

# Symmetry-Surfing the Moduli Space of Kummer K3s

Anne Taormina and Katrin Wendland

*Dedicated to Prof. Dr. Friedrich Hirzebruch, October 17, 1927 - May 27, 2012,  
in admiration and gratitude:*

*To an extraordinary scientist, an unforgettable teacher, and a model of altruism.*

**ABSTRACT.** A maximal subgroup of the Mathieu group  $M_{24}$  arises as the combined holomorphic symplectic automorphism group of all Kummer surfaces whose Kähler class is induced from the underlying complex torus. As a subgroup of  $M_{24}$ , this group is the stabilizer group of an octad in the Golay code. To meaningfully combine the symmetry groups of distinct Kummer surfaces, we introduce the concepts of Niemeier markings and overarching maps between pairs of Kummer surfaces. The latter induce a prescription for symmetry-surfing the moduli space, while the former can be seen as a first step towards constructing a vertex algebra that governs the elliptic genus of K3 in an  $M_{24}$ -compatible fashion. We thus argue that a geometric approach from K3 to Mathieu Moonshine may bear fruit.

## Introduction

This work is motivated by several mysteries related to the Mathieu Moonshine phenomenon. Central to this phenomenon is the elliptic genus of K3, which encodes topological data on K3 surfaces and at the same time is expected to organise a selection of states in  $N = (4, 4)$  superconformal field theories (SCFTs) on K3 into representations of the Mathieu group  $M_{24}$ . The existence of the relevant representations follows from Gannon’s result [Gan12], which in turn builds on the work of Cheng, Gaberdiel-Hohenegger-Volpato and Eguchi-Hikami [Che10, GHV10b, GHV10a, EH11]. The precise construction of those representations in terms of conformal field theory data, however, has been completely elusive so far, since the detailed nature of the states governing the elliptic genus has not been pinned down. Indeed, the elliptic genus is a topological invariant generalizing the genera of multiplicative sequences that were introduced by F. Hirzebruch [Hir66]. It can be viewed as the regularized index of a  $U(1)$ -equivariant Dirac operator on the loop space of K3 [AKMW87, Wit87]. It also arises from the supertrace over the subsector of Ramond-Ramond states of every superconformal field theory on K3, and hence it counts states with signs [EOTY89, Kap05]. That the net contribution should yield a well-defined representation of any group, let alone of  $M_{24}$ , is mysterious. However, from the properties of twining and twisted-twining genera it has been argued that one should actually expect this representation to be realized in terms of a vertex algebra  $\hat{\mathcal{X}}$  [GPRV12]. We share that view, although not the recent claim by some

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experts exclusively expecting holomorphic vertex algebras in this context, and casting doubts on whether K3 surfaces bear any key to the Mathieu Moonshine Mysteries [Gan12, GPV13].

In fact, we argue that the resolution of certain aspects of Mathieu Moonshine might benefit from deepening our understanding of the implications of Mukai’s work [Muk88], and from building on the insights offered by Kondo [Kon98]. Of course, Mukai has proved in [Muk88] that every holomorphic symplectic symmetry group of a K3 surface is a subgroup of the group  $M_{24}$ . But he also proved that all these symmetry groups are smaller than  $M_{24}$  by orders of magnitude. In fact, all of them are subgroups of  $M_{23}$ . In [TW11] we advertised the idea that presumably,  $M_{24}$  could be obtained by combining the holomorphic symplectic symmetry groups of distinct K3 surfaces at different points of the moduli space. As a test bed, we proved the existence of an overarching map  $\Theta$  which allows to combine the holomorphic symplectic symmetry groups of two special, distinct Kummer surfaces in terms of their induced actions on the Niemeier lattice  $N$  of type  $A_1^{24}$ . We also proved that this combined action on  $N$  yields the largest possible group that can arise by means of such an overarching map. This group is  $(\mathbb{Z}_2)^4 \rtimes A_7$ , which we therefore called the **OVERARCHING FINITE SYMMETRY GROUP** of Kummer surfaces. It contains as proper subgroups all holomorphic symplectic symmetry groups of Kummer surfaces which are equipped with the dual Kähler class induced from the underlying torus.

In this note, in Section 1 we briefly recall the Kummer construction and gather the information appearing in [TW11] that is useful for the present work. In Section 2, we introduce the concept of Niemeier markings and generalize the ideas summarized above by showing that the technique introduced for two specific examples of Kummer surfaces in [TW11], namely the tetrahedral and the square Kummer K3, generalizes to other pairs of Kummer surfaces. As an application of this technique, Section 3 constructs three overarching maps for three pairs of Kummer surfaces with maximal symmetry. Section 4 shows that for any pair of Kummer K3s, one can find representatives in the smooth universal cover of the moduli space of hyperkähler structures such that there exists an overarching map analogous to the one constructed in [TW11]. Moreover, there always exists a continuous path between the two representatives of our Kummer surfaces, such that  $\Theta$  is compatible with all holomorphic symplectic symmetries along the path. This is the idea of symmetry-surfing the moduli space, alluded to in the title of the present paper.

Our surfing procedure allows us to combine the action of all holomorphic symplectic symmetry groups of Kummer surfaces with induced dual Kähler class by means of their induced actions on the lattice  $N$ . In fact, this action is independent of all choices of overarching maps. We also prove in Section 4 that the combined action of all these groups is given by a faithful representation of  $(\mathbb{Z}_2)^4 \rtimes A_8$  on  $N$ . The subgroup  $(\mathbb{Z}_2)^4 \rtimes A_7$ , i.e. the overarching finite symmetry group of Kummer surfaces, is the stabilizer subgroup of  $(\mathbb{Z}_2)^4 \rtimes A_8$  for one root in the Niemeier lattice  $N$ , just as the subgroup  $M_{23}$  of the Mathieu group  $M_{24}$  is the stabilizer subgroup of  $M_{24}$ , which naturally acts on  $N$ , for one root in  $N$ . We view this as evidence that the Mathieu Moonshine phenomenon is tied to the largest Mathieu group  $M_{24}$  rather than  $M_{23}$ , as also argued by Gannon [Gan12].

In Section 5, we highlight the relevance of our geometric approach, and in particular of the Niemeier markings, in the quest for a vertex algebra that governs the elliptic genus of K3 at lowest order. To this effect, we establish a link between our work on Kummer surfaces and a special class of  $N = (4, 4)$  SCFTs at central charge  $c = \bar{c} = 6$ , namely  $\mathbb{Z}_2$ -orbifolds of toroidal conformal field theories<sup>1</sup>. This necessitates a transition from geometry to superconformal theory language, which we describe in Appendix A. The upshot is that our surfing idea is natural: the symmetry groups act on the twisted ground states of the  $\mathbb{Z}_2$ -orbifold conformal field theories, and that action completely determines these symmetries. The twisted ground states can be viewed as a stable part of the Hilbert space when one surfs between  $\mathbb{Z}_2$ -orbifolds. As such the twisted ground states collect the various symmetry groups just like the Niemeier lattice does by means of our Niemeier markings. In passing we explain how the very idea of constructing a vertex algebra from the field content of

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<sup>1</sup>To avoid clumsy terminology, we simply refer to those SCFTs  $\mathcal{C}$  on K3 which are obtained by the standard  $\mathbb{Z}_2$ -orbifold procedure from a toroidal theory as “ $\mathbb{Z}_2$ -orbifolds”.

SCFTs on K3, which simultaneously governs the elliptic genus and symmetries, motivates why we restrict our attention to symmetry groups that are induced from geometric symmetries in some geometric interpretation, that is, to subgroups of  $M_{24}$ .

### 1. Kummer surfaces and quaternions

An interesting class of K3 surfaces is obtained through the KUMMER CONSTRUCTION, which amounts to taking a  $\mathbb{Z}_2$ -orbifold of any complex torus  $T$  of dimension 2, and minimally resolving the singularities that arise from the orbifold procedure. More specifically, let  $T = T(\Lambda) = \mathbb{C}^2/\Lambda$  with  $\Lambda \subset \mathbb{C}^2$  denote a lattice of rank 4 over  $\mathbb{Z}$ , and with generators  $\vec{\lambda}_i$ ,  $i \in \{1, \dots, 4\}$ . The group  $\mathbb{Z}_2$  acts naturally on  $\mathbb{C}^2$  by  $(z_1, z_2) \mapsto (-z_1, -z_2)$  and thereby on  $T(\Lambda)$ . Using Euclidean coordinates  $\vec{x} = (x_1, x_2, x_3, x_4)$ , where  $z_1 = x_1 + ix_2$  and  $z_2 = x_3 + ix_4$ , points on the quotient  $T(\Lambda)/\mathbb{Z}_2$  are identified according to

$$\vec{x} \sim \vec{x} + \sum_{i=1}^4 n_i \vec{\lambda}_i, \quad n_i \in \mathbb{Z}, \quad \vec{x} \sim -\vec{x}.$$

Hence  $T(\Lambda)/\mathbb{Z}_2$  has 16 singularities of type  $A_1$ , located at the fixed points of the  $\mathbb{Z}_2$ -action. These fixed points are conveniently labelled by the HYPERCUBE  $\mathbb{F}_2^4 \cong \frac{1}{2}\Lambda/\Lambda$ , where  $\mathbb{F}_2 = \{0, 1\}$  is the finite field with two elements, as

$$(1.1) \quad \vec{F}_{\vec{a}} := \left[ \frac{1}{2} \sum_{i=1}^4 a_i \vec{\lambda}_i \right] \in T(\Lambda)/\mathbb{Z}_2, \quad \vec{a} = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4.$$

DEFINITION 1.1. The complex surface  $X_\Lambda$  obtained by minimally resolving the 16 singularities of  $T(\Lambda)/\mathbb{Z}_2$  is a K3 surface (see e.g. [Nik75]) called a KUMMER SURFACE<sup>2</sup>.

According to the above definition, the Kummer surface  $X_\Lambda$  carries the complex structure induced from the universal cover  $\mathbb{C}^2$  of  $T$ . It may also be equipped with a Kähler structure<sup>3</sup>, and this is natural if one is interested in the description of *finite* groups of symplectic automorphisms of Kummer surfaces. We specify such a Kähler structure by choosing a so-called DUAL KÄHLER CLASS  $\omega$ , that is, a homology class which is Poincaré dual to a Kähler class. Indeed, first recall the following:

DEFINITION 1.2. Consider a K3 surface  $X$ . A map  $f: X \rightarrow X$  of finite order is called a SYMPLECTIC AUTOMORPHISM if and only if  $f$  is biholomorphic and it induces the identity map on  $H^{2,0}(X, \mathbb{C})$ .

If  $\omega$  is a dual Kähler class on  $X$  and the induced map  $f_*: H_*(X, \mathbb{R}) \rightarrow H_*(X, \mathbb{R})$  leaves  $\omega$  invariant, then  $f$  is a HOLOMORPHIC SYMPLECTIC AUTOMORPHISM with respect to  $\omega$ .

When a dual Kähler class  $\omega$  on  $X$  has been specified, then the group of holomorphic symplectic automorphisms of  $X$  with respect to  $\omega$  is called the SYMMETRY GROUP of  $X$ .

As an application of the Torelli theorem for K3 surfaces, the discussion of holomorphic symplectic automorphisms  $f$  of a K3 surface  $X$  can be entirely rephrased in terms of the induced lattice automorphisms  $f_*$  of the full integral homology lattice  $H_*(X, \mathbb{Z})$  (these and other results on geometry and symmetries of Kummer K3s are standard; for a summary, see e.g. [TW11, Thm. 3.2.2]). Then (see [TW11, Prop. 3.2.4] for a proof),

PROPOSITION 1.3. Consider a K3 surface  $X$ , and denote by  $G$  a group of symplectic automorphisms of  $X$ . Then  $G$  is finite if and only if  $X$  possesses a dual Kähler class which is invariant under  $G$ .

Throughout this work, we focus on Kummer surfaces  $X_{\Lambda, \omega_0}$ , by which we mean that as Kähler structure on  $X_\Lambda$  we choose the one induced from the standard Kähler structure of the torus  $T(\Lambda)$  inherited from the Euclidean metric on its universal cover  $\mathbb{C}^2$ . Here,  $\omega_0$  denotes the corresponding

<sup>2</sup>We denote by  $\pi: T \dashrightarrow X$  the corresponding rational map of degree 2, and by  $\pi_*: H_*(T, \mathbb{Z}) \rightarrow H_*(X, \mathbb{Z})$  the induced map on homology.

