Narrowing the Complexity Gap for Colouring (C_s, P_t)-Free Graphs

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Abstract. Let k be a positive integer. The k-Colouring problem is to decide whether a graph has a k-colouring. The k-Precolouring Extension problem is to decide whether a colouring of a subset of a graph's vertex set can be extended to a k-colouring of the whole graph. A k-list assignment of a graph is an allocation of a list — a subset of $\{1, \dots, k\}$ — to each vertex, and the List k-Colouring problem asks whether the graph has a k-colouring in which each vertex is coloured with a colour from its list. We prove a number of new complexity results for these three decision problems when restricted to graphs that do not contain a cycle on s vertices or a path on t vertices as induced subgraphs (for fixed positive integers s and t).

1 Introduction

It is well-known deciding whether a graph can be coloured with at most k colours is NP-complete even if $k = 3$ [18], and so the problem has been studied for special graph classes; see the surveys of Randerath and Schiermeyer [21] and Tuza [23], and the very recent survey of Golovach, Johnson, Paulusma and Song [8]. In this paper, we consider the computational complexity of several graph colouring problems for graph classes defined in terms of forbidden induced subgraphs. We introduce some notation and terminology before stating our results.

Terminology. Let $G = (V, E)$ be a graph. A *colouring* of G is a mapping $c : V \to \{1, 2, \dots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We call $c(u)$ the *colour* of u. A *k-colouring* of G is a colouring with $1 \leq c(u) \leq k$ for all $u \in V$. We study the following decision problem:

k-Colouring
Instance : A graph G.
Question : Is G k-colourable?

A *k-precolouring* of $G = (V, E)$ is a mapping $c_W : W \to \{1, 2, \dots k\}$ for some subset $W \subseteq V$. A k-colouring c is an *extension* of c_W if $c(v) = c_W(v)$ for each $v \in W$. Another decision problem:

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k-Precolouring Extension
Instance: A graph G and a k-precolouring c_W of G.
Question: Can c_W be extended to a k-colouring of G?

A *list assignment* of a graph $G = (V, E)$ is a function L that assigns a list $L(u)$ of *admissible* colours to each $u \in V$. If $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then L is also called a *k-list assignment*. A colouring c *respects* L if $c(u) \in L(u)$ for all $u \in V$. Here is our next decision problem:

List k-Colouring
Instance: A graph G and a k-list assignment L for G.
Question: Is there a colouring of G that respects L?

Note that k-Colouring can be viewed as a special case of k-Precolouring Extension which is, in turn, a special case of List k-Colouring.

Let G be a graph and $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that G is $(H_1, \ldots, H_p)$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we write H_1-free instead of (H_1)-free. We denote the cycle, complete graph and path, each on r vertices, by C_r, K_r and P_r, respectively. The *complement* of a graph $G = (V, E)$, denoted by $\overline{G}$, has vertex set V and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The disjoint union of two graphs G and H is denoted $G + H$, and the disjoint union of r copies of G is denoted rG.

Our Results. Several papers [4, 9, 13] have considered the computational complexity of the three decision problems defined above when restricted to (C_s, P_t)-free graphs. In this paper, we continue this investigation. Our first contribution is to state the following theorem that provides a complete summary of our current knowledge. In Section 5, we prove the theorem by providing references for results that demonstrate or imply each case. The cases marked with an asterisk are new results presented in this paper. We use p-time to mean polynomial-time throughout the paper.

Theorem 1. *Let k, s, t be three positive integers. The following statements hold for (C_s, P_t)-free graphs.*

(i) List k-Colouring *is* NP*-complete if*

1.* $k \geq 4$, $s = 3$ *and* $t \geq 8$
2.* $k \geq 4$, $s \geq 5$ *and* $t \geq 6$.

List k-Colouring *is p-time solvable if*

3. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
4. $k = 3$, $s = 3$ *and* $t \leq 6$
5. $k = 3$, $s = 4$ *and* $t \geq 1$
6. $k = 3$, $s \geq 5$ *and* $t \leq 6$
7. $k \geq 4$, $s = 3$ *and* $t \leq 6$
8. $k \geq 4$, $s = 4$ *and* $t \geq 1$
9. $k \geq 4$, $s \geq 5$ *and* $t \leq 5$.

