# Complex hyperbolic free groups with many parabolic elements

John R Parker and Pierre Will

Abstract. We consider in this work representations of the of the fundamental group of the 3-punctured sphere in $\mathrm{PU}(2,1)$ such that the boundary loops are mapped to $\mathrm{PU}(2,1)$. We provide a system of coordinates on the corresponding representation variety, and analyse more specifically those representations corresponding to subgroups of $(3,3,\infty)$-groups. In particular we prove that it is possible to construct representations of the free group of rank two $\langle a,b\rangle$ in $\mathrm{PU}(2,1)$ for which $a$, $b$, $ab$, $ab^{-1}$, $ab^2$, $a^2b$ and $[a,b]$ all are mapped to parabolics.

## 1. Introduction

In this paper we consider representations of $F_2 = \langle a, b \,|\, \rangle$, the free group of rank two, into $\mathrm{SU}(2,1)$. The latter group is a three-fold covering of $\mathrm{PU}(2,1)$, which is the holomorphic isometry group of complex hyperbolic two-space $\mathbf{H}^2_{\mathbb{C}}$. Specifically, we consider the deformation space of such representations, that is the space of conjugacy classes of representations:

$$\mathcal{R} = \mathrm{Hom}\big(F_2, \mathrm{SU}(2,1)\big)//\mathrm{SU}(2,1).$$

It is not hard to see that the dimension of this space is the same as that of $\mathrm{SU}(2,1)$, namely four complex dimensions or eight real dimensions. We will be particularly interested in those representations with many parabolic elements. The locus of points in $\mathcal{R}$ where a given group element is parabolic is an algebraic real hypersurface.

We will very often use the alternative presentation $F_2 = \langle a, b, c \,|\, abc = 1\rangle$, which gives an identification of $F_2$ with the fundamental group of the 3-holed sphere, the generators corresponding to the three peripheral loops. We will be especially interested in representations $\rho \in \mathcal{R}$ for which $A = \rho(a)$, $B = \rho(b)$ and $C = \rho(c)$ are all parabolic. We say that such a representation of $F_2$ to $\mathrm{SU}(2,1)$ is *parabolic*. Viewing $F_2$ as the fundamental group of the three-holed sphere, parabolic representations map peripheral loops to parabolic maps. It is a well known fact that there is only one such representation in $\mathrm{PSL}(2,\mathbb{C})$ up to conjugacy. We will describe here the corresponding deformation space for $\mathrm{SU}(2,1)$. In particular, the conditions that $\rho(a)$, $\rho(b)$ and $\rho(c)$ are parabolic are independent and each give a single real equation. Since $\mathcal{R}$ has (real) dimension eight, the space of parabolic representations has dimension five.

Before giving our main results, we now indicate our motivation. There is a beautiful description of the $\mathrm{SU}(2,1)$ representation space of closed surface groups due to Goldman [**Go1, Go2**], Toledo [**T**] and Xia [**X**]. Of particular interest are *complex hyperbolic quasi-Fuchsian* representations of a surface group to $\mathrm{SU}(2,1)$; see Parker-Platis [**PP**] for a survey on this topic. In particular, Parker and Platis, Problem 6.2 of [**PP**], ask whether the boundary of complex hyperbolic quasi-Fuchsian space comprises representations with parabolic elements and they ask which parabolic maps can arise. We can consider a decomposition of the surface into three-holed spheres and then allow the three boundary curves to be pinched, so they are represented by parabolic elements. The fundamental group of a three holed sphere is a free group on two generators $F_2 = \langle a, b, c \,|\, abc = 1\rangle$. The condition that the three boundary curves are pinched is exactly that $A = \rho(a)$, $B = \rho(b)$ and $C = \rho(c)$ should all be parabolic.

If $C = (AB)^{-1}$ is parabolic then, of course, the product $BA$ is parabolic as well. The fixed points $p_A$, $p_B$, $p_{AB}$ and $p_{BA}$ of $A$, $B$, $AB$ and $BA$ give an ideal tetrahedron in $\mathbf{H}^2_{\mathbb{C}}$ (an ordered quadruple of boundary

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points). The shape of the tetrahedron $\tau_\rho = (p_A, p_B, p_{AB}, p_{BA})$ is a conjugacy invariant of the representation $\rho$ that we are going to use to give a coordinate system on the family of conjugacy classes of representations. Moreover the shape of a tetrahedron $\tau_\rho$ for a parabolic representation $\rho$ can not be arbitrary. Indeed we prove that if $\rho$ is a parabolic representation of $F_2$ to $\mathrm{SU}(2,1)$ then $\tau_\rho$ is *balanced*. An ideal tetrahedron $(p_1, p_2, p_3, p_4)$ is balanced when $p_3$ and $p_4$ are mapped to the same point by the orthogonal projection onto the geodesic connecting $p_1$ and $p_2$. To see this, we connect the shape of the tetrahedron to the conjugacy classes of $\rho(a)$, $\rho(b)$ and $\rho(ab)$ via the complex cross-ratio $\mathbb{X}(p_A, p_B, p_{AB}, p_{BA})$ (see [**KR**]). More precisely, we prove in Corollary 2.6 that when $\rho$ is parabolic we have:

$$\mathbb{X}(p_A, p_B, p_{AB}, p_{BA}) = \lambda_A\lambda_B\lambda_C, \tag{1.1}$$

where $C = (AB)^{-1}$ and $\lambda_A$, $\lambda_B$ and $\lambda_C$ are respectively the eigenvalues associated to the boundary fixed points of $A$, $B$ and $C$. As $A$, $B$ and $C$ are parabolic, these eigenvalues all have unit modulus, which implies that the cross-ratio also has unit modulus. This condition is equivalent to saying that the tetrahedron $\tau_\rho$ is balanced, as proved in Section 2.3.

The next question is the converse. Given a balanced ideal tetrahedron $\tau$, and given three unit complex numbers $\lambda_A$, $\lambda_B$ and $\lambda_C$ such that (1.1) holds, can we construct a parabolic representation $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$ such that $\tau = \tau_\rho$ as before? The answer is yes, if we allow that $A$, $B$ and $C$ may be parabolic or complex reflections. This ambiguity comes from the fact that an isometry having a boundary fixed point with unit modulus eigenvalue can be either parabolic or a complex reflection (see section 2.1). This is Proposition 3.2.

We focus next on the case where the three (unit modulus) eigenvalues $\lambda_A$, $\lambda_B$ and $\lambda_C$ all are equal. From (1.1) they are necessarily all the same cube root of the cross ratio. We show that such a representation admits a three fold symmetry. In particular, it is a subgroup of a $(3, 3, \infty)$ group generated by two regular elliptic maps $J_1$ and $J_2$ or order 3 whose product $J_1J_2$ is parabolic. Specifically, we prove (Theorem 4.2):

THEOREM. *Suppose that* $\rho : F_2 = \langle a, b, c \,|\, abc = id\rangle \longrightarrow \mathrm{SU}(2,1)$ *has the property that* $\rho(a)$, $\rho(b)$, $\rho(c)$ *are all parabolic and have the same eigenvalues. Then* $\rho(F_2)$ *is an index 3 subgroup of a* $\mathrm{SU}(2,1)$ *representation of the* $(3, 3, \infty)$ *group.*

This leads to our main result connecting the representation to geometry of complex hyperbolic space, (Theorem 4.6):

THEOREM. *There is a bijection between the set of* $\mathrm{PU}(2,1)$*-orbits of non-degenerate balanced ideal tetrahedra, and the set of* $\mathrm{PU}(2,1)$*-conjugacy classes of* $(3, 3, \infty)$ *groups in* $\mathrm{PU}(2,1)$.

Using a normalisation of balanced tetrahedra, we obtain an explicit parametrisation of the order 3 generators of a $(3, 3, \infty)$ group. Next, we investigate when more group elements are parabolic. In particular, we can prove (Corollary 4.8):

THEOREM. *There is a one parameter family of groups generated $A$ and $B$ in* $\mathrm{PU}(2,1)$ *so that $A$, $B$, $AB$, $AB^{-1}$, $AB^2$, $A^2B$ and $[A, B]$ are all parabolic.*

It would be very interesting to find out whether any (or all) of these representations are discrete and free, and also whether or not it is possible to find any more parabolic elements.

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## 2. Fixed point tetrahedra of thrice punctured sphere groups

We refer the reader to [**ChG, Go3, P1**] for basic material on the complex hyperbolic space. We will denote by $\mathbb{A}$ and $\mathbb{X}$ respectively the Cartan invariant (see Chapter 7 of [**Go3**]) and the complex cross-ratio (see [**KR**], and Chapter 7 of [**Go3**]).

**2.1. Conjugacy classes in** $\mathrm{PU}(2,1)$**.** We recall that the group of holomorphic isometries of the complex hyperbolic is $\mathrm{PU}(2,1)$. Elements of $\mathrm{PU}(2,1)$ are classified by the usual trichotomy: loxodromic, elliptic and parabolic isometries. This trichotomy may be refined in various ways. In particular, an elliptic isometry $A$ is called *regular* if and only if any lift $\mathbf{A}$ to $\mathrm{SU}(2,1)$ has three pairwise distinct eigenvalues. Whenever an elliptic isometry is not regular it is called a *complex reflection*. The set of fixed points in $\mathbf{H}^2_{\mathbb{C}}$ of a complex reflection can be either a point or complex line (see [**Go3**] for details). Note that a complex reflection does not necessarily have finite order, in contrast to the usual terminology in real spaces.

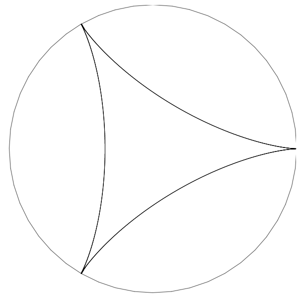


FIGURE 1. The null locus of $f$ and the circle $\{|z| = 3\}$.

As in the classical cases of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ it is possible to detect the types using the trace of a lift of an element of $\mathrm{PU}(2,1)$ to $\mathrm{SU}(2,1)$. However certain subtleties arise here that we would like to describe as they will play a role in our work. Let us first recall the trace classification of isometries (Theorem 6.4.2 of [**Go3**]).

