \quad n \in \mathbb{N}.$$

Using Proposition 3.3 and Theorem 3.4, we give sufficient conditions in which the T-periodic Markov kernel $P(s,s+T,\cdot,\cdot)$ of an SDE satisfies the local Doeblin condition (3.8). Specifically, we sufficiently show (local) irreducibility and existence of a jointly continuous density with respect to the Lebesgue measure Λ (a Haar measure on the $\mathbb{R}^d$, see Remark 3.5). It is possible to show both properties simultaneously. For instance, heat kernel estimates such as the classical one by Aronson [3] sufficiently imply the Proposition 3.3 for non-autonomous SDEs with bounded drift and non-degenerate bounded diffusion.

Relaxing the non-degeneracy and boundedness assumption of [3], it is well-known that autonomous SDEs satisfying (4.8) and Hörmander's condition possesses a smooth density (globally with respect to Λ) for the Markov transition kernel [38,29,53]. However it is generally insufficient to yield irreducibility i.e. Hörmander's condition does not imply the process can reach any given non-empty open set with positive probability. We refer readers to Remark 2.2 of [21] for a counterexample. This suggests some degree of non-degeneracy is required to imply irreducibility. We emphasise that in existing literature, Hörmander's condition is often applied for autonomous SDEs with relatively few existing results for the non-autonomous case. Observe also that Theorem 3.4 requires density of the transition kernel to exist locally rather than globally. Recent advances by Höpfner, Löcherbach and Thieullen gave the existence of a smooth local density of non-autonomous SDEs under a time-dependent Hörmander's condition in [28].

Since the intention of this paper is to introduce main ideas and approach to deduce the existence and uniqueness of periodic measures, we shall show (global) irreducibility under the assumption that the diffusion matrix and its inverse are bounded and utilise the results of [28] for a local density. It will be the subject of future works to generalise the results in the direction of local time-dependent Hörmander's condition and relaxing the non-degeneracy assumption to attain a local irreducibility.

Consider the following associated control system to (4.5)

$$\begin{cases} dZ_t = \varphi(t)dt + \sigma(t,Z_t)dW_t, \quad t \geq s, \\ Z_s = x, \end{cases} \tag{4.9}$$

for some bounded adapted process $\varphi : \mathbb{R}^+ \to \mathbb{R}^d$. Inspired by the irreducibility argument of [8], we have the following lemma:

Lemma 4.1. *Assume b and σ are locally Lipschitz and moreover σ satisfies (4.1) and (4.2). Assume further that there exists a norm-like function V and constant $c > 0$ such that*

$$\tilde{\mathcal{L}} V \leq cV. \tag{4.10}$$

Let $X_t = X_t^{s,x}$ and $Z_t = Z_t^{s,x}$ satisfy (4.5) and (4.9) respectively. Then the laws of X_t and Z_t are equivalent.

Proof. By Theorem 3.5 of [24], locally Lipschitz coefficients and (4.10) yield that X_t exists and is unique. Since φ is a bounded adapted process and σ is locally Lipschitz with linear growth, by Theorem 3.1 of [39], Z_t also exists and is unique. Set $\tau_n = \inf_{t\geq s}\{\|Z_t\| \geq n\}$, $Z_t^n = Z_{t\wedge\tau_n}$ and

$$\mathbb{P}^n(d\omega) = \mathbb{P}(d\omega) M_t^n,$$

where

$$M_t^n = \exp\left(-\frac{1}{2}\int_s^{t\wedge\tau_n} \alpha^2(r)dr - \int_s^{t\wedge\tau_n} \alpha(r)dW_r\right),$$

and $\alpha(r) = \sigma^{-1}(r, Z_r)[\varphi(r) - b(r, Z_r)]$. It is clear that $\alpha(r)$ is bounded for $s \leq r \leq \tau_n$, hence Novikov condition is satisfied. Then Girsanov theorem implies

$$\widetilde{W}_t^n = W_t + \int_s^t \alpha(r)dr$$

is a Brownian motion on $\mathbb{R}^d$ under the probability measure $\mathbb{P}^n$. It is clear that $d\widetilde{W}_r^n = dW_r + \alpha(r)dr$ and $\varphi(t) = \sigma(t, Z_t)\alpha(t) + b(t, Z_t)$ so

$$\begin{aligned}
Z_t^n &= x + \int_s^{t\wedge\tau_n} \varphi(r)dr + \int_s^{t\wedge\tau_n} \sigma(r, Z_r^n)dW_r \\
&= x + \int_s^{t\wedge\tau_n} \left[\sigma(r, Z_r^n)\alpha(r) + b(r, Z_r^n)\right]dr + \int_s^{t\wedge\tau_n} \sigma(r, Z_r^n)\left[d\widetilde{W}_r^n - \alpha(r)dr\right] \\
&= x + \int_s^{t\wedge\tau_n} b(r, Z_r^n)dr + \int_s^{t\wedge\tau_n} \sigma(r, Z_r^n)d\widetilde{W}_r^n,
\end{aligned}$$

i.e. Z_t^n is a solution of (4.5) on $(\Omega, \mathcal{F}, \mathbb{P}^n)$. As the law of the solution does not depend on the choice of probability space, we have that

$$\mathbb{P}(X_t^n \in \Gamma) = \mathbb{P}^n(Z_t^n \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As $\mathbb{P}$ and $\mathbb{P}^n$ are equivalent, the laws of X_t^n and Z_t^n are equivalent. This implies that

$$\mathbb{P}^n(\tau_n > t) = \mathbb{P}^n\left(\sup_{s\leq r\leq t}\|Z_r\| \leq n\right) = \mathbb{P}\left(\sup_{s\leq r\leq t}\|X_r\| \leq n\right) \to 1 \quad \text{as } n \to \infty.$$

Define

$$M_t = \exp\left(-\frac{1}{2} \int_s^t \alpha^2(r)dr - \int_s^t \alpha(r)dW_r \right).$$

Then

$$\mathbb{E}[M_t] \geq \mathbb{E}[M_t^n I_{\{\tau_n > t\}}] = \mathbb{P}^n(\tau_n > t) \to 1 \quad \text{as } n \to \infty.$$

Moreover, we can prove that $\mathbb{P}(\tau_n > t) \to 1$ as $n \to \infty$. This suggests from Borel-Cantelli Lemma that there is a subsequence n_k such that $\tau_{n_k} \to \infty$ almost surely where $n_k \to \infty$ as $k \to \infty$. Thus

$$M_t^{n_k} \to M_t, \quad \text{as } k \to \infty$$

almost surely. Now, by Fatou's lemma

$$\lim_{k\to\infty} \mathbb{E}\left[M_t^{n_k}\right] \geq \mathbb{E}\left[\lim_{k\to\infty} M_t^{n_k}\right] = \mathbb{E}[M_t],$$

and $\mathbb{E}[M_t^{n_k}] = 1$ for each k since $M_t^{n_k}$ is a martingale. So $\mathbb{E}[M_t] \leq 1$. Thus, we have that $\mathbb{E}[M_t] = 1$. Now we apply Girsanov theorem [7] to yield that

$$\widetilde{W}_t = W_t + \int_s^t \alpha(r)dr$$

is a Brownian motion on $\mathbb{R}^d$ under the probability measure $\widetilde{\mathbb{P}}$ where $\widetilde{\mathbb{P}}(d\omega) = \mathbb{P}(d\omega)M_t$. As before,

$$Z_t = x + \int_s^t \varphi(r)dr + \int_s^t \sigma(r, Z_r)dW_r = x + \int_s^t b(r, Z_r)dr + \int_s^t \sigma(r, Z_r)d\widetilde{W}_r$$

is a solution to (4.5) on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. As the law of the solution does not depend on the choice of probability space, we have that

$$\mathbb{P}(X_t \in \Gamma) = \widetilde{\mathbb{P}}(Z_t \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, the laws of X_t and Z_t are equivalent. □

Theorem 4.2. *Consider SDE (4.5) and assume the same conditions as Lemma 4.1. Then the Markov transition kernel $P(s,t,\cdot,\cdot)$ for $s < t < \infty$ is irreducible i.e. $P(s,t,x,\Gamma) > 0$ for all $x \in \mathbb{R}^d$ and non-empty open $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.*