(ii) k-Precolouring Extension *is* NP*-complete if*

1. $k = 4$, $s = 3$ *and* $t \geq 10$
2. $k = 4$, $s = 5$ *and* $t \geq 7$
3. $k = 4$, $s = 6$ *and* $t \geq 7$
*4.** $k = 4$, $s = 7$ *and* $t \geq 8$
5. $k = 4$, $s \geq 8$ *and* $t \geq 7$
6. $k \geq 5$, $s = 3$ *and* $t \geq 10$
*7.** $k \geq 5$, $s \geq 5$ *and* $t \geq 6$.

k-Precolouring Extension *is p-time solvable if*

8. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
9. $k = 3$, $s = 3$ *and* $t \leq 6$
10. $k = 3$, $s = 4$ *and* $t \geq 1$
11. $k = 3$, $s \geq 5$ *and* $t \leq 6$
12. $k \geq 4$, $s = 3$ *and* $t \leq 6$
13. $k \geq 4$, $s = 4$ *and* $t \geq 1$
14. $k \geq 4$, $s \geq 5$ *and* $t \leq 5$.

(iii) k-Colouring *is* NP*-complete if*

*1.** $k = 4$, $s = 3$ *and* $t \geq 39$
2. $k = 4$, $s = 5$ *and* $t \geq 7$
3. $k = 4$, $s = 6$ *and* $t \geq 7$
4. $k = 4$, $s = 7$ *and* $t \geq 9$
5. $k = 4$, $s \geq 8$ *and* $t \geq 7$
6. $k \geq 5$, $s = 5$ *and* $t \geq 7$
7. $k \geq 5$, $s \geq 6$ *and* $t \geq 6$.

k-Colouring *is p-time solvable if*

8. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
9. $k = 3$, $s = 3$ *and* $t \leq 7$
10. $k = 3$, $s = 4$ *and* $t \geq 1$
11. $k = 3$, $s \geq 5$ *and* $t \leq 7$
12. $k = 4$, $s = 3$ *and* $t \leq 6$
13. $k = 4$, $s = 4$ *and* $t \geq 1$
14. $k = 4$, $s = 5$ *and* $t \leq 6$
15. $k = 4$, $s \geq 6$ *and* $t \leq 5$
16. $k \geq 5$, $s = 3$ *and* $t \leq k + 2$
17. $k \geq 5$, $s = 4$ *and* $t \geq 1$
18. $k \geq 5$, $s \geq 5$ *and* $t \leq 5$.

We describe the rest of the paper.

In Section 2, we consider List k-Colouring restricted to (C_s, P_t)-free graphs and prove two results. We first show that List 4-Colouring is NP-complete for $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}, P_6)$-free graphs, thus strengthening the NP-completeness result of List 4-Colouring for P_6-free graphs [10]. (We observe that $\overline{P_1 + 2P_2}$ is also known as the 5-vertex wheel and $\overline{P_1 + P_4}$ is sometimes called the gem or the 5-vertex fan.) We also show that List 4-Colouring is NP-complete for P_8-free bipartite graphs.

In Section 3, we show that for all $k \geq 4$, k-Precolouring Extension is NP-complete for P_{10}-free bipartite graphs extending a result of Kratochvíl [17] who showed that 5-Precolouring Extension is NP-complete for P_{13}-free bipartite graphs. We also prove that 4-Precolouring Extension is NP-complete for $(C_5, C_6, C_7, C_8, P_8)$-free graphs and that for all $k \geq 5$, k-Precolouring Extension is NP-complete for (C_s, P_t)-free graphs if $s \geq 5$ and $t \geq 6$.

In Section 4, we show that 4-Colouring is NP-complete for (C_3, P_{39})-free graphs improving a result of Golovach et al. [9] who showed that 4-Colouring is NP-complete for (C_3, P_{164})-free graphs.

In Section 5, we prove Theorem 1 by combining a number of previously known results with our new results, and in Section 6 we summarize the open cases and pose a number of related open problems.