PROPOSITION 2.1. *Let $A$ be a non-trivial element of* $\mathrm{PU}(2,1)$ *and* $\mathbf{A}$ *a lift of it to* $\mathrm{SU}(2,1)$. *We denote by $f$ the polynomial function given by* $f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27$. *Then*

(1) *The isometry $A$ is loxodromic if and only if* $f(\mathrm{tr}\mathbf{A}) > 0$.
(2) *It is regular elliptic if and only if* $f(\mathrm{tr}\mathbf{A}) < 0$.
(3) *It is parabolic or a complex reflection if and only if* $f(\mathrm{tr}\mathbf{A}) = 0$.

A parabolic isometry $P$ is called unipotent whenever it admits a unipotent lift $\mathbf{P} \in \mathrm{SU}(2,1)$. There are two types of unipotent parabolics, namely 2-step or 3-step unipotents, depending on the nilpotency index of $\mathbf{P} - I$ (moreover, a 2-step unipotent map is not conjugate to its own inverse, and so there are three conjugacy classes). A non-unipotent parabolic map is called *screw-parabolic*. The spectrum of the lift of a parabolic is always of the kind $\{e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha}\}$ for some $\alpha \in \mathbb{R}$. When $\alpha = 0$, the parabolic is unipotent. Therefore the traces of parabolic isometries form a curve in $\mathbb{C}$, given by $\{2e^{i\alpha} + e^{-2i\alpha}, \alpha \in \mathbb{R}\}$, which is depicted in Figure 1. We will often refer to this curve as *the deltoid*. In view of Proposition 2.1, this curve is the zero-locus of the polynomial $f$. However Proposition 2.1 tells us that if $f(\mathrm{tr}\mathbf{A}) = 0$, then we need more information to know the type of the isometry $A$, as it could be a complex reflection. This can be done by using the fact that lifts of complex reflections are semi-simple whereas those of parabolics are not (see the proof of Proposition 3.1).

### 2.2. Fixed points, eigenvalues, cross-ratios.

DEFINITION 2.2. We will call *parabolic* any representation $\rho : F_2 = \langle a, b, c \,|\, abc = id \rangle \longrightarrow \mathrm{PU}(2,1)$ which maps $a$, $b$ and $c$ (thus $ab$ and $ba$) to parabolic isometries. We will denote by $\mathcal{P}$ the set of parabolic representations of $F_2$.

Given a parabolic representation $\rho$, we will denote by $A$, $B$, $AB = C^{-1}$ and $BA$ the images under $\rho$ of $a$, $b$, $ab$ and $ba$, and by $p_A$, $p_B$, $p_{AB} = p_C$ and $p_{BA}$ their boundary fixed points.

DEFINITION 2.3. Let $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$ be a parabolic representation. We will call *fixed point tetrahedron of* $\rho$ and denote by $\tau_\rho$ the ideal tetrahedron $(p_A, p_B, p_{AB}, p_{BA})$.

If $p \in \mathbf{H}^2_{\mathbb{C}}$, in particular if $p$ is a fixed point of $A \in \mathrm{PU}(2,1)$, we will denote by the same letter in bold font $\mathbf{p}$ a lift of $p$ to $\mathbb{C}^3$.

DEFINITION 2.4. If $A \in \mathrm{SU}(2,1)$ projectively fixes $p_A$, we say that $\lambda_A$ is the eigenvalue of $A$ associated to $p$ if $A\mathbf{p}_A = \lambda_A \mathbf{p}_A$.

The following lemma provides an identity connecting eigenvalues with cross ratios and angular invariants of fixed points that will play an important role in our discussion. We refer the reader to [**KR**] or to Chapter 7 of [**Go3**] for the basic definitions concerning the Korányi-Riemann cross-ratio of four points, which we will denote by $\mathbb{X}$ and the Cartan angular invariant of three points, which we denote by $\mathbb{A}$ .

LEMMA 2.5. *Let $A$ and $B$ be in $\mathrm{PU}(2,1)$. Let $p_A$ and $p_B$ be fixed points of $A$ and $B$ with eigenvalues $\lambda_A$ and $\lambda_B$. Let $p_{AB}$ and $p_{BA}$ be fixed points of $AB$ and $BA$ such that $Ap_{BA}=p_{AB}$. Denote by $\lambda_{AB}$ the corresponding eigenvalue of $AB$. Assume that the four points $(p_A,p_B,p_{AB},p_{BA})$ are pairwise distinct.*

(1) *The eigenvalues of $AB$ and $BA$ associated with $p_{AB}$ and $p_{BA}$ are equal.*
(2) *The four points $p_A$, $p_B$, $p_{AB}$, $p_{BA}$ satisfy the following cross-ratio identity.*

$$\mathbb{X}(p_A,p_B,p_{AB},p_{BA})=\frac{1}{\overline{\lambda}_A\overline{\lambda}_B\lambda_{AB}} \tag{2.1}$$

(3) *Taking the principal determination of the argument, we have*

$$arg\left(\mathbb{X}(p_A,p_B,p_{AB},p_{BA})\right)=\mathbb{A}(p_A,p_B,p_{AB})-\mathbb{A}(p_A,p_B,p_{BA}) \pmod{2\pi}.$$

The last part of Lemma 2.5 has nothing to do with $A$ and $B$, and is a general property of ideal tetrahedra in $\mathbf{H}^2_{\mathbb{C}}$. One should be careful to write this equality only up to a multiple of $2\pi$, as noted by Cunha and Gusevkii in [**CuG**].

PROOF. The first part of the Lemma is a direct consequence of $AB=A(BA)A^{-1}$. Because $Ap_{BA}=p_{AB}$ and $Bp_{AB}=p_{BA}$, there exists complex numbers $\mu$ and $\nu$ such that

$$A\mathbf{p}_{BA}=\mu\mathbf{p}_{AB} \text{ and } B\mathbf{p}_{AB}=\nu\mathbf{p}_{BA}.$$

But any lift $\mathbf{p}_{AB}$ of $p_{AB}$ satisfies $AB\mathbf{p}_{AB}=\lambda_{AB}\mathbf{p}_{AB}$. This implies that $\lambda_{AB}=\mu\nu$. Let us compute the cross ratio. We use the fact that $A$ and $B$ preserve the Hermitian form.

$$\begin{aligned}\mathbb{X}(p_A,p_B,p_{AB},p_{BA}) &= \frac{\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle\langle\mathbf{p}_{BA},\mathbf{p}_B\rangle}{\langle\mathbf{p}_{AB},\mathbf{p}_B\rangle\langle\mathbf{p}_{BA},\mathbf{p}_A\rangle}\\ &= \frac{\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle\langle\mathbf{p}_{BA},\mathbf{p}_B\rangle}{\langle B\mathbf{p}_{AB},B\mathbf{p}_B\rangle\langle A\mathbf{p}_{BA},A\mathbf{p}_A\rangle}\\ &= \frac{\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle\langle\mathbf{p}_{BA},\mathbf{p}_B\rangle}{\overline{\lambda}_A\overline{\lambda}_B\mu\nu\langle\mathbf{p}_{BA},\mathbf{p}_B\rangle\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle}\\ &= \frac{1}{\overline{\lambda}_A\overline{\lambda}_B\lambda_{AB}}.\end{aligned}$$

Finally,

$$\mathbb{X}(p_A,p_B,p_{AB},p_{BA})=\frac{|\langle\mathbf{p}_{BA},\mathbf{p}_B\rangle|^2\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle\langle\mathbf{p}_A,\mathbf{p}_B\rangle\langle\mathbf{p}_B,\mathbf{p}_{AB}\rangle}{|\langle\mathbf{p}_{AB},\mathbf{p}_B\rangle|^2\langle\mathbf{p}_{BA},\mathbf{p}_A\rangle\langle\mathbf{p}_A,\mathbf{p}_B\rangle\langle\mathbf{p}_B,\mathbf{p}_{BA}\rangle}.$$

The result follows by taking argument on both sides since, by definition we have:

$$\begin{aligned}\mathbb{A}(p_A,p_B,p_{AB}) &= \arg\big(-\langle\mathbf{p}_{AB},\mathbf{p}_A\rangle\langle\mathbf{p}_A,\mathbf{p}_B\rangle\langle\mathbf{p}_B,\mathbf{p}_{AB}\rangle\big),\\ \mathbb{A}(p_A,p_B,p_{BA}) &= \arg\big(-\langle\mathbf{p}_{BA},\mathbf{p}_A\rangle\langle\mathbf{p}_A,\mathbf{p}_B\rangle\langle\mathbf{p}_B,\mathbf{p}_{BA}\rangle\big).\end{aligned}$$

□

Let us rephrase Lemma 2.5 for a parabolic representation.

COROLLARY 2.6. *Let $A$ and $B$ be two parabolic isometries such that $AB$ (and thus $BA$) are both parabolic with fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$ $p_A$, $p_B$, $p_{AB}$ and $p_{BA}$. Then*

(1) *The cross ratio $\mathbb{X}(p_A,p_B,p_{AB},p_{BA})$ has unit modulus.*
(2) *Moreover, setting $C=(AB)^{-1}$ and denoting by $\lambda_C$ the eigenvalue of $C$ associated with $p_{AB}$ then $\mathbb{X}(p_A,p_B,p_{AB},p_{BA})=\lambda_A\lambda_B\lambda_C$.*

PROOF. It is a direct consequence of $\lambda_C=\lambda_{AB}^{-1}$ and of the fact that eigenvalues of parabolics are unit complex numbers. □

### 2.3. Balanced ideal tetrahedra.

Definition 2.7. Let $\tau = (p_1, p_2, p_3, p_4)$ be an ideal tetrahedron and $\pi_{12}$ be the orthogonal projection onto the (real) geodesic $\gamma_{12} = (p_1p_2)$. We will say that $\tau$ is *balanced* whenever the images of $p_3$ and $p_4$ under $\pi_{12}$ are equal.

Definition 2.8. We denote by $\mathcal{B}$ the set of balanced ideal tetrahedra.