<sup>3</sup>For most parts of our work, the Kähler class is degenerate in the sense that it corresponds to an orbifold limit of Kähler metrics.

dual Kähler class on  $X_\Lambda$ . This restricts the symmetry groups of Kummer surfaces that can be obtained, but is sufficient to argue for the existence of a combined symmetry group  $(\mathbb{Z}_2)^4 \rtimes A_8$  in Section 4.

The generic structure of the symmetry group  $G$  of the Kummer surface  $X_{\Lambda, \omega_0}$  is a semi-direct product  $G = G_t \rtimes G_T$  (see, for example, [TW11, Prop. 3.3.4]). The normal subgroup  $G_t \cong (\mathbb{Z}_2)^4$  of  $G$  is the so-called translational automorphism group which is induced from the shifts by half lattice vectors  $\frac{1}{2}\vec{\lambda}$ ,  $\vec{\lambda} \in \Lambda$ , on the underlying torus  $T = T(\Lambda)$ . The group  $G_T$  is the normalizer of  $G_t$  in  $G$ . It is the group of symmetries of the Kummer surface induced by the holomorphic symplectic automorphisms of the torus  $T$  fixing  $0 \in \mathbb{C}^2/\Lambda = T$ . That is,  $G_T \cong G'_T/\mathbb{Z}_2$ , where  $G'_T$  is the group of linear holomorphic symplectic automorphisms of  $T$ . These groups and their possible actions on a torus  $T$  have been classified by Fujiki [Fuj88], who proves that  $G'_T$  is isomorphic to a subgroup of one of the following groups: the cyclic groups  $\mathbb{Z}_4, \mathbb{Z}_6$ , the binary dihedral groups  $\mathcal{O}$  and  $\mathcal{D}$  of order 8 and 12, and the binary tetrahedral group  $\mathcal{T}$ . This actually implies that the symmetry group  $G$  is a subgroup of  $(\mathbb{Z}_2)^4 \rtimes A_6$ , where  $A_6$  is the alternating group on six elements. Moreover<sup>4</sup>,  $\mathcal{T}$  acts only on the so-called tetrahedral torus, while  $\mathcal{D}$  acts only on the so-called triangular torus.  $\mathcal{O}$  can act on the square torus or on the tetrahedral torus, where it is realized as a subgroup of  $\mathcal{T}$ . Finally the action of the cyclic groups  $\mathbb{Z}_4$  and  $\mathbb{Z}_6$  agrees with that of a cyclic subgroup of  $\mathcal{O}$ ,  $\mathcal{D}$  or  $\mathcal{T}$ , possibly on a torus that does not enjoy the full dihedral or tetrahedral symmetry. In summary, the maximal groups that can occur are  $\mathcal{O}$ ,  $\mathcal{D}$  and  $\mathcal{T}$ .

By definition, any element of  $G$  must leave the complex structure and the dual Kähler class  $\omega_0$  of the Kummer surface  $X_{\Lambda, \omega_0}$  invariant. Hence in terms of real local coordinates  $\vec{x} = (x_1, x_2, x_3, x_4)$  as above and with respect to standard real coordinate vector fields  $\vec{e}_1, \dots, \vec{e}_4$ , using the notations of [TW11, Section 3],  $G$  must preserve each of the following 2-cycles in  $H_2(X_{\Lambda, \omega_0}, \mathbb{R})$ ,

$$(1.2) \quad \Omega_1 = e_1 \vee e_3 - e_2 \vee e_4, \quad \Omega_2 = e_1 \vee e_4 + e_2 \vee e_3 \quad \text{and} \quad \omega_0 = e_1 \vee e_2 + e_3 \vee e_4.$$

Equivalently, every symmetry group  $G$  must preserve the hyperkähler structure which is specified by the nowhere vanishing holomorphic 2-form and the Kähler class on  $X_{\Lambda, \omega_0}$ . We can work with local holomorphic coordinates  $(z_1, z_2)$  that are induced from the underlying torus. The invariant classes hence are given by  $dz_1 \wedge dz_2$ , and  $\frac{1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ . Moreover,  $G_T \cong G'_T/\mathbb{Z}_2$  where  $G'_T$  acts linearly. In other words,  $G'_T$  is a finite subgroup of  $SU(2)$ . Once a group  $G'_T \subset SU(2)$  preserving the lattice  $\Lambda$  has been identified such that  $\mathbb{Z}_2 \subset G'_T$ , then  $G_T \cong G'_T/\mathbb{Z}_2$  acts faithfully on the Kummer surface  $X_{\Lambda, \omega_0}$ .

It is not surprising that quaternions provide an elegant framework to describe the groups  $G_T \cong G'_T/\mathbb{Z}_2$  we are interested in when symmetry-surfing [Fuj88, Bri98]. Indeed, we recall a formalism taken from [Bri98] which is tailored to recover the maximal groups  $G'_T$  classified by Fujiki, i.e.  $G'_T \cong \mathcal{O}, \mathcal{D}, \mathcal{T}$ . It moreover provides a unified description of the lattice  $\Lambda$  for each torus on which one of these groups can act as automorphism group. In fact, each lattice  $\Lambda$  is given in terms of unit quaternion generators, and the automorphisms act by quaternionic left multiplication.

The link between the skew field of quaternions  $\mathbb{H}$  and lattices  $\Lambda \subset \mathbb{R}^4$  is through the natural isomorphism

$$(1.3) \quad \mathbb{R}^4 \longrightarrow \mathbb{H}, \quad q = (q_0, q_1, q_2, q_3) \longmapsto q_0 + q_1i + q_2j - q_3k,$$

with  $\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k \mid q_\mu \in \mathbb{R}, \mu \in \{0, \dots, 3\}\}$ . The unit quaternions form a group which is isomorphic to  $SU(2)$ , and under the identification (1.3) its regular representation on  $\mathbb{R}^4 \cong \mathbb{C}^2$  is realized by left multiplication on  $\mathbb{H} \cong \mathbb{R}^4$ . One immediately checks that with this faithful representation, every unit quaternion leaves the standard holomorphic two-form  $dz_1 \wedge dz_2$  and Kähler class  $\frac{1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$  on  $\mathbb{R}^4 \cong \mathbb{C}^2$  invariant. Hence this identification allows us to realize each of our groups  $G'_T$  in terms of a finite group of unit quaternions.

Assume now that  $\Lambda \subset \mathbb{R}^4 \cong \mathbb{H}$  is a lattice of rank 4 which carries the faithful action of an automorphism group  $G'_T \subset SU(2)$ , where  $G'_T$  is one of the maximal groups  $\mathcal{O}, \mathcal{D}, \mathcal{T}$  from Fujiki's

<sup>4</sup>See the end of this section, items 1.-3., for the precise definitions of the relevant lattices and group actions.

classification. By the properties of these maximal groups, we can assume without loss of generality that  $G'_T$  has generators  $a, b, c$  that are represented by unit quaternions of the form

$$(1.4) \quad \begin{aligned} \hat{a} &= \cos\left(\frac{\pi}{m}\right) - i \sin\left(\frac{\pi}{r}\right) + j \cos\left(\frac{\pi}{n}\right), \\ \hat{b} &= j, \\ \hat{c} &= \cos\left(\frac{\pi}{n}\right) + j \cos\left(\frac{\pi}{m}\right) + k \sin\left(\frac{\pi}{r}\right), \end{aligned}$$

with the constraint  $\cos^2\left(\frac{\pi}{m}\right) + \cos^2\left(\frac{\pi}{n}\right) = \cos^2\left(\frac{\pi}{r}\right)$ , where the numbers  $m, n, r \in \mathbb{Z}$  determine the group  $G'_T$  [Cox74]. Moreover, for the lattice  $\Lambda \subset \mathbb{R}^4 \cong \mathbb{H}$  we can choose the unit quaternion generators  $1, \hat{a}, \hat{b}, \hat{c}$ . Hence in terms of  $\mathbb{R}^4$ , we let

$$\begin{aligned} \vec{\lambda}_1 &= (1, 0, 0, 0), & \vec{\lambda}_2 &= \left(\cos\left(\frac{\pi}{m}\right), -\sin\left(\frac{\pi}{r}\right), \cos\left(\frac{\pi}{n}\right), 0\right), \\ \vec{\lambda}_3 &= (0, 0, 1, 0), & \vec{\lambda}_4 &= \left(\cos\left(\frac{\pi}{n}\right), 0, \cos\left(\frac{\pi}{m}\right), -\sin\left(\frac{\pi}{r}\right)\right), \end{aligned}$$

be the generators of  $\Lambda$ .

We now summarise the data needed for symmetry-surfing the moduli space of Kummer surfaces. We describe the three maximal symmetry groups  $G_T \cong G'_T/\mathbb{Z}_2$  of Kummer surfaces induced by the holomorphic symplectic automorphisms of some torus  $T = T(\Lambda)$  fixing  $0 \in \mathbb{C}^2/\Lambda = T$ , along with the possible lattices  $\Lambda$ :

(1) **Dihedral group  $\mathbf{D}_2 \cong \mathcal{O}/\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$**

Take the lattice  $\Lambda$  to be  $\Lambda_0 := \text{span}_{\mathbb{Z}}\{1, \hat{a} = i, \hat{b} = j, \hat{c} = k\}$ , with  $\{\hat{a}, \hat{b}, \hat{c}\}$  generating the quaternionic group  $G'_T \cong Q_8$  of order 8. It is immediate that  $Q_8$  is the automorphism group of  $\Lambda_0$ , which is the lattice yielding the square Kummer surface  $X_0$  in [TW11]. There, an equivalent description of the generators of the binary dihedral group  $\mathcal{O}$  was given by

$$(1.5) \quad \alpha_1: (z_1, z_2) \mapsto (iz_1, -iz_2), \quad \alpha_2: (z_1, z_2) \mapsto (-z_2, z_1),$$

both of which are of order 4.

(2) **Alternating group  $\mathbf{A}_4 \cong \mathcal{T}/\mathbb{Z}_2$**

The lattice  $\Lambda$  may be generated by  $\{1, \hat{a} = \cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{5\pi}{4}\right) + j \cos\left(\frac{\pi}{3}\right), \hat{b} = j, \hat{c} = \cos\left(\frac{\pi}{3}\right) + j \cos\left(\frac{\pi}{3}\right) + k \sin\left(\frac{5\pi}{4}\right)\}$ , hence the four lattice vectors that generate  $\Lambda$  may be chosen as  $\vec{\lambda}_1 = (1, 0, 0, 0)$ ,  $\vec{\lambda}_2 = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right)$ ,  $\vec{\lambda}_3 = (0, 0, 1, 0)$  and  $\vec{\lambda}_4 = \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ .

One shows that the orbit of  $\vec{\lambda}_1$  under the group  $G'_T = \mathcal{T}$  yields 24 unit lattice vectors. This lattice is isometric to the lattice  $\Lambda_1 := \Lambda_{D_4}$  used in [TW11] to construct the tetrahedral Kummer surface  $X_1 = X_{D_4}$  from the torus  $T(\Lambda_{D_4})$ . We will use this Kummer surface in what follows, hence we recall the generators of  $\Lambda_{D_4}$ :

$$(1.6) \quad \vec{\lambda}_1 = (1, 0, 0, 0), \quad \vec{\lambda}_2 = (0, 1, 0, 0), \quad \vec{\lambda}_3 = (0, 0, 1, 0), \quad \vec{\lambda}_4 = \frac{1}{2}(1, 1, 1, 1).$$

Generators of the binary tetrahedral group  $\mathcal{T}$  may be taken to be

$$(1.7) \quad \begin{aligned} \gamma_1: (z_1, z_2) &\mapsto (iz_1, -iz_2), \\ \gamma_2: (z_1, z_2) &\mapsto (-z_2, z_1), \\ \gamma_3: (z_1, z_2) &\mapsto \frac{i+1}{2}(i(z_1 - z_2), -(z_1 + z_2)). \end{aligned}$$

These generators satisfy the relations  $\gamma_1^4 = \gamma_2^4 = \mathbb{1}$  and  $\gamma_3^3 = \mathbb{1}$ . Note that the minimum number of generators for the group  $\mathcal{T}$  is 2, and indeed, one has  $\gamma_2 = \gamma_1^2 \gamma_3 \gamma_1 (\gamma_3)^{-1}$ .