Related Work. In this paper, we focus on (C_s, P_t)-free graphs. We comment that this can be seen as a natural continuation of investigations into the complexity of k-Colouring and List k-Colouring for P_r-free graphs (see [8]). The sharpest results are the following. Hoàng et al. [14] proved that, for all $k \geq 1$, List k-Colouring is p-time solvable on P_5-free graphs. Huang [15] proved that 4-Colouring is NP-complete for P_7-free graphs and that 5-Colouring is NP-complete for P_6-free graphs. Recently, Chudnovsky, Maceli and Zhong [5, 6] announced a p-time algorithm for solving 3-Colouring on P_7-free graphs. Broersma et al. [3] proved that List 3-Colouring is p-time solvable for P_6-free graphs. Golovach, Paulusma and Song [10] proved that List 4-Colouring is NP-complete for P_6-free graphs. These results lead to the following table (in which the open cases are denoted by "?").

	k-Colouring				k-Precolouring Extension				List k-Colouring			
	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$
$r \leq 5$	P	P	P	P	P	P	P	P	P	P	P	P
$r = 6$	P	?	NP-c	NP-c	P	?	NP-c	NP-c	P	NP-c	NP-c	NP-c
$r = 7$	P	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c
$r \geq 8$	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c

Table 1. The complexity of k-Colouring, k-Precolouring Extension and List k-Colouring for P_r-free graphs.

2 New Results for List Colouring

We start by proving that List 4-Colouring is NP-complete for the class of $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4})$-free graphs. This result will follow from a closer analysis of the hardness reduction for List 4-Colouring for P_6-free graphs [10], which is from the problem Not-All-Equal 3-Sat with positive literals only. This problem was shown to be NP-complete by Schaefer [22], and is defined as follows. The input I consists of a set $X = \{x_1, x_2, \ldots, x_n\}$ of variables, and a set $\mathcal{C} = \{D_1, D_2, \ldots, D_m\}$ of 3-literal clauses over X in which all literals are positive. The question is whether there exists a truth assignment for X such that each D_i contains at least one true literal and at least one false literal. We may assume without loss of generality (see, for example, [10]) that each D_i contains either two or three literals and that each literal occurs in at most three different clauses. Given such an instance, Golovach et al. [10] define the following graph J_I and 4-list assignment L.

- a-type and b-type vertices: for each clause D_j, there are two *clause components* D_j and D'_j each isomorphic to P_5. Considered along the paths the vertices in D_j are $a_{j,1}, b_{j,1}, a_{j,2}, b_{j,2}, a_{j,3}$ with lists of admissible colours $\{2,4\}, \{3,4\}, \{2,3,4\}, \{3,4\}, \{2,3\}$, respectively, and the vertices in D'_j are $a'_{j,1}, b'_{j,1}, a'_{j,2}, b'_{j,2}, a'_{j,3}$ with lists of admissible colours $\{1,4\}$, $\{3,4\}$, $\{1,3,4\}$, $\{3,4\}$, $\{1,3\}$, respectively.

- x-type vertices: for each variable x_i, there is a vertex x_i with list of admissible colours $\{1, 2\}$.
- For every clause D_j with variables $x_{i_1}, x_{i_2}, x_{i_3}$, there are edges $a_{j,h}x_{i_h}$ and $a'_{j,h}x_{i_h}$ for $h = 1, 2, 3$.
- There is an edge from every x-type vertex to every b-type vertex.

See Figure 1 for an example of the graph J_I. In this figure, D_j is a clause with ordered variables $x_{i_1}, x_{i_2}, x_{i_3}$. The thick edges indicate the connection between these vertices and the a-type vertices of the two copies of the clause gadget. Indices from the labels of the clause gadget vertices have been omitted to increase visibility.

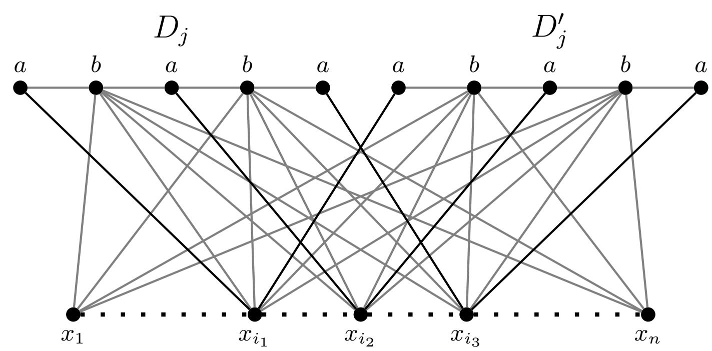

Fig. 1. An example of a graph J_I, as shown in [10]. Only the clause $D_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ is displayed.