We will use the following choice for the cross ratio:

$$\mathbb{X}(p_1, p_2, p_3, p_4) = \frac{\langle \mathbf{p}_3, \mathbf{p}_1\rangle\langle \mathbf{p}_4, \mathbf{p}_2\rangle}{\langle \mathbf{p}_3, \mathbf{p}_2\rangle\langle \mathbf{p}_4, \mathbf{p}_1\rangle}.$$

Proposition 2.9. *An ideal tetrahedron $\tau$ is balanced if and only if the cross-ratio $\mathbb{X}(p_1, p_2, p_3, p_4)$ has unit modulus.*

An immediate corollary of Proposition 2.9 and Lemma 2.5 is:

Corollary 2.10. *Let $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$ be a parabolic representation. The tetrahedron $\tau_\rho$ is balanced.*

Proof. (Proposition 2.9.). Choose lifts $\mathbf{p}_1$ and $\mathbf{p}_2$ of $p_1$ and $p_2$ in such a way that $\langle \mathbf{p}_1, \mathbf{p}_2\rangle = -1$. Then the projection of a point $z$ in the closure of $\mathbf{H}^2_{\mathbb{C}}$ onto $\gamma_{12}$ is given by

$$\pi_{12}(z) = \sqrt{\frac{|\langle \mathbf{z}, \mathbf{p}_2\rangle|}{|\langle \mathbf{z}, \mathbf{p}_1\rangle|}}\mathbf{p}_1 + \sqrt{\frac{|\langle \mathbf{z}, \mathbf{p}_1\rangle|}{|\langle \mathbf{z}, \mathbf{p}_2\rangle|}}\mathbf{p}_2. \tag{2.2}$$

Note that this expression does not depend on the chosen lift for $z$. Therefore the condition $\pi_{12}(p_3) = \pi_{12}(p_4)$ is equivalent to the two relations obtained by identifying the $\mathbf{p}_1$ and $\mathbf{p}_2$ components of $\pi_{12}(p_3)$ and $\pi_{12}(p_4)$. This gives (after squaring both sides of the equality)

$$\frac{|\langle \mathbf{p}_3, \mathbf{p}_2\rangle|}{|\langle \mathbf{p}_3, \mathbf{p}_1\rangle|} = \frac{|\langle \mathbf{p}_4, \mathbf{p}_2\rangle|}{|\langle \mathbf{p}_4, \mathbf{p}_1\rangle|} \text{ and } \frac{|\langle \mathbf{p}_3, \mathbf{p}_1\rangle|}{|\langle \mathbf{p}_3, \mathbf{p}_2\rangle|} = \frac{|\langle \mathbf{p}_4, \mathbf{p}_1\rangle|}{|\langle \mathbf{p}_4, \mathbf{p}_2\rangle|}$$

These two relations are clearly both equivalent to

$$\left|\frac{\langle \mathbf{p}_3, \mathbf{p}_1\rangle\langle \mathbf{p}_4, \mathbf{p}_2\rangle}{\langle \mathbf{p}_3, \mathbf{p}_2\rangle\langle \mathbf{p}_4, \mathbf{p}_1\rangle}\right| = |\mathbb{X}(p_1, p_2, p_3, p_4)| = 1$$

□

By applying an element of $\mathrm{SU}(2,1)$ if necessary, we may assume that

$$\mathbf{p}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix}-e^{2i\theta}\\ \sqrt{2\cos(2\theta)}e^{i\theta-i\psi}\\ 1\end{bmatrix}, \quad \mathbf{p}_4 = \begin{bmatrix}-r^2e^{-2i\phi}\\ r\sqrt{2\cos(2\phi)}e^{-i\phi+i\psi}\\ 1\end{bmatrix} \tag{2.3}$$

where $r > 0$, $2\theta \in [-\pi/2, \pi/2]$, $2\phi \in [-\pi/2, \pi/2]$ and $\psi \in [0, \pi/2]$. The tetrahedron is completely determined up to $\mathrm{PU}(2,1)$ equivalence by the parameters $r$, $\theta$, $\phi$ and $\psi$. We want to now give an invariant interpretation of these parameters. First observe that $2\theta = \mathbb{A}(p_2, p_1, p_3)$ and $2\phi = \mathbb{A}(p_1, p_2, p_4)$.

Lemma 2.11. *In the above normalisation* (2.3)*, the tetrahedron $(p_1, p_2, p_3, p_4)$ is balanced if and only if $r = 1$.*

Proof. Computing the cross ratio in this case, we obtain $\mathbb{X}(p_1, p_2, p_3, p_4) = r^2e^{-2i\theta-2i\phi}$. □

Note that this implies that an ideal tetrahedron $(p_1, p_2, p_3, p_4)$ is balanced if and only if the two points $p_3$ and $p_4$ both lie on the boundary of a bisector $\mathcal{B}$ whose complex spine is the complex line spanned by $p_1$ and $p_2$ and whose real spine is a geodesic orthogonal to $(p_1p_2)$ (see Chapter 5 of [**Go3**] for definitions of these notions).

Definition 2.12. We denote by $\tau(\theta, \phi, \psi)$ the tetrahedron given by (2.3), where $r$ is replaced by 1 in $p_4$.

Definition 2.13. Let $(p_1, p_2, p_3, p_4)$ be an ideal tetrahedron for which neither of the triples $(p_1, p_2, p_3)$ and $(p_1, p_2, p_4)$ lie in a complex line . Denote by $c_{12}$ a polar vector to the complex line spanned by $p_1$ and $p_2$. The following quantity is well-defined and is called the *bending parameter*.

$$\mathbb{B}(p_1, p_2, p_3, p_4) = \mathbb{X}(p_4, p_3, p_1, c_{12}) \cdot \mathbb{X}(p_4, p_3, p_2, c_{12}) \tag{2.4}$$

To check that $\mathbb{B}$ is well-defined, note fist that it does not depend on the choice of lifts for the $p_i$'s, nor on the choice of $c_{12}$. Secondly, the Hermitian products involving $c_{12}$ in the two cross-ratios are $\langle c_{12}, p_3\rangle$ and $\langle c_{12}, p_4\rangle$ which are non-zero in view of the assumption made, therefore the two cross-ratios $\mathbb{X}(p_4, p_3, p_1, c_{12})$ and $\mathbb{X}(p_4, p_3, p_2, c_{12})$ are well-defined.

Example 1. In the normalised form given above by (2.3), we see that

$$\mathbb{B}(p_1, p_2, p_3, p_4) = \frac{\cos(2\theta)}{\cos(2\phi)} e^{4i\psi}.$$

Assume that $\mathbb{A}(p_2, p_1, p_3) = \pm\pi/2$ or $\mathbb{A}(p_1, p_2, p_4) = \pm\pi/2$. This means that $p_3$ or $p_4$ respectively lies on the complex line through $p_1$ and $p_2$. Using the standard form (2.3) we see that the middle entry of $\mathbf{p}_3$ or $\mathbf{p}_4$ is zero. Therefore the angle $\psi$ is not well defined in that case.

The following proposition is a straightforward consequence of the above normalisation.

Proposition 2.14. *A balanced tetrahedron $(p_1, p_2, p_3, p_4)$ such that neither of the triples $(p_1, p_2, p_3)$ and $(p_1, p_2, p_4)$ lie in a complex line is uniquely determined up to* PU(2,1) *by the three quantities $\mathbb{A}(p_1, p_2, p_3)$, $\mathbb{A}(p_1, p_2, p_4)$ and $\mathbb{B}(p_1, p_2, p_3, p_4)$.*

## 3. Constructing thrice punctured sphere groups from tetrahedra

We wish now to work in the converse direction: given a balanced ideal tetrahedron $(p_1, p_2, p_3, p_4)$, is it possible to construct a parabolic representation $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$, such that $\rho(a)$ fixes $p_1$, $\rho(b)$ fixes $p_2$, $\rho(ab)$ fixes $p_3$ and $\rho(ba)$ fixes $p_4$.

**3.1. Mappings of boundary points.** In this section we will consider configurations of distinct points on $\partial\mathbf{H}^2_{\mathbb{C}}$, and use them to construct maps in $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ with certain properties. Consider a matrix $A$ in SU(2,1). We know that the eigenvectors of $A$ in $V_-$ and $V_0$ correspond to fixed points of $A$ in $\mathbf{H}^2_{\mathbb{C}}$ and $\partial\mathbf{H}^2_{\mathbb{C}}$ respectively. We say that $p \in \partial\mathbf{H}^2_{\mathbb{C}}$ is a *neutral fixed point* of $A$ if the corresponding eigenvector $\mathbf{p}$ has an eigenvalue $\lambda$ with $|\lambda| = 1$. Note that a matrix $A$ with a neutral fixed point in $\partial\mathbf{H}^2_{\mathbb{C}}$ must be either parabolic or a complex reflection.

In particular, we consider triples of points $p$, $q$ and $r$ of $\partial\mathbf{H}^2_{\mathbb{C}}$. Our goal will be to show that there is a unique holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ which sends $q$ to $r$ and with $p$ as a neutral fixed point with a prescribed eigenvalue. Moreover, we will show how to determine when such an isometry is parabolic and when it is a complex reflection.

Proposition 3.1. *Let $p$, $q$, $r$ be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ and let $\lambda$ be a complex number of unit modulus. Then there exists a unique holomorphic isometry $A$ sending $q$ to $r$ and for which $p$ is a neutral fixed point with associated eigenvalue $\lambda$. Moreover,*

1. *If $\lambda^3 = -e^{2i\mathbb{A}(p,q,r)}$ and $p$, $q$ and $r$ do not lie in a complex line, then $A$ is elliptic.*
2. *Otherwise $A$ is parabolic.*

Proof. First, such an isometry is unique if it exists. Indeed, if there were two such isometries, say $f_1$ and $f_2$, then $f_1 \circ f_2^{-1}$ would fix both $p$ and $r$. Moreover the eigenvalue of $f_1 \circ f_2^{-1}$ associated with $p$ would be 1 (or a cube root of 1). Thus $f_1 \circ f_2^{-1}$ would be the identity.

To prove existence, let us fix lifts $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ for the three points $(p, q, r)$.