Proof. By Lemma 4.1, as $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, it is sufficient to show that for any $\delta > 0$ and $x, a \in \mathbb{R}^d$ that

$$\mathbb{P}(\|Z_t^{s,x} - a\| < \delta) > 0.$$

We consider the auxiliary system

$$\begin{cases} dY_t = \sigma(t, Y_t)dW_t, \\ Y_s = x. \end{cases} \tag{4.11}$$

Since σ is locally Lipschitz, then (4.11) has a unique solution Y_t satisfying

$$Y_t = x + \int_s^t \sigma(r, Y_r)dW_r. \tag{4.12}$$

For $u \in [s, t)$, $R > 0$ and $\widetilde{a} \in \mathbb{R}^d$ all to be chosen later, pick a bounded function $f : [u, t] \times \mathbb{R}^d \to \mathbb{R}^d$ such that f is Lipschitz and

$$f(r, y) = \begin{cases} 0 & \text{if } \|y\| > 2R, \\ \frac{\widetilde{a} - y}{t - u} & \text{if } \|y\| \le R. \end{cases}$$

Define the integral

$$I_1(y) = y + \int_u^t f(r, y)dr, \quad y \in \mathbb{R}^d.$$

Observe that if $\|y\| \le R$,

$$I_1(y) = y + \frac{1}{t - u} \int_u^t (\widetilde{a} - y)dr = \widetilde{a}. \tag{4.13}$$

Set

$$\varphi(r) = \begin{cases} 0 & \text{if } r \in [s, u), \\ f(r, Y_u) & \text{if } r \in [u, t]. \end{cases}$$

Then it is clear that $Z_r^{s,x} = Y_r$ for $r \in [s, u)$. Hence, by sample-path continuity of Y_t, Z_t can be represented as an initial-valued SDE in terms of Y_u namely

$$Z_t^{s,x} = Y_u + \int_u^t f(r, Y_u)dr + \int_u^t \sigma(r, Z_r)dW_r.$$

Let $I_1 = I_1(Y_u)$ and $I_2 = \int_u^t \sigma(r, Z_r)dW_r$. Then $Z_t^{s,x} = I_1 + I_2$. Choose any fixed $\widetilde{a} \in \mathbb{R}^d$ such that

$$\|a - \widetilde{a}\| \leq \frac{\delta}{3}.$$

Suppose the events $\{I_1 = \widetilde{a}\}$ and $\{\|I_2\| \leq \frac{\delta}{3}\}$ hold then

$$\|Z_t^{s,x} - a\| = \|(I_1 - \widetilde{a}) + (I_2 + \widetilde{a} - a)\| \leq \|I_2\| + \|\widetilde{a} - a\| \leq \frac{2}{3}\delta.$$

Hence

$$P(\|Z_t^{s,x} - a\| \leq \delta) \geq P(I_1 = \widetilde{a} \text{ and } \|I_2\| \leq \frac{\delta}{3}) \geq \mathbb{P}(I_1 = \widetilde{a}) - \mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right), \tag{4.14}$$

where we used the elementary inequality $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) - \mathbb{P}(B^c)$ for any event $A, B \in \mathcal{F}$. Thus the proof is complete provided the right hand side of inequality (4.14) is positive. First it is easy to prove by Itô's formula and BDG inequality that $\mathbb{E}\sup_{u \leq r \leq t} |X_t|^2 \leq K$ for some constant K. By Chebyshev's inequality and Itô's isometry,

$$\mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) \leq \frac{9}{\delta^2}\mathbb{E}[\|I_2\|^2] = \frac{9}{\delta^2}\mathbb{E}[\int_u^t \|\sigma(r, Z_r)\|^2 dr] \leq \frac{9}{\delta^2} C \int_u^t (1 + \mathbb{E}Z_r^2)dr.$$

Hence, one can fix a $u \in [s, t)$ such that

$$\mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) \leq \frac{1}{4}.$$

Similarly, for the fixed u and any $R > 0$, from (4.12) we have

$$\begin{aligned}\mathbb{P}(\|Y_u\| > R) &\leq \frac{1}{R^2}\mathbb{E}[\|Y_u\|^2] = \frac{2}{R^2}\left(\|x\|^2 + \mathbb{E}[\int_u^s \|\sigma(r, Y_r)dr\|^2]\right) \\ &\leq \frac{2}{R^2}\left[\|x\|^2 + C(K+1)(u-s)^2\right].\end{aligned}$$

Hence one can fix a sufficiently large $R > 0$ such that

$$\mathbb{P}(\|Y_u\| \leq R) \geq \frac{3}{4}. \tag{4.15}$$

By (4.13), we have the inclusion $\{\|Y_u\| \leq R\} \subset \{I_1 = \widetilde{a}\}$. Hence by (4.15)

$$\mathbb{P}(I_1 = \widetilde{a}) \geq \mathbb{P}(\|Y_u\| \leq R) \geq \frac{3}{4}.$$

The proof is complete by the following inequality for irreducibility

$$\mathbb{P}(\|Z_t^{s,x} - a\| \le \delta) = \mathbb{P}(I_1 = \widetilde{a}) - \mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) \ge \frac{3}{4} - \frac{1}{4} = \frac{1}{2}. \quad \square$$

In the next theorem, we apply Theorem 1 of [28] to attain a smooth density of transition probabilities in extension of classical results by Aronson [3] for parabolic equations with bounded time-dependent coefficients. We assume that σ is time-independent as in [28]. It would be of future works to study the possible generalisation of Theorem 1 of [28] for T-periodic σ.

Theorem 4.3. *Consider SDE (4.5) and assume the same conditions as Lemma 4.1. Assume that (4.3) and (4.4) holds. Assume further that there exists a compact set $K \in \mathcal{B}(\mathbb{R}^d)$ such that (3.2) and (3.3) hold. Then the results of Theorem 3.2 hold.*

Proof. The invertibility of σ implies linearly independent columns hence our collective assumptions satisfy Theorem 1 of [28]. Hence there exists a smooth density $p(s,t,x,y)$ with respect to Λ. Then using Theorem 4.2, we have that Theorem 3.4 holds. Hence the assumptions of Theorem 3.2 are satisfied. $\square$

4.2. Geometric ergodicity of periodic measures

In the previous section, we studied limiting periodic measures in a qualitative manner. We extend this for geometrically ergodic periodic measures. That is, the convergence towards the periodic measure is exponentially fast. We recall the geometric drift condition for SDE.

Definition 4.4. The SDE (4.5) is said to satisfy the geometric drift condition if there exists a norm-like function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^+)$ and constants $C \ge 0$ and $\lambda > 0$ such that

$$\tilde{\mathcal{L}} V \le C - \lambda V \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \tag{4.16}$$

where $\tilde{\mathcal{L}}$ is given by (4.6).

Note that if (4.16) is satisfied then the SDE is regular. Specifically, since $V \ge 0$ and $\tilde{\mathcal{L}}[\text{const}] = 0$, it is easy to see that

$$\tilde{\mathcal{L}}(V+1) \le C - \lambda V \le C \le C(V+1),$$

hence the regularity condition (4.10) is satisfied.

Using the geometric drift condition, we give one of the main results on the existence, uniqueness and geometric ergodicity of a periodic measure. It is worth noting that if the SDE coefficients have a trivial period, then the theorem recovers known results of invariant measures. Hence, the results here presented can be regarded as time-periodic generalisations of such theorems of invariant measures for autonomous SDEs.

Theorem 4.5. *Assume T-periodic SDE (4.5) coefficients satisfies (4.1), (4.2), (4.3) and (4.4). Assume further that there exists a T-periodic norm-like $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^+)$ satisfying the geometric drift condition (4.16). Then Theorem 3.6 follows.*

Proof. By Itô's formula and the regularity of V, one has

$$d\left(e^{\lambda t}V(t,X_t)\right)=e^{\lambda t}(\lambda V+\tilde{\mathcal{L}}V)dt+e^{\lambda t}\langle\nabla V,\sigma dW_t\rangle,$$

hence by the geometric drift condition

$$\begin{aligned}V(t,X_t)&=e^{-\lambda(t-s)}V(s,X_s)+\int_s^t e^{-\lambda(t-r)}(\lambda V+\tilde{\mathcal{L}}V)dr+\int_s^t e^{-\lambda(t-r)}\langle\nabla V,\sigma dW_r\rangle\\&\le e^{-\lambda(t-s)}V(s,X_s)+\frac{C}{\lambda}\left(1-e^{-\lambda(t-s)}\right)+\int_s^t e^{-\lambda(t-r)}\langle\nabla V,\sigma dW_r\rangle.\end{aligned}\tag{4.17}$$

By (4.8) and the regularity of V, $\int_s^t e^{-\lambda(t-r)}\langle\sigma^T(r,X_r)\nabla V(r,X_r),dW_r\rangle_{\mathbb{R}^d}$ is a martingale. Hence

$$\mathbb{E}^{s,x}[V(t,X_t)]\le e^{-\lambda(t-s)}V(s,x)+\frac{C}{\lambda}(1-e^{-\lambda(t-s)})\quad s\le t.\tag{4.18}$$