The following two lemmas are known.

Lemma 1 ([10]). *The graph J_I has a colouring that respects L if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemma 2 ([10]). *The graph J_I is P_6-free.*

We are now ready to prove our main result.

Theorem 2. *The* List 4-Colouring *problem is* NP*-complete for the class of* $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}, P_6)$*-free graphs.*

Proof. Lemma 1 shows that the List 4-Colouring problem is NP-hard for the class of graphs J_I, where $I = (X, \mathcal{C})$ is an instance of Not-All-Equal 3-Sat with positive literals only, in which every clause contains either two or three literals and in which each literal occurs in at most three different clauses. Lemma 2 shows that each J_I is P_6-free. As the List 4-Colouring problem is readily seen to be in NP, it remains to prove that each J_I is $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4})$-free. For contradiction, assume that some J_I has an induced subgraph H isomorphic to a graph in $\{C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$.

First suppose that $H \in \{C_5, C_6\}$. The total number of x-type and b-type vertices can be at most 3, as otherwise H contains an induced C_4 or a vertex of degree at least 3, which is not possible. Because $|V(H)| \geq 5$ and the subgraph of H induced by its b-type and x-type vertices is connected, H must contain at least two adjacent a-type vertices. This is not possible.

Now suppose that $H = K_4$. Because the b-type and x-type vertices induce a bipartite graph, H must contain an a-type vertex. Every a-type vertex has degree at most 3. If it has degree 3, then it has two non-adjacent neighbours (which are of b-type). Hence, this is not possible.

Finally suppose that $H \in \{\overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$. Let u be the vertex that has degree 4 in H. Then u cannot be of a-type, because no a-type vertex has more than three neighbours in J_I. Suppose u is of b-type. Then every other vertex of H is either of a-type or of x-type. Because vertices of the same type are not adjacent, H must contain two a-type vertices and two x-type vertices. Then an a-type vertex is adjacent to two x-type vertices. This is not possible. Suppose u is of x-type. Then every other vertex of H is either of a-type or of b-type. Because vertices of the same type are non-adjacent, H must contain two a-type vertices and two b-type vertices. However, then u is adjacent to two a-type vertices in the same clause-component. This is not possible. □

Our second hardness result is also based on the hardness reduction of List 4-Colouring for P_6-free graphs. Let J_I be defined as before. We subdivide every edge between an a-type vertex and an x-type vertex and give each new vertex the list $\{1, 2\}$ (we say that these new vertices are of c-type). This results in a new graph J'_I with list assignment L' which extends the original list assignment L for J_I.

Lemma 3. *The graph J'_I is P_8-free and bipartite.*

Proof. The graph J'_I is readily seen to be bipartite. Below we prove that J'_I P_8-free (but not P_7-free).

Let P be an induced path in J'_I. If P contains no x-type vertex, then P contains vertices of at most one clause-component together with at most two c-type vertices. This means that $|V(P)| \leq 7$. If P contains no b-type vertex, then P can contain at most one x-type vertex (as any two x-type vertices can only be connected by a path that uses at least one b-type vertex). Consequently, P can have at most two a-type vertices and at most two c-type vertices. Hence, $|V(P)| \leq 5$ in this case. From now on assume that P contains at least one b-type vertex and at least one x-type vertex. Also note that P can contain in total at most three vertices of b-type and x-type.

First suppose that P contains exactly three vertices of b-type and x-type. Then these vertices form a 3-vertex subpath in P of types b, x, b or x, b, x. In both cases we can extend both ends of the subpath only by an a-type vertex and an adjacent c-type vertex, which means that $|V(P)| \leq 7$. Now suppose that P contains exactly two vertices of b-type and x-type. Because these vertices are of different type, they are adjacent and we can extend both ends of the corresponding 2-vertex subpath of P only by an a-type vertex and an adjacent c-type vertex. This means that $|V(P)| \leq 6$. This completes our proof. □

The following lemma can be proven by exactly the same arguments that were used to prove Lemma 1.