- Assume first that $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ is a basis, that is $(p, q, r)$ do not lie on a common complex line. The following matrix, written in the basis $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ has eigenvalue $\lambda$ associated to $\mathbf{p}$ and projectively maps $q$ to $r$.

$$M_1 = \begin{bmatrix} \lambda & 0 & \lambda\dfrac{\langle\mathbf{r},\mathbf{q}\rangle}{\langle\mathbf{p},\mathbf{q}\rangle} + \overline{\lambda}^2\dfrac{\langle\mathbf{r},\mathbf{p}\rangle\langle\mathbf{q},\mathbf{r}\rangle}{\langle\mathbf{p},\mathbf{r}\rangle\langle\mathbf{q},\mathbf{p}\rangle} \\ 0 & 0 & -\overline{\lambda}^2\dfrac{\langle\mathbf{r},\mathbf{p}\rangle}{\langle\mathbf{q},\mathbf{p}\rangle} \\ 0 & \lambda\dfrac{\langle\mathbf{q},\mathbf{p}\rangle}{\langle\mathbf{r},\mathbf{p}\rangle} & \lambda + \overline{\lambda}^2 \end{bmatrix}. \tag{3.1}$$

It is not hard to check that $M_1$ preserves the Hermitian form. Furthermore, this isometry is elliptic if and only if the matrix $M_1 - \lambda \cdot I$ has rank one. Now,

$$M_1 - \lambda \cdot I = \begin{bmatrix} 0 & 0 & \lambda\dfrac{\langle \mathbf{r},\mathbf{q}\rangle}{\langle \mathbf{p},\mathbf{q}\rangle} + \overline{\lambda}^2 \dfrac{\langle \mathbf{r},\mathbf{p}\rangle\langle \mathbf{q},\mathbf{r}\rangle}{\langle \mathbf{p},\mathbf{r}\rangle\langle \mathbf{q},\mathbf{p}\rangle} \\ 0 & -\lambda & -\overline{\lambda}^2 \dfrac{\langle \mathbf{r},\mathbf{p}\rangle}{\langle \mathbf{q},\mathbf{p}\rangle} \\ 0 & \lambda\dfrac{\langle \mathbf{q},\mathbf{p}\rangle}{\langle \mathbf{r},\mathbf{p}\rangle} & \overline{\lambda}^2. \end{bmatrix} \tag{3.2}$$

Since the bottom right $2 \times 2$ minor of $M_1 - \lambda \cdot I$ vanishes, we see that $M_1$ is elliptic if and only if the top right entry of $M_1 - \lambda \cdot I$ vanishes, which gives after a little rewriting

$$\lambda^3 = -\frac{\langle \mathbf{p},\mathbf{q}\rangle\langle \mathbf{q},\mathbf{r}\rangle\langle \mathbf{r},\mathbf{p}\rangle}{\langle \mathbf{p},\mathbf{r}\rangle\langle \mathbf{r},\mathbf{q}\rangle\langle \mathbf{q},\mathbf{r}\rangle} = -e^{2i\mathbb{A}(p,q,r)}.$$

- If $(\mathbf{p},\mathbf{q},\mathbf{r})$ is not a basis of $\mathbb{C}^3$, that is if $(p,q,r)$ lie on a complex line $L$, then any isometry fixing $p$ and mapping $q$ to $r$ preserves $L$. If $\mathbf{n}$ is polar to $L$, then $(\mathbf{p},\mathbf{n},\mathbf{q})$ is a basis of $\mathbb{C}^3$. In this basis, the vector $\mathbf{r}$ is given by

$$\mathbf{r} = \frac{\langle \mathbf{r},\mathbf{q}\rangle}{\langle \mathbf{p},\mathbf{q}\rangle}\mathbf{p} + \frac{\langle \mathbf{r},\mathbf{p}\rangle}{\langle \mathbf{q},\mathbf{p}\rangle}\mathbf{q}$$

The matrix $M_2$ given in (3.3) represents a holomorphic isometry mapping $q$ to $r$ and with $p$ a neutral fixed point:

$$M_2 = \begin{bmatrix} \lambda & 0 & \lambda\dfrac{\langle \mathbf{r},\mathbf{q}\rangle\langle \mathbf{q},\mathbf{p}\rangle}{\langle \mathbf{r},\mathbf{p}\rangle\langle \mathbf{p},\mathbf{q}\rangle} \\ 0 & 1/\lambda^2 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{3.3}$$

Because the three points $p$, $q$, $r$ are distinct, the top-right coefficient is never zero thus $M_2 - \lambda \cdot I$ always has rank 2. Hence $M_2$ represents a parabolic isometry.

□

As a direct application of Proposition 3.1, we can associate parabolic (or boundary elliptic) representations to balanced ideal tetrahedra.

Proposition 3.2. *Let $(p_1,p_2,p_3,p_4)$ be a balanced ideal tetrahedron and let $\lambda_A$ and $\lambda_B$ be two complex numbers of modulus 1. There exists a unique representation $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$ such that*

- *$A = \rho(a)$ fixes $p_1$ with eigenvalue $\lambda_A$ and $B = \rho(b)$ fixes $p_2$ with eigenvalue $\lambda_B$.*
- *$AB = \rho(ab)$ and $BA = \rho(ba)$ are parabolic or boundary elliptic and fix respectively $p_3$ and $p_4$.*

Proof. Define $A = \rho(a)$ and $B = \rho(b)$ using Proposition 3.1: $A$ is the unique isometry with fixing $p_1$ with eigenvalue $\lambda_A$ and mapping $p_4$ to $p_3$, and $B$ is the unique isometry fixing $p_2$ with eigenvalue $\lambda_B$ and mapping $p_3$ to $p_4$. From this definition, we see that $AB$ fixes $p_3$ and $BA$ fixes $p_4$. It remains to check that the eigenvalue $\lambda_3$ of $AB$ associated to $p_3$ (which is the same as the eigenvalue $\lambda_4$ of $BA$ associated to $p_4$) has unit modulus. From Lemma 2.5 we have

$$\lambda_3 = \frac{\lambda_A\lambda_B}{\mathbb{X}(p_1,p_2,p_3,p_4)}.$$

Since the tetrahedron is balanced, we have $\left|\mathbb{X}(p_1,p_2,p_3,p_4)\right| = 1$ and the result follows. □

Remark 1. The function mapping $(\tau,\lambda_A,\lambda_B)$ to the representation $\rho$ given by Proposition 3.2 is not a bijection. Indeed in the case where one of $\rho(a)$, $\rho(b)$ or $\rho(c)$ is a complex reflections it does not have a unique fixed point, and so different ideal tetrahedra can give the same representation.

**3.2. A specific normalisation.** We now give the parabolic representation of $F_2$ in $\mathrm{PU}(2,1)$ corresponding the balanced tetrahedron $\tau(\theta,\phi,\psi)$ given in Definition 2.12. This means that

$$\mathbf{p}_A = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{p}_B = \begin{bmatrix}0\\0\\1\end{bmatrix},\quad \mathbf{p}_{AB} = \begin{bmatrix}-e^{2i\theta}\\ \sqrt{2\cos(2\theta)}e^{i\theta-i\psi}\\ 1\end{bmatrix},\quad \mathbf{p}_{BA} = \begin{bmatrix}-e^{-2i\phi}\\ \sqrt{2\cos(2\phi)}e^{-i\phi+i\psi}\\ 1\end{bmatrix} \tag{3.4}$$

where

$$\begin{aligned} 2\theta &= \mathbb{A}(p_B,p_A,p_{AB}) \in [-\pi/2,\pi/2] \\ 2\phi &= \mathbb{A}(p_A,p_B,p_{BA}) \in [-\pi/2,\pi/2] \\ 4\psi &= \arg\left(\mathbb{B}(p_A,p_B,p_{AB},p_{BA})\right) \in [0,2\pi). \end{aligned}$$

Writing $c_1 = \sqrt{2\cos(2\theta)}$ and $c_2 = \sqrt{2\cos(2\phi)}$, the matrices $A$ and $B$ in $\mathrm{SU}(2,1)$ giving the parabolic representation are

$$A = \begin{bmatrix}\lambda_A & -\overline{\lambda}_A^2 c_1\, e^{-i\theta+i\psi} + \lambda_A c_2\, e^{i\phi-i\psi} & -\lambda_A e^{2i\theta} - \lambda_A e^{2i\phi} + \overline{\lambda}_A^2 c_1 c_2\, e^{-i\theta-i\phi+2i\psi} \\ 0 & \overline{\lambda}_A^2 & \lambda_A c_1\, e^{i\theta-i\psi} - \overline{\lambda}_A^2 c_2\, e^{-i\phi+i\psi} \\ 0 & 0 & \lambda_A\end{bmatrix}, \tag{3.5}$$

$$B = \begin{bmatrix}\lambda_B & 0 & 0 \\ \overline{\lambda}_B^2 c_1\, e^{-i\theta-i\psi} - \lambda_B c_2\, e^{i\phi+i\psi} & \overline{\lambda}_B^2 & 0 \\ -\lambda_B e^{2i\theta} - \lambda_B e^{2i\phi} + \overline{\lambda}_B^2 c_1 c_2\, e^{-i\theta-i\phi-2i\psi} & -\lambda_B c_1\, e^{i\theta+i\psi} + \overline{\lambda}_B^2 c_2\, e^{-i\phi-i\psi} & \lambda_B\end{bmatrix}. \tag{3.6}$$

## 4. Thrice punctured sphere groups with a three-fold symmetry.

In this section we restrict our attention to the case where there is a three-fold symmetry of the parabolic representation $\rho(F_2) = \langle A, B\rangle$. Consider the eigenvalues $\lambda_A$, $\lambda_B$ and $\lambda_C$ of $A$, $B$ and $C = B^{-1}A^{-1}$ at $p_A$, $p_B$ and $p_{AB}$. Specifically, we show that if these are equal then $\langle A, B\rangle$ is an index 3 subgroup of a $(3,3,\infty)$ group $\langle J_1, J_2\rangle$. Moreover, this can be interpreted geometrically, for there is a bijection between $(3,3,\infty)$ groups and balanced ideal tetrahedra.

We go on to give conditions under which further elements of this group are pinched, that is they have become parabolic. In doing so, we rule out the case where they are complex reflections. Therefore pinching a single element is equivalent to satisfying a single real algebraic equation (Proposition 2.1) this defines a real hypersurface. Our main result is that for the $(3,3,\infty)$ group it is possible to simultaneously pinch $J_1J_2^{-1}$ and $[J_1, J_2]$. Indeed there is a 1 parameter way of doing this. This means that for the thrice punctured sphere group, it is possible to pinch four conjugacy classes in addition to the three boundary curves.

