Enriched BEM for fracture in anisotropic materials

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Abstract. This paper develops the formulation of the enriched Boundary Element Method (BEM) for the analysis of fracture applications in anisotropic materials. The formulation is based in the Partition of Unity Method (PUM), via the implementation of ad-hoc enrichment functions that describe the displacement field in the vicinity of the crack tip. Numerical results are presented in order to validate the enriched BEM formulation and a comparison with the results obtained using other techniques is further performed and discussed. Namely, both results using the eXtended Finite Element Method (X-FEM) as well as the dual BEM (in combination with discontinuous quarter-point elements) are considered for comparison purposes.

Introduction

Fracture mechanics of anisotropic materials has received great attention in the latter years due to the increasing use of such materials in many engineering applications. For instance, composite materials are widely employed in aircraft and space structures, automobiles, wind power generators or sporting goods. When dealing with fracture problems, the BEM has been shown as a powerful and effective tool in comparison to other computational techniques [1, 2].

In this paper we develop the enriched BEM formulation for anisotropic fracture problems. To this end, the PUM [3] is implemented in a dual BEM context, by extending the formulation previously presented for isotropic materials [4] to the more general anisotropic case. The corresponding enrichment functions are derived and further details on the implementation of the enriched BEM are briefly discussed. In particular, additional collocation points have to be used in order to accommodate the extra unknowns that the enrichment introduces. The formulation is validated by several numerical examples involving stress intensity factor computations for mixed-mode cracks.

Furthermore, we compare the results of the enriched BEM with those obtained by alternative numerical techniques: dual BEM implemented in combination with discontinuous quarter-point elements [2] and X-FEM with anisotropic enrichment functions [5]. Computational cost and precision of results obtained from each of these methods is discussed to close the paper.

Governing equations

Consider an anisotropic elastic domain Ω, the static equilibrium equations in the presence of body forces $\mathbf{b}$ are defined as

$$\sigma_{ij,j} + b_i = 0 \tag{1}$$

Symmetry holds for the stress and strain tensors, i.e.: $\sigma_{ij} = \sigma_{ji}$; $\varepsilon_{ij} = \varepsilon_{ji}$, where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

The linear constitutive equations are given by the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

where C_{ijkl} define the material constants tensor, satisfying the following symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{3}$$

Enriched BEM Formulation

The boundary element method (BEM) has been established as a reference when dealing with linear elastic fracture mechanics problems [1], being more accurate and robust than domain discretization methods such as the finite element method (FEM).

In a FEM context, the partition of unity [3] was applied by Belytschko and Black [6] to capture the displacements asymptotic behavior around the crack tip, so that the crack is no longer part of the geometry, being represented by a set of enrichment functions at the elements containing the crack. This approach is now known as the extended finite element method (X-FEM) and has been subject of research in a variety of fields.

The dual BEM is the usual choice when dealing with fracture mechanics problems. It can be summarised by a displacement boundary integral equation (DBIE)

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = \int_\Gamma u^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{4}$$

and a traction boundary integral equation (TBIE), obtained by the derivation of (4) and further substitution in (2)

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_\Gamma s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = N_r\int_\Gamma d^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{5}$$

where Γ represents all the boundaries (including crack boundaries) of domain Ω; N_r is the outward unit normal to the boundary at the collocation point $\boldsymbol{\xi}$; c_{ij} is the free term deriving from the Cauchy Principal Value integration of the strongly singular kernels p^*_{ij}; u^*_{ij} and p^*_{ij} are the displacement and traction fundamental solutions; d^*_{rij} and s^*_{rij} follow from derivation and substitution into the generalized Hooke's law of u^*_{ij} and p^*_{ij}, respectively. Explicit expressions of the kernels u^*_{ij}, p^*_{ij}, d^*_{rij} and s^*_{rij} are given in [2].