(3) **Permutation group  $\mathbf{S}_3 \cong \mathcal{D}/\mathbb{Z}_2$**

Take the lattice  $\Lambda_2$  generated by  $\{1, \hat{a} = -\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right), \hat{b} = j, \hat{c} = -j \cos\left(\frac{\pi}{3}\right) - k \sin\left(\frac{\pi}{3}\right)\}$ , hence the four lattice vectors that generate  $\Lambda_2$  may be chosen as

$$(1.8) \quad \vec{\lambda}_1 = (1, 0, 0, 0), \quad \vec{\lambda}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0\right), \quad \vec{\lambda}_3 = (0, 0, 1, 0), \quad \vec{\lambda}_4 = \left(0, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

The orbit of  $\vec{\lambda}_1$  under the binary dihedral group  $G'_T \cong \mathcal{D}$  yields 12 unit vectors in  $\Lambda_2$ . The Kummer surface obtained from  $T(\Lambda_2)$  is the triangular Kummer surface  $X_2$ . The

generators of  $\mathcal{D}$  have order 3 and 4, respectively, and they are given by

$$(1.9) \quad \begin{aligned} \beta_1: (z_1, z_2) &\longmapsto (\zeta z_1, \zeta^{-1} z_2), \\ \beta_2: (z_1, z_2) &\longmapsto (-z_2, z_1), \end{aligned}$$

where  $\zeta := e^{2\pi i/3}$ .

## 2. Overarching maps and Niemeier markings

The description of symmetries of K3 surfaces is most efficient in terms of lattices. To this end, recall that the geometric action of a symmetry group  $G$  of a K3 surface  $X$  is fully captured by its action on the lattice  $L_G = (L^G)^\perp \cap H_*(X, \mathbb{Z})$ , where  $L^G := H_*(X, \mathbb{Z})^G$ . This follows from the Torelli theorem (see the discussion of Def. 1.2) and the very definition of  $L^G$  as the sublattice of  $H_*(X, \mathbb{Z})$  on which  $G$  acts trivially.

On the other hand, if  $X_{\Lambda, \omega_0}$  is a Kummer surface with its induced dual Kähler class, then the induced action of  $G$  on the Kummer lattice  $\Pi \subset H_*(X, \mathbb{Z})$  bears all information about the action of  $G$  (see [TW11, Prop. 3.3.3]):

**PROPOSITION 2.1.** Consider a Kummer surface  $X_{\Lambda, \omega_0}$  with its induced dual Kähler class. Let  $\Pi \subset H_*(X, \mathbb{Z})$  denote the Kummer lattice, that is, the smallest primitive sublattice of the integral K3 homology which contains the 16 classes  $E_{\vec{a}}$ ,  $\vec{a} \in \mathbb{F}_2^4$ , that are obtained from blowing up the fixed points  $\vec{F}_{\vec{a}}$  of the  $\mathbb{Z}_2$ -action on the underlying torus (1.1).

Then every symmetry of  $X$  induces a permutation of the  $E_{\vec{a}}$ . This permutation is given by an affine linear transformation of the labels  $\vec{a} \in \mathbb{F}_2^4$ , which in turn uniquely determines the symmetry.

In the case of Kummer surfaces we thus have two competing lattices  $\Pi$  and  $L_G$  which conveniently encode the action of the symmetry group  $G$  of  $X_{\Lambda, \omega_0}$ . In [TW11] we argue that neither does  $L_G$  contain the rank 16 Kummer lattice, nor does, in general, the Kummer lattice contain  $L_G$ . Instead, combining the two, in [TW11, Prop. 3.3.6] we introduce the lattice  $M_G$ , which is generated by  $L_G$  and  $\Pi$  along with the vector  $v_0 - v$ , where  $v_0, v$  are generators of  $H_0(X, \mathbb{Z})$  and  $H_4(X, \mathbb{Z})$  with<sup>5</sup>  $\langle v_0, v \rangle = 1$ . We argue that in the Kummer case we can generalize and improve some extremely useful techniques introduced by Kondo [Kon98] to this enlarged lattice  $M_G$ . Indeed, we prove that this lattice allows a primitive embedding into the Niemeier lattice  $N(-1)$  with root lattice  $A_1^{24}$  [TW11, Thm. 3.3.7], where the decoration  $(-1)$  indicates that the roots of  $N(-1)$  have length square  $-2$ . This embedding allows us to view the symmetry group  $G$  as a group of lattice automorphisms of  $N(-1)$ : the action of  $G$  on  $N(-1)$  is defined such that the embedding  $\iota_G: M_G \hookrightarrow N(-1)$  is  $G$ -equivariant, and  $G$  acts trivially on the orthogonal complement of  $\iota_G(M_G)$  in  $N(-1)$ . Since the automorphism group of  $N(-1)$ , up to reflections in the roots of  $N(-1)$ , is the Mathieu group  $M_{24}$ , this conveniently realizes every symmetry group  $G$  of a Kummer K3 as a subgroup of  $M_{24}$ .

In what follows, we use the notations and conventions of [TW11] throughout. In particular, we fix the Kummer lattice  $\Pi$  within the abstract lattice  $H_*(X, \mathbb{Z})$  as well as its image under  $\iota_G$  in  $N(-1)$  for every Kummer surface, independently of the parameters of the underlying torus. More precisely, we fix a unique MARKING for all our Kummer surfaces, that is, an explicit isometry of the lattice  $H_*(X, \mathbb{Z})$  with a standard even, unimodular lattice of signature  $(4, 20)$ . As is explained in [TW11, Sect. 2.2], the Kummer construction induces a natural such marking, which in particular fixes the position of  $\Pi$  within the lattice  $H_*(X, \mathbb{Z})$ . In this setting, among the data specifying each Kummer surface we have to include the choice of generators  $\vec{\lambda}_1, \dots, \vec{\lambda}_4 \in \mathbb{R}^4$  for the lattice  $\Lambda$  of the underlying torus  $T = T(\Lambda)$ . Note that the choice of such a fixed marking amounts to the transition to a smooth universal cover of the moduli space of hyperkähler structures on K3. Similarly to  $\Pi \subset H_*(X, \mathbb{Z})$ , we also fix the position of  $\tilde{\Pi}(-1) := \iota_G(\Pi)$  in  $N(-1)$  such that  $\tilde{\Pi}$  is common to all Kummer surfaces. To do so, in [TW11, (2.14)] we construct a bijection  $I: \mathcal{I} \setminus \mathcal{O}_9 \rightarrow \mathbb{F}_2^4$  between the 16 elements of the set  $\mathcal{I} := \{1, 2, \dots, 24\}$  that do not belong to our choice of reference octad  $\mathcal{O}_9 := \{3, 5, 6, 9, 15, 19, 23, 24\}$  from the Golay code and the vertices of the hypercube  $\mathbb{F}_2^4$ . In

<sup>5</sup>On  $H_*(X, \mathbb{Z})$ , we use the standard quadratic form which is induced by the intersection form.

[TW11, Prop. 2.3.4] we prove that the  $\mathbb{Q}$ -linear extension of  $\iota_G(E_{\vec{a}}) := f_{I^{-1}(\vec{a})}$  yields an isometry between  $\Pi$  and  $\tilde{\Pi}(-1)$ , where  $\{f_n, n \in \mathcal{I}\}$ , denotes a root basis of the root lattice  $A_1^{24}$  in  $N(-1)$ . Thus we have fixed the position of  $\tilde{\Pi}$  within  $N$  for all Kummer surfaces, similarly to fixing the position of  $\Pi$  within the abstract lattice  $H_*(X, \mathbb{Z})$ . This motivates the

**DEFINITION 2.2.** With notations as above, for a Kummer surface  $X_{\Lambda, \omega_0}$  with symmetry group  $G$ , an isometric embedding  $\iota_G: M_G \hookrightarrow N(-1)$  such that  $\iota_G(E_{\vec{a}}) = f_{I^{-1}(\vec{a})}$  for all  $\vec{a} \in \mathbb{F}_2^4$  is called a NIEMEIER MARKING.

By the above, every Kummer surface  $X$  allows a Niemeier marking [TW11, Prop. 4.1.1]. In general, the embedding  $\iota_G$  is not uniquely determined. However, the action of  $G$  on  $N$ , which is induced by the requirement that  $\iota_G$  is  $G$ -equivariant, is independent of all choices: indeed,  $\iota_G(E_{\vec{a}}) = f_{I^{-1}(\vec{a})} \forall \vec{a} \in \mathbb{F}_2^4$  fixes the action of  $G$  on the lattice  $\tilde{\Pi} \subset N$ , and by the arguments presented in the discussion of [TW11, Cor. 3.3.8] this already uniquely determines the action of  $G$  on all of  $N$ .

In particular, consider the translational symmetry group  $G_t \cong (\mathbb{Z}_2)^4$  discussed in Section 1. Its action on the roots  $f_n, n \in \mathcal{I}$ , of  $N$ , which is common to all Kummer surfaces, is generated by the following permutations [TW11, Prop. 4.1.1]:

$$(2.1) \quad G_t := (\mathbb{Z}_2)^4 : \begin{cases} \iota_1 &= (1, 11)(2, 22)(4, 20)(7, 12)(8, 17)(10, 18)(13, 21)(14, 16), \\ \iota_2 &= (1, 13)(2, 12)(4, 14)(7, 22)(8, 10)(11, 21)(16, 20)(17, 18), \\ \iota_3 &= (1, 14)(2, 17)(4, 13)(7, 10)(8, 22)(11, 16)(12, 18)(20, 21), \\ \iota_4 &= (1, 17)(2, 14)(4, 12)(7, 20)(8, 11)(10, 21)(13, 18)(16, 22). \end{cases}$$

Now recall Mukai's seminal result [Muk88] that the symmetry group of every K3 surface is isomorphic to a subgroup of one of eleven subgroups of the Mathieu group  $M_{24}$ , the largest one of which has 960 elements. Hence symmetry groups of K3 surfaces are by orders of magnitude smaller than the group  $M_{24}$ , whose appearance one expects from Mathieu Moonshine. Therefore, in [TW11] we propose to use Niemeier markings to combine the symmetry groups of distinct Kummer surfaces by means of their actions on the Niemeier lattice  $N$ . To underpin this idea by lattice identifications, we propose to extend a given Niemeier marking  $\iota_G$  to a linear bijection  $\Theta: H_*(X, \mathbb{Z}) \rightarrow N(-1)$ , which restricts to an isometry on the largest possible sublattice of  $H_*(X, \mathbb{Z})$ . More precisely, we propose to construct a map  $\Theta$  which induces Niemeier markings of all K3 surfaces along a smooth path in the smooth universal cover of the moduli space of hyperkähler structures on K3. If this path connects two distinct Kummer K3s  $X_A$  and  $X_B$ , then we call  $\Theta_{AB}$  an OVERARCHING MAP for  $X_A$  and  $X_B$ . This is the key to exhibit an overarching symmetry in the moduli space of Kummer K3s. We say that an overarching map  $\Theta_{AB}$  for Kummer surfaces  $X_A$  and  $X_B$  allows us to SURF from one of the corresponding Kummer surfaces to the other IN MODULI SPACE.

For two Kummer surfaces  $X_A, X_B$  with complex and Kähler structures induced from the underlying torus and with symmetry groups  $G_A, G_B$ , respectively, we will argue below that the following holds: under appropriate additional assumptions, one can construct an overarching map  $\Theta_{AB}$  which restricts to a Niemeier marking, that is to an isometric  $G_k$ -equivariant embedding  $\iota_{G_k}: M_{G_k} \hookrightarrow N(-1)$ , for both  $k = A$  and  $k = B$ , just like the map  $\Theta$  constructed in [TW11] for the tetrahedral Kummer surface  $X_1 = X_{D_4}$  and the square Kummer surface  $X_0$ . That  $\Theta$  restricts to the desired Niemeier markings is sufficient to ensure that  $\Theta_{AB}$  is an overarching map according to the above definition. Indeed, we can always find a path in the smooth universal cover of the moduli space which connects  $X_A$  and  $X_B$ , such that all intermediate points of the path are Kummer surfaces with the minimal symmetry group  $G = G_t \cong (\mathbb{Z}_2)^4$ . The group  $G_t$  is compatible with  $\Theta_{AB}$  by construction. See [TW11, Thm. 4.4.2] for an example – one solely needs to ensure that  $\text{span}_{\mathbb{C}}\{\Omega_1, \Omega_2, \omega_0\}^\perp \cap \pi_* H_2(T, \mathbb{Z}) = \{0\}$  along the path.

To determine sufficient conditions on the existence of  $\Theta_{AB}$ , first note that by the above, see also [TW11, Thm. 3.3.7], the lattices  $M_{G_k}$  share the Kummer lattice  $\Pi$  and the vector  $v_0 - v$ . By the Definition 2.2 of Niemeier markings  $\iota_G: M_G \hookrightarrow N(-1)$ , we require  $\Theta_{AB}(E_{\vec{a}}) = f_{I^{-1}(\vec{a})}$

for all  $\vec{a} \in \mathbb{F}_2^4$ . As mentioned above,  $G_k$ -equivariance of  $\iota_{G_k}$  then already fixes the action of  $G_k$  on  $N$ . An overarching map  $\Theta_{AB}$  hence only exists if there is an index  $n_0 \in \mathcal{O}_9$ , such that  $f_{n_0}$  is invariant under the action of both groups  $G_A, G_B$ , such that  $\Theta_{AB}(v_0 - v) = f_{n_0}$  is consistent with  $G_k$ -equivariance.