This is in strong contrast to the classical case. Every thrice punctured sphere groups in $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$ admits a three-fold symmetry, that is, it is an index three subgroup of a $(3,3,\infty)$ group. However, it is not possible to make any more elements of this group parabolic.

### 4.1. Existence of a three-fold symmetry.

Definition 4.1. Consider a balanced tetrahedron with vertices $p_A$, $p_B$, $p_{AB}$ and $p_{BA}$, all lying in $\partial\mathbf{H}^2_{\mathbb{C}}$. We define the following elements of $\mathrm{PU}(2,1)$ (see Figure 2):

- $J_1$ is the order 3 isometry cyclically permuting $p_B$, $p_A$ and $p_{AB}$.
- $J_2$ is the order 3 isometry cyclically permuting $p_A$, $p_B$ and $p_{BA}$.

When these triples of points do not lie in a complex line, such an isometry is unique.

Using the lifts of the vertices given in (3.4), the maps $J_1$ and $J_2$ from Definition 4.1 are given as matrices in $\mathrm{SU}(2,1)$ by

$$J_1 = \begin{bmatrix} e^{4i\theta/3} & \sqrt{2\cos(2\theta)}e^{i\theta/3+i\psi} & -e^{-2i\theta/3} \\ -\sqrt{2\cos(2\theta)}e^{i\theta/3-i\psi} & -e^{4i\theta/3} & 0 \\ -e^{-2i\theta/3} & 0 & 0\end{bmatrix}, \tag{4.1}$$

$$J_2 = \begin{bmatrix} 0 & 0 & -e^{-2i\phi/3} \\ 0 & -e^{4i\phi/3} & \sqrt{2\cos(2\phi)}e^{i\phi/3+i\psi} \\ -e^{-2i\phi/3} & -\sqrt{2\cos(2\phi)}e^{i\phi/3-i\psi} & e^{4i\phi/3}\end{bmatrix}. \tag{4.2}$$

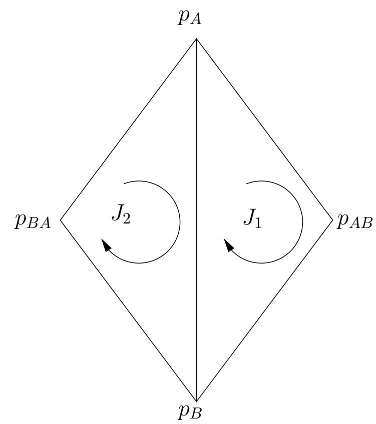



FIGURE 2. Action of $J_1$ and $J_2$ on the fixed points of $A$, $B$, $AB$ and $BA$.

The ambiguity in the lift from $\mathrm{PU}(2,1)$ to $\mathrm{SU}(2,1)$ is precisely the same as the choice of cube root of $e^{i\theta}$ and $e^{i\phi}$. Then we immediately have

$$J_1^{-1} = \begin{bmatrix} 0 & 0 & -e^{2i\theta/3} \\ 0 & -e^{-4i\theta/3} & \sqrt{2\cos(2\theta)}e^{-i\theta/3-i\psi} \\ -e^{2i\theta/3} & -\sqrt{2\cos(2\theta)}e^{-i\theta/3+i\psi} & e^{-4i\theta/3} \end{bmatrix}. \tag{4.3}$$

THEOREM 4.2. *Let $\rho : F_2 \longrightarrow \mathrm{PU}(2,1)$ be a representation so that $A = \rho(a)$, $B = \rho(b)$ and $AB = \rho(c^{-1})$ are all parabolic and let $p_A$, $p_B$ and $p_{AB}$ be their fixed points. Let $J_1$ be the order three map cyclically permuting $p_B$, $p_A$ and $p_{AB}$. Let $p_{BA}$ be the fixed point of $BA$ and let $J_2$ be the order three map cyclically permuting $p_A$, $p_B$ and $p_{BA}$. Then the following are equivalent:*

(i) *$A = J_1J_2$ and $B = J_2J_1$.*
(ii) *$\lambda_A$ and $\lambda_B$ are equal to the same cube root of the cross ratio $\mathbb{X}(p_A, p_B, p_{AB}, p_{BA})$.*

PROOF. Suppose that $A = J_1J_2$ and $B = J_2J_1$. Then $(AB)^{-1} = J_1^{-1}J_2J_1^{-1} = J_1^{-1}BJ_1 = J_1AJ_1^{-1}$. Therefore, $A$, $B$ and $C = B^{-1}A^{-1}$ are all conjugate, and so $\lambda_A = \lambda_B = \lambda_C$. Using Corollary 2.6 they must be all equal to the same cube root of $\mathbb{X}(p_A, p_B, p_{AB}, p_{BA})$.

Conversely, assume that $\lambda_A = \lambda_B$ and $\lambda_A^3 = \mathbb{X}(p_A, p_B, p_{AB}, p_{BA}) = e^{-2i\theta-2i\phi}$, and consider the two isometries

$$A' = J_1J_2 \quad \text{and} \quad B' = J_2J_1.$$

Clearly $A'$ and $B'$ are conjugate. Moreover, they are also conjugate to

$$C' = (A'B')^{-1} = J_1^{-1}J_2J_1^{-1} = J_1A'J_1^{-1}.$$

From the definition of $J_1$ and $J_2$ we see that $A'(p_A) = J_1J_2(p_A) = J_1(p_B) = p_A$ so $A'$ fixes $p_A$. Similarly $B'$ fixes $p_B$, $A'B'$ fixes $p_{AB}$ and $B'A'$ fixes $p_{BA}$. As a consequence of Lemma 2.5, we see that the eigenvalues $\lambda_{A'}$, $\lambda_{B'}$, $\lambda_{C'}$ satisfy

$$\mathbb{X}(p_A, p_B, p_{AB}, p_{BA}) = \frac{1}{\overline{\lambda}_{A'}\overline{\lambda}_{B'}\overline{\lambda}_{C'}}.$$

As the cross ratio has unit modulus, it implies that the three eigenvalues have unit modulus. As they are equal (the three isometries are conjugate), they are all equal to the same cube root of $\mathbb{X}(p_A, p_B, p_{AB}, p_{BA})$. Using Proposition 3.1, this implies that $A = A'$ and $B = B'$. □

The following proposition is a straightforward corollary.

COROLLARY 4.3. *The following two conditions are equivalent.*

(1) *The eigenvalue $\lambda$ of $J_1J_2$ associated with $p_1$ has unit modulus.*
(2) *The tetrahedron $(p_1, p_2, p_3, p_4)$ is balanced.*

*In this case, $\mathbb{X}(p_1, p_2, p_3, p_4) = \lambda^3$.*

Because $J_1$ and $J_2$ have order three, we see that

$$AB^{-1} = J_1J_2J_1^{-1}J_2^{-1} = [J_1, J_2], \tag{4.4}$$
$$[A, B] = ABA^{-1}B^{-1} = (J_1J_2)(J_2J_1)(J_2^{-1}J_1^{-1})(J_1^{-1}J_2^{-1}) = (J_1J_2^{-1})^3. \tag{4.5}$$

**4.2. Parameters .** We have seen, Proposition 2.14, that a balanced tetrahedron with ideal vertices $p_1$, $p_2$, $p_3$ and $p_4$ is determined up to $\mathrm{PU}(2,1)$ equivalence by

$$2\theta = \mathbb{A}(p_2, p_1, p_3), \quad 2\phi = \mathbb{A}(p_1, p_2, p_4), \quad 4\psi = \arg\big(\mathbb{B}(p_1, p_2, p_3, p_4)\big).$$

In the next sections we write certain traces in terms of these parameters $\theta$, $\phi$, $\psi$. We will then obtain equations in these variables that determine when certain words in the group $\Gamma = \langle J_1, J_2\rangle$ are parabolic or unipotent. It turns out that many of these computations become easier if we switch to the following real variables.

$$x = 4\sqrt{\cos(2\theta)\cos(2\phi)}\cos(2\psi), \quad y = 4\sqrt{\cos(2\theta)\cos(2\phi)}\sin(2\psi), \quad z = 4\cos(\theta - \phi). \tag{4.6}$$

Recall that $\theta \in [-\pi/4, \pi/4]$, $\phi \in [-\pi/4, \pi/4]$ and $\psi \in [0, \pi/2]$. Note that $z \geq 0$ with equality if and only if $\phi = -\theta = \pm\pi/4$ and $z \leq 4$ with equality if and only if $\phi = \theta$. Furthermore, note that

$$\begin{aligned} 2\cos^2(\theta - \phi) &= 1 + \cos(2\theta - 2\phi) \\ &\geq \cos(2\theta + 2\phi) + \cos(2\theta - 2\phi) \\ &= 2\cos(2\theta)\cos(2\phi). \end{aligned}$$

This implies that $z^2 \geq x^2 + y^2$ with equality if and only if $\phi = -\theta$. The latter inequality implies $-z \leq x \leq z$ and $-z \leq y \leq z$. Note for later use that, in particular, the condition $x = z$ implies that $\phi = -\theta$ and $\psi = 0$. The Jacobian associated to the change of variable (4.6) is $\mathcal{J} = 128\sin(2\theta + 2\phi)\sin(\theta - \phi)$. Therefore, this change of variables is a local diffeomorphism at all points where $\theta \neq \pm\phi$.

**4.3. Ruling out complex reflections.** The goal of this section is to describe the isometry type of certain elements of the group $\langle J_1, J_2\rangle$, and show that they can not be complex reflections. More precisely, we are going to prove that if $J_1J_2$, $J_1J_2^{-1}$ or $[J_1, J_2]$ has a neutral fixed point, then it is either parabolic of the identity. We begin by studying the product $J_1J_2$. It is possible to find an expression for $A = J_1J_2$ and $B = J_2J_1$ by plugging $\lambda_A = \lambda_B = e^{-2i\theta/3-2i\phi/3}$ in (3.5) and (3.6). This leads to

$$\mathrm{tr}(J_1J_2) = 2e^{-2i\theta/3-2i\phi/3} + e^{4i\theta/3+4i\phi/3}. \tag{4.7}$$

In particular $\mathrm{tr}(J_1J_2)$ lies on the deltoid curve described in Section 2.1 (see Figure 1), and we have to decide if $J_1J_2$ is parabolic, a complex reflection or the identity.

