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On the use of Reuleaux plasticity for geometric non-linear analysis

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ABSTRACT

Three dimensional analyses including geometric and material non–linearity require robust, efficient constitutive models able to simulate engineering materials. However, many existing constitutive models have not gained widespread use due to their computational burden and lack of guidance on choosing appropriate material constants. Here we offer a simple cone-type elasto-plastic formulation with a new deviatoric yielding criterion based on a modified Reuleaux triangle. The perfect plasticity model may be thought of as a hybrid between Drucker-Prager (D-P) and Mohr-Coulomb (M-C) that provides control over the internal friction angle independent of the shape of the deviatoric section. This surface allows an analytical backward Euler stress integration on the curved surface and exact integration in the regions where singularities appear. The attraction of the proposed algorithm is the improved fit to deviatoric yielding and the one–step integration scheme, plus a fully defined consistent tangent. The constitutive model is implemented within a lean 3D geometrically non-linear finite-element program. By using an updated Lagrangian logarithmic strain–Kirchhoff stress implementation, existing infinitesimal constitutive models can be incorporated without modification.

1 MODIFIED REULEUAX

Frictional isotropic yield surfaces may be defined using Haigh-Westergaard cylindrical coordinates ξ, ρ and θ, utilising the normalised deviatoric radius, $\bar{\rho} = \rho/\rho_c$; where ρ_c is the radius on the compression meridian ($\theta = \pi/6$) and ρ is a function of the Lode angle

$$\theta = \frac{1}{3} \arcsin\left((-3\sqrt{3}J_3)/(2J_2^{3/2})\right), \qquad \in [-\pi/6, \pi/6]. \tag{1}$$

Here $J_2 = (\text{tr}[s]^2)/2$, $J_3 = (\text{tr}[s]^3)/3$, $[s] = [\sigma] - \xi[I]/\sqrt{3}$ and $\xi = \text{tr}[\sigma]/\sqrt{3}$. From geometric considerations, the modified Reuleaux (MR) Lode angle dependency may be obtained as

$$\bar{\rho}(\theta) = \sqrt{\bar{a}^2 + \bar{r}^2 - 2\bar{a}\,\bar{r}\cos(\phi)}, \tag{2}$$

where

$$\bar{r} = \left(\bar{\rho}_e^2 - \bar{\rho}_e + 1\right)/\left(2\bar{\rho}_e - 1\right), \qquad \bar{a} = \bar{r} - \bar{\rho}_e \qquad \text{and} \qquad \bar{\rho}_e = \rho_e/\rho_c. \tag{3}$$

$\bar{\rho}_e \in [0.5, 1]$ gives the relative size of the radius under triaxial extension ($\sigma_1 = \sigma_2 < \sigma_3$) with respect to that under triaxial compression ($\sigma_1 < \sigma_2 = \sigma_3$). The arc angle, see Figure 1(i), is defined as

$$\phi = \pi/6 + \theta - \arcsin\left(\bar{a}\sin(5\pi/6 - \theta)/\bar{r}\right). \tag{4}$$

If the arc centres[1] coincide with the singularities on the compression meridians (that is, if $\bar{r} = 1 + \bar{\rho}_e$ so that $\bar{a} = 1$) then the shape of the deviatoric section is a Reuleaux triangle. Allowing the location of the arc centres to vary along projections of the compression meridians gives rise to the *modified Reuleaux triangle*, see Figure 1(i).[2] The MR cone can be defined as