Specifically,

$$\mathbb{E}^{s,x}[V(s+T,X_{s+T})]\le e^{-\lambda T}V(s,x)+\frac{C}{\lambda}(1-e^{-\lambda T}).\tag{4.19}$$

Define the functions $U_s(\cdot):=V(s,\cdot)\ge 0$. Since V is T-periodic, we have that (4.19) is equivalent to

$$P(s,s+T)U_s(x)\le e^{-\lambda T}U_s(x)+\frac{C}{\lambda}(1-e^{-\lambda T})\tag{4.20}$$

That is to say (2.6) is satisfied for each $s\ge 0$. Subtracting $U_s(x)$ from (4.20) yields

$$P(s,s+T)U_s(x)-U_s(x)\le(1-e^{-\lambda T})\left(\frac{C}{\lambda}-U_s(x)\right).$$

Since U_s is norm-like assumption, define for $\epsilon>0$

$$K=\bigcap_{s\in[0,T]}K_s,\quad\text{where } K_s:=\left\{x\in\mathbb{R}^d|U_s(x)\le\frac{C}{\lambda}+\frac{\epsilon}{1-e^{-\lambda T}}\right\}.$$

For sufficiently large ϵ, K is non-empty compact set. Since the SDE is regular, the same proof from Theorem 4.3 implies that Theorem 3.4 holds i.e. $P(s,s+T,x,\cdot)$ satisfies the local Doeblin condition for each $s\ge 0$. Thus the conditions of Theorem 3.6 are met. $\square$

Theorem 4.5 depends crucially on finding a suitable Foster-Lyapunov function V. Dissipative SDEs are special cases where the Euclidean norm is a such Foster-Lyapunov function. This has the advantage that it can be simpler to verify that the geometric drift condition is satisfied. The definition of dissipativity below coincides with that of Hale [23] when the SDE is deterministic ($\sigma=0$).

Definition 4.6. SDE (4.5) is weakly dissipative if there exist constants $c, \lambda > 0$ such that

$$2\langle b(t,x), x\rangle + \sum_{i,j=1}^{d} a_{ij}(t,x) \leq c - \lambda\|x\|^2 \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \tag{4.21}$$

and dissipative if $c = 0$.

Corollary 4.7. *Assume T-periodic SDE (4.5) coefficients satisfies (4.1), (4.2), (4.3) and (4.4) and is weakly dissipative. Then Theorem 3.6 holds.*

Proof. By Theorem 4.5, it suffices to show $V(t,x) = \|x\|^2$ satisfies the geometric drift condition. We compute that

$$\tilde{\mathcal{L}}\|x\|^2 = 2\langle b(t,x), x\rangle + \sum_{i,j=1}^{d} a_{ij}(t,x) \leq c - \lambda\|x\|^2,$$

i.e. $\|x\|^2$ satisfies the geometric drift condition. □

Theorem 4.8. *Consider T-periodic SDE (4.5) with σ satisfying (4.1), (4.2) and (4.3) and drift*

$$b(t,x) = \begin{pmatrix} \sum_{k=0}^{2p_1-1} S_k^1(t) x_1^k \\ \vdots \\ \sum_{k=0}^{2p_d-1} S_k^d(t) x_d^k \end{pmatrix},$$

where $\{p_i\}_{i=1}^d \in \mathbb{N}\backslash\{0\}$, $\{S_k^i\}_{i=1\ldots d}^{k=1\ldots 2p_i-2}$ are continuously differentiable T-periodic functions and constants $S_{2p_i-1}^i < 0$. Then Theorem 4.5 holds.

Proof. Clearly b satisfies (4.4). Hence by Corollary 4.7, it suffices to show that the SDE is weakly dissipative. We compute that

$$\langle b(t,x), x\rangle = \sum_{i=1}^{d} \sum_{k=1}^{2p_i} S_k^i x_i^k.$$

For each fixed $1 \leq i \leq d$, $\sum_{k=1}^{2p_i} S_k^i x_i^k$ is an even degree polynomial with leading negative coefficient. By assumption, $\{S_k^i\}$ are all bounded hence, fixing a $\lambda \in (0, -\min_i S_{2p_i-1}^i)$, define the constants

$$\widetilde{c}_i := \sup_{x_i \in \mathbb{R}, t \in [0,T]} \left(\sum_{k=1}^{2p_i} S_k^i x_i^k + \lambda x_i^{2p_i} \right) < \infty, \quad c_i := \widetilde{c}_i + \sup_{x_i \in \mathbb{R}} \lambda \left(x_i^2 - x_i^{2p_i} \right) < \infty,$$

then we deduce the SDE is weakly dissipative by

$$\langle b(t,x),x\rangle \le \sum_{i=1}^{d}\left(\widetilde{c}_i-\lambda x_i^{2p_i}\right)\le \sum_{i=1}^{d}\left(c_i-\lambda x_i^2\right)=\sum_{i=1}^{d}c_i-\lambda\|x\|^2. \quad \square$$

As it would be more apparent in the next section of gradient SDEs, Theorem 4.8 has many physical applications. They model multi-stable systems such as modulated Josephson-junctions systems, superionic conductors, excited chicken hearts to the dithered ring lasers as well as other laser systems. We refer to [57] and references therein for further details of these applications.

We give two specific examples of Theorem 4.8. First, we consider periodically forced mean-reverting Ornstein-Uhlenbeck processes. In this example, we compute the density of the process, periodic measure and the exponential convergence rate explicitly. While the computations are straightforward, it appears that the periodic measure and its geometric convergence for this system has not been previously noted in literature. The classical Ornstein-Uhlenbeck process is mean-reverting and has a geometric invariant measure. This contrasts with periodically forced Ornstein-Uhlenbeck processes which does not have limiting invariant measure. Instead, the system has a limiting periodic measure and mean-reverting to a periodic mean. We expect this to be useful for systems possessing periodic mean reversion due to factors such as seasonality.

Example 4.9. Consider the following multidimensional Ornstein-Uhlenbeck equation

$$dX_t=(S(t)-AX_t)\,dt+\sigma\,dW_t, \tag{4.22}$$

where $A=M^{-1}DM\in\mathbb{R}^{d\times d}$ for some $M\in GL(\mathbb{R}^d)$ and $D=\mathrm{diag}(\lambda_1,\cdots,\lambda_d)$ is a diagonal matrix with positive eigenvalues $\{\lambda_n\}_{n=1}^d$, $\sigma\in GL(\mathbb{R}^d)$ and $S(t):\mathbb{R}^+\to\mathbb{R}^d$ be a T-periodic continuously differentiable function. By applying Itô's formula on $e^{tA}X_t$ or by a variation of constants formula, we have

$$X_t=e^{-(t-s)A}X_s+\int_s^t e^{-(t-r)A}S(r)dr+\int_s^t e^{-(t-r)A}\sigma\,dW_r \quad t\ge s. \tag{4.23}$$

Observe that $\xi(t):=\int_{-\infty}^t e^{-A(t-r)}S(r)dr$ satisfies $\partial_r(e^{Ar}\xi)=e^{Ar}S(r)$ and is continuous and T-periodic. Define

$$J(s,t):=\int_s^t e^{-(t-r)A}S(r)dr=\xi(t)-e^{-(t-s)A}\xi(s). \tag{4.24}$$

By the T-periodicity of ξ, it is clear $\lim_{t\to\infty}J(s,t)$ does not converge. Instead, it converges along integral multiples of the period in the following way: let Id be the identity matrix on $\mathbb{R}^d$ and define

$$\xi_n(s):=J(s,s+nT)=(Id-e^{-nTA})\xi(s), \quad n\in\mathbb{N}.$$

Then $\xi(s)=\lim_{n\to\infty}\xi_n(s)$. We shall see $\xi(s)$ as the "long term periodic mean". From (4.23), it is easy to see that X_t is normally distributed. Specifically, we can compute

$$\mathbb{E}^{s,x}[X_t] = e^{-(t-s)A}x + J(s,t).$$

Since $A = M^{-1}DM$ then $e^{-(t-r)A} = M^{-1}e^{-(t-r)D}M$, thus denoting $N = M\sigma$, component-wise, we have $(e^{-(t-r)D}NdW_r)_i = e^{-(t-r)\lambda_i}\sum_{k=1}^{d} N_{ik}dW_r^k$. Hence by independence of Brownian motion and properties of Itô's inner-product, we have

$$\begin{aligned}
C_{ij}(s,t) &:= \mathbb{E}^{s,x}\left[\left(\int_s^t (e^{-(t-r)D}NdW_r)_i\right)\left(\int_s^t (e^{-(t-r)D}NdW_r)_j\right)\right] \\
&= \sum_{k,k'=1}^{d} N_{ik}N_{jk'}\mathbb{E}^{s,x}\left[\left(\int_s^t e^{-(t-r)\lambda_i}dW_r^k\right)\left(\int_s^t e^{-(t-r)\lambda_j}dW_r^{k'}\right)\right] \\
&= \sum_{k=1}^{d} N_{ik}N_{jk}\mathbb{E}^{s,x}\left[\int_s^t e^{-(t-r)(\lambda_i+\lambda_j)}dr\right] \\
&= \frac{(M\sigma\sigma^T M^T)_{ij}}{\lambda_i+\lambda_j}\left(1-e^{-(t-s)(\lambda_i+\lambda_j)}\right).
\end{aligned}$$