Proposition 4.4. *The map $J_1J_2$ is always parabolic unless $p_{AB} = p_{BA}$, in which case it is the identity. In particular, it cannot be a non-trivial reflection.*

Proof. Using Proposition 3.1 with $p = p_A$, $q = p_{BA}$ and $r = p_{AB}$, we see that $J_1J_2$ is a complex reflection if and only if its eigenvalue $\lambda_A$ associated to $p_A$ satisfies $\lambda_A^3 = -\exp\big(2i\mathbb{A}(p_A, p_{BA}, p_{AB})\big)$. But we know from Corollary 2.6 and the three-fold symmetry that

$$\lambda_A^3 = \mathbb{X}(p_A, p_B, p_{AB}, p_{BA}).$$

Combining these two relations, taking argument on both sides, and using part 3 of Lemma 2.5, we obtain that

$$\mathbb{A}(p_A, p_B, p_{AB}) - \mathbb{A}(p_A, p_B, p_{BA}) = \pi + 2\mathbb{A}(p_A, p_{BA}, p_{AB}) \mod 2\pi. \tag{4.8}$$

On the other hand, the cocycle relation of the Cartan invariant (Corollary 7.1.12 of [**Go3**]) gives us

$$\mathbb{A}(p_A, p_B, p_{AB}) - \mathbb{A}(p_A, p_B, p_{BA}) + \mathbb{A}(p_A, p_{AB}, p_{BA}) - \mathbb{A}(p_B, p_{AB}, p_{BA}) = 0. \tag{4.9}$$

Summing equations (4.8) and (4.9) gives

$$\mathbb{A}(p_A, p_{AB}, p_{BA}) + \mathbb{A}(p_B, p_{AB}, p_{BA}) = \pi \mod 2\pi.$$

As these two Cartan invariants belong to $[-\pi/2, \pi/2]$ (see Chapter 7 of [**Go3**]), they must be either both equal $\pi/2$ or both equal $-\pi/2$. This means that the four points $p_A$, $p_B$, $p_{AB}$ and $p_{BA}$ belongs to a common complex line $L$ (Corollary 7.1.13 of [**Go3**]). Moreover the fact that $\mathbb{A}(p_A, p_{AB}, p_{BA})$ and $\mathbb{A}(p_B, p_{AB}, p_{BA})$ have the same sign means that $p_A$ and $p_B$ lie on the same side of the geodesic connecting $p_{AB}$ and $p_{BA}$. As the tetrahedron $(p_A, p_B, p_{AB}, p_{BA})$ is balanced, $p_{AB}$ and $p_{BA}$ orthogonally project onto the same point of the geodesic $(p_A p_B)$. This is only possible when $p_{AB} = p_{BA}$. This implies that $J_2 = J_1^{-1}$. □

In $(\theta, \phi, \psi)$-coordinates, it is straightforward to check that $p_{AB} = p_{BA}$ if and only if $\psi = 0$ and $\theta = -\phi$. Therefore we see that $J_1 J_2$ can only be a complex reflection when $\phi = -\theta$ and $\psi = 0$. Plugging these values in (4.2) and (4.3), we see that this implies $J_2 = J_1^{-1}$.

Remark 2. Note that in $(x, y, z)$ coordinates the relation $2\cos(\theta - \phi) - 2\sqrt{\cos(2\theta)\cos(2\phi)}\cos(2\psi) = 0$ simply becomes $x = z$. The previous discussion shows thus that $x = z$ implies that $J_1 J_2$ is the identity.

Corollary 4.5. *The maps $J_1 J_2^{-1}$ and $[J_1 J_2]$ are never complex reflections.*

Proof. In Proposition 4.4 the only facts we have used about $J_1$ and $J_2$ are that $J_1$ and $J_2$ have order three and their product has a neutral fixed point on the boundary. By changing $J_2$ to $J_2^{-1}$ or $J_2 J_1 J_2^{-1}$ respectively, we see that if $J_1 J_2^{-1}$ or $[J_1, J_2]$ has a neutral fixed point on the boundary then it is parabolic or the identity. □

The following result is a straightforward consequence of the previous Proposition 4.4 (note that a $(3, 3, \infty)$-group is a group generated by two order three elements of which product is parabolic).

Theorem 4.6. *There is a bijection between the set of* $\mathrm{PU}(2,1)$*-orbits of non-degenerate balanced tetrahedra, and the set of* $\mathrm{PU}(2,1)$*-conjugacy classes of* $(3, 3, \infty)$*-groups in* $\mathrm{PU}(2,1)$.

Here by non-degenerate, we mean the the four vertices of the tetrahedron are pairwise distinct.

Remark 3. It follows from Corollary 4.5 that whenever $f(\mathrm{tr}(J_1 J_2^{-1})) = 0$, then $J_1 J_2^{-1}$ is parabolic or the identity. For later use, we compute $f(\mathrm{tr}(J_1 J_2^{-1}))$. First, a simple computation shows

$$J_1 J_2^{-1} = e^{i\theta/3 - i\phi/3} \begin{bmatrix} e^{i\theta - i\phi} - c_1 c_2 e^{2i\psi} + e^{-i\theta + i\phi} & -c_1 e^{-i\phi + i\psi} + c_2 e^{i\theta - i\psi} & -e^{i\theta + i\phi} \\ -c_1 e^{-i\phi - i\psi} + c_2 e^{i\theta + i\psi} & e^{i\theta - i\phi} - c_1 c_2 e^{-2i\psi} & c_1 e^{i\phi - i\psi} \\ -e^{-i\theta - i\phi} & -c_2 e^{-i\theta - i\psi} & e^{-i\theta + i\phi} \end{bmatrix}. \tag{4.10}$$

Therefore

$$\begin{aligned} \mathrm{tr}(J_1 J_2^{-1}) &= e^{i\theta/3 - i\phi/3}\Big(4\cos(\theta - \phi) - 4\sqrt{\cos(2\theta)\cos(2\phi)}\cos(2\psi)\Big) \\ &= e^{i\theta/3 - i\phi/3}\,(z - x) \end{aligned} \tag{4.11}$$

Plugging this value into Proposition 2.1, we obtain after rearranging that

$$f\left(\mathrm{tr}(J_1 J_2^{-1})\right) = (x - z)^2\left(x^2 - z^2 + 18\right) - 27. \tag{4.12}$$

The hypersurface defined by this equation is shown (in $(\theta, \phi, \psi)$-coordinates) in black in Figure 4. It is interesting to note that if $J_1 J_2^{-1}$ is parabolic, then the above quantity must be non-zero and thus $x - z \neq 0$. This implies that when $J_1 J_2^{-1}$ is parabolic, so is $J_1 J_2$.

**4.4. Super-pinching.** In this section we show that it is possible to have a one parameter family of representations of $F_2$ to $\mathrm{SU}(2,1)$ with seven primitive conjugacy classes of parabolic map. Because we also impose 3-fold symmetry, this is the same as saying that we have a one parameter family of representations of $\mathbb{Z}_3 * \mathbb{Z}_3$ with three primitive parabolic conjugacy classes.

Theorem 4.7. *There is a one parameter family of groups generated by $J_1$ and $J_2$ in* $\mathrm{SU}(2,1)$ *with the following properties:*

- *$J_1$ and $J_2$ are both elliptic maps of order 3;*
- *$J_1 J_2$, $J_1 J_2^{-1}$ and $[J_1, J_2]$ are all parabolic.*

Passing to the subgroup generated by $A = J_1 J_2$ and $B = J_2 J_1$, this implies

Corollary 4.8. *There is a one parameter family of groups generated by $A$ and $B$ in* $\mathrm{SU}(2,1)$ *with $A$, $B$, $AB$, $AB^{-1}$, $AB^2$, $A^2 B$ and $[A, B]$ all parabolic.*

Proof. In the groups from Theorem 4.7 we write $A = J_1 J_2$, $B = J_2 J_1$, leading to $AB = J_1 A^{-1} J_1^{-1}$, so these maps are all parabolic. Furthermore, using (4.4) we see that $AB^{-1} = [J_1, J_2]$ is parabolic, and so is $BAB = J_1^{-1} A B^{-1} J_1$ and $A^2 B = J_1 B A^{-1} J_1^{-1}$. Finally, using (4.5) we see $[A, B] = (J_1 J_2^{-1})^3$ is also parabolic. □

LEMMA 4.9. *In $(x,y,z)$-coordinates the trace for the commutator $[J_1,J_2]$ is given by*

$$\mathrm{tr}[J_1,J_2]=3+\frac{(x-z)(3x-z)+y^2+2i(x-z)y}{4} \tag{4.13}$$

PROOF. By direct computation from the expressions for $J_1J_2$ and $J_1^{-1}J_2^{-1}$ above we find:

$$\begin{aligned}\mathrm{tr}[J_1,J_2] &= 5+8\cos(2\theta)\cos(2\phi)+2\cos(2\theta-2\phi)\\ &\quad -12\sqrt{\cos(2\theta)\cos(2\phi)}\cos(\theta-\phi)e^{2i\psi}-4\sqrt{\cos(2\theta)\cos(2\phi)}\cos(\theta-\phi)e^{-2i\psi}\\ &\quad +4\cos(2\theta)\cos(2\phi)e^{4i\psi}.\end{aligned}$$

Simplifying and changing variables gives the result. □

PROOF. (Theorem 4.7.) We again use the change of variables (4.6), namely

$$x=4\sqrt{\cos(2\theta)\cos(2\phi)}\cos(2\psi),\quad y=4\sqrt{\cos(2\theta)\cos(2\phi)}\sin(2\psi),\quad z=4\cos(\theta-\phi).$$

By construction, we know that $J_1$ and $J_2$ are both regular elliptic maps of order three and that $J_1J_2$ is parabolic or a complex reflection. Moreover, we know from Remark 3 that if $J_1J_2^{-1}$ is parabolic, so is $J_1J_2$. Let us assume that both are parabolic and consider the commutator $[J_1J_2]$. Rewriting condition (4.12), we obtain

$$2z(x-z)=\frac{27-(x-z)^4-18(x-z)^2}{(x-z)^2}. \tag{4.14}$$

Substituting this identity into the expression (4.13) for $\mathrm{tr}[J_1,J_2]$ and simplifying, yields:

$$\mathrm{tr}[J_1,J_2]=\frac{2(x-z)^4-6(x-z)^2+27+(x-z)^2y^2+2i(x-z)^3y}{4(x-z)^2}.$$

Our goal will be to substitute this expression into Proposition 2.1. Specifically, using Corollary 4.5, if $f\big(\mathrm{tr}[J_1,J_2]\big)=0$ then $[J_1,J_2]$ will be parabolic. Such solutions will be exactly the groups we are looking for.