$$f = \rho - \alpha\bar{\rho}(\xi - \xi_c) = 0 \tag{5}$$

where α is the opening angle of the cone. $\alpha = -\tan(\psi_{MC})$, given the M-C internal friction angle ψ_{MC} under triaxial compression. ξ_c locates the intersection of the yield surface with the hydrostatic axis. Thus (5) defines a cone with a MR deviatoric section and linear meridians, pinned at ξ_c with the space diagonal ($\sigma_1 = \sigma_2 = \sigma_3$) as the cone's axis, see Figure 1(ii) (in this case with $\xi_c = 0$). The MR cone can be seen as a hybrid surface, lying between the D-P and M-C envelopes, allowing some control over the shape of the deviatoric section, independent of the cone opening angle.
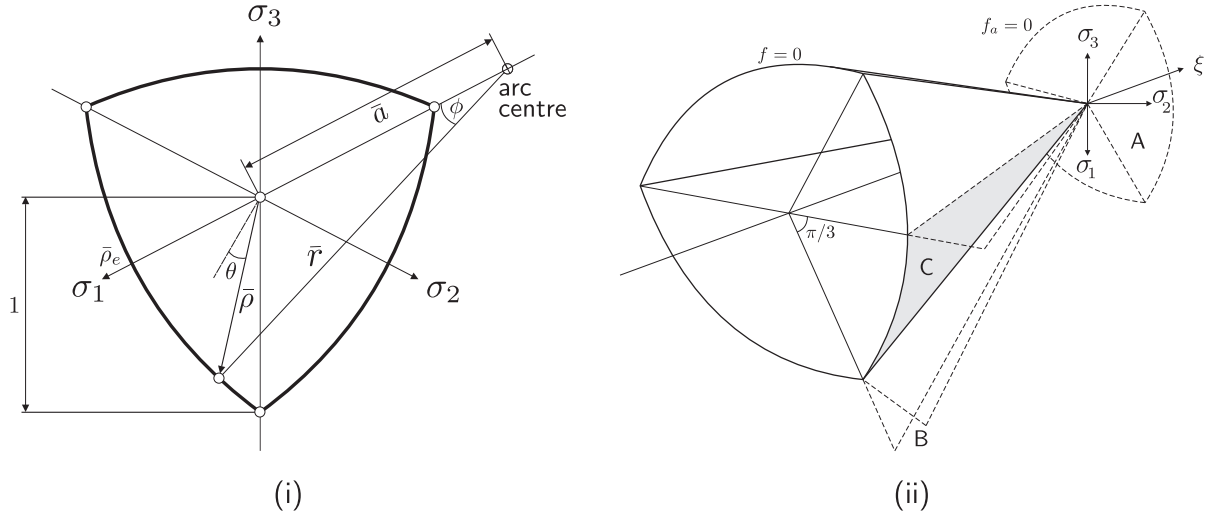

Figure 1: (i) Modified Reuleaux deviatoric section (ii) Modified Reuleaux cone stress return regions.

1.1 Analytical backward Euler stress return

Simo and Hughes [2] showed that for associated flow the backward Euler integration corresponds to the minimisation of

$$\{\{\sigma_r\} - \{\sigma_t\}\}^T [C^e] \{\{\sigma_r\} - \{\sigma_t\}\}, \tag{6}$$

with respect to the return stress $\{\sigma_r\}$ (where $[C^e]$ is the elastic compliance matrix), which represents a Closest Point Projection (CPP). The minimisation is subject to the following constraints: $f \leq 0, \dot{\gamma} \geq 0$ and $f\dot{\gamma} = 0$. $(\cdot)_t$ and $(\cdot)_r$ denote quantities associated with the trial state and the return state respectively, where f is the yield function and $\dot{\gamma}$ is the plastic multiplier. The return stress is not generally the closest point geometrically in standard stress space, but rather the stress that minimises the energy square norm (6). When returning to the MR cone we make use of energy-mapped space, allowing us to find the closest point geometrically (see [1] or [3] for more details).

Consider the trial elastic stress lying outside the yield surface ($f > 0$). For this state there are three distinct stress return regions associated with the MR cone, as shown in Figure 1 (ii), namely: A stress origin (point), B compression meridian (line) or C non-planar surface. The CPP and consistent tangent are considered for each region in [1]. The approach, when returning to the compression meridian or tensile apex, follows Clausen *et al.'s* method [4]. When returning elsewhere, finding the closest point to the surface in energy-mapped space [3] requires the solution of a quartic. Errors associated with the integration scheme have been shown to be less than 3%, whilst providing a 2.5-5.0 times speed up over the conventional iterative backward Euler stress return method [1].

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[1]There are three arc centres in a principal stress space deviatoric plane; isotropy requires only one to be considered.