Hence the covariance matrix $\mathrm{Cov}(X_t|X_s = x) := \mathbb{E}^{s,x}[X_tX_t^T] - \mathbb{E}^{s,x}[X_t]\mathbb{E}^{s,x}[X_t^T] = M^{-1}C(s,t)M$, where $C(s,t)$ has entries $C_{ij}(s,t)$ as defined above. Thus, denoting $\mathcal{N}$ for the multivariate normal distribution, the Markov transition kernel of (4.22) is given by

$$P(s,t,x,\cdot) = \mathcal{N}\left(e^{-(t-s)A}x + J(s,t), M^{-1}C(s,t)M\right)(\cdot), \tag{4.25}$$

Since $\lim_{t\to\infty} J(s,t)$ does not converge (for any fixed s), (4.25) does not converge. This implies there does not exist a limiting invariant measure for this periodically forced Ornstein-Uhlenbeck process. This contrasts with the classical Ornstein-Uhlenbeck process (when $S(t) =$ const), where one often takes $t \to \infty$ to yield a (unique) limiting invariant measure. On the other hand, for every fixed s, along integral multiple of the period i.e. $t = s + nT$, one has directly from (4.25)

$$\begin{aligned}
P(s, s+nT, x,\cdot) &= \mathcal{N}\left(e^{-nTA}x + \xi_n(s), M^{-1}C(s,s+nT)M\right)(\cdot) \\
&\to \mathcal{N}\left(\xi(s), M^{-1}CM\right)(\cdot) =: \rho_s(\cdot),
\end{aligned} \tag{4.26}$$

as $n \to \infty$ where C is the matrix with entries $C_{ij} = \frac{(M\sigma\sigma^T M^T)_{ij}}{\lambda_i+\lambda_j}$. That is to say that the long-time behaviour is characterised by ρ_s for every fixed $s \geq 0$. Since ξ is T-periodic, ρ is also T-periodic. Moreover, it is easy to explicitly verified that ρ is periodic measure of the system. It is worth noting that for every s, $\mathrm{supp}(\rho_s) = \mathbb{R}^d$ and that the periodic measure (4.26) is unique. This contrasts with the time-homogeneous Markovian systems, where the uniqueness of periodic measure (if exist) holds only if it is supported by disjoint Poincaré sections [13].

The above calculations give the existence and uniqueness of a periodic measure. However, it does not immediately give a convergence rate. For simplicity, we show the convergence and its rate for the one-dimensional case. We first recall that the Kullback-Leibler divergence,

$D_{KL}(\cdot||\cdot)$, is pre-metric on $\mathcal{P}(\mathbb{R}^d)$. Let $P, Q \in \mathcal{P}(\mathbb{R}^d)$ with densities $p, q \in L^1(\mathbb{R}^d)$ respective, the Kullback-Leibler divergence can be defined by

$$D_{KL}(P||Q) := \int_{\mathbb{R}^d} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx.$$

For vectors $\mu_i \in \mathbb{R}^d$ and matrices $\sigma_i \in GL(\mathbb{R}^d)$ where $i = 1, 2$, we have specifically the following explicit expression for normal densities.

$$\begin{aligned} &D_{KL}(\mathcal{N}(\mu_1, \sigma_1)||\mathcal{N}(\mu_2, \sigma_2)) \\ &= \frac{1}{2}\left(\mathrm{Tr}(\sigma_2^{-1}\sigma_1) + (\mu_2 - \mu_1)^T \sigma_2^{-1}(\mu_2 - \mu_1) - d + \ln\left(\frac{\det(\sigma_2)}{\det(\sigma_1)}\right)\right). \end{aligned}$$

Moreover, Pinsker's inequality states

$$\|P - Q\|_{TV}^2 \le \frac{1}{2} D_{KL}(P||Q), \quad P, Q \in \mathcal{P}(E).$$

We recall the elementary identity $\ln(1 - y) = -\sum_{k=1}^{\infty} \frac{y^k}{k}$ for any fixed $y \in (-1, 1)$. Hence, the following elementary inequality holds by a geometric sum

$$-(y + \ln(1 - y)) = \sum_{k=2}^{\infty} \frac{y^k}{k} \le \frac{y}{2} \sum_{k=1}^{\infty} y^k \le \frac{y}{2}\frac{1}{1 - y}, \quad y \in (0, 1).$$

Now, since both ρ_{s+t} and $P(s, s, +t, x, \cdot)$ are normally distributed, by Pinsker's inequality and (4.24), for all $t \ge \delta$ and for any fixed $\delta > 0$, $x \in \mathbb{R}^d$

$$\begin{aligned} &\|P(s, s + t, x, \cdot) - \rho_{s+t}\|_{TV}^2 \\ &\le \frac{1}{2} D_{KL}(P(s, s + t, x, \cdot)||\rho_{s+t}) \\ &= \frac{1}{4}\left(1 - e^{-2tA} + \frac{(\xi(s + t) - e^{-tA}x - J(s, s + t))^2}{\sigma^2/2\alpha} - 1 - \ln(1 - e^{-2tA})\right) \\ &= \frac{1}{4}\left(\frac{e^{-2tA}(\xi(s) - x)^2}{\sigma^2/2\alpha} + \frac{1}{2}\frac{e^{-2tA}}{1 - e^{-2tA}}\right) \\ &\le \frac{e^{-2tA}}{\sigma^2}\frac{A}{2}\left((\xi(s) - x)^2 + \frac{\sigma^2}{4A}\frac{1}{1 - e^{-2\delta A}}\right). \end{aligned}$$

Deducing indeed the convergence is geometric. We go a little further solely to align with Theorem 4.5. For every fixed $s \in [0, T)$ and for any fixed $\gamma > 0$, there exists a constant $r_s = r_s(\gamma) > 0$ such that

$$(x - \xi(s))^2 - (1 + \gamma)x^2 = -2\xi(s)x + \xi^2(s) - \gamma x^2 \le r_s.$$

Hence $(x-\xi(s))^2 \leq (1+\gamma)x^2 + r_s$. Define $R_s := \max\left\{1+\gamma, r_s + \frac{\sigma^2}{4A}\frac{1}{1-e^{-2A\delta}}\right\} > 1$, then

$$\|P(s,s+t,x,\cdot)-\rho_{s+t}\|_{TV}^2 \leq \frac{e^{-2At}}{\sigma^2}\frac{A}{2}R_s\left(x^2+1\right) \leq e^{-2At}\left(\sqrt{\frac{\alpha}{2}}\frac{R_s}{\sigma}\right)^2\left(x^2+1\right)^2,$$

where we trivially squared the last two terms. Specifically by letting $t = nT$, we have geometric ergodicity of the grid chain

$$\|P(s,s+nT,x,\cdot)-\rho_s\|_{TV}^2 \leq e^{-2nTA}\left(\sqrt{\frac{A}{2}}\frac{R_s}{\sigma}\right)^2\left(x^2+1\right)^2, \quad n \in \mathbb{N}.$$

For computationally inclined readers, we give explicit formula for ξ_t in the one dimensional case. Multidimensional case can be computed similarly. By Fourier Series, for any $S \in L^2[0,T]$, S can be represented by

$$S(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi}{T}t - n\pi\right) + B_n \sin\left(\frac{2n\pi}{T}t - n\pi\right),$$

with the usual Fourier coefficients for $n \in \mathbb{N}\backslash\{0\}$

$$A_n = \frac{2}{T}\int_0^T S(t)\cos\left(\frac{2n\pi}{T}t - n\pi\right)dt, \quad B_n = \frac{2}{T}\int_0^T S(t)\sin\left(\frac{2n\pi}{T}t - n\pi\right)dt.$$

It is trivial to see $\xi_0 := \frac{1}{A}\frac{A_0}{2}$ satisfies $\partial_t(e^{At}\xi_0) = \frac{A_0}{2}e^{tA}$. Similarly,

$$\xi_n^{cos}(t) := \frac{1}{A}\frac{T^2\cos\left(n\pi - \frac{2n\pi}{T}t\right) - 2n\pi T\sin\left(n\pi - \frac{2n\pi}{T}t\right)}{4\pi^2n^2+T^2}$$

satisfies $\partial_t(e^{tA}\xi_n^{cos}(t)) = e^{tA}\cos\left(\frac{2n\pi}{T}t - n\pi\right)$ and

$$\xi_n^{sin}(t) := -\frac{1}{A}\frac{T^2\sin\left(n\pi - \frac{2n\pi}{T}t\right) + 2n\pi T\cos\left(n\pi - \frac{2n\pi}{T}t\right)}{4\pi^2n^2+T^2}$$

satisfies $\partial_t(e^{tA}\xi_n^{sin}(t)) = e^{tA}\sin\left(\frac{2n\pi}{T}T - n\pi\right)$. Clearly ξ_n^{cos} and ξ_n^{sin} are both T-periodic and $\xi(t) := \xi_0 + \sum_{i=1}^{\infty} A_n\xi_n^{cos}(t) + B_n\xi_n^{sin}(t)$ is the desired T-periodic continuous (hence) bounded function satisfying $\partial_t(e^{tA}\xi) = e^{tA}S$.