For the complementary lattices  $\widehat{K}_{G_k} := ((\pi_*(H_2(T, \mathbb{Z}))^{(G_k)T})^\perp \cap \pi_*H_2(T, \mathbb{Z}))$  for  $k \in \{A, B\}$  introduced in [TW11, Thm. 3.3.7], choose bases  $I_{i_k, k}^\perp$ ,  $i_k \in \{1, \dots, N_k\}$ , where  $N_k \leq 3$  by construction. If all the vectors  $I_{1, A}^\perp, \dots, I_{N_A, A}^\perp, I_{1, B}^\perp, \dots, I_{N_B, B}^\perp$  are linearly independent, then we claim that under one final assumption we can find an overarching map  $\Theta_{AB}$  for  $X_A$  and  $X_B$  as desired<sup>6</sup>. Indeed, as in [TW11§4.1], for each of the six two-cycles<sup>7</sup>  $\lambda_{ij}$ , we first choose a set  $Q_{ij} \subset \mathcal{I}$  of four labels, such that

$$\Theta_{AB}(\pi_*\lambda_{ij}) = \sum_{n \in Q_{ij}} f_n \pmod{2N(-1)}$$

is compatible with the required  $G_k$ -equivariance. In fact, for each  $\lambda_{ij}$ , this constraint only leaves a choice between two complementary sets  $Q_{ij} \subset \mathcal{O}_9$  which are explicitly listed in [TW11, (4.3)]. Choose these quadruplets of labels such that for each  $Q_{ij}$ ,  $n_0 \notin Q_{ij}$ . Analogously to [TW11, Prop. 4.2.5] this defines a map  $\bar{I}$  through  $\bar{I}(\pi_*\lambda_{ij}) := Q_{ij}$  and  $\bar{I}(\lambda + \lambda') := \bar{I}(\lambda) + \bar{I}(\lambda')$  by symmetric differences of sets. Since isometric embeddings  $\iota_{G_k}: M_{G_k} \hookrightarrow N(-1)$  exist by [TW11, Prop. 4.1.1], we can now find appropriate candidates  $\Theta_{AB}(I_{i_k, k}^\perp) \in N(-1)$  such that  $\Theta_{AB}$  restricts to an isometry on both lattices  $\widehat{K}_{G_k}$ . Indeed, up to contributions of the form  $2\Delta$  with  $\Delta \in N(-1)$ , each  $\Theta_{AB}(I_{i_k, k}^\perp)$  is a linear combination of roots  $f_j$  with  $j \in \bar{I}(I_{i_k, k}^\perp)$ . Under the final assumption that all the  $\Theta_{AB}(I_{i_k, k}^\perp)$  constructed in this manner are linearly independent, clearly  $\Theta_{AB}$  can be extended to an overarching map as desired.

All our assumptions hold true in two of the three cases for which we shall construct overarching maps and exhibit overarching symmetries in Section 3 below. In one case, the vectors  $I_{1, A}^\perp, \dots, I_{N_A, A}^\perp, I_{1, B}^\perp, \dots, I_{N_B, B}^\perp$  fail to be linearly independent. However, the linear dependence results from a repetition of vectors,  $I_{a, A}^\perp = I_{b, B}^\perp$ , so by listing every vector only once, linear independence is achieved, and the argument goes through as above.

This technique allows us to find overarching maps between any two Kummer surfaces, as we shall see in the next two sections. More precisely, for any pair of Kummer surfaces we can find representatives  $X_A$  and  $X_B$  in the smooth universal cover of the moduli space of hyperkähler structures, such that an overarching map for  $X_A$  and  $X_B$  exists. Hence we can surf between any two points in moduli space.

### 3. Construction of overarching maps

In Section 1, we have identified three distinct Kummer surfaces  $X_k$ ,  $k \in \{0, 1, 2\}$ , whose associated tori  $T = \mathbb{C}^2/\Lambda$  have maximal symmetry. In order to explore the overarching symmetry for the moduli space of Kummer surfaces by surfing from  $X_0$  to  $X_1$  and  $X_2$ , and from  $X_1$  to  $X_2$ , we apply the recipe given in Section 2 to construct three overarching maps  $\Theta_{k\ell}$ ,  $0 \leq k < \ell \leq 2$ , that yield overarching symmetry groups for the three pairs of Kummer surfaces  $(X_k, X_\ell)$ . As was explained in Section 2, the construction of an overarching map requires the existence of a root  $f_{n_0} \in N(-1)$ ,  $n_0 \in \mathcal{O}_9$ , that is invariant under the action of  $G_k$  and  $G_\ell$ . In the cases of interest to us here, the value of  $n_0$  varies from map to map, but we carefully note down all possible choices, since this will be crucial in the subsequent section. We first summarize the construction of the overarching map  $\Theta_{01}$  valid for the square and tetrahedral Kummer surfaces, which appeared with some additional details in [TW11]. Then we proceed to the construction of the other two maps,  $\Theta_{02}$  and  $\Theta_{12}$ , which are new. This exercise paves the way to Section 4, where we argue that one

<sup>6</sup>We will see below that the assumption of linear independence can be relaxed, but for simplicity of exposition we first consider this case.

<sup>7</sup>Recall that for  $T = T(\Lambda)$ ,  $\lambda_{ij} := \lambda_i \vee \lambda_j \in H_2(T, \mathbb{Z})$  denotes the integral two-cycle specified by the lattice vectors  $\vec{\lambda}_i, \vec{\lambda}_j \in \Lambda$ .



can combine various overarching groups and obtain an action of a maximal subgroup  $(\mathbb{Z}_2)^4 \rtimes A_8$  of  $M_{24}$  on the Niemeier lattice  $N(-1)$ , overarching the entire Kummer moduli space.

**3.1. Overarching the square and tetrahedral Kummer K3s.** The full symmetry group of the square Kummer surface  $X_0$  is the group  $G_0 := (\mathbb{Z}_2)^4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$  of order 64, while that of the tetrahedral Kummer surface  $X_1 := X_{D_4}$  is the group  $G_1 := (\mathbb{Z}_2)^4 \rtimes A_4$  of order 192. By the discussion in the previous section, there exist Niemeier markings  $\iota_{G_k}$ ,  $k \in \{0, 1\}$ , which allow the definition of induced actions of the groups  $G_k$  on the Niemeier lattice  $N(-1)$ , independently of all choices. Indeed, for the respective generators listed at the end of Section 1, according to [TW11, Sects. 4.2, 4.3] we obtain

$$(3.1) \quad (G_0)_T := \mathbb{Z}_2 \times \mathbb{Z}_2 : \begin{cases} \alpha_1 = (4, 8)(6, 19)(10, 20)(11, 13)(12, 22)(14, 17)(16, 18)(23, 24), \\ \alpha_2 = (2, 21)(3, 9)(4, 8)(10, 12)(11, 14)(13, 17)(20, 22)(23, 24), \end{cases}$$

$$(3.2) \quad (G_1)_T := A_4 : \begin{cases} \gamma_1 = (2, 8)(7, 18)(9, 24)(10, 22)(11, 13)(12, 17)(14, 20)(15, 19), \\ \gamma_2 = (2, 18)(7, 8)(9, 19)(10, 17)(11, 14)(12, 22)(13, 20)(15, 24), \\ \gamma_3 = (2, 12, 13)(4, 16, 21)(7, 17, 20)(8, 22, 14)(9, 19, 24)(10, 11, 18). \end{cases}$$

The construction of the map  $\Theta_{01}$  requires that one root  $f_{n_0}$  with  $n_0 \in \mathcal{O}_9$  is invariant under  $G_0$  and  $G_1$ . One checks that indeed  $n_0 := 5$  is the only label in  $\mathcal{O}_9$  which is fixed by both groups.

According to [TW11, (4.9), (4.21)], the generators of the rank 3 lattices  $\widehat{K}_{G_1}$  and  $\widehat{K}_{G_0}$  are

$$(3.3) \quad \begin{aligned} I_{1,1}^\perp &= \pi_* \lambda_{14} + \pi_* \lambda_{24} - \pi_* \lambda_{23}, & I_{1,0}^\perp &= \pi_* \lambda_{14} - \pi_* \lambda_{23}, \\ I_{2,1}^\perp &= \pi_* \lambda_{13} + \pi_* \lambda_{24} + \pi_* \lambda_{34}, & I_{2,0}^\perp &= \pi_* \lambda_{13} + \pi_* \lambda_{24}, \\ I_{3,1}^\perp &= -\pi_* \lambda_{12} + \pi_* \lambda_{14} + \pi_* \lambda_{34}, & I_{3,0}^\perp &= \pi_* \lambda_{34} - \pi_* \lambda_{12}. \end{aligned}$$

From [TW11, (4.3)] we read that  $n_0 = 5 \notin Q_{ij}$  implies

$$(3.4) \quad \begin{aligned} Q_{12} &= \{3, 6, 15, 19\}, & Q_{13} &= \{6, 15, 23, 24\}, & Q_{14} &= \{3, 9, 15, 24\}, \\ Q_{34} &= \{6, 9, 15, 19\}, & Q_{24} &= \{15, 19, 23, 24\}, & Q_{23} &= \{3, 9, 15, 23\}. \end{aligned}$$

Hence the map  $\bar{I}$  described in Section 2 is

$$(3.5) \quad \begin{aligned} \bar{I}(I_{1,1}^\perp) &= \{15, 19\}, & \bar{I}(I_{2,1}^\perp) &= \{9, 15\}, & \bar{I}(I_{3,1}^\perp) &= \{15, 24\}, \\ \bar{I}(I_{1,0}^\perp) &= \{23, 24\}, & \bar{I}(I_{2,0}^\perp) &= \{6, 19\}, & \bar{I}(I_{3,0}^\perp) &= \{3, 9\}. \end{aligned}$$

Our choice of images of the generators (3.3) under  $\Theta_{01}$  must ensure that  $\Theta_{01}$  restricts to an isometry on both lattices  $\widehat{K}_{G_k}$ . Therefore, note that the quadratic form on  $\widehat{K}_{G_1}$  with respect to the basis  $I_{i,1}^\perp$ ,  $i \in \{1, 2, 3\}$ , and that on  $\widehat{K}_{G_0}$  with respect to the basis  $I_{i,0}^\perp$ ,  $i \in \{1, 2, 3\}$ , are

$$(3.6) \quad \widehat{K}_{G_1} : \begin{pmatrix} -4 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -4 \end{pmatrix}, \quad \widehat{K}_{G_0} : \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

according to [TW11, (4.20)] and [TW11, (4.27)]. Then the following gives linearly independent candidates for the  $\Theta_{01}(I_{i,k}^\perp) \in N(-1)$  as desired:

$$(3.7) \quad \Theta_{01} : \begin{cases} I_{1,1}^\perp \mapsto f_{19} - f_{15}, & I_{2,1}^\perp \mapsto f_9 - f_{15}, & I_{3,1}^\perp \mapsto f_{24} - f_{15}, \\ I_{1,0}^\perp \mapsto f_{24} - f_{23}, & I_{2,0}^\perp \mapsto f_{19} - f_6, & I_{3,0}^\perp \mapsto f_9 - f_3. \end{cases}$$

Equivalently,

$$(3.8) \quad \Theta_{01} : \begin{cases} \pi_* \lambda_{12} \mapsto 2q_{12} = f_3 + f_6 - f_{15} - f_{19}, \\ \pi_* \lambda_{34} \mapsto 2q_{34} = f_6 + f_9 - f_{15} - f_{19}, \\ \pi_* \lambda_{13} \mapsto 2q_{13} = -f_6 + f_{15} - f_{23} + f_{24}, \\ \pi_* \lambda_{24} \mapsto 2q_{24} = -f_{15} + f_{19} + f_{23} - f_{24}, \\ \pi_* \lambda_{14} \mapsto 2q_{14} = f_3 - f_9 - f_{15} + f_{24}, \\ \pi_* \lambda_{23} \mapsto 2q_{23} = f_3 - f_9 - f_{15} + f_{23}. \end{cases}$$

On the Kummer lattice  $\Pi$ , we set  $\Theta_{01}(E_{\bar{a}}) = f_{I^{-1}(\bar{a})}$ , as always. Finally, a consistent choice for the images of  $v, v_0$  is

$$\Theta_{01} : \begin{cases} v_0 \mapsto \frac{1}{2}(f_3 + f_5 + f_6 + f_9 - f_{15} - f_{19} - f_{23} - f_{24}), \\ v \mapsto \frac{1}{2}(f_3 - f_5 + f_6 + f_9 - f_{15} - f_{19} - f_{23} - f_{24}). \end{cases}$$

This completes the construction of the map  $\Theta_{01}$  which is compatible with the symmetry groups of the square ( $G_0$ ) and tetrahedral ( $G_1$ ) Kummer surfaces. Viewed as a linear bijection  $\Theta_{01} : H_*(X_0, \mathbb{Z}) \rightarrow N(-1)$ , its restriction  $\Theta_{01}|_{M_{G_0}}$  yields a  $G_0$ -equivariant and isometric embedding of  $M_{G_0}$  in  $N(-1)$ . Viewed instead as a linear bijection  $\Theta_{01} : H_*(X_1, \mathbb{Z}) \rightarrow N(-1)$ , its restriction  $\Theta_{01}|_{M_{G_1}}$  yields a  $G_1$ -equivariant and isometric embedding of  $M_{G_1}$  in  $N(-1)$ . This property of the overarching map  $\Theta_{01}$  gives us ground to argue that there is an overarching symmetry group for the square and tetrahedral Kummer surfaces, whose action is encoded in the same Niemeier lattice  $N(-1)$  through the generators (2.1) of the translational symmetry group  $G_t$  common to all Kummer surfaces, in addition to the generators (3.1) and (3.2). The group generated this way is a copy of  $(\mathbb{Z}_2)^4 \rtimes A_7 \subset M_{24}$ .