[2]As $\bar{\rho}_e \to 0.5$ both $\bar{r}$ and $\bar{a}$ tend to ∞ and the deviatoric section becomes an equilateral triangle. If $\bar{\rho}_e = 1$ then $\bar{\rho} = 1$ and we recover a circular deviatoric section centred on the hydrostatic axis (as found in the D-P model).

2 NON-LINEAR FINITE-ELEMENT ANALYSIS

Finite-Element (FE) programs offering the capability to include three dimensional finite deformation inelastic continuum analysis are typically expressed in thousands of lines. A number of open source codes encourage researchers to extend or modify the basic algorithms, however due to size of these codes, this requires a significant time investment from potential new developers. Researchers are faced with writing their own algorithms from scratch or mastering lengthy codes which are typically understandable only by those close to the original development. However, high level computational environments, such as MATLAB, allow engineers, scientists and mathematicians to produce powerful numerical analysis scripts rapidly. By using lean, efficient algorithms and subfunctions, it is possible to write the main routine of an elasto-plastic finite deformation FE program within a single page. Once a program spills onto multiple pages the ability to easily visualise the program structure is lost and the opportunity for error detection is reduced. Transparent programs facilitate re-analysis, adjustment, improvement and experimentation, resulting in polished robust algorithms.

A three dimensional MATLAB finite deformation updated Lafrangian FE code has been developed by the first author at Durham University, with the intention of analysing geotechnical problems subject to large deformations and strains. The constitutive model described in Section 1 was implemented within the FE code.

Unlike infinitesimal theory, within a finite deformation framework there exists a choice for the stress and strain measures. However, certain combinations provide advantages when moving between infinitesimal and large strain theories. The implemented FE code uses a logarithmic strain–Kirchhoff stress relationship along with an exponential map for the plastic flow equation to allow the implementation of standard small strain constitutive algorithms within a finite deformation framework without modification.

3 NUMERICAL ANALYSIS

In this section we present the analysis of the expansion of a thick-walled soil cylinder under internal
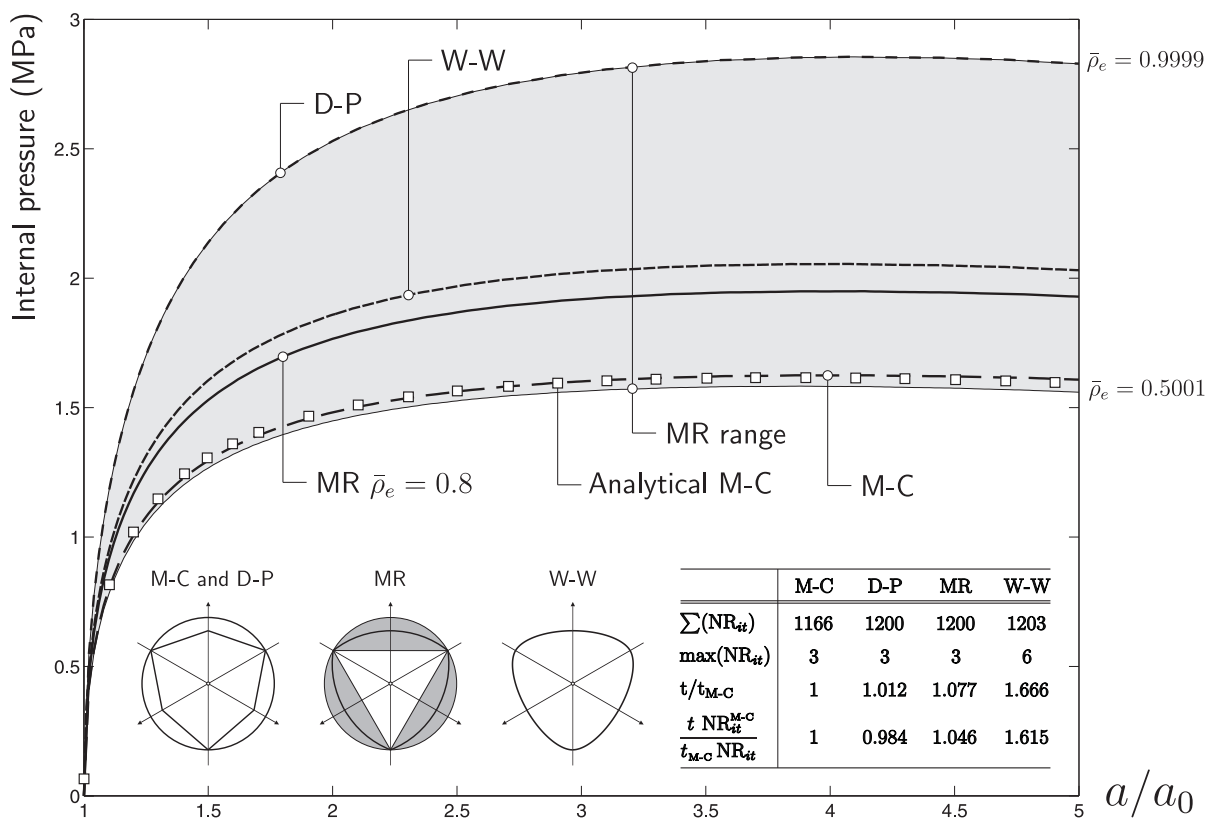