Example 4.10. The stochastic overdamped Duffing Oscillator has many physical applications including the phenomena of stochastic resonance in climate dynamics modelling of ice age [4,49,31] as mentioned in detail in the introduction. We also expect applications to periodically-forced model of price dynamics in the financial markets akin to [36] with a similar interpretation. The Duffing Oscillator is given by

$$dX_t = \left[-X_t^3 + X_t + A\cos(\omega t)\right]dt + \sigma dW_t, \tag{4.27}$$

where $A, \omega \in \mathbb{R}$ and $\sigma \neq 0$ are (typically small) parameters. In the Benzi-Parisi-Sutera-Vulpiani climate change stochastic resonance model, $\omega = 2\pi/10^5$ and the two stable equilibrium climates are distanced by $10K$. The stochastic differential equation (4.27) is a normalised equation of the Benzi-Parisi-Sutera-Vulpiani model. According to Corollary 4.7, there exists a unique periodic measure which is geometric ergodic.

Remark 4.11. Through the theory of non-autonomous RDS, [6] gave the existence and uniqueness of the periodic measure for (4.27) in one dimension. Note that Theorem 4.8 goes further than [6] to infer that the convergence is actually geometric. Moreover, Theorem 4.8 gives the other types of converges presented in Theorem 3.6. To our knowledge, this paper contains the first proof of the geometric ergodicity of the stochastic overdamped Duffing Oscillator. We note also that the approach we have taken works in multidimensional case and is completely different to the that of [6]. As mentioned in the introduction, we expect our approach can be extended to the infinite dimensional setting of SPDEs.

4.2.1. Gradient systems

In this section, we give results for the existence and uniqueness of geometric periodic measures for stochastic T-periodic gradient systems. These are SDEs of the form

$$dX_t = -\nabla V(t, X_t)dt + \sigma(X_t)dW_t, \tag{4.28}$$

where $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ is T-periodic, $\nabla = (\partial_1, \cdots, \partial_d)$ is the spatial gradient operator, W_t denotes a d-dimensional Brownian motion and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. Note that the T-periodicity of V implies the T-periodicity of ∇V is T-periodic hence the gradient SDE (4.28) is T-periodic.

Gradient systems arise naturally in physical applications, where V is referred to as the potential function [19,43,52]. Indeed examples of T-periodic gradient systems, include the periodic forced Ornstein-Uhlenbeck from Example 4.9 derived from $V(t,x) = \frac{\alpha}{2}\left(x - \frac{S(t)}{\alpha}\right)^2$ and the Duffing Oscillator from Example 4.10 derived from double-well potential $V(t,x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + A\cos(\omega t)x$. In fact, it is easy to verify that the Theorem 4.8 is a special case of gradient SDEs derived from potential $V(t,x) = \sum_{i=1}^{d}\sum_{k=1}^{2p_i}\frac{S_k^i(t)}{k+1}x_i^{k+1}$. While these examples are weakly dissipative where the Euclidean norm is a suitable Foster-Lyapunov function satisfying (4.16), in general, finding a Foster-Lyapunov function satisfying (4.16) for a given SDE is generally non-trivial (if at all possible) particularly in higher dimensions. A mathematical advantage of gradient systems is that V itself is a natural choice of Foster-Lyapunov function to satisfy (4.16). This is apparent by observing the generator of (4.28) is given by

$$\tilde{\mathcal{L}}V(t,x) = \partial_t V(t,x) - \|\nabla V(t,x)\|^2 + \frac{1}{2}\sum_{i,j=1}^{d}\left(\sigma\sigma^T(x)\right)_{ij}\partial_{ij}^2 V(t,x), \tag{4.29}$$

and exploiting the norm term.

For autonomous gradient SDEs derived from a norm-like potential $V(t,x) = V(x)$ and noise proportional to the identity $\sigma \in \mathbb{R}^+\backslash\{0\}$, it is well-known [40,19,52] that the invariant measure has a particularly simple form and is given by (upon normalisation) $\pi(\Gamma) = \int_\Gamma \exp\left(-\frac{2V(x)}{\sigma^2}\right)dx$ for $\Gamma \in \mathcal{B}$. However, due to the intricate interplay between stochasticity and periodicity, periodic measures (with a minimal positive period) do not have such simple expression. Indeed the periodic measure (4.26) from Example 4.9 does not take this simple form i.e. $\rho_s(\Gamma) \neq \int_\Gamma \exp\left(-\frac{2V(s,x)}{\sigma^2}\right) dx$.

The following corollary of Theorem 4.5 is generally simple to verify to yield gradient SDEs with a geometric periodic measure.

Corollary 4.12. *Assume σ satisfy (4.1), (4.2) and (4.3). Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ be a norm-like function such that for all $n \in \mathbb{N}$*

$$\partial^\alpha V \quad \textit{bounded on } \mathbb{R}^+ \times B_n, \alpha \in \mathbb{N}^{d+1}, |\alpha| \in \{1, d+1\},$$

and (4.16) holds, where $\tilde{\mathcal{L}}$ is given by (4.29). Then the results of Theorem 3.6 hold for SDE (4.28).

While Corollary 4.12 covers all the examples considered thus far, it applies to a wider class of SDEs than that of weakly dissipative systems. In the next proposition, we use Corollary 4.12 to extend the case of Theorem 4.8 when $p_i = \text{const}$ for all i and allowing for products of the spatial variables. It does not aim to be most general however suffices a range of applications. We shall employ more standard multi-index notation: for spatial variables $x = (x_1, \cdots, x_d)$ and multi-index $\alpha \in \mathbb{N}^d$, define $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For $\alpha, \beta \in \mathbb{N}^d$, we have the partial ordering $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for each $1 \leq i \leq d$. We define the standard tuple basis $e_i = (0, \cdots, 1, \cdots, 0)$ where the 1 appears on the i'th index. For fixed $\beta \in \mathbb{N}^d$, we define $\sum_{\alpha \geq \beta}^N C_\alpha := \sum_{\alpha \geq \beta}^{|\alpha| \leq N} C_\alpha$. Recall standard asymptotic notation where for functions $f_1, f_2, g : \mathbb{R}^d \to \mathbb{R}$, we write $\max\{f_1, f_2\} = o(g)$ if $\lim_{\|x\| \to \infty} \frac{\max\{|f_1(x)|, |f_2(x)|\}}{g(x)} = 0$. This implies that for any $\epsilon > 0$, there exists $R > 0$ such that

$$\max\{|f_1(x)|, |f_2(x)|\} \leq \epsilon |g(x)|, \quad x \in B_R^c. \tag{4.30}$$

Proposition 4.13. *Assume σ satisfy (4.1), (4.2) and (4.3). Let $\{S_\alpha(t)\}_{\alpha \in \mathbb{N}^d}$ be continuously differentiable T-periodic functions and $\{S_i\}_{i=1}^d$ are strictly positive constants. Then the gradient system (4.28) with potential*

$$V(t, x) = \sum_{i=1}^d S_i x_i^p + \sum_{|\alpha|=0}^{p-1} S_\alpha(t) x^\alpha, \quad p \in 2\mathbb{N} := \{2, 4, ..., \},$$

satisfies Corollary 4.12 hence the results of Theorem 3.6 holds.