Lemma 4. *The graph J'_I has a colouring that respects L' if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemmas 3 and 4 imply the last result of this section.

Theorem 3. List 4-Colouring *is* NP*-complete for P_8-free bipartite graphs.*

3 New Results for Precolouring Extension

In this section we give three results on the k-Precolouring Extension problem.

Let $k \geq 4$. Consider the bipartite graph J'_I with its list assignment L' from Section 2. The list of admissible colours $L'(u)$ of each vertex u is a subset of $\{1, 2, 3, 4\}$. We add $k - |L'(u)|$ pendant vertices to u and precolour these vertices with different colours from $\{1, \ldots, k\} \setminus L'(u)$. This results in a graph J''_I with a k-precolouring c_W, where W is the set of all the new pendant vertices.

Lemma 5. *The graph J''_I is P_{10}-free and bipartite.*

Proof. Because J'_I is P_8-free and bipartite by Lemma 3, and moreover, we only added pendant vertices, J''_I is P_{10}-free and bipartite. □

The following lemma can be proven by exactly the same arguments that were used to prove Lemma 1.

Lemma 6. *The graph J''_I has a k-colouring that is an extension of c_W if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemmas 5 and 6 imply the first result of this section.

Theorem 4. *For all $k \geq 4$, k-*Precolouring Extension *is* NP*-complete for the class of P_{10}-free bipartite graphs.*

Here is our second result.

Theorem 5. *The* 4-Precolouring Extension *problem is* NP*-complete for the class of $(C_5, C_6, C_7, C_8, P_8)$-free graphs.*

Proof. Let J_I be the instance with list assignment L as constructed in Section 2. Instead of considering lists, we introduce new vertices, which we precolour (we do not precolour any old vertices). For each clause D_j we add five new vertices, s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$. We add edges $a_{j,1}s_j$, $a_{j,3}t_j$ and $a_{j,h}u_{j,h}$ for $h = 1, \ldots 3$. We precolour s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ by colours 3, 4, 1, 1, 1, respectively. For each clause D'_j we add five new vertices, s'_j, t'_j, $u'_{j,1}$, $u'_{j,2}$, $u'_{j,3}$. We add edges $a'_{j,1}s'_j$, $a'_{j,3}t'_j$ and $a'_{j,h}u'_{j,h}$ for $h = 1, \ldots 3$. We precolour s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ by colours 3, 4, 2, 2, 2, respectively. Finally, we add two new vertices c_1, c_2, which we make adjacent to all x-type vertices, and two new vertices y_1, y_2, which we make adjacent to all b-type vertices. We colour c_1, c_2, y_1, y_2 with colours 3, 4, 1, 2, respectively. This results in a new graph J_I^*. Because y_1, y_2 can be viewed as x-type vertices and c_1, c_2 as b-type vertices, because every other new vertex is a pendant vertex and because J_I is (C_5, C_6, P_6)-free (by Theorem 2), we find that J_I^* is $(C_5, C_6, C_7, C_8, P_8)$-free. Moreover, our precolouring forces the lists $L(v)$ upon every vertex v of J_I. Hence, J_I^* has a 4-colouring extending this precolouring if and only if J_I has a colouring that respects L. By Lemma 1 the latter is true if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal. □

Broersma et al. [3] showed that 5-Precolouring Extension for P_6-free graphs is NP-complete. It can be shown that the gadget constructed in their NP-hardness reduction is C_s-free for all $s \geq 5$. By adding $k-5$ dominating vertices, precoloured with colours $6, \ldots, k$, to each vertex in their gadget, we can extend their result from $k = 5$ to $k \geq 5$. This leads to the following theorem.

Theorem 6. *For all $k \geq 5$, k-Precolouring Extension is NP-complete for (C_s, P_t)-free graphs if $s \geq 5$ and $t \geq 6$.*

4 New Results for Colouring

In this section, we prove that 4-Colouring is NP-complete for (C_3, P_{39})-free graphs. We do this by modifying the graph J''_I from Section 3 when $k = 4$.