The extended boundary element method (X-BEM) was first proposed by Simpson and Trevelyan [4] for fracture mechanics problems in isotropic materials. The main idea is to model the asymptotic behavior of the displacements around the crack tips by introducing new degrees of freedom. The displacements $\mathbf{u}^h(\mathbf{x})$ are thus redefined as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i\in\mathcal{N}} N_i(\mathbf{x})\mathbf{u}_i + \sum_{k\in\mathcal{N}^{\mathcal{CT}}} N_k(\mathbf{x})\sum_\alpha F_\alpha(\mathbf{x})\mathbf{a}^\alpha_k \tag{6}$$

where $\mathcal{N}$ and $\mathcal{N}^{\mathcal{CT}}$ are the sets with nonenriched and enriched nodes, respectively, N_i is the standard Lagrangian shape function associated with node i, $\mathbf{u}_i$ is the vector of nodal degrees of freedom, and $\mathbf{a}^\alpha_k$ represents the enriched basis functions which capture the asymptotic behavior around the crack tips. In elastic materials, $\mathbf{a}^\alpha_k$ is an 8-component vector for two-dimensional problems, since only two nodal variables (u_1, u_2) and four enrichment functions are needed to describe all the possible deformation states in the vicinity of the crack-tip [5].

In this work, we use the anisotropic enrichment functions obtained by Hattori et al. [5] for the X-FEM:

$$F_l(r,\theta) = \sqrt{r}\begin{pmatrix} \Re\{A_{11}B^{-1}_{11}\beta_1 + A_{12}B^{-1}_{21}\beta_2\} \\ \Re\{A_{11}B^{-1}_{12}\beta_1 + A_{12}B^{-1}_{22}\beta_2\} \\ \Re\{A_{21}B^{-1}_{11}\beta_1 + A_{22}B^{-1}_{21}\beta_2\} \\ \Re\{A_{21}B^{-1}_{12}\beta_1 + A_{22}B^{-1}_{22}\beta_2\} \end{pmatrix} \tag{7}$$

where $\beta_i = \sqrt{\cos\theta + \mu_i \sin\theta}$, r is the distance between the crack tip and an arbitrary position, θ is the orientation measured from a coordinate system centered at the crack tip, and $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\mu}$ are obtained from the following eigenvalue problem:

$$\left(\begin{array}{c|c} -\mathbf{L}^{-1}\mathbf{M} & -\mathbf{L}^{-1} \\ \hline \mathbf{Z} - \mathbf{M}^T\mathbf{L}^{-1}\mathbf{M} & -\mathbf{M}^T\mathbf{L}^{-1} \end{array}\right) \left(\begin{array}{c} \mathbf{A}_m \\ \hline \mathbf{B}_m \end{array}\right) = \mu_m \left(\begin{array}{c} \mathbf{A}_m \\ \hline \mathbf{B}_m \end{array}\right) \quad \text{(no sum on } m\text{)} \tag{8}$$

with

$$\mathbf{Z} := \mathbf{C}_{1ij1}; \quad \mathbf{M} := \mathbf{C}_{2ij1}; \quad \mathbf{L} := \mathbf{C}_{2ij2} \tag{9}$$

Let us emphasize that the anisotropic enrichment functions can also be used for isotropic materials, since this is a degenerated case from anisotropic materials. For more details please refer to reference [5].

The enriched anisotropic BEM formulation is similar to the one used by Simpson and Trevelyan [4] for isotropic materials. The extended DBIE and the TBIE can be restated as:

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + \int_{\Gamma_c} p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})F_\alpha(\boldsymbol{x})\mathbf{a}^\alpha_k d\Gamma = \int_\Gamma u^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{10}$$

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_\Gamma s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + N_r\int_{\Gamma_c} s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})F_\alpha(\boldsymbol{x})\mathbf{a}^\alpha_k d\Gamma = N_r\int_\Gamma d^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{11}$$

where $\Gamma_c = \Gamma_+ \cup \Gamma_-$ stands for the crack surfaces Γ_+ and Γ_-. In this work, only the element containing the crack tip receives the enrichment function. Thus, if the element does not belong to the crack tip the dual BEM formulation is the same as stated previously in Eqs. (4) and (5). Let us remind that strongly singular and hypersingular terms arise from the integration of the p^*_{ij}, d^*_{rij} and s^*_{rij} kernels and they are regularised in the same way as shown in [2].

Numerical results

In this section we compare the numerical results from the extended BEM, the dual BEM with quarter-point elements [2] and the X-FEM. For validation purposes, we will focus on isotropic materials for now.

The Stress Intensity Factors (SIF) are calculated differently for each method: for the X-BEM, a J-integral is carried out; for the quarter-point dual BEM, a direct extrapolation is performed from the displacements at the crack tip; and for the X-FEM, the interaction integral is used [2, 4, 6].

