Classifying the Clique-Width of H-Free Bipartite Graphs*

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Abstract. Let G be a bipartite graph, and let H be a bipartite graph with a fixed bipartition (B_H, W_H). We consider three different, natural ways of forbidding H as an induced subgraph in G. First, G is H-free if it does not contain H as an induced subgraph. Second, G is strongly H-free if G is H-free or else has no bipartition (B_G, W_G) with $B_H \subseteq B_G$ and $W_H \subseteq W_G$. Third, G is weakly H-free if G is H-free or else has at least one bipartition (B_G, W_G) with $B_H \not\subseteq B_G$ or $W_H \not\subseteq W_G$. Lozin and Volz characterized all bipartite graphs H for which the class of strongly H-free bipartite graphs has bounded clique-width. We extend their result by giving complete classifications for the other two variants of H-freeness.

1 Introduction

The *clique-width* of a graph G, is a well-known graph parameter that has been studied both in a structural and in an algorithmic context. It is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;
(ii) taking the disjoint union of two labelled graphs G_1 and G_2;
(iii) joining each vertex with label i to each vertex with label j $(i \neq j)$;
(iv) renaming label i to j.

We refer to the surveys of Gurski [13] and Kamiński, Lozin and Milanič [14] for an in-depth study of the properties of clique-width.

We say that a class of graphs has *bounded* clique-width if every graph from the class has clique-width at most p for some constant p. As many NP-hard graph problems can be solved in polynomial time on graph classes of bounded clique-width [10,15,20,21], it is natural to determine whether a certain graph class has bounded clique-width and to find new graph classes of bounded clique-width. In particular, many papers determined the clique-width of graph classes characterized by one or more forbidden induced subgraphs [1,2,3,4,5,6,7,8,9,11,16,17,18,19].

In this paper we focus on classes of bipartite graphs characterized by a forbidden induced subgraph H. A graph G is H-free if it does not contain H as

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an induced subgraph. If G is bipartite, then when considering notions for H-freeness, we may assume without loss of generality that H is bipartite as well. For bipartite graphs, the situation is more subtle as one can define the notion of freeness with respect to a fixed bipartition (B_H, W_H) of H. This leads to two other notions (see also Section 2 for formal definitions). We say that a bipartite graph G is strongly H-free if G is H-free or else has no bipartition (B_G, W_G) with $B_H \subseteq B_G$ and $W_H \subseteq W_G$. Strongly H-free graphs have been studied with respect to their clique-width, although under less explicit terminology (see e.g. [14,17,18]). In particular, Lozin and Volz [18] completely determined those bipartite graphs H, for which the class of strongly H-free graphs has bounded clique-width (we give an exact statement of their result in Section 3). If G is H-free or else has at least one bipartition (B_G, W_G) with $B_H \not\subseteq B_G$ or $W_H \not\subseteq W_G$, then G is said to be weakly H-free. As far as we are aware this notion has not been studied with respect to the clique-width of bipartite graphs.

Our Results: We completely classify the classes of H-free bipartite graphs of bounded clique-width. We also introduce the notion of weak H-freeness for bipartite graphs and characterize those classes of weakly H-free bipartite graphs that have bounded clique-width. In this way, we have identified a number of new graph classes of bounded clique-width. Before stating our results precisely in Section 3, we first give some terminology and examples in Section 2. In Section 4 we give the proofs of our results.

2 Terminology and Examples

We first give some terminology on general graphs, followed by terminology for bipartite graphs. We illustrate the definitions of H-freeness, strong H-freeness and weak H-freeness of bipartite graphs with some examples. As we will explain, these examples also make clear that all three notions are different from each other.

General graphs: Let G and H be graphs. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. A bijection of the vertices $f : V_G \to V_H$ is called a *(graph) isomorphism* when $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. If such a bijection exists then G and H are *isomorphic*. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. A graph G is $(H_1, \ldots, H_p)$*-free* if no H_i is an induced subgraph of G. If $p = 1$ we may write H_1-free instead of (H_1)-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote the disjoint union of r copies of G by rG.

Bipartite graphs: A graph G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. Let H be a bipartite graph. We say that H is a *labelled* bipartite graph if we are also given a *black-and-white labelling* ℓ, which is a labelling that assigns either the colour "black" or the colour "white" to each vertex of H in such a way that the two resulting monochromatic colour classes B_H^ℓ and W_H^ℓ form a partition of H into two (possibly empty) independent sets. From now on we denote a graph H with such a labelling ℓ by

$H^\ell = (B_H^\ell, W_H^\ell, E_H)$. Here the pair (B_H^ℓ, W_H^ℓ) is *ordered*, that is, $(B_H^\ell, W_H^\ell, E_H)$ and $(W_H^\ell, B_H^\ell, E_H)$ are different labelled bipartite graphs.