First we review a well-known piece of graph theory. The *Mycielski construction* of a graph $G = (V, E)$ is the new graph G' constructed from G by adding a new vertex v' for each $v \in V$ that is adjacent to every neighbour of v in G, followed by adding a further new vertex u adjacent to every new vertex v'. By repeating this construction from K_2, a sequence of graphs $M_2, M_3, \ldots$ is obtained. Here, $M_2 = K_2$, $M_3 = C_5$ and M_4 is the well-known Grötzsch graph. Mycielski [19] showed that every M_k is C_3-free and has chromatic number k. Moreover, any proper subgraph of M_k is $(k-1)$-colourable (see for example [1]).

We focus on M_5. For any pair of adjacent vertices p and r, $M_5 - pr$ is 4-colourable and, in every 4-colouring, p and r are coloured alike (else a 4-colouring of M_5 has been found). We let M_{pq} be the graph obtained from $M_5 - pr$ by adding a new vertex q and making it adjacent to r only. Note that M_{pq} is 4-colourable and that, in any 4-colouring of M_{pq}, the vertices p and q must have different colours.

Let G be a graph with $e = xy \in E(G)$. The *M-identification* of e in G is the following operation: delete the edge $e = xy$ and add a copy of M_{pq} between x and y by identifying $p \in M_{pq}$ and $q \in M_{pq}$ with x and y, respectively. We denote this copy of M_{pq} by M_e.

We are now ready to explain how we modify the graph J_I''. Recall that $k = 4$. First we take a complete graph on four new vertices $t_1, \ldots, t_4$. We perform an M-identification of every edge $t_i t_j$. Recall that we had defined a precolouring W for a subset $W \subseteq V(J_I'')$. We add an edge between a vertex t_i and a vertex $u \in W$ if and only if $c_W(u) \neq i$. This results in a new graph J_I'''.

In the next three lemmas we show three properties of J_I'''. The proof of the third lemma has been omitted due to page restrictions.

Lemma 7. *The graph J_I''' is 4-colourable if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Proof. We claim that J_I''' is 4-colourable if and only if J_I'' has a 4-colouring that is an extension of c_W. This follows by construction and from the fact that p and q have different colours in any 4-colouring of M_{pq}. In order to prove the lemma it remains to apply Lemma 6. □

Lemma 8. *The graph J_I''' is C_3-free.*

Proof. The graph J_I''' is C_3-free because of the following three reasons. Firstly, M_{pq} is C_3-free. Secondly, we applied an M-identification for every edge $t_i t_j$. So, the vertices $t_1, \ldots, t_4$ form an independent set of J_I'''. Thirdly, the neighbours of $t_1, \ldots, t_4$ in J_I'' are all in W, and W is an independent set of J_I'', and thus of J_I'''. □

Lemma 9. *The graph J_I''' is P_{39}-free.*

The main result of this section now follows from Lemmas 7–9.

Theorem 7. 4-Colouring *is* NP*-complete for (C_3, P_{39})-free graphs.*

5 Proof of Theorem 1

To prove Theorem 1 we need first to discuss some additional results. Kobler and Rotics [16] showed that for any constants p and k, List k-Colouring is p-time solvable on any class of graphs that have clique-width at most p, assuming that a p-expression is given. Oum [20] showed that a $(8^p - 1)$-expression for any n-vertex graph with clique-width at most p can be found in $O(n^3)$ time. Combining these two results leads to the following theorem.

Theorem 8. *Let $\mathcal{G}$ be a graph class of bounded clique-width. For all $k \geq 1$,* List k-Colouring *can be solved in p-time on $\mathcal{G}$.*

We also need the following result due to Gravier, Hoáng and Maffray [11] who slightly improved upon a bound of Gyárfás [12] who showed that every (K_s, P_t)-free graph can be coloured with at most $(t-1)^{s-2}$ colours.

Theorem 9 ([11]). *Let $s, t \geq 1$ be two integers. Then every (K_s, P_t)-free graph can be coloured with at most $(t-2)^{s-2}$ colours.*

We now prove Theorem 1 by considering each case. For each we either refer back to an earlier result, or give a reference; the results quoted can clearly be seen to imply the statements of the theorem.

We first consider the intractable cases of List k-Colouring and note that (i).1 follows from Theorem 3, and Theorem 2 implies that List 4-Colouring is NP-complete for the class of (C_5, C_6, P_6)-free graphs which proves (i).2.