**3.2. Overarching the square and the triangular Kummer K3s.** The full symmetry group of the triangular Kummer surface  $X_2$  is the group  $G_2 := (\mathbb{Z}_2)^4 \rtimes S_3$  of order 96, see (1.8) and (1.9). Independently of the choice of Niemeier marking, the induced action of  $G_2$  on the Niemeier lattice is generated by

$$(3.9) \quad (G_2)_T := S_3 : \begin{cases} \beta_1 = (2, 17, 14)(4, 7, 8)(10, 16, 12)(11, 13, 21)(18, 20, 22)(5, 24, 23), \\ \beta_2 = (2, 21)(3, 9)(4, 8)(10, 12)(11, 14)(13, 17)(20, 22)(23, 24). \end{cases}$$

The construction of an overarching map  $\Theta_{02}$  for  $X_0$  and  $X_2$  requires a root  $f_{n_0}$  with  $n_0 \in \mathcal{O}_9$  which is invariant under  $G_0$  and  $G_2$ . From (3.1) and (3.9) we observe that  $\alpha_2 = \beta_2$  and that  $n_0 = 15$  is the only label in  $\mathcal{O}_9$  which is invariant under both groups.

To calculate the generators of the lattice  $\widehat{K}_{G_2}$  following the techniques explained in [TW11], we first need to determine generators of the lattice  $(\pi_* H_2(T, \mathbb{Z}))^{(G_2)_T}$ . With the basis  $\vec{\lambda}_1, \dots, \vec{\lambda}_4$  for the triangular lattice given in (1.8), we obtain primitive generators of that lattice as

$$\pi_* \lambda_{13} - \pi_* \lambda_{24}, \quad \pi_* \lambda_{13} + \pi_* \lambda_{23} + \pi_* \lambda_{14}, \quad \pi_* \lambda_{12} + \pi_* \lambda_{34},$$

and hence the orthogonal complement  $\widehat{K}_{G_2}$  of  $(\pi_* H_2(T, \mathbb{Z}))^{(G_2)_T}$  in  $\pi_* H_2(T, \mathbb{Z})$  is generated by the lattice vectors

$$(3.10) \quad I_{1,2}^\perp := \pi_* \lambda_{12} - \pi_* \lambda_{34}, \quad I_{2,2}^\perp := \pi_* \lambda_{13} + \pi_* \lambda_{23} + \pi_* \lambda_{24}, \quad I_{3,2}^\perp := \pi_* \lambda_{14} - \pi_* \lambda_{23}.$$

From [TW11, (4.3)] we read that  $n_0 = 15 \notin Q_{ij}$  implies

$$(3.11) \quad \begin{aligned} Q_{12} &= \{5, 9, 23, 24\}, & Q_{13} &= \{3, 5, 9, 19\}, & Q_{14} &= \{5, 6, 19, 23\}, \\ Q_{34} &= \{3, 5, 23, 24\}, & Q_{24} &= \{3, 5, 6, 9\}, & Q_{23} &= \{5, 6, 19, 24\}. \end{aligned}$$

Hence the map  $\bar{I}$  described in Section 2 is as in (3.5) for  $I_{i,0}^\perp$ ,  $i \in \{1, 2, 3\}$ , and furthermore,

$$\bar{I}(I_{1,2}^\perp) = \{3, 9\}, \quad \bar{I}(I_{2,2}^\perp) = \{5, 24\}, \quad \bar{I}(I_{3,2}^\perp) = \{23, 24\}.$$

We now need to choose the images in  $N(-1)$  of the generators (3.10) under  $\Theta_{02}$  such that  $\Theta_{02}$  restricts to an isometry on the lattices  $\widehat{K}_{G_0}$  and  $\widehat{K}_{G_2}$ . To do so, note that the quadratic form

on  $\widehat{K}_{G_0}$  with respect to the basis  $I_{i,0}^\perp$ ,  $i \in \{1, 2, 3\}$ , and that of  $\widehat{K}_{G_2}$  with respect to the basis  $I_{i,1}^\perp$ ,  $i \in \{1, 2, 3\}$ , by (3.6) and (3.10) are

$$(3.12) \quad \widehat{K}_{G_0} : \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad \widehat{K}_{G_2} : \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -4 \end{pmatrix}.$$

Moreover, we have  $I_{1,0}^\perp = I_{3,2}^\perp$  and  $I_{3,0}^\perp = I_{1,2}^\perp$ , such that we can find candidates for  $\Theta_{02}(I_{i,k}^\perp) \in N$  as desired:

$$(3.13) \quad \Theta_{02} : \begin{cases} I_{1,0}^\perp \mapsto f_{24} - f_{23}, & I_{2,0}^\perp \mapsto f_6 - f_{19}, & I_{3,0}^\perp \mapsto f_3 - f_9, \\ I_{1,2}^\perp \mapsto f_3 - f_9, & I_{2,2}^\perp \mapsto f_5 - f_{24}, & I_{3,2}^\perp \mapsto f_{24} - f_{23}. \end{cases}$$

For example, we can choose the following map in order to induce (3.13):

$$(3.14) \quad \Theta_{02} : \begin{cases} \pi_* \lambda_{12} \mapsto 2q_{12} = f_5 + f_9 - f_{23} - f_{24}, \\ \pi_* \lambda_{34} \mapsto 2q_{34} = f_3 + f_5 - f_{23} - f_{24}, \\ \pi_* \lambda_{13} \mapsto 2q_{13} = f_3 + f_5 - f_9 - f_{19}, \\ \pi_* \lambda_{24} \mapsto 2q_{24} = -f_3 - f_5 + f_6 + f_9, \\ \pi_* \lambda_{14} \mapsto 2q_{14} = f_5 - f_6 + f_{19} - f_{23}, \\ \pi_* \lambda_{23} \mapsto 2q_{23} = f_5 - f_6 + f_{19} - f_{24}. \end{cases}$$

On the Kummer lattice  $\Pi$ , we set  $\Theta_{02}(E_{\bar{a}}) = f_{I^{-1}(\bar{a})}$ , as before. Finally, a consistent choice for the images of  $v$ ,  $v_0$  is

$$(3.15) \quad \Theta_{02} : \begin{cases} v_0 \mapsto \frac{1}{2}(f_3 + f_5 + f_6 - f_9 + f_{15} - f_{19} - f_{23} - f_{24}), \\ v \mapsto \frac{1}{2}(f_3 + f_5 + f_6 - f_9 - f_{15} - f_{19} - f_{23} - f_{24}). \end{cases}$$

This completes the construction of the overarching map  $\Theta_{02}$  for the square and the triangular Kummer surfaces. Again, the overarching map  $\Theta_{02}$  leads to an overarching symmetry group, whose action is encoded in the same Niemeier lattice  $N(-1)$  through the generators (2.1) of the translational symmetry group  $G_t$  common to all Kummer surfaces, in addition to the generators (3.1) and (3.9). The resulting group is a copy of  $(\mathbb{Z}_2)^4 \rtimes \mathcal{D} \subset M_{24}$ , where  $\mathcal{D}$  denotes the binary dihedral group of order 12, as before.

**3.3. Overarching the tetrahedral and triangular Kummer K3s.** The construction of an overarching map  $\Theta_{12}$  for  $X_1$  and  $X_2$  requires a root  $f_{n_0}$  with  $n_0 \in \mathcal{O}_9$  which is invariant under  $G_1$  and  $G_2$ , whose generators are given in (3.9) and (3.2). The only label in  $\mathcal{O}_9$  which is invariant under both these groups is  $n_0 = 6$ .

The generators of the rank 3 lattice  $\widehat{K}_{G_1}$  are given in (3.3), and those of the lattice  $\widehat{K}_{G_2}$  by (3.10). From [TW11, (4.3)] we read that  $n_0 = 6 \notin Q_{ij}$  implies

$$(3.16) \quad \begin{aligned} Q_{12} &= \{5, 9, 23, 24\}, & Q_{13} &= \{3, 5, 9, 19\}, & Q_{14} &= \{3, 9, 15, 24\}, \\ Q_{34} &= \{3, 5, 23, 24\}, & Q_{24} &= \{15, 19, 23, 24\}, & Q_{23} &= \{3, 9, 15, 23\}. \end{aligned}$$

Hence the map  $\bar{I}$  described in Section 2 is

$$\begin{aligned} \bar{I}(I_{1,1}^\perp) &= \{15, 19\}, & \bar{I}(I_{2,1}^\perp) &= \{9, 15\}, & \bar{I}(I_{3,1}^\perp) &= \{15, 24\}, \\ \bar{I}(I_{1,2}^\perp) &= \{3, 9\}, & \bar{I}(I_{2,2}^\perp) &= \{5, 24\}, & \bar{I}(I_{3,2}^\perp) &= \{23, 24\}. \end{aligned}$$

We now need to choose the images in  $N$  of the generators  $I_{i,1}^\perp$ ,  $i \in \{1, 2, 3\}$  and  $I_{i,2}^\perp$ ,  $i \in \{1, 2, 3\}$  under  $\Theta_{12}$  such that  $\Theta_{12}$  restricts to an isometry on the lattices  $\widehat{K}_{G_1}$  and  $\widehat{K}_{G_2}$ . Given the quadratic form (3.6) for  $\widehat{K}_{G_1}$  and (3.12) for  $\widehat{K}_{G_2}$ , the following gives linearly independent candidates for  $\Theta_{12}(I_{i,k}^\perp) \in N$ :

$$(3.17) \quad \Theta_{12} : \begin{cases} I_{1,1}^\perp \mapsto f_{19} - f_{15}, & I_{2,1}^\perp \mapsto f_9 - f_{15}, & I_{3,1}^\perp \mapsto f_{24} - f_{15}, \\ I_{1,2}^\perp \mapsto f_3 - f_9, & I_{2,2}^\perp \mapsto f_5 - f_{24}, & I_{3,2}^\perp \mapsto f_{24} - f_{23}. \end{cases}$$

Equivalently,

$$(3.18) \quad \Theta_{12}: \begin{cases} \pi_* \lambda_{12} \mapsto & 2q_{12} + 2f_3 - 2f_{15} = & -f_5 - f_9 + f_{23} + f_{24} + 2f_3 - 2f_{15}, \\ \pi_* \lambda_{34} \mapsto & 2q_{34} - 2f_{15} = & f_3 - f_5 + f_{23} + f_{24} - 2f_{15}, \\ \pi_* \lambda_{13} \mapsto & 2q_{13} + 2f_{15} - 2f_{23} = & -f_3 + f_5 + f_9 - f_{19} + 2f_{15} - 2f_{23}, \\ \pi_* \lambda_{24} \mapsto & 2q_{24} = & -f_{15} + f_{19} + f_{23} - f_{24}, \\ \pi_* \lambda_{14} \mapsto & 2q_{14} = & f_3 - f_9 - f_{15} + f_{24}, \\ \pi_* \lambda_{23} \mapsto & 2q_{23} = & f_3 - f_9 - f_{15} + f_{23}. \end{cases}$$

On the Kummer lattice  $\Pi$ , we set  $\Theta_{12}(E_{\bar{a}}) = f_{I^{-1}(\bar{a})}$ . Finally, a consistent choice for the images of  $v, v_0$  is

$$(3.19) \quad \Theta_{12}: \begin{cases} v_0 \mapsto & \frac{1}{2}(f_3 + f_5 + f_6 - f_9 - f_{15} - f_{19} + f_{23} - f_{24}), \\ v \mapsto & \frac{1}{2}(f_3 + f_5 - f_6 - f_9 - f_{15} - f_{19} + f_{23} - f_{24}). \end{cases}$$

This completes the construction of the overarching map  $\Theta_{12}$  which is compatible with the symmetry groups of the tetrahedral ( $G_1$ ) and triangular ( $G_2$ ) Kummer surfaces. Hence there is an overarching symmetry group for the tetrahedral and the triangular Kummer surfaces, whose action is encoded in the same Niemeier lattice  $N(-1)$  through the generators (2.1) of the translational symmetry group  $G_t$  common to all Kummer surfaces, in addition to the generators (3.2) and (3.9). The group thus generated is a copy of  $(\mathbb{Z}_2)^4 \rtimes A_7 \subset M_{24}$ .