Proof. We compute

$$\begin{cases} \partial_t V = \sum_{|\alpha|=0}^{p-1} \dot{S}_\alpha x^\alpha, \\ \partial_i V = S_i p x_i^{p-1} + \sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha - e_i}, \\ \partial_{ii}^2 V = p(p-1) S_i x_i^{2p-2} + \sum_{\alpha \geq 2e_i}^{p-1} \alpha_i (\alpha_i - 1) S_\alpha x^{(\alpha - 2e_i)}, \\ \partial_{ij}^2 V = \sum_{\alpha \geq e_i + e_j}^{p-1} S_\alpha \alpha_i \alpha_j x^{\alpha - e_i - e_j}, & i \neq j. \end{cases}$$

So

$$\|\nabla V\|^2 = \sum_{i=1}^{d} (\partial_i V)^2 = \sum_{i=1}^{d} \left[S_i^2 p^2 x_i^{2p-2} + 2 S_i p \sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha+(p-2)e_i} + \left(\sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha - e_i} \right)^2 \right].$$

Note that V, $\partial_t V$, $\partial_{ij}^2 V$ and $\left(\|\nabla V\|^2 - \sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right)$ has maximum order p, $p-1$, $p-3$ and $2p-3$ respectively. Our assumptions ensure that $\max_{\alpha\in\mathbb{N}^d}(\sup_{t\in\mathbb{R}}|S_\alpha(t)|) < \infty$ and $\max_{i,j}\sup_{x\in\mathbb{R}^d} a_{ij}(x) < \infty$. Since higher even powers dominate lower powers i.e. $x^\alpha = o(\sum_{i=1}^d c_i x_i^{2n})$ where $c_i > 0$ and $|\alpha| < 2n$ where $n \in \mathbb{N}$, we have for any $\lambda > 0$

$$\max\left\{\lambda V, \partial_t V, \partial_{ij}^2 V, \left(\|\nabla V\|^2 - \sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right)\right\} = o\left(\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right), \quad 2 < p \in 2\mathbb{N}.$$

Then for $2 < p \in 2\mathbb{N}$, by (4.30), for any $\epsilon \in (0, \frac{1}{4})$, there exists $R > 0$ such that

$$\tilde{\mathcal{L}} V + \lambda V \leq |\partial_t V| - \|\nabla V\|^2 + \frac{1}{2} |\sum_{i,j=1}^d a_{ij} V| + \lambda V \leq (4\epsilon - 1)\left(\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right) \leq 0,$$
$$x \in B_R^c.$$

By continuity, $\tilde{\mathcal{L}} V + \lambda V$ is bounded on B_R. Hence (4.16) is satisfied. For $p = 2$ where V and $\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}$ are of the same order, the same calculations hold provided one restricts $0 < \lambda < 4\min_i S_i^2$. □

In physics literature, "periodically forced" or "periodically driven" generally refers to the addition of a periodic term on the drift which otherwise be autonomous i.e. $b(t, x) = b_0(x) + S(t)$ for some periodic function S and drift b_0 independent of t. Particular instances of Proposition 4.13 include periodically-forced systems such Example 4.9 and Example 4.10. Mentioned examples so far are systems with polynomial potentials. While polynomial approximation of potentials (by Weierstrass approximation theorem for instance) can be effective for practical reasons, we consider periodically forced gradient systems that need not be derived from a polynomial potential. We remark that periodically forced gradient SDEs occurs in physical applications and phenomena as we have already seen in previous examples. For further discussions on periodically forced stochastic systems, we refer readers to [31] for theory and applications.

Consider the following (autonomous) gradient SDE on $\mathbb{R}^d$

$$dX_t = -\nabla U(X_t) + \sigma(X_t) dW_t, \tag{4.31}$$

where σ satisfy (4.1), (4.2) and (4.3) and $U \in C^2(\mathbb{R}^d, \mathbb{R}^+)$ satisfies the (autonomous) geometric drift condition

$$LU \leq C - \lambda U \quad \text{on } \mathbb{R}^d, \tag{4.32}$$

where $C \geq 0, \lambda > 0$ are constants and $\mathcal{L}$ is the infinitesimal generator of (4.31) given by

$$Lf(x) = -\langle \nabla U(x), \nabla f(x) \rangle + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 f(x), \quad f \in C^2(\mathbb{R}^d).$$

This classical geometric drift condition yields the existence, uniqueness and ergodicity of an invariant measure. The context of the next lemma sufficiently yields a geometric periodic measure when the autonomous gradient system is periodically forced. Essentially, the autonomous system retains its stability up to replacing its invariant measure for a periodic measure with a minimal positive period. Note that we do not impose any particular form imposed on the potential, hence more general than polynomials. We note that the assumptions are easily satisfied for many practical systems.

Proposition 4.14. *Let $U \in C^2(\mathbb{R}^d, \mathbb{R}^+)$ be a norm-like potential satisfying (4.32) and that for any $c_1, c_2 > 0$, there exists a compact set $K = K(c_1, c_2) \subset \mathbb{R}^d$ such that*

$$c_1 \|x\| \le c_2 U(x) \quad x \in K^c.$$

Then for any T-periodic ($T > 0$) continuously differentiable function $S : \mathbb{R}^+ \to \mathbb{R}^d$, the periodically forced gradient SDE

$$dX_t = -[\nabla U(X_t) + S(t)]dt + \sigma(X_t)dW_t$$

possesses a unique geometric periodic measure.

Proof. By Theorem 4.5, we verify $V(t,x) = U(x) - \langle S(t), x \rangle$ satisfies (4.16). By the assumptions on U and S, it is clear that $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ is a T-periodic norm-like potential satisfying the regularity assumptions of Corollary 4.12. Since $\partial_{ij}^2 V = \partial_{ij}^2 U$, we compute that

$$\begin{aligned}
\tilde{\mathcal{L}} V &= -\langle \dot{S}, x \rangle - \langle \nabla U(X_t) + S(t), \nabla U(X_t) - S(t) \rangle + \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 V \\
&= -\langle \dot{S}, x \rangle - \|\nabla U\|^2 + \|S\|^2 + \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 U \\
&= \|S\|^2 - \langle \dot{S}, x \rangle + LU.
\end{aligned}$$

As U satisfies the geometric drift condition, by picking any fixed $\lambda^- \in (0, \lambda)$, we have

$$\begin{aligned}
\tilde{\mathcal{L}} V &\le \|S\|^2 - \langle \dot{S}, x \rangle + C - \lambda U \\
&= \|S\|^2 - \langle \dot{S}, x \rangle + C - (\lambda - \lambda^-)U - \lambda^- U + (\lambda - \lambda^-)\langle S, x \rangle - (\lambda - \lambda^-)\langle S, x \rangle \\
&= \|S\|^2 - \langle \dot{S} + (\lambda - \lambda^-)S, x \rangle + C - (\lambda - \lambda^-)V - \lambda^- U \\
&\le \|S\|^2 + \|\dot{S} + (\lambda - \lambda^-)S\|_\infty \|x\| + C - (\lambda - \lambda^-)V - \lambda^- U,
\end{aligned}$$

where $\|\dot{S} + (\lambda - \lambda^-)S\|_\infty := \sup_{s\in[0,T]} \|\dot{S}(s) + (\lambda - \lambda^-)S(s)\| < \infty$ as S and $\dot{S}$ are bounded. Then, by assumption with $c_1 = \|\dot{S} + (\lambda - \lambda^-)S\|_\infty$ and $c_2 = \lambda^-$, we have a compact set $K \subset \mathbb{R}$ such that

$$c := \sup_{x\in K} \left(\|\dot{S} + (\lambda - \lambda^-)S\|_\infty \|x\| - \lambda^- U \right) < \infty.$$

Hence $\tilde{\mathcal{L}}V \leq \left(C + c + \|S\|^2\right) - \left(\lambda - \lambda^-\right) V$ i.e. the geometric drift condition (4.16) is satisfied. □

4.3. Langevin dynamics

Langevin equations originated to model noisy molecular systems and many other physical phenomena. As such, we expect applications to the physical sciences. In fact, we shall see it extends easily from stochastic gradient systems in an “overdamped” limit and applies immediately to the stochastic periodically-forced harmonic oscillator. We refer the reader to [58,52] for further applications, details and derivations of Langevin equations. Akin to earlier sections, we give sufficient conditions for the existence, uniqueness and geometric convergence of a periodic measure for T-periodic Langevin equations. We study Langevin equations of the form

$$m d\dot{q}_t = (F(t, q_t) - \gamma \dot{q}_t)\, dt + \sigma dW_t, \tag{4.33}$$

with position $q_t \in \mathbb{R}^d$, velocity $\dot{q}_t \in \mathbb{R}^d$, acceleration $\ddot{q}_t \in \mathbb{R}^d$, constant mass $m > 0$, time-dependent force $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, d-dimensional Brownian motion W_t and constant matrix $\sigma \in GL(\mathbb{R}^d)$. For $\gamma \geq 0$, $\gamma \dot{q}_t$ is understood as the frictional force of the system. The proportional constant γ is referred as the damping constant. Without loss of generality, we take mass to be unit i.e. $m = 1$.