Figure 2: Internal expansion of a thick walled soil cylinder.

pressure. This is a one-dimensional axi-symmetric problem but here we use a 2D version of our 3D code to make comparisons with an analytical solution. $3°$ of a cylinder with initial internal radius (a_0) of 1m and an external radius (b_0) of 500m was discretised using 100 4-noded quadrilateral elements[3]. The following material parameters were used: Young's modulus of 100MPa, Poisson's ratio of 0.3, cohesion c of 70kPa, friction angle of $20°$ and $\bar{\rho}_e = 0.8$ (to coincide with $\bar{\rho}_e$ for M-C). The internal radius was expanded to 5m via 400 equal displacement-controlled increments. Figure 2 presents the pressure-internal expansion plots for the four constitutive models: D-P, M-C[4], Willam-Warnke (W-W) [6] cone and MR. a/a_0 is the ratio of the current to the original internal radius. The M-C numerical solution displays excellent agreement with the analytical solution [7]. Results for the MR cone using $\bar{\rho}_e = 0.5001$ and $\bar{\rho}_e = 0.9999$ demonstrate the model's ability to provide solutions spanning between those provided by the M-C and D-P cones. With $\bar{\rho}_e = 0.8$ the W-W and MR cones produced a stiffer response when compared against the M-C solution.

Figure 2 also gives run time comparisons between the different constitutive models. $\sum(\text{NR}_{it})$ is the total number of global Newton-Raphson (N-R) iterations, $\max(\text{NR}_{it})$ is the maximum number of N-R iterations for any loadstep, t/t_{M-C} is the run time normalised with respect to the M-C run time and the ratio $(t\text{NR}_{it})/(t_{\text{M-C}}\text{NR}_{it}^{\text{M-C}})$ gives the time per iteration normalised with respect to the M-C iteration time. The W-W formulation, which produced similar results to the MR cone, required a 58.9% increase in the run-time, with MR having a run time only 7.7% greater than M-C.

4 CONCLUSION

This paper presents a simple cone-type elasto-plastic formulation with a deviatoric yielding criterion based on a modified Reuleaux triangle and its implementation within a finite deformation FE code. The perfect plasticity model allows an analytical backward Euler stress integration on the curved surface and exact integration in the regions where singularities appear [1].

By utilising lean efficient algorithms and subfunctions, MATLAB has allowed the main file of the implemented FE program to fit comfortably on one page, being only 70 lines in length[5]. Through the use of an updated Lagrangian logarithmic strain–Kirchhoff stress formulation (with an exponential map for the plastic flow equation) the code allows for the incorporation of existing isotropic small strain constitutive models without modification.

The numerical analysis of the internal expansion of a thick walled cylinder has been presented. Results obtained with the MR cone are compared with results from M-C, D-P and W-W cones. It has been shown that the computational advantages over a Willam-Warnke cone model are significant and that the proposed model provides an attractive alternative to the Drucker-Prager and Mohr-Coulomb models.

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[3] The size of the elements were progressively increased by a factor 1.2 from the inner to the outer surface.

[4] See [5], amongst others for more details on D-P and M-C.

[5] The complete MATLAB .m files are available from the first author on request, email w.m.coombs@durham.ac.uk.