#### 4. Overarching the moduli space of Kummer K3s by $(\mathbb{Z}_2)^4 \rtimes \mathbf{A}_8$

In this section we argue that our surfing procedure allows us to surf between any two points of the moduli space of Kummer K3s. More precisely, for any two Kummer surfaces with induced dual Kähler class, we can find representatives in the smooth universal cover of the moduli space of hyperkähler structures, such that an overarching map between the two representatives exists. This allows us to combine all symmetry groups of such Kummer surfaces to a larger, overarching group.

To see this, let us first consider an arbitrary Kummer surface  $X_{\tilde{\Lambda}, \omega_0}$  with induced dual Kähler class, and let  $\tilde{G}$  denote its symmetry group. According to our discussion in Section 1,  $\tilde{G} = (\mathbb{Z}_2)^4 \rtimes (\tilde{G}'_T / \mathbb{Z}_2)$ , where  $\tilde{G}'_T \subset SU(2)$  is the linear automorphism group of the lattice  $\tilde{\Lambda}$ . Moreover,  $\tilde{G}'_T$  is a subgroup of one of the three maximal linear automorphism groups of complex tori, the binary tetrahedral group  $\mathcal{T}$  or one of the dihedral groups  $\mathcal{D}, \mathcal{O}$  of order 12 and 8.

Let  $G'_T = \mathcal{O}, \mathcal{T}$  or  $\mathcal{D}$ , such that  $\tilde{G}'_T \subset G'_T$ , and let  $\Lambda = \Lambda_0, \Lambda_1$  or  $\Lambda_2$  denote the corresponding choice of lattice from Section 1 which has  $G'_T$  as its linear automorphism group. Fujiki's classification [Fuj88] implies that we can choose  $G'_T$  and  $\Lambda$  in such a way that there is a smooth deformation of  $\Lambda$  into  $\tilde{\Lambda}$ , call it  $\Lambda^t$  with  $t \in [0, 1]$  and  $\Lambda^0 = \Lambda, \Lambda^1 = \tilde{\Lambda}$ , such that the linear automorphism group of each  $\Lambda^t$  with  $t \neq 0$  is  $\tilde{G}'_T$ . The quaternionic language introduced in Section 1 is particularly useful to check this. For example, if  $\tilde{G}'_T = \mathbb{Z}_4$ , then by Fujiki's results we can choose coordinates such that the action of this group on  $\mathbb{C}^2$  is generated by our symmetry  $\alpha_1$  of (1.5), and we can choose  $G'_T = \mathcal{O}$  with  $\Lambda = \Lambda_0$  the lattice of the square torus. One finds generators  $\vec{\lambda}_1^t, \dots, \vec{\lambda}_4^t$  for the lattices  $\Lambda^t$  as desired such that  $\vec{\lambda}_2^t = \alpha_1(\vec{\lambda}_1^t)$  and  $\vec{\lambda}_3^t = \alpha_1(\vec{\lambda}_4^t)$  for every  $t \in [0, 1]$ .

This deformation argument implies that by use of our fixed marking, the invariant sublattices of the integral torus homology,  $L^{\tilde{G}'_T} = H_*(T, \mathbb{Z})^{\tilde{G}'_T}$  and  $L^{G'_T} = H_*(T, \mathbb{Z})^{G'_T}$ , obey  $L^{G'_T} \subset L^{\tilde{G}'_T}$ . Hence for the symmetry group  $G$  of  $X_{\Lambda, \omega_0}$  and for the lattices that yield our Niemeier markings we have  $M_{\tilde{G}} \subset M_G$ , see Def. 2.2 and the discussion preceding it. From this it follows that one can find a representative of  $X_{\tilde{\Lambda}, \omega_0}$  in the smooth universal cover of the moduli space of hyperkähler structures such that every Niemeier marking  $\iota_G: M_G \hookrightarrow N(-1)$  of the maximally symmetric Kummer surface  $X_{\Lambda, \omega_0}$  restricts to a Niemeier marking  $\iota_{\tilde{G}} := \iota_G|_{M_{\tilde{G}}}$  of the Kummer surface  $X_{\tilde{\Lambda}, \omega_0}$ . Hence any overarching map  $\Theta$  for the maximally symmetric Kummer surface  $X_{\Lambda, \omega_0}$  and any other

Kummer K3  $X$  also allows to surf from  $X_{\tilde{\Lambda}, \omega_0}$  to  $X$ .

Now consider two distinct Kummer surfaces  $\tilde{X}_A$  and  $\tilde{X}_B$  with their induced dual Kähler classes. By the above, we can choose maximally symmetric Kummer surfaces  $X_A$  and  $X_B$  from the square, the tetrahedral and the triangular Kummer surfaces, such that the following holds: there are representatives of  $\tilde{X}_A$  and  $\tilde{X}_B$  in the smooth universal cover of the moduli space of hyperkähler structures such that any Niemeier marking of  $X_A$  restricts to a Niemeier marking of  $\tilde{X}_A$ , and analogously for  $X_B$  and  $\tilde{X}_B$ . Then by the above, the overarching map  $\Theta_{AB}$  for  $X_A$  and  $X_B$  which was constructed<sup>8</sup> in Section 3 also overarches  $\tilde{X}_A$  and  $\tilde{X}_B$ . In other words, we can surf from  $\tilde{X}_A$  to  $\tilde{X}_B$ .

We conclude that by means of our overarching maps we can surf the entire moduli space of hyperkähler structures of Kummer surfaces. In particular, we can combine the actions of all symmetry groups of Kummer surfaces with induced dual Kähler class by means of their action on the Niemeier lattice  $N$ . Recall from Section 2 that by construction, every overarching map  $\Theta_{AB}$  between Kummer surfaces  $X_A$  and  $X_B$  with symmetry groups  $G_A$  and  $G_B$  assigns a fixed root  $\Theta_{AB}(v_0 - v) = f_{n_0} \in N(-1)$  to the root  $v_0 - v \in H_*(X, \mathbb{Z})$ , where  $n_0 \in \mathcal{O}_9$  is a label in our reference octad from the Golay code<sup>9</sup>. This root  $f_{n_0}$  is fixed under the induced actions of both  $G_A$  and  $G_B$ . For the overarching group  $G_{AB}$  obtained from  $G_A$  and  $G_B$ , which by construction is a subgroup of the stabilizer group  $(\mathbb{Z}_2)^4 \rtimes A_8$  of the octad  $\mathcal{O}_9$ , this implies that  $G_{AB}$  additionally fixes one label  $n_0 \in \mathcal{O}_9$ . Hence  $G_{AB}$  is a subgroup of  $(\mathbb{Z}_2)^4 \rtimes A_7$ , the group which we call the OVERARCHING SYMMETRY GROUP OF KUMMER K3S. In Section 3 we have seen that for two pairs of distinct Kummer surfaces with maximal symmetry, the overarching group yields  $G_{AB} = (\mathbb{Z}_2)^4 \rtimes A_7$ . The third pair has overarching group  $(\mathbb{Z}_2)^4 \rtimes \mathcal{D}$ . Moreover, in each case there exists precisely one label in  $\mathcal{O}_9$  which is fixed by both  $G_A$  and  $G_B$ . This label, however, is different for each of the three pairs of Kummer K3s with maximal symmetry. It follows that the combined symmetry group for all Kummer K3s with induced dual Kähler class is  $(\mathbb{Z}_2)^4 \rtimes A_8$ .

## 5. Interpretation and outlook

Let us now explain how our construction fits into the quest for the expected representation of  $M_{24}$  on a vertex algebra which governs the elliptic genus of K3. As mentioned in the Introduction, the elliptic genus arises from the contribution to the partition function of any superconformal field theory on K3 which counts states in the Ramond-Ramond sector with signs according to fermion numbers. This part of the partition function is modular invariant on its own, inducing the well-known modularity properties of the elliptic genus. The very construction of the elliptic genus, in addition, amounts to a projection onto those states which are Ramond ground states on the antiholomorphic side. The usual rules for fermion numbers imply that the OPE between any two fields in the Ramond sector yields contributions from the Neveu-Schwarz sector only. Hence the expected vertex algebra can certainly not arise in the Ramond-Ramond sector. Of course we can spectral flow the relevant fields into the Neveu-Schwarz sector, where (prior to all projections) they indeed form a closed vertex algebra<sup>10</sup>  $\hat{\mathcal{X}}$ . Note that the choice of a spectral flow requires the choice of a holomorphic and an antiholomorphic  $U(1)$ -current within the superconformal algebra of our SCFT. For definiteness, we use the spectral flow which maps Ramond-Ramond ground states to (chiral, chiral) states.

The resulting vertex algebra  $\hat{\mathcal{X}}$  certainly governs the elliptic genus. Its space of states contains the states underlying the well-known (chiral, chiral) algebra  $\mathcal{X}$  of Lerche-Vafa-Warner [LVW89], which accounts for the contributions to the lowest order terms of the elliptic genus. In Appendix A we describe the (chiral, chiral) algebra  $\mathcal{X}$  (see (A.1)) more concretely in the context relevant to

<sup>8</sup>If  $X_A = X_B$ , then there is nothing left to be shown.

<sup>9</sup>This fixed label  $n_0$  is responsible for the fact that each  $G_k$  is a subgroup of  $M_{23}$ , as we emphasized in [TW11].

<sup>10</sup>Here and in the following, we loosely refer to the space of fields which create states in the Neveu-Schwarz sector, equipped with the OPE, as a “vertex algebra”, which however is *not* a holomorphic VOA.

this work, namely in  $\mathbb{Z}_2$ -orbifold conformal field theories  $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$  on K3, where  $\mathcal{T}$  denotes the underlying toroidal theory. As expanded upon in Appendix A, the very truncation to the (chiral, chiral) algebra  $\mathcal{X}$  makes  $\mathcal{X}$  completely independent of all moduli. In principle, this is a desired effect when aiming at constructing a vertex algebra which governs the elliptic genus, since the latter is independent of all moduli. However, from the action of a linear map on  $\mathcal{X}$  (generated by the fields in (A.1), independently of all moduli), it is not clear whether or not it is a symmetry, while the Mathieu Moonshine phenomenon dictates that we consider symmetries of some underlying vertex algebra.

We shall come back to this ‘bottom up’ discussion further down, but we first take a closer look at the ‘top-down’ approach, and consider the action of symmetries of  $\mathcal{C}$  on the (chiral, chiral) algebra  $\mathcal{X}$  generated by (A.1). We impose a number of rather severe assumptions on such symmetries, in order to ensure that they descend to symmetries of a candidate vertex algebra that governs the elliptic genus. As mentioned in the Introduction, this graded vertex algebra at leading order is the (chiral, chiral) algebra  $\mathcal{X}$ . Following [LVW89] we identify  $\mathcal{X}$  with the cohomology of a K3 surface  $X$ . Associated to every Calabi-Yau manifold  $Y$ , there is the chiral de Rham complex [MSV99] which furnishes a sheaf of vertex algebras governing the elliptic genus of  $Y$  and containing the usual de Rham complex of  $Y$  at leading order [BL00, Bor01]. We thus find it natural<sup>11</sup> to restrict our attention to symmetries of  $\mathcal{C}$  that descend to the chiral de Rham complex of  $X$ . To this end, we assume that our SCFT  $\mathcal{C}$  comes with a choice of generators of the  $N = (4, 4)$  superconformal algebra, which in particular fixes the  $U(1)$ -currents and a preferred  $N = (2, 2)$  subalgebra. As mentioned above, this is already necessary when we choose the spectral flow to  $\mathcal{X}$ . Recall that the choice of  $U(1)$ -currents amounts to the choice of a complex structure in any geometric interpretation of  $\mathcal{C}$  [AM94]. We furthermore use the notion advertised by [GPRV12], which requires symmetries to fix the superconformal algebra of  $\mathcal{C}$  pointwise.<sup>12</sup> To identify  $\mathcal{X}$  with the cohomology of a K3 surface  $X$ , we need to perform a large volume limit [Wit82, LVW89]. More generally, according to [Kap05], the space of states singled out by the elliptic genus is mapped to the appropriate cohomology of the chiral de Rham complex of  $X$  only in the large volume limit. In order to perform such a large volume limit, we need to choose a geometric interpretation of  $\mathcal{C}$ .