Denote momentum $p_t = \dot{q}_t$, then (4.33) can be rewritten as a system of first order SDEs

$$\begin{cases} dq_t = p_t dt, \\ dp_t = (-\gamma p_t + F(t, q_t))\, dt + \sigma dW_t. \end{cases} \tag{4.34}$$

In phase space coordinates $X_t = (q_t, p_t) \in \mathbb{R}^{2d}$ this can be rewritten as

$$dX_t = b(t, X_t)dt + \Sigma d\mathcal{W}_t, \tag{4.35}$$

where

$$b(t,x) = b(t,q,p) = \begin{pmatrix} p \\ -\gamma p + F(t,q) \end{pmatrix} \in \mathbb{R}^{2d}, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \in \mathbb{R}^{2d\times 2d}, \quad \mathcal{W}_t = \begin{pmatrix} 0 \\ W_t \end{pmatrix}. \tag{4.36}$$

On a physical level, observe that the noise is degenerate in that the noise affects q_t only through p_t. Formally, Langevin SDE (4.35) is degenerate since $\Sigma \notin GL(\mathbb{R}^{2d})$. Resultantly, Theorem 4.2 does not apply. Hence in this current paper, we only study Langevin dynamics with only additive noise. It will be of future works to study the situation with multiplicative noise.

Written in phase space coordinates, it is clear that (4.33) has unique solution provided b and σ are Lipschitz. Labelling $x = (q, p) = (x_1, .., x_{2d})$, the infinitesimal generator is given by

$$\tilde{\mathcal{L}} f(t,x) = \partial_t f + \langle p, \nabla_q f\rangle + \langle -\gamma p + F, \nabla_p f\rangle + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}\partial^2_{p_i p_j} f,$$
$$f \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^{2d}), \tag{4.37}$$

where $\nabla_q := (\partial_{q_1}, \cdots, \partial_{q_d})^T$ and similarly $\nabla_p := (\partial_{p_1}, \cdots, \partial_{p_d})^T$.

Remark 4.15. We remark that in physical applications concerning small particles, the mass is typically small. This suggests the inertia term $m\ddot{q}_t$ can be neglected. Hence, not rigorously, the dynamics (4.33) can be well-approximated by

$$0 = F(t, q_t) - \gamma \dot{q}_t + \sigma dW_t,$$

i.e. reduced to SDEs studied earlier in this section. Suggesting that Langevin equations may be studied with multiplicative noise in the context of small particles. A particular source of interesting dynamics and applications is the case when $F(t,q) = -\nabla_q V(t,q)$ for some potential $V(t,q)$ and so the Langevin equations are gradient systems (provided $\gamma > 0$). Such systems without inertia are called overdamped Langevin dynamics.

With the inapplicability of Theorem 4.2, we the following irreducibility lemma for non-autonomous Langevin equation. This can be done by a similar method as in [43], so it is omitted here.

Lemma 4.16. *Consider T-periodic Langevin equation (4.34) with locally Lipschitz F. Assume there exists a norm-like function V satisfying (4.10). Then the Markov transition kernel satisfies $P(s,t,x,\Gamma) > 0$ for any $s < t < \infty$, $x \in \mathbb{R}^{2d}$ and non-empty open $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$.*

Thus we have the following Langevin counterpart of Theorem 4.5.

Theorem 4.17. *Consider T-periodic Langevin equation (4.35) with F satisfying (4.4) (in place of b). Assume there exists a norm-like function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^{2d}, \mathbb{R}^+)$ satisfying (4.16) where $\mathcal{L}$ is given by (4.37). Then there exists a unique geometric periodic measure $\rho : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^{2d})$ satisfying the convergences from Theorem 3.6.*

Proof. Let σ_i denote the i'th column of σ then $\Sigma_i = (0, \sigma_i)^T$ denote the i'th column of Σ. Denoting $Id \in \mathbb{R}^{d\times d}$ to be the identity matrix, observe that Lie bracket

$$[\Sigma_i, b] = (Db)\Sigma_i = \begin{pmatrix} 0 & Id \\ -d^2F(t,q) & -\gamma Id \end{pmatrix}\begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} = \begin{pmatrix} \sigma_i \\ -\gamma\sigma_i \end{pmatrix}.$$

Since $\sigma \in GL\,(\mathbb{R}^d)$, the columns σ_i are linear independent. Hence the Lie algebra generated by Σ_i and b spans $\mathbb{R}^{2d}$. By the assumptions on F, b satisfies (4.4). Hence together with Foster-Lyapunov function V, there exists a smooth density $p(s,t,x,y)$ with respect to Λ by Theorem 1 of [28] is satisfied. F is locally Lipschitz hence with Lemma 4.16, Theorem 3.4 holds. Hence the assumptions of Theorem 3.6 are satisfied. □

5. Density of periodic measures

Similar to invariant measures, it is interesting and important to know when periodic measures possess a density with respect to Lebesgue measure. We show that for T-periodic SDEs where the periodic measure exists and the Markov transition possesses a density, then the density of the periodic measure exists and provide a formula for it. We show also that the density of the periodic measure necessarily and sufficiently satisfies a Fokker-Planck PDE of an "initial-terminal" kind. We also provide an explicit example for the periodically forced Ornstein-Uhlenbeck process. In previous sections, we have predominantly been focused on initial state, here we change our perspective to the forward spatial variable. As such, at the risk of confusion, we interchange the roles of x and y i.e. we take $y \in \mathbb{R}^d$ to be the initial state and $x \in \mathbb{R}^d$ to be the forward spatial variable.

We already know the conditions to guarantee the existence of the density $p(s,t,y,x)$ of the two-parameter Markov transition kernel, $P(s,t,y,\cdot)$ and the existence of a periodic measure from our results in the last section. Let $(\rho_t)_{t\in\mathbb{R}^+}$ be a family of probability measures satisfying $\rho_t = P^*(s,t)\rho_s$ for $s \le t$, then Fubini's theorem yields

$$\rho_t(\Gamma) = \int_{\mathbb{R}^d} P(s,t,y,\Gamma)\rho_s(dy) = \int_{\Gamma} \left(\int_{\mathbb{R}^d} p(s,t,y,x)\rho_s(dy) \right) dx, \quad \Gamma \in \mathcal{B}(\mathbb{R}^d), s \le t. \tag{5.1}$$

If ρ is a periodic measure, then specifically

$$\rho_t(\Gamma) = \int_{\Gamma} \left(\int_{\mathbb{R}^d} p(t,t+T,y,x)\rho_t(dy) \right) dx,$$

that is to say $q(t,x) = \int_{\mathbb{R}^d} p(t,t+T,y,x)\rho_t(dy)$ is the density of ρ_t as observed in [11]. Given the existence of q, it is clear by (5.1) that q satisfies

$$q(t,x) = \int_{\mathbb{R}^d} p(s,t,y,x)q(s,y)dy, \tag{5.2}$$

indeed this property holds for any family of measures with densities. Moreover, it is easy to see that $\lim_{t\downarrow s} q(t,\cdot) = q(s,\cdot)$. In this section we will prove that q satisfies the following Fokker-Planck equation

$$\partial_t q = \mathcal{L}^*(t)q,$$

where $\mathcal{L}^*(t)$ is the Fokker-Planck operator given by

$$\mathcal{L}^*(t)q = -\sum_{i=1}^{d} \partial_{x_i}(b_i(t,x)q) + \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{x_i x_j}\left(\left(\sigma\sigma^T(t,x) \right)_{ij} q \right). \tag{5.3}$$

In this section, we will always assume that the operator $\mathcal{L}^*(t)$ is uniformly elliptic i.e. there exists $\lambda > 0$ such that $\langle \xi, \sigma\sigma^T(t,x)\xi\rangle \geq \lambda\|\xi\|^2$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$.

In the following, we shall use the notation $X \sim q$ to mean the random variable X is distributed by probability density $q \in L^1(\mathbb{R}^d)$. For random variables X^0 and X^1, we write $X^0 \sim X^1$ if they have the same distribution. We state and prove the following useful lemma.