We say that two labelled bipartite graphs H_1^ℓ and $H_2^{\ell^*}$ are *isomorphic* if the (unlabelled) graphs H_1 and H_2 are isomorphic, and if in addition there exists an isomorphism $f : V_{H_1} \to V_{H_2}$ such that for all $u \in V_{H_1}$, $u \in W_{H_1}^\ell$ if and only if $f(u) \in W_{H_2}^{\ell^*}$. Moreover, if $H_1 = H_2$, then ℓ and ℓ^* are said to be *isomorphic* labellings. For example, the bipartite graphs $(\{u, v\}, \emptyset)$ and $(\{x, y\}, \emptyset)$ are isomorphic, and the labelled bipartite graph $(\{u, v\}, \emptyset, \emptyset)$ is isomorphic to the labelled bipartite graph $(\{x, y\}, \emptyset, \emptyset)$. However, $(\{x, y\}, \emptyset, \emptyset)$ is neither isomorphic to $(\emptyset, \{x, y\}, \emptyset)$ nor to $(\{x\}, \{y\}, \emptyset)$ (also see Fig. 1).

We write $H_1^\ell \subseteq_{li} H_2^{\ell^*}$ if $H_1 \subseteq_i H_2$, $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$ and $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$. In this case we say that H_1^ℓ is a *labelled* induced subgraph of $H_2^{\ell^*}$. Note that the two labelled bipartite graphs $H_1^{\ell_1}$ and $H_2^{\ell_2}$ are isomorphic if and only if $H_1^{\ell_1}$ is a labelled induced subgraph of $H_2^{\ell_2}$, and vice versa.

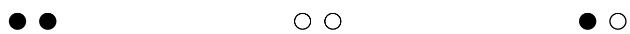


Fig. 1: The graph $2P_1$ partitioned into three ways; none of these three labelled bipartite graphs are isomorphic to each other.

Let G be an (unlabelled) bipartite graph, and let H^ℓ be a labelled bipartite graph. We say that G contains H^ℓ as a *strongly labelled* induced subgraph if $H^\ell \subseteq_{li} (B_G, W_G, E_G)$ for some bipartition (B_G, W_G) of G. If not, then G is said to be *strongly H^ℓ-free*. We say that G contains H^ℓ as a *weakly labelled* induced subgraph if $H^\ell \subseteq_{li} (B_G, W_G, E_G)$ for all bipartitions (B_G, W_G) of G. If not, then G is said to be *weakly H^ℓ-free*. Equivalently, G is strongly H^ℓ-free if for every labelling ℓ^* of G, G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph and G is weakly H^ℓ-free if there is a labelling ℓ^* of G such that G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. Note that these two notions of freeness are only defined for (unlabelled) bipartite graphs. Let $\{H_1^{\ell_1}, \dots, H_p^{\ell_p}\}$ be a set of labelled bipartite graphs. Then a graph G is *strongly (weakly) $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$-free* if G is strongly (weakly) $H_i^{\ell_i}$-free for $i = 1, \dots, p$.

The following lemma shows that for all labelled bipartite graphs H^ℓ, the class of H-free graphs is a (possibly proper) subclass of the class of strongly H^ℓ-free bipartite graphs and that the latter graph class is a (possibly proper) subclass of the class of weakly H^ℓ-free bipartite graphs.

Lemma 1. *Let G be a bipartite graph and H^ℓ be a labelled bipartite graph. The following two statements hold:*

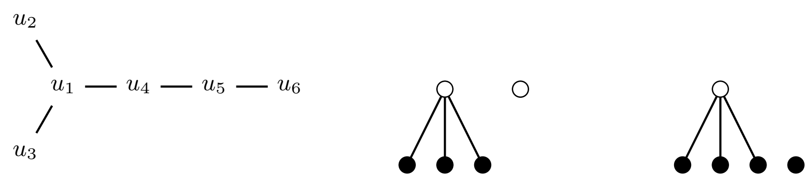
(i) If G is H-free, then G is strongly H^ℓ-free.
(ii) If G is strongly H^ℓ-free, then G is weakly H^ℓ-free.

Moreover, the two reverse statements are not necessarily true.

Proof. Statements (i) and (ii) follow by definition. The following two examples, which are also depicted in Fig. 2, show that the reverse statements may not necessarily be true. Let G be isomorphic to $S_{1,1,3}$ with $V_G = \{u_1, \dots, u_6\}$ and $E_G = \{u_1u_2, u_1u_3, u_1u_4, u_4u_5, u_5u_6\}$. Let $H = K_{1,3} + P_1$. We denote the vertex set and edge set of H by $V_H = \{x_1, x_2, x_3, x_4, x_5\}$ and $E_H = \{x_1x_2, x_1x_3, x_1x_4\}$.