Summarising, in view of constructing a vertex algebra from the fields in  $\mathcal{C}$ , such that  $\mathcal{X}$  governs the leading order terms of the elliptic genus, we restrict our attention to symmetries that are induced from geometric symmetries. This justifies why so far, in our work, we have searched for explanations of Mathieu Moonshine phenomena within the context of geometric symmetries only.

As a further potential justification for this restriction recall the notion of ‘exceptional’ symmetry groups of sigma models on K3, that is, symmetry groups of such SCFTs which are not realizable as subgroups of  $M_{24}$ , obtained from the classification in [GHV10a]. According to [GV12], in many cases ‘exceptional’ symmetry is linked to certain quantum symmetries which as we shall argue cannot be induced from any classical geometric symmetries. Indeed, these symmetries in [GV12] are characterized by the property that they generate a group  $G$ , such that orbifolding the K3 model by  $G$  yields a toroidal SCFT. We remark that there is no geometric counterpart of such an orbifold construction, which would have to yield a complex four-torus as an orbifold of a K3 surface. Indeed, the odd cohomology of a complex four-torus cannot be restored by blowing up quotient singularities in an orbifold by a symplectic automorphism group of a K3 surface. However, this is only a potential justification for our restriction to geometric symmetries since, according to [GV12], ‘exceptional’ symmetry groups also occur in a few cases where to date it is not known whether or not such purely non-geometric quantum symmetries are responsible for the ‘exceptionality’ of the symmetry group. Although the group  $M_{24}$  itself contains elements that can never act in terms of a geometric symmetry on K3, we are optimistic that every element of  $M_{24}$  can be

<sup>11</sup>By [BL00, FS07], the CFT orbifold procedure descends to the chiral de Rham complex; this should be the source for the behavior of the twining genera in Mathieu Moonshine, at least for those symmetries that are induced from geometric ones.

<sup>12</sup>This, for instance, excludes equivalences of SCFTs induced by mirror symmetry, which acts as an outer automorphism on the superconformal algebra.

obtained as a composition of ‘geometric’ symmetries.

We wish to emphasize that it is immediately clear that the (chiral, chiral) algebra  $\mathcal{X}$  cannot carry a representation of  $M_{24}$ . Indeed, (A.2) is the basis of a four-dimensional subspace of the 24-dimensional space  $\mathcal{X}$  which is invariant under all symmetries that are of interest here, but by the known properties of representations of  $M_{24}$ , this group can only act trivially on the remaining 20-dimensional space. Hence a vertex algebra which governs the massless leading order terms of the elliptic genus, and which at the same time carries the expected representation of  $M_{24}$ , must be related to  $\mathcal{X}$  by some nontrivial map. The Niemeier markings and the overarching maps which were constructed in [TW11] should be viewed as a first approach towards constructing such a map. This claim is based on the observation that, from a geometric viewpoint, the introduction of Niemeier markings is necessary to combine symmetry groups of Kummer surfaces into larger groups. Indeed, it follows from Mukai’s results that for any finite group  $\widehat{G}$  of lattice automorphisms of  $H_*(X, \mathbb{Z})$  that is not a subgroup of one of the eleven maximal groups listed in [Muk88], the lattice  $L_{\widehat{G}} := (H_*(X, \mathbb{Z})^{\widehat{G}})^{\perp} \cap H_*(X, \mathbb{Z})$  is indefinite and thus violates the signature requirements for symmetries of K3 surfaces. Therefore, we never expected  $M_{24}$  to act on  $H_*(X, \mathbb{Z})$  either. It would be interesting to see if the massive sector of the elliptic genus is also subject to a ‘no-go theorem’ when working in the framework of  $\mathbb{Z}_2$ -orbifold CFTs on K3. A priori, the situation could be different, as the original Mathieu Moonshine observation [EOT11] states that in the elliptic genus, the multiplicities of *massive* characters of the  $N = 4$  superconformal algebra yield dimensions of representations of  $M_{24}$ . In a forthcoming work [TW13] we present evidence in favour of our expectation that the *massive* fields which contribute to the elliptic genus are related to a representation of  $M_{24}$  in a much more immediate fashion.

We now return to the ‘bottom-up’ approach, and investigate more closely the action of symmetry groups on the vertex algebra  $\mathcal{X}$ , to explain in terms of CFT data how our Niemeier markings and overarching maps are relevant in the context of SCFTs on K3. To this end note that the entire group  $\mathrm{SL}(2, \mathbb{C})$  acts naturally on the truncated vertex algebra  $\mathbb{C} \otimes \mathcal{X}$  of (A.1), preserving  $U(1)$ -charges. However, a given element of  $\mathrm{SL}(2, \mathbb{C})$  may not have an extension to a symmetry of the full SCFT  $\mathcal{C}$ . Whether or not this is the case cannot be determined from the action on the fields listed in (A.1). Indeed, this depends on the moduli of  $\mathcal{C}$ , but the vertex algebra  $\mathcal{X}$  has lost its dependence on all moduli due to the truncation, as described earlier. However, as we explain in Appendix A, one may introduce the analog  $\mathcal{X}^{\mathbb{Z}}$  of the lattice of integral homology in the vector space  $\mathcal{X}$ , and use its structure to determine whether or not an element of  $\mathrm{SL}(2, \mathbb{C})$  acts as a symmetry of  $\mathcal{C}$ .

By the above, we are only interested in symmetry groups  $G$  that are induced by geometric symmetries, and in line with our work so far, we restrict our attention to those that are induced<sup>13</sup> from the underlying toroidal CFT  $\mathcal{T}$ . By definition, a symmetry of a SCFT must be compatible with all OPEs in that theory. In particular, the standardized OPE (A.4) must be preserved. Following the arguments presented in Appendix A, this implies that each of our symmetry groups  $G$  acts as a group of lattice automorphisms on  $\mathcal{X}^{\mathbb{Z}}$ , such that this lattice of fields in our SCFT contains a sublattice  $\mathcal{X}_G^{\mathbb{Z}}$  which bears all relevant information about the  $G$ -action on our SCFT. This lattice can be identified with the lattice  $M_G$  which is central to our construction, in that our Niemeier markings isometrically replicate it as a sublattice of the Niemeier lattice  $N(-1)$ . This allows a more elegant description of  $G$  as a subgroup of  $M_{24}$ , and it enables us to combine the symmetry groups from distinct K3 theories to a larger, overarching group. In other words, our Niemeier marking describes precisely the action of geometric symmetry groups on the vertex algebra which governs the elliptic genus to leading order terms. This justifies the relevance of our construction in the context of our quest to unravel some of the mysteries of the Mathieu Moonshine phenomenon.

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<sup>13</sup>This includes the symmetries induced by half lattice shifts in the underlying toroidal theory  $\mathcal{T}$ .

The picture that we offer here shows how the beautiful interplay between geometry and conformal field theory may yield some keys to the Mathieu Moonshine Mysteries. Such an interplay is expected. On the one hand, the elliptic genus is a purely geometric quantity. On the other hand, this quantity also appears in the context of SCFTs on K3, where its decomposition into  $N = 4$  characters is natural. Notably, it is only after decomposing the elliptic genus into  $N = 4$  characters that one observes the Mathieu Moonshine phenomenon [EOT11].

We expect that order by order, the elliptic genus dictates the construction of representations of  $M_{24}$  on appropriately truncated vertex algebras arising from SCFTs on K3. In other words, the very representations of  $M_{24}$  that are observed in the elliptic genus are intrinsic to these SCFTs. The reason why the emerging group is  $M_{24}$  is still unclear, but we expect it to be rooted in the structure of these SCFTs, where geometry dictates the symmetries which can act on these representations. By symmetry-surfing the moduli space of SCFTs on K3, we expect that the natural representations of geometric symmetry groups on these vertex algebras combine to the action of  $M_{24}$ .

Our construction of overarching maps in [TW11] should be viewed as a very first step towards finding such vertex algebras for the leading order terms of the elliptic genus. In the present work, we show that our overarching maps indeed allow us to combine all relevant symmetry groups, as long as we restrict to  $\mathbb{Z}_2$ -orbifold conformal field theories on K3 and their geometric interpretations on Kummer K3s, and to symmetries that are induced geometrically from the underlying toroidal theories. Indeed, since one can easily associate a vertex algebra to the Niemeier lattice  $N$ , one could claim that we have solved the problem of constructing a vertex algebra that furnishes the expected symmetries. However, of course we pay dearly since this vertex algebra does not govern the leading order terms of the elliptic genus in any obvious way. Still our approach paves the way to defining the desired vertex algebra. As we have explained above, we expect vertex algebras associated with all remaining orders of the elliptic genus to relate directly to the respective representations of  $M_{24}$ , and we present evidence in favour of this expectation in [TW13].

### Appendix A. Transition to superconformal field theory

Throughout our work, we use homological data to describe geometric symmetries of K3 surfaces. This is natural, since the techniques are well-established in algebraic geometry, but also since the well-known properties of (chiral, chiral) algebras [Wit82, LVW89] recover (co)homological data from sigma model interpretations of SCFTs. This is particularly straightforward for the  $\mathbb{Z}_2$ -orbifold conformal field theories which are relevant to our investigations. Since our work is motivated by Mathieu Moonshine [EOT11], which is rooted in conformal field theory, and since the role of the *integral* (co)homology in (chiral, chiral) algebras seems not so well established, we gather in this appendix the tools needed to make a smooth transition to superconformal field theory.

We first need to fix some notations. Every toroidal conformal field theory  $\mathcal{T}$  possesses two free Dirac fermions on the holomorphic side, which we denote by  $\chi_+^1(z), \chi_+^2(z)$ . The complex conjugate fields are denoted  $\chi_-^1(z), \chi_-^2(z)$ , such that

$$\chi_+^i(z)\chi_-^j(w) \sim \frac{\delta_{ij}}{z-w}, \quad i, j \in \{1, 2\},$$

while the antiholomorphic counterparts are denoted  $\bar{\chi}_\pm^1(\bar{z}), \bar{\chi}_\pm^2(\bar{z})$ . The corresponding holomorphic - antiholomorphic combinations are more appropriate for our purposes,

$$\xi_1 := \frac{1}{2}(\chi_+^1 + \bar{\chi}_+^1), \quad \xi_2 := \frac{1}{2i}(\chi_+^1 - \bar{\chi}_+^1), \quad \xi_3 := \frac{1}{2}(\chi_+^2 + \bar{\chi}_+^2), \quad \xi_4 := \frac{1}{2i}(\chi_+^2 - \bar{\chi}_+^2).$$

Moreover, in every  $\mathbb{Z}_2$ -orbifold conformal field theory  $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$  on K3, there is a 16-dimensional space of twisted ground states, generated by fields  $T_{\vec{a}}$  in the Ramond-Ramond sector, where the label  $\vec{a} \in \mathbb{F}_2^4$  refers to the fixed point  $\vec{F}_{\vec{a}}$  as in (1.1) at which the respective field is localized. For ease of notation we denote by  $\tilde{T}_{\vec{a}}, \vec{a} \in \mathbb{F}_2^4$ , the (chiral, chiral) fields which the  $T_{\vec{a}}$  flow to under our



choice of spectral flow. Then the following is a list of 24 fields which generate the (chiral, chiral) algebra in every theory  $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$  on K3:

$$(A.1) \quad \xi_1 \xi_2 \xi_3 \xi_4, \quad \xi_i \xi_j \quad (1 \leq i < j \leq 4), \quad \mathbb{1}; \quad \widetilde{T}_{\vec{a}} \quad (\vec{a} \in \mathbb{F}_2^4),$$

where  $\mathbb{1}$  denotes the vacuum field, and where we may restrict our attention to the real vector space  $\mathcal{X}$  generated by these 24 fields. After truncation of the OPE to chiral primaries [LVW89], the fields listed in (A.1) form a closed vertex algebra  $\mathcal{X}$  over  $\mathbb{R}$ . Note that this very truncation makes the vertex algebra completely independent of all moduli.