To simplify the expressions as much as possible, we make a further change of variables, namely we write $X=(x-z)^2$ and $Y=(x-z)y$. With respect to these new variables, we have:

$$\mathrm{tr}[J_1,J_2]=\frac{2X^2-6X+27+Y^2+2iXY}{4X}.$$

Plugging this into Proposition 2.1 and simplifying, we find that

$$256X^4 f\big(\mathrm{tr}[J_1,J_2]\big)=P(X,Y)$$

where

$$\begin{aligned}P(X,Y) &= Y^8+4(4X^2-14X+27)Y^6+6(12X^4-8X^3+360X^2-756X+729)Y^4\\ &\quad +4(16X^6-24X^5+1404X^4-4536X^3+20412X^2-30618X+19683)Y^2\\ &\quad +(2X^2-2X+27)(2X^2-18X+27)^3\end{aligned}$$

Therefore, in order to find groups where $[J_1,J_2]$ is parabolic or a complex reflection, we must identify those values of $X$ for which there exists $Y$ with $P(X,Y)=0$. It is clear that for a given value of $X$ and large enough values of $Y$ we must have $P(X,Y)>0$. Therefore for each $X$ such that $P(X,0)<0$ there exists $Y$ such that $P(X,Y)=0$. But $P(X,0)=(2X^2-2X+27)(2X^2-18X+27)^3$, and $(2X^2-2X+27)>0$ on $\mathbb{R}$. It follows from this fact that $P(X,0)\leq 0$ if and only if

$$\frac{9-3\sqrt{3}}{2}\leq X\leq\frac{9+3\sqrt{3}}{2} \tag{4.15}$$

Therefore, for this range of $X$ there exists a $Y$ with $P(X,Y)=0$. In Figure 3 we illustrate the locus $P(X,Y)=0$ in this range □

REMARK 4. Computing the resultant of $P(X,Y)$ and $\partial P/\partial Y$ with respect to $X$, it is possible to verify that the curve depicted on Figure 3 is in fact the full zero locus of $P$ on $\mathbb{R}^+\times\mathbb{R}$. This can be done easily using computation software such as MAPLE. This indicates that the set of classes of groups $\langle J_1,J_2\rangle$ having these property is reduced to this (topological) circle.

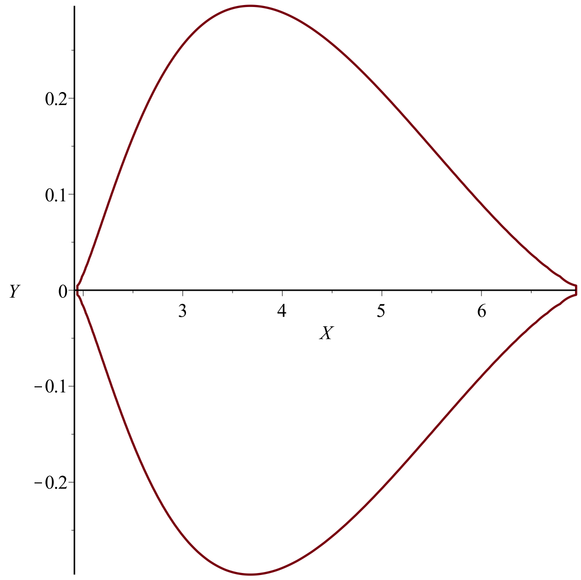



FIGURE 3. The locus $P(X,Y)=0$ in the range $\big(9-3\sqrt{3}\big)/2 \le X \le \big(9+3\sqrt{3}\big)/2$.

## 5. Discreteness

So far we have not discussed discreteness. However, there are certain subfamilies in our parameter space which have been studied before, and where the range of discreteness is known. We discuss these case by case.

**5.1. Finite: $\theta=-\phi$, $\psi=0$.** This is a simple case. It is easy to see that they imply $p_{AB}=p_{BA}$ and hence $J_2=J_1^{-1}$. In this case, the group has collapsed to a finite group. Therefore, though discrete, this group is far from being faithful.

**5.2. Ideal triangle groups: $\theta=-\phi$, $\psi=\pi/2$.** The condition $\theta=-\phi$ implies that $J_1J_2$ is unipotent. Furthermore, consider $I_0$, the complex reflection of order 2 in the complex line spanned by $\infty=p_A$ and $o=p_B$. That is

$$I_0=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

Observe that, as well as fixing $p_A$ and $p_B$, the involution $I_0$ swaps $p_{AB}$ and $p_{BA}$.

Using the definitions of $J_1$ and $J_2$, this immediately implies $J_2=I_0J_1^{-1}I_0$. Writing $I_1=J_1I_0J_1^{-1}$ and $I_2=J_1^{-1}I_0J_1$ we see that $J_1J_2=I_1I_0$, $J_2J_1=I_0I_2$ and $J_1^{-1}J_2J_1^{-1}=I_2I_1$ are all unipotent. Therefore these groups are complex hyperbolic ideal triangle groups, as studied by Goldman and Parker [**GoP**] and by Schwartz [**S1, S2, S3**]. Schwartz's theorem is that such a group is discrete provided $(I_1I_2I_0)^2=(J_1J_2^{-1})^3$ is not elliptic. We have

$$\mathrm{tr}(J_1J_2^{-1})=8\cos(2\theta)e^{2i\theta/3}.$$

It is straightforward to check when the right hand side lies outside the deltoid. Therefore we get the following reformulation of Schwartz's result:

THEOREM 5.1. *[Schwartz] If $\theta=-\phi$, $\psi=\pi/2$ the group $\langle J_1,J_2\rangle$ is discrete and isomorphic to $\mathbb{Z}_3\star\mathbb{Z}_3$ if and only if*

$$\cos(2\theta)\ge\frac{\sqrt{3}}{8\sqrt{2}}.$$

*Moreover, for the value of $\theta$ where equality is attained, the map $J_1J_2^{-1}$ is parabolic*

**5.3. Modular group deformations 1: $\theta=\phi$, $\psi=0$.** Let $I_0$ be the following complex reflection in a complex line that swaps $\infty=p_A$ and $o=p_B$:

$$I_0=\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

It is not hard to see that, as well as swapping $p_A$ and $p_B$, the involution $I_0$ swaps $p_{AB}$ and $p_{BA}$. Thus we have $J_2=I_0J_1I_0$. Hence $J_1J_2=(J_1I_0)^2$. This means that $J_1I_0$ is also parabolic. Since $I_0$ is a complex reflection fixing a complex line, these groups belong to the family of representations of the modular group considered by Falbel and Parker [**FP**]. Their main result, Theorem 1.2 of [**FP**] is that such groups are discrete and faithful provided $J_1I_0J_1^{-1}I_0=J_1J_2^{-1}$ is not elliptic. we have

$$\mathrm{tr}(J_1J_2^{-1})=4-4\cos(2\theta).$$

Therefore we can restate their result as:

THEOREM 5.2. *[Falbel-Parker] If $\theta=\phi$, $\psi=0$ the group $\langle J_1,J_2\rangle$ is discrete and isomorphic to $\mathbb{Z}_3\star\mathbb{Z}_3$ if and only if*

$$\cos(2\theta)\leq\frac{1}{4}.$$

*Moreover, for the value of $\theta$ where equality is attained, the map $J_1J_2^{-1}$ is parabolic*

**5.4. Modular group deformations 2: $\theta=\phi$, $\psi=\pi/2$.** Now we take $I_0$ to be a complex reflection in a point that swaps $\infty=p_A$ and $o=p_B$. Namely:

$$I_0=\begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}.$$

Once gain, $I_0$ swaps $p_{AB}$ and $p_{BA}$ and so $J_2=I_0J_1I_0$ and $J_1I_0$ is parabolic. But since $I_0$ now fixes just a point, we are in the family of representations of the modular group considered by Falbel and Koseleff [**FK**] and by Gusevskii and Parker [**GuP**]. The main result of these papers is that such groups are discrete and faithful for all values of $\theta$. We can restate this as:

THEOREM 5.3. *[Falbel-Koseleff, Gusevskii-Parker] If $\theta=\phi$, $\psi=\pi/2$ the group $\langle J_1,J_2\rangle$ is discrete and isomorphic to $\mathbb{Z}_3\star\mathbb{Z}_3$ for all $\theta\in[-\pi/4,\pi/4]$.*

**5.5. Bending: $\theta=\phi=0$.** We now consider the case where $\theta=\phi=0$ but $\psi$ is allowed to vary. Since $0=2\theta=\mathbb{A}(p_B,p_A,p_{AB})$ and $0=2\phi=\mathbb{A}(p_A,p_B,p_{BA})$ then the triples $(p_B,p_A,p_{AB})$ and $(p_A,p_P,p_{BA})$ each lie on an $\mathbb{R}$-circle. These are the bending deformations of $\mathbb{R}$-Fuchsian groups constructed by Will in [**W1, W2**]. The main result of [**W2**], which holds for any cusped surface group, is that these groups obtained by bending are discrete for a range of values of $\psi\in[0,\pi/4]$. Recently, these results have been extended in the case of the 3-punctured sphere by Parker and Will in [**PW**]. The main result of the latter paper comprises the fact that these groups are discrete and isomorphic to $F_2$ whenever $J_1J_2^{-1}$ is not elliptic. In the case where $\theta=\phi=0$, we have

$$\mathrm{tr}(J_1J_2^{-1})=8\sin^2(\psi).$$

The main result of [**PW**] implies thus the following:

THEOREM 5.4. *[Will, Parker-Will] If $\theta=\phi=0$ the group $\langle J_1,J_2\rangle$ is discrete and isomorphic to $\mathbb{Z}_3\star\mathbb{Z}_3$ if and only if*

$$\sin(\psi)\geq\sqrt{\frac{3}{8}}.$$

*Moreover, for the value of $\psi$ where equality is attained, the map $J_1J_2^{-1}$ is parabolic.*