Now the tractable cases. Erdös, Rubin and Taylor [7] and Vizing [24] observed that 2-List Colouring is p-time solvable on general graphs implying (i).3. Broersma et al. [3] showed that List 3-Colouring is p-time solvable for P_6-free graphs from which we can infer (i).4 and (i).6. Golovach et al. [9] proved that for all $k, r, s, t \geq 1$, List k-Colouring can be solved in linear time for $(K_{r,s}, P_t)$-free graphs. By taking $r = s = 2$, we obtain (i).5 and (i).8. The class of (C_3, P_6)-free graphs was shown to have bounded clique-width by Brandstädt, Klembt and Mahfud [2]; using Theorem 8 we see that List k-Colouring is p-time solvable on (C_3, P_6)-free graphs for all $k \geq 1$ demonstrating (i).7. Hoàng, Kamiński, Lozin, Sawada, and Shu [14] proved that for all $k \geq 1$, List k-Colouring is p-time solvable on P_5-free graphs proving (i).9.

We now consider k-Precolouring Extension. The tractable cases all follow from the results on List k-Colouring just discussed. So we are left to consider the NP-complete cases. Theorem 4 implies (ii).1 and (ii).6. Theorems 5 and 6 imply (ii).4 and (ii).7 And (ii).2, (ii).3 and (ii).5 follow immediately from corresponding results for k-Colouring proved by Hell and Huang [13].

Finally, we consider k-Colouring; first the NP-complete cases. Theorem 7 gives us (iii).1. Golovach, Paulusma and Song [9] proved that for all $s \geq 5$, there exists a constant $t(s)$ such that 4-Colouring is NP-complete for $(C_5, \ldots, C_s, P_{t(s)})$-free graphs. In particular, they showed that 4-Colouring is NP-complete for (C_5, P_{23})-free graphs, and this result has been strengthened by Hell and Huang [13] who proved all the other NP-completeness subcases.

Chudnovsky, Maceli and Zhong [5, 6] announced that 3-Colouring is p-time solvable on P_7-free graphs, and Chudnovsky, Maceli, Stacho and Zhong [4] announced that 4-Colouring is p-time solvable for (C_5, P_6)-free graphs. Theorem 9 gives us (iii).16. All other tractable cases follow from the corresponding tractable cases for List k-Colouring. □

6 Open Problems

From Theorem 1, we see that the following cases are open in the classification of the complexity of graph colouring problems for (C_s, P_t)-free graphs:

(i) For List k-Colouring the following cases are open:

- $k = 3$, $s = 3$ and $t \geq 7$
- $k = 3$, $s \geq 5$ and $t \geq 7$
- $k \geq 4$, $s = 3$ and $t = 7$.

(ii) For k-Precolouring Extension the following cases are open:

- $k = 3$, $s = 3$ and $t \geq 7$
- $k = 3$, $s \geq 5$ and $t \geq 7$
- $k = 4$, $s = 3$ and $7 \leq t \leq 9$
- $k = 4$, $s \geq 5$ and $t = 6$
- $k = 4$, $s = 7$ and $t = 7$
- $k \geq 5$, $s = 3$ and $7 \leq t \leq 9$

(iii) For k-Colouring the following cases are open:

- $k = 3$, $s = 3$ and $t \geq 8$
- $k = 3$, $s \geq 5$ and $t \geq 8$
- $k = 4$, $s = 3$ and $7 \leq t \leq 38$
- $k = 4$, $s \geq 6$ and $t = 6$
- $k = 4$, $s = 7$ and $7 \leq t \leq 8$
- $k \geq 5$, $s = 3$ and $t \geq k + 3$
- $k \geq 5$, $s = 5$ and $t = 6$.

Besides solving these missing cases (and the missing cases from Table 1) we pose the following problems specifically. First, does there exist a graph H and an integer $k \geq 3$ such that List k-Colouring is NP-complete and k-Colouring is p-time solvable for H-free graphs? Theorem 1 shows that if we forbid two induced subgraphs then the complexity of these two problems *can* be different: take $k = 4$, $H_1 = C_5$ and $H_2 = P_6$. Second, is List 4-Colouring NP-complete for P_7-free bipartite graphs? This is the only missing case of List 4-Colouring for P_t-free bipartite graphs due to Theorems 1 and 3.

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