We remark that the real and imaginary parts<sup>14</sup> of the four fields with  $U(1)$ -charges  $(2, 2)$ ,  $(2, 0)$ ,  $(0, 2)$ ,  $(0, 0)$  in (A.1),

$$(A.2) \quad \xi_1 \xi_2 \xi_3 \xi_4, \quad \xi_1 \xi_3 - \xi_2 \xi_4, \quad \xi_1 \xi_4 + \xi_2 \xi_3, \quad \mathbb{1},$$

remain invariant under every symmetry of  $\mathcal{C}$ . These fields are naturally identified with the cycles  $\pi_* v^T$ ,  $\Omega_1, \Omega_2$ ,  $\pi_* v_0^T \in \pi_* H_*(T, \mathbb{R})$  on K3, with  $\Omega_1, \Omega_2$  as in (1.2) and  $v^T, v_0^T$  generators of  $H_4(T, \mathbb{Z})$  and  $H_0(T, \mathbb{Z})$  such that  $\langle v^T, v_0^T \rangle = 1$ . The invariance of  $\Omega_1, \Omega_2$  under symmetries means that in a given geometric interpretation, one restricts attention to symplectic automorphisms (see [TW11] for further details). In the description of the moduli space of SCFTs on a K3 surface  $X$  of [AM94], our SCFT  $\mathcal{C}$  is specified by the relative position of a positive definite fourplane in  $H^*(X, \mathbb{R})$  with respect to  $H^*(X, \mathbb{Z})$ . The two-forms  $\Omega_1, \Omega_2$  generate a two-dimensional subspace of that fourplane, while the choice of  $v^T$  and  $v_0^T$  amounts to the choice of a geometric interpretation of the toroidal theory  $\mathcal{T}$  which induces a natural geometric interpretation of its  $\mathbb{Z}_2$ -orbifold  $\mathcal{C}$  (see [NW01, Wen01]). Here, the four fields in (A.2) are the real and imaginary parts of the images of the four charged Ramond-Ramond ground states under our choice of spectral flow. These four Ramond-Ramond ground states also furnish a fourplane that can be used to describe the moduli space of superconformal field theories on K3 [NW01]. Note however that the fourplane of [AM94] is not the one generated by the four vectors in (A.2).

The vector space  $\mathcal{X}$  can be identified with the real K3 homology  $H_*(X, \mathbb{R})$ , where the  $\xi_i \xi_j$  with  $1 \leq i < j \leq 4$  are mapped to our generators  $e_i \vee e_j$  of  $\pi_* H_2(T, \mathbb{R})$ , and the  $\widetilde{T}_{\vec{a}}$  are in 1:1 correspondence with the cycles  $E_{\vec{a}}$  that arise from the minimal resolution of  $T/\mathbb{Z}_2$  (see [NW03] for the subtleties in this identification, due to the B-field that is induced by orbifolding).

One may, in addition, introduce the analog of the lattice of integral homology for the vector space  $\mathcal{X}$ , thereby recovering the dependence on the moduli. To appreciate this, note that before truncation the OPE between twist fields  $T_{\vec{b}}$  and  $T_{\vec{b}'}$  with  $\vec{b}, \vec{b}' \in \mathbb{F}_2^4$  yields, to leading order, a primary field  $W_{\vec{b}-\vec{b}'}(z, \bar{z})$  which does depend on the moduli. This is best measured by means of the OPE between the free bosonic superpartners of the Dirac fermions  $\xi_1, \dots, \xi_4$  and  $W_{\vec{b}-\vec{b}'}(z, \bar{z})$ . For convenience of notation, we introduce real, holomorphic  $U(1)$ -currents  $j_1(z), \dots, j_4(z)$ , which arise as the superpartners of the real and the imaginary parts of  $2\chi_+^1(z)$ ,  $2\chi_+^2(z)$ , respectively, and note that the relevant OPE then has the form

$$j_k(z) W_{\vec{a}}(w, \bar{w}) \sim \frac{W_{\vec{a}}(w, \bar{w})}{z-w} \sum_{l=1}^4 a_l \lambda_k^l \quad \text{for } \vec{a} = (a_1, \dots, a_4) \in \mathbb{F}_2^4.$$

Here,  $\lambda_1^l, \dots, \lambda_4^l$  are the Euclidean coordinates of generators  $\vec{\lambda}_1, \dots, \vec{\lambda}_4$  of a rank 4 lattice  $\Lambda \subset \mathbb{R}^4$ , if the underlying toroidal SCFT  $\mathcal{T}$  has a geometric interpretation on the torus  $T = \mathbb{R}^4/\Lambda$ , where we identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  as usual. We observe that in the truncation procedure yielding the (chiral, chiral) algebra  $\mathcal{X}$  of (A.1), the moduli-dependent fields  $W_{\vec{a}}(z, \bar{z})$  are projected to zero, and therefore the dependence on the moduli disappears from  $\mathcal{X}$ . However, one may introduce new fields

$$(A.3) \quad J_k(z) := \sum_{l=1}^4 \mu_k^l j_l(z) \quad \text{for } k \in \{1, \dots, 4\},$$

<sup>14</sup>Here and in the following, for a field  $\eta \in \mathbb{C} \otimes \mathcal{X}$  with  $\eta = \eta_1 + i\eta_2$  and  $\eta_1, \eta_2 \in \mathcal{X}$ , we call  $\eta_1, \eta_2$  the real and the imaginary part of  $\eta$ .

where  $\mu_1^l, \dots, \mu_4^l$  are the Euclidean coordinates of the basis  $\vec{\mu}_1, \dots, \vec{\mu}_4$  dual<sup>15</sup> to  $\vec{\lambda}_1, \dots, \vec{\lambda}_4$ , such that the OPEs with the fields  $W_{\vec{a}}(w, \bar{w})$  take the standardized ‘‘integral’’ form

$$(A.4) \quad J_k(z)W_{\vec{a}}(w, \bar{w}) \sim \frac{a_k}{z-w}W_{\vec{a}}(w, \bar{w}), \quad k \in \{1, \dots, 4\}.$$

The fermionic superpartners  $\tilde{\Psi}_1(z), \dots, \tilde{\Psi}_4(z)$  of the new fields  $J_1(z), \dots, J_4(z)$  and their antiholomorphic counterparts  $\bar{\Psi}_1(\bar{z}), \dots, \bar{\Psi}_4(\bar{z})$  yield a lattice with generators

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_3\tilde{\Psi}_4, \tilde{\Psi}_k\tilde{\Psi}_l\bar{\Psi}_m\bar{\Psi}_n, \tilde{\Psi}_k\bar{\Psi}_l, \dots$$

over  $\mathbb{Z}$ . However, to determine a lattice which plays the role of the integral homology of the Kummer surface  $X$ , one needs to recall that the identification<sup>16</sup> of  $\mathcal{X}$  with  $H^*(X, \mathbb{R})$  rests on the correspondence  $\chi_+^k \leftrightarrow dz_k, \bar{\chi}_+^k \leftrightarrow d\bar{z}_k$  for  $k \in \{1, 2\}$ , with local holomorphic coordinates  $z_1, z_2$  on  $X$ . This correspondence holds exactly on flat manifolds and in a large radius limit [Wit82, LVW89]. Hence at large radii, the *real*<sup>17</sup> fermionic fields  $\tilde{\Psi}_k$  are identified with the  $\bar{\Psi}_k$ , and thus with

$$\Psi_k := \sum_{l=1}^4 \mu_k^l \xi_l \quad \text{for } k \in \{1, \dots, 4\}.$$

This leaves us with the lattice  $\mathcal{Y}^{\mathbb{Z}}$  generated over  $\mathbb{Z}$  by

$$\Psi_1\Psi_2\Psi_3\Psi_4, \Psi_i\Psi_j, (1 \leq i < j \leq 4), \mathbb{1},$$

which is the analog of the lattice  $\pi_*H_*(T, \mathbb{Z}) \subset H_*(X, \mathbb{R})$ . Using the twist fields  $\tilde{T}_{\vec{a}}, \vec{a} \in \mathbb{F}_2^4$ , as additional generators that correspond to the vectors  $E_{\vec{a}}$  in the Kummer lattice, and then performing the usual gluing procedure, one obtains a lattice  $\mathcal{X}^{\mathbb{Z}}$  which can be identified with  $H_*(X, \mathbb{Z}) \subset H_*(X, \mathbb{R})$ . In particular, the relative position of  $\mathcal{X}^{\mathbb{Z}}$  with respect to the basis (A.1) of  $\mathcal{X}$  determines the respective point in the moduli space. For the SCFT associated with the square Kummer surface<sup>18</sup>, we can choose the eight fields  $\xi_1\xi_2\xi_3\xi_4, \xi_i\xi_j$  ( $1 \leq i < j \leq 4$ ),  $\mathbb{1}$  as generators of the lattice  $\mathcal{Y}^{\mathbb{Z}}$ .

Now note that each symmetry of a Kummer surface  $X_{\Lambda, \omega_0}$  as studied in our work induces a symmetry of a SCFT  $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ , with  $\mathcal{T}$  a toroidal theory<sup>19</sup> associated with the torus  $\mathbb{R}^4/\Lambda$ . By construction, our geometric symmetry groups  $G$  enjoy an induced action as groups of lattice automorphisms on the lattice  $\mathcal{X}^{\mathbb{Z}}$ . By definition, the symmetries of a CFT are compatible with all OPEs, hence they must in particular leave the standardized OPEs (A.4) invariant. Since our symmetries are induced by geometric symmetries of the toroidal theory  $\mathcal{T}$ , they act linearly on the  $J_k(z)$  and they permute the fields  $\pm W_{\vec{a}}(z, \bar{z})$ . It follows that such symmetries act as lattice automorphisms on the lattice generated by the  $J_k(z)$ . The same thus holds for the lattice generated by their superpartners  $\Psi_k(z)$  and for the lattice  $\mathcal{Y}^{\mathbb{Z}}$  mentioned above. Since our symmetries also permute the twist fields  $\pm \tilde{T}_{\vec{a}}$  amongst each other in a manner compatible with gluing, altogether it follows that they must act as automorphisms of the lattice  $\mathcal{X}^{\mathbb{Z}}$ . By the above, the vector space  $\mathcal{X}$  can be identified with the K3 homology, and  $\mathcal{X}^{\mathbb{Z}}$  can be identified with the integral homology. In particular, the lattice  $\mathcal{X}^{\mathbb{Z}}$  possesses a sublattice  $\mathcal{X}_G^{\mathbb{Z}}$  which can be identified with the lattice  $M_G$  that is so crucial to our construction, see Def. 2.2. The action of  $G$  on  $\mathcal{X}_G^{\mathbb{Z}}$  bears all relevant information about the  $G$ -action on our SCFT. Our construction hence realizes the very representation of  $G$  on  $\mathcal{X}$  in terms of the action of a subgroup  $G$  of  $M_{24}$  on the Niemeier lattice

<sup>15</sup>Here, we identify  $\mathbb{R}^4 \cong (\mathbb{R}^4)^*$  by means of the standard Euclidean scalar product.

<sup>16</sup>From [LVW89], we obtain an immediate identification with cohomology, which however is equivalent to homology by Poincaré duality.

<sup>17</sup>For open strings, one can view  $\chi_+^k$  and its antiholomorphic partner  $\bar{\chi}_+^k$  as complex conjugates, where the left and right modes combine into standing waves. In this language, we are simply reviewing the emergence of charge lattices for D-branes.

<sup>18</sup>with vanishing B-field on the underlying toroidal theory

<sup>19</sup>This leaves a choice of the B-field  $B_{\mathcal{T}}$  in the toroidal theory  $\mathcal{T}$ , which must be invariant under our symmetry; of course  $B_{\mathcal{T}} = 0$  is always admissible.

$N$ . In other words, our Niemeier marking describes precisely the action of the relevant symmetry groups on the (chiral, chiral) algebra.

Certainly from the description of the moduli space of SCFTs in terms of cohomological data [AM94, NW01], we are lead to expect that the role of the (chiral, chiral) algebra  $\mathcal{X}$  along with the lattice  $\mathcal{X}^{\mathbb{Z}}$  in its underlying vector space generalizes to arbitrary K3 theories.

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CENTRE FOR PARTICLE THEORY, DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, DURHAM, DH1 3LE, U.K.

*E-mail address:* [anne.taormina@durham.ac.uk](mailto:anne.taormina@durham.ac.uk)

MATHEMATICS INSTITUTE, UNIVERSITY OF FREIBURG, D-79104 FREIBURG, GERMANY.

*E-mail address:* [katrin.wendland@math.uni-freiburg.de](mailto:katrin.wendland@math.uni-freiburg.de)