Note that $\pi/4\sim 0.659$ and $\arcsin(\sqrt{3/8})\sim 0.784$

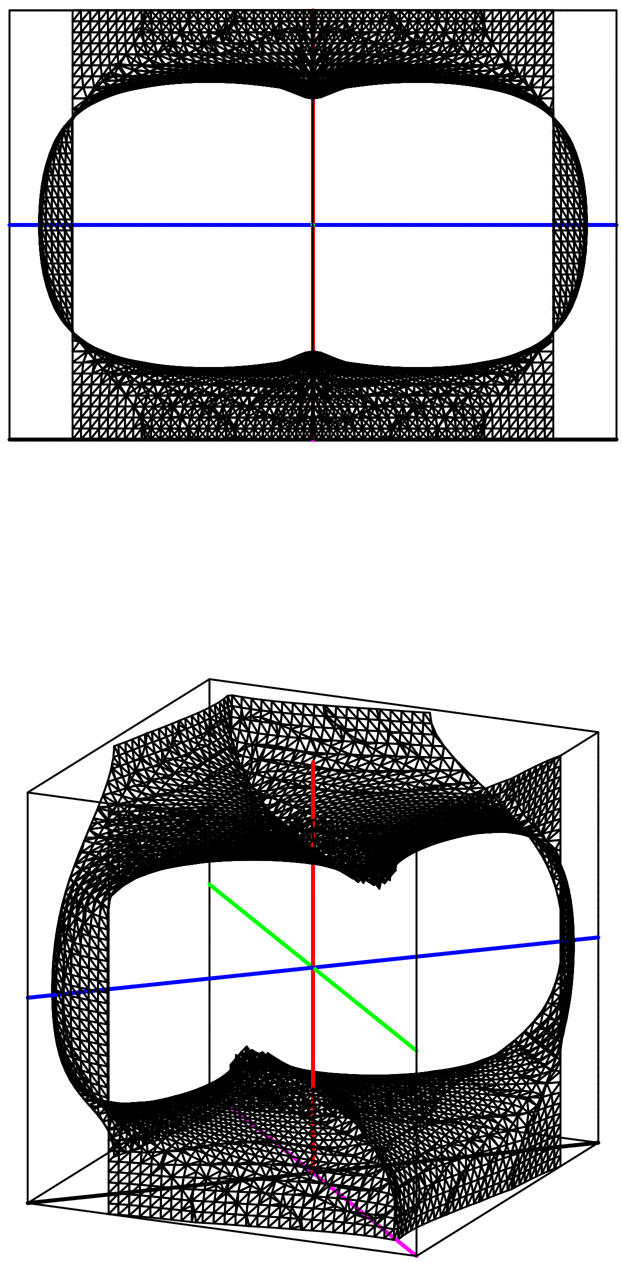
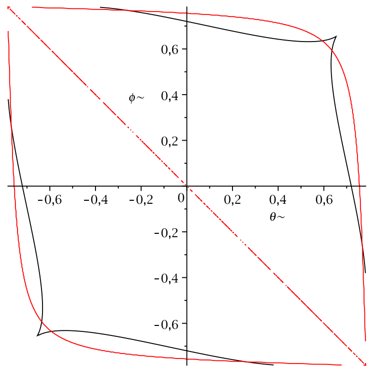
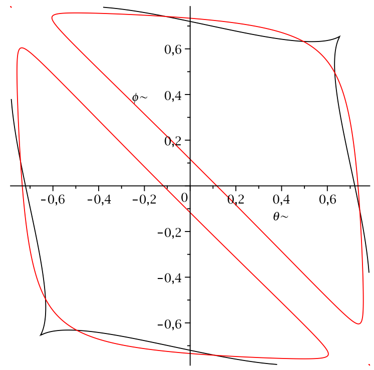


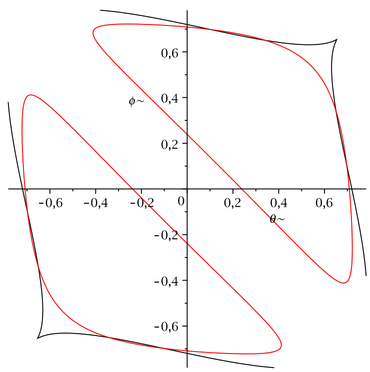
Figure 4. Two views of the parabolicity locus of $J_1 J_2^{-1}$ and the special families. The colours are as follows : the black surface is the locus where $J_1 J_2^{-1}$ is parabolic, the vertical red segment is the bending family, the black segment correspond to finite groups, the blue segment is the ideal triangle group case, the green and magenta segments are the two families corresponding to representations of the modular group.



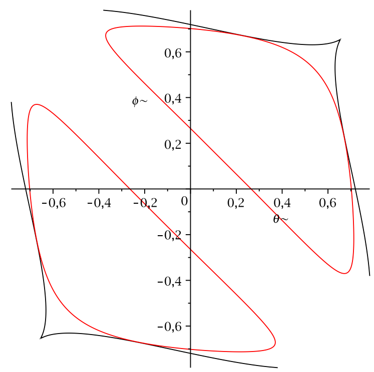

$\psi = 0$



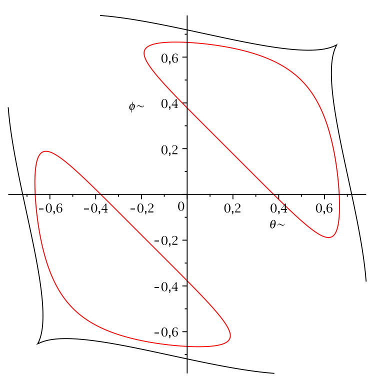

$\psi = 0.02$



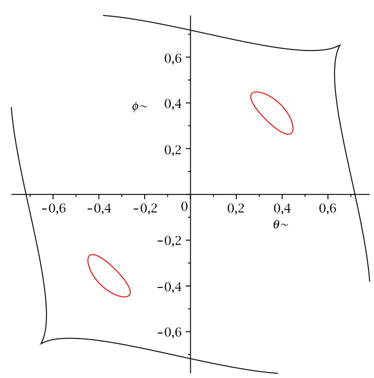

$\psi = 0.04$




$\psi = 0.044$




$\psi = 0.06$




$\psi = 0.085$

FIGURE 5. The horizontal slice $\psi = \psi_0$ for $\psi_0 = 0$, 0.02, 0.04, 0.044, 0.06 and 0.085. The black (resp. red) curve is the intersection of the locus where $J_1J_2^{-1}$ (resp. $[J_1, J_2]$) is parabolic. Each intersection point corresponds therefore to a group $\langle J_1, J_2\rangle$ where $J_1J_2$, $J_1J_2^{-1}$ and $[J_1, J_2]$ are parabolic. These pictures indicate that such groups exists for values of $\psi$ between 0 and 0.044.

## References


ChG. S. Chen, L. Greenberg; *Hyperbolic spaces*, in Contributions to Analysis. Academic Press, New York (1974), 49–87.

CuG. H. Cunha and N Gusevskii; *The moduli space of points in the boundary of complex hyperbolic space*. J. Geom. Anal. **22** (2012) no.1, 1–11.

FK. E. Falbel and P.V. Koseleff; *A circle of modular groups in* $\mathrm{PU}(2,1)$. Math. Res. Lett. **9** (2002) no.2-3, 379–391.

FP. E. Falbel, J.R. Parker; *The moduli space of the modular group in complex hyperbolic geometry*. Invent. Math. **152** (2003), no. 1, 57–88 .

Go1. W.M. Goldman, *Representations of fundamental groups of surfaces*. In “Geometry and Topology”, ed J. Alexander and J. Harer. Lecture Notes in Mathematics **1167** (1985), 95–117.

Go2. W.M. Goldman, *Convex real projective structures on compact surfaces*. J. Diff. Geom. **31** (1990) 791–845.

Go3. W.M. Goldman; Complex Hyperbolic Geometry. Oxford Mathematical Monographs. Oxford University Press (1999).

GoP. W.M. Goldman and J.R. Parker, *Complex hyperbolic ideal triangle groups*. J. reine angew. Math. **425** (1992), 71–86.

GuP. N. Gusevskii, J.R. Parker, *Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant*. Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999). Geom. Ded. **97** (2003) 151–185.

KR. A. Korányi, H.M. Reimann; *The complex cross-ratio on the Heisenberg group*. Enseign. Math. **33** (1987), 291–300.

P1. J.R. Parker; *Complex Hyperbolic Kleinian Groups*, Cambridge University Press, to appear.

PP. J.R. Parker and I. Platis; *Complex Hyperbolic quasi-Fuchsian groups*, In “Geometry of Riemann surfaces”, ed F.P Gardiner, G. González-Diez and Ch. Kourouniotis. London Math. Soc. Lecture Notes, **368** (2010), 309–355.

PW. J.R. Parker, P. Will; In preparation.

S1. R.E. Schwartz; *Ideal triangle groups, dented tori and numerical analysis*, Ann. of Math.(2) **153** (2001), 533–598 .

S2. R.E. Schwartz; *Degenerating the complex hyperbolic ideal triangle groups*, Acta. Math. **186** (2001), no. 1, 105–154.

S3. R.E. Schwartz; *A better proof of the Goldman-Parker conjecture*, Geom. Topol. **9** (2005), 1539–1601.

T. D. Toledo; *Representations of surfaces groups in complex hyperbolic space*. J. Differential Geom. **29** (1989), 125–133.

W1. P. Will; *The punctured torus and Lagrangian triangle groups in* $\mathrm{PU}(2,1)$. J. reine angew. Math. **602** (2007), 95–121.

W2. P. Will; *Bending Fuchsian representations of fundamental groups of cusped surfaces in* $\mathrm{PU}(2,1)$. J. Diff. Geom., **90**(3) (2012) 473–520

X. E. Xia; *The moduli of flat* $\mathrm{PU}(2,1)$ *structures on Riemann surfaces*. Pacific J. Maths. **195** (2000), 231–256.



Department of Mathematical Sciences, Durham University, South Road, Durham 3H1 3LE, England
*E-mail address*: j.r.parker@durham.ac.uk

Institut Fourier, Université Grenoble I, 100 rue des Maths, 38042 St-Martin d'Hères, France
*E-mail address*: pierre.will@ujf-grenoble.fr