Edge crack

Figure 1 illustrates a square plate ($h/w = 0.5$) with a single edge crack of length a under a uniform loading σ. The size of the crack is defined by $a/w = 0.5$. Results for the X-FEM are obtained using topological and geometrical enrichment, with a fixed area of $r_e/a = 0.2$. For more information about both adopted enrichment types please refer to [5] for instance.

For the dual BEM and the X-BEM a 10 continuous quadratic element per side mesh was used. The crack is discretized with 10 discontinuous elements. The dual BEM presents a quarter-point in the element at the crack tip, while the X-BEM presents an enriched element. Figure 2 illustrates the SIFs obtained from all the compared numerical approaches. The number of elements is given for the X-FEM only. The normalized mode I reference is taken from [7].

It is evident that the BEM solutions are more accurate than both solutions obtained with different enrichment types.

Centered crack

A rectangular plate $h/w = 2$ with a centered crack of length $2a$ under a uniform loading is represented in Figure 5, where two different θ values were evaluated: $\theta = 0^o$ and $\theta = 45^o$. The size of the crack is

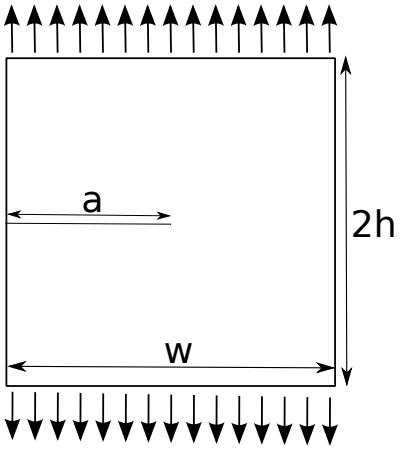


Figure 1: Edge crack problem.

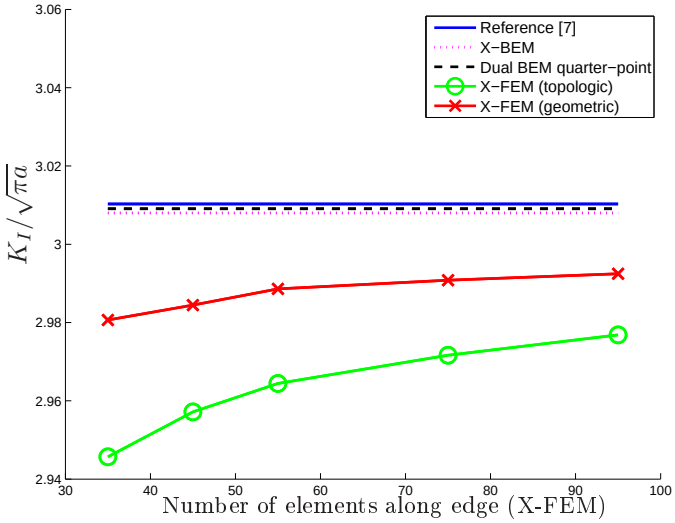


Figure 2: Mode I SIFs for the edge crack problem.

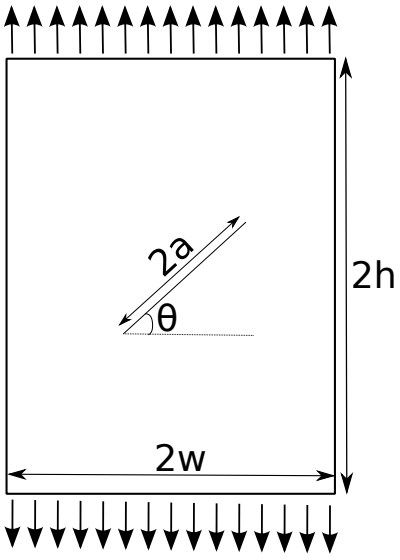


Figure 3: Centered crack problem.

$a/w = 0.5$ in both cases . The BEM meshes present 6 quadratic elements per side, and 10 discontinuous elements at the crack.

Figure 4 shows the mode I for the dual BEM, X-BEM and X-FEM, this latter with several results according to the used number of elements per side. Reference [8] was employed to validate the numerical results. It is evident that the dual BEM and the X-BEM approaches present better accuracy than the X-FEM approach.