Lemma 5.1. *Assume $(X_t^0)_{t\geq s}$, $(X_t^1)_{t\geq s+T}$ are two processes satisfying the T-periodic SDE (4.5). If $X_s^0 \sim X_{s+T}^1$ then $X_{s+t}^0 \sim X_{s+T+t}^1$ for all $t \geq 0$.*

Proof. For concreteness, let $X_s^0 \sim X_{s+T}^1 \sim q \in L^1(\mathbb{R}^d)$ and $p^0(s+t,\cdot)$ denote the distribution of X_{s+t}^0 and similarly $p^1(s+T+t,\cdot)$ for X_{s+T+t}^1. Then p^k satisfies the Fokker-Planck equation i.e. for $k=0,1$ and $t \geq 0$

$$\begin{cases} \partial_t p^k(t+kT,x) = \mathcal{L}^*(t+kT)p^k(t+kT,x), \\ p^k(s+kT,\cdot) = q. \end{cases}$$

It is clear that $\mathcal{L}^*(t) = \mathcal{L}^*(t+T)$ by the T-periodic coefficients. By the linearity of the Fokker-Planck operator, it is easy to see that $\hat{p}(t,\cdot) := p^0(s+t,\cdot) - p^1(s+t+T,\cdot)$ satisfies

$$\begin{cases} \partial_t \hat{p} = \mathcal{L}^*(t)\hat{p} \quad t \geq 0, \\ \hat{p}(0,\cdot) = 0. \end{cases}$$

Then an application of parabolic maximum principle or otherwise yields that $\hat{p}(t,\cdot) = 0$ for all $t \geq 0$ is the only physical solution. Hence concluding $p^0(s+t,\cdot) = p^1(s+T+t,\cdot)$ for all $t \geq 0$. $\square$

With Lemma 5.1, we are now ready to state the main result of this section.

Theorem 5.2. *Consider T-periodic SDE (4.5) with continuous coefficients. For $q \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ define $\rho : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^d)$ by*

$$\rho_t(\Gamma) = \frac{1}{\|q(t,\cdot)\|_{L^1(\mathbb{R}^d)}} \int_\Gamma q(t,x)dx, \quad t \geq 0.$$

Then ρ is a T-periodic measure if and only if

$$\partial_t q = \mathcal{L}^*(t)q, \quad q(0,\cdot) = q(T,\cdot). \tag{5.4}$$

Hence, if (5.4) has a unique solution then there is a unique periodic measure with density q.

Proof. For notational convenience and without loss of generality, we let $q(t,x)$ be normalised. Assume ρ is a T-periodic measure, then by definition, $\rho_t = \rho_{t+T}$ for all $t \geq 0$ i.e.

$$\int_\Gamma q(t,x)dx = \rho_t(\Gamma) = \rho_{t+T}(\Gamma) = \int_\Gamma q(t+T,x)dx, \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As this holds for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, it follows that $q(t,\cdot) = q(t+T,\cdot)$. On the other hand, it is well known that $p(s,t,y,x)$ satisfies the Fokker-Planck equation

$$\partial_t p(s,t,y,x) = \mathcal{L}^*(t) p(s,t,y,x).$$

We take derivative with respect to t on both sides of (5.2), we have

$$\begin{aligned}
\partial_t q(t,x) &= \int_{\mathbb{R}^d} \partial_t p(s,t,y,x) q(s,y) dy \\
&= \int_{\mathbb{R}^d} \mathcal{L}^*(t) p(s,t,y,x) q(s,y) dy \\
&= \int_{\mathbb{R}^d} -\sum_{i=1}^d \partial_{x_i}(b_i(t,x) p(s,t,y,x)) q(s,y) dy \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \left(\left(\sigma\sigma^T(t,x) \right)_{ij} p(s,t,y,x) \right) q(s,y) dy \\
&:= A + B.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
A &= -\sum_{i=1}^d \int_{\mathbb{R}^d} \left[\partial_{x_i}(b_i(t,x)) p(s,t,y,x) + b_i(t,x) \partial_{x_i}(p(s,t,y,x)) \right] q(s,y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x)) \int_{\mathbb{R}^d} p(s,t,y,x) q(s,y) dy - \sum_{i=1}^d b_i(t,x) \partial_{x_i} \int_{\mathbb{R}^d} p(s,t,y,x) q(s,y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x)) q(t,x) - \sum_{i=1}^d b_i(t,x) \partial_{x_i} q(t,x) \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x) q(t,x)).
\end{aligned}$$

Similarly, for the second term, we have

$$B = \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \left(\left(\sigma\sigma^T(t,x) \right)_{ij} q(t,x) \right).$$

Therefore, the density function $q(\cdot,\cdot)$ satisfies

$$q(t,\cdot) = q(t+T,\cdot), \quad \partial_t q = \mathcal{L}^*(t) q, \quad \text{for all } s \geq 0.$$

By Lemma 5.1, it suffices that this PDE holds specifically for $t = 0$ hence we have (5.4).

To prove the converse, we first note that Lemma 5.1 yields that $q(t,\cdot) = q(t+T,\cdot)$ for all $t \geq 0$. Thus ρ is T-periodic. By (5.2) and Fubini's theorem, it is clear that

$$P^*(s,t)\rho_s(\Gamma) = \int_{\mathbb{R}^d} \left[\int_\Gamma p(s,t,y,x)dx \right] q(s,y)dy = \int_\Gamma q(t,x)dx = \rho_t(\Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d),$$

concluding that ρ is a T-periodic measure. □

There is an "alternative" way to arrive the PDE of Theorem 5.2 as seen in [30]. By considering lifted coordinates (t, X_t), one can consider stationary solutions of the lifted Fokker-Planck operator $\tilde{\mathcal{L}}^*$ i.e. $q(t,x)$ satisfying

$$\tilde{\mathcal{L}}^* q := -\partial_t q(t,x) + \mathcal{L}^*(t) q(t,x) = 0. \tag{5.5}$$

This is equivalent to (5.4) upon rearranging. However, this approach does not naturally impose any boundary conditions, hence is not sufficient for q to be the density of the periodic measure. Theorem 5.2 states that the boundary condition is necessary. While [30] imposes the periodic boundaries, the reasoning does not seem apparent. We shall show in the example below that, despite $\mathcal{L}^*(t)$ is T-periodic, a solution to the PDE need not be periodic and relaxing such condition, perhaps expectedly, one can have infinitely many solutions.

Example 5.3. The one-dimensional periodically-forced Ornstein-Uhlenbeck process from Example 4.9 has its Fokker-Planck operator given explicitly by

$$\tilde{\mathcal{L}}^* q = -\partial_t q - \partial_x((S(t) - \alpha x)q) + \frac{\sigma^2}{2}\partial_x^2 q = -\partial_t q - S(t)\partial_x q + \alpha q + \alpha x \partial_x q + \frac{\sigma^2}{2}\partial_x^2 q$$

and the periodic measure is $\rho_t = \mathcal{N}\left(\xi(t), \frac{\sigma^2}{2\alpha}\right)$, where $\xi(t) = e^{-\alpha t}\int_{-\infty}^t e^{\alpha r} S(r)dr$. Here, the density of the periodic measure is given by

$$q(t,x) = \frac{1}{\sqrt{\pi\sigma^2/\alpha}} \exp\left(-\frac{(x-\xi(s))^2}{\sigma^2/\alpha} \right).$$

We compute

$$\dot{\xi} = -\alpha\xi + S(t), \quad \partial_t q = 2\frac{\alpha}{\sigma^2}\dot{\xi}(x-\xi)q, \quad \partial_x q = -2\frac{\alpha}{\sigma^2}(x-\xi)q,$$

and

$$\partial_x^2 q = -2\frac{\alpha}{\sigma^2}\left[\partial_x(xq) - \xi\partial_x q\right] = -2\frac{\alpha}{\sigma^2}\left[1 - 2\frac{\alpha}{\sigma^2}(x-\xi)^2\right] q.$$

Hence, substituting directly,

$$\begin{aligned}\frac{\tilde{\mathcal{L}}^* q}{q} &= -2\frac{\alpha}{\sigma^2}\dot{\xi}(x-\xi) + 2\frac{\alpha}{\sigma^2}S(t)(x-\xi) + \alpha - 2\frac{\alpha^2}{\sigma^2}x(x-\xi) - \alpha\left[1 - 2\frac{\alpha}{\sigma^2}(x-\xi)^2\right] \\ &= 2\frac{\alpha^2}{\sigma^2}\xi(x-\xi) - 2\frac{\alpha^2}{\sigma^2}x(x-\xi) + 2\frac{\alpha^2}{\sigma^2}(x-\xi)^2 \\ &= 0.\end{aligned}$$

Thus indeed the q satisfies (5.4). We show that the periodic condition of (5.4) cannot simply be dropped because of periodic coefficients. From (4.25), the transition density is explicitly given by

$$p(s,t,y,x) = \frac{1}{\sqrt{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha(t-s)})}} \exp\left(-\frac{(x - e^{-\alpha(t-s)}y - J(s,t))}{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha(t-s)})}\right),$$

satisfies $-\partial_t p(t,x) + \mathcal{L}^*(t)p(t,x) = 0$ for every fixed initial time s and point y. However, p is not periodic as J is not periodic. Since there is a non-periodic solution for every $y \in \mathbb{R}$, there are, in fact, infinite number of solutions to the PDE if one relaxed the demand of periodicity.

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