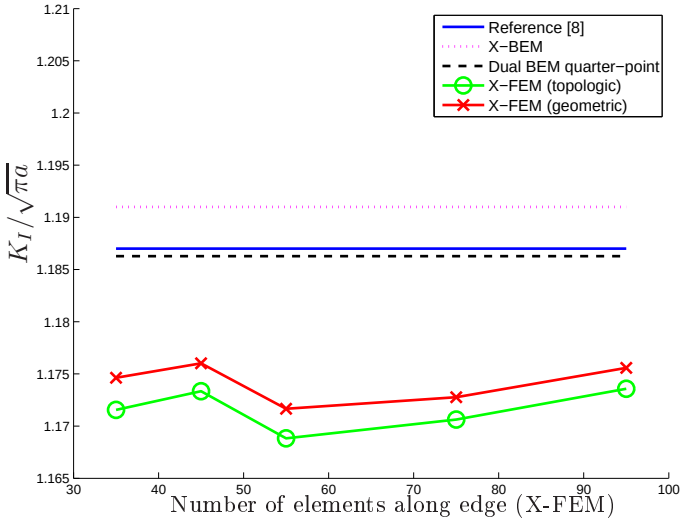
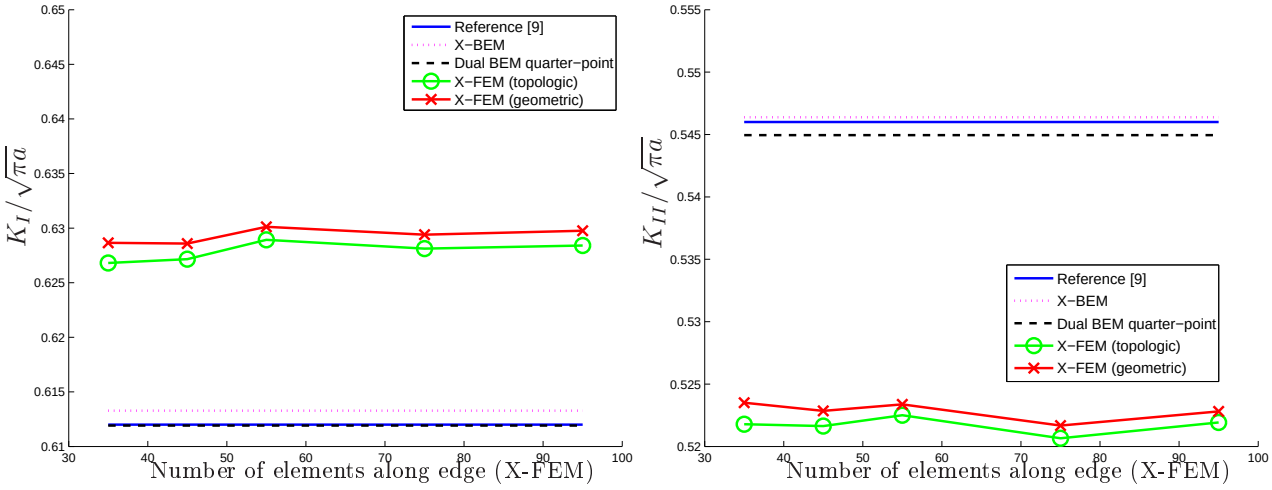


Figure 4: Mode I SIFs for the centered crack problem - $\theta = 0^o$.

Now we analyse a fracture problem presenting mixed mode types. Figures 5(a) and 5(b) illustrate the mode I and mode II, respectively, when $\theta = 45^o$. The references solutions were obtained from the reference [9]. In this case, the best results are obtained with the dual BEM for the mode I, and with the X-BEM for mode II, while the X-FEM results still conserves a relatively low error.



(a) Mode I (b) Mode II

Figure 5: SIFs for the centered crack problem - $\theta = 45^o$.

Summary

An enriched boundary element method was proposed in this work. This method uses the benefits of the partition of unit to model in a more general way the asymptotic displacements around the crack tip. Existing anisotropic enrichment functions for the X-FEM were employed at the proposed method. Some

numerical examples were evaluated, and the results compared to a well established BEM formulation and the X-FEM . It was observed that the dual BEM and the X-BEM have superior performance comparing to the X-FEM. Nevertheless, the dual BEM has obtained slightly better accuracy than the X-BEM. The X-BEM has been proven to be a viable alternative to the dual BEM with quarter-point elements.

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