Let $H^\ell = (\{x_2, x_3, x_4\}, \{x_1, x_5\}, E_H)$. We first notice that G is not H-free, because $G[u_1, u_2, u_3, u_4, u_6]$ is isomorphic to $K_{1,3} + P_1$. However, we do have that G is strongly H^ℓ-free, because H^ℓ is neither a labelled induced subgraph of $(\{u_1, u_5\}, \{u_2, u_3, u_4, u_6\}, E_G\}$ nor of $(\{u_2, u_3, u_4, u_6\}, \{u_1, u_5\}, E_G\}$.

Let $H^{\ell^*} = (\{x_2, x_3, x_4, x_5\}, \{x_1\}, E_H)$. Then G is not strongly H^{ℓ^*}-free, because $(\{u_2, u_3, u_4, u_6\}, \{u_1\}, \{u_1u_2, u_1u_3, u_1u_4\})$ is isomorphic to H^{ℓ^*}. However, G is weakly H^{ℓ^*}-free, because H^{ℓ^*} is not a labelled induced subgraph of $(\{u_1, u_5\}, \{u_2, u_3, u_4, u_6\}, E_G\})$. $\square$



(a) G (b) H^ℓ (c) H^{ℓ^*}

Fig. 2: The graphs G, H^ℓ and H^{ℓ^*} from the proof of Lemma 1.

Special Graphs: For $r \geq 1$, the graphs C_r, K_r, P_r denote the cycle, complete graph and path on r vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. If $r = 3$, the graph $K_{1,r}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ denote the tree that has only one vertex x of degree 3 and that has exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{h,i,j}$ is called a *subdivided claw*.

Let $H^\ell = (B_H^\ell, W_H^\ell, E_H)$ be a labelled bipartite graph. The *opposite* of H^ℓ is defined as the labelled bipartite graph $H^{\overline{\ell}} = (W_H^\ell, B_H^\ell, E_H)$. We say that $\overline{\ell}$ is the *opposite* black-and-white labelling of ℓ. Suppose that H is a bipartite graph such that among all its black-and-white labellings, all those that maximize the number of black vertices are isomorphic. In this case we pick one of such labelling and call it b.

3 The Classifications

A full classification of the boundedness of the clique-width of strongly H^ℓ-free bipartite graphs was given by Lozin and Voltz [18], except that in their result the trivial case when $H^\ell = (sP_1)^b$ or $H^\ell = (sP_1)^{\overline{b}}$ for some $s \geq 1$ was missing. Their proof is correct except that it overlooked this case, which occurs when one of the colour classes of the labelled graph H^ℓ is empty. However, strongly $(sP_1)^b$-free bipartite graphs can have at most $2s-2$ vertices, and as such form a class of bounded clique-width. Below we state their result after incorporating this small correction, followed by our results for the other two variants of freeness. We refer to Fig. 3 for pictures of the labelled bipartite graphs used in Theorems 1 and 3.

Theorem 1 ([18]). *Let H^ℓ be a labelled bipartite graph. The class of strongly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H^\ell = (sP_1)^b$ *or* $H^\ell = (sP_1)^{\overline{b}}$ *for some* $s \geq 1$
- $H^\ell \subseteq_{li} (K_{1,3}+3P_1)^b$ *or* $H^\ell \subseteq_{li} (K_{1,3}+3P_1)^{\overline{b}}$
- $H^\ell \subseteq_{li} (K_{1,3}+P_2)^b$ *or* $H^\ell \subseteq_{li} (K_{1,3}+P_2)^{\overline{b}}$
- $H^\ell \subseteq_{li} (P_1+S_{1,1,3})^b$ *or* $H^\ell \subseteq_{li} (P_1+S_{1,1,3})^{\overline{b}}$
- $H^\ell \subseteq_{li} (S_{1,2,3})^b$ *or* $H^\ell \subseteq_{li} (S_{1,2,3})^{\overline{b}}$.

Theorem 2. *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H = sP_1$ *for some* $s \geq 1$
- $H \subseteq_i K_{1,3}+3P_1$
- $H \subseteq_i K_{1,3}+P_2$
- $H \subseteq_i P_1+S_{1,1,3}$
- $H \subseteq_i S_{1,2,3}$.

Theorem 3. *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H^\ell = (sP_1)^b$ *or* $H^\ell = (sP_1)^{\overline{b}}$ *for some* $s \geq 1$
- $H^\ell \subseteq_{li} (P_1+P_5)^b$ *or* $H^\ell \subseteq_{li} (P_1+P_5)^{\overline{b}}$
- $H \subseteq_i P_2+P_4$
- $H \subseteq_i P_6$.

4 The Proofs of Our Results

We first recall a number of basic facts on clique-width known from the literature. We then state a number of other lemmas which we use to prove Theorems 2 and 3.

4.1 Facts about Clique-width

The *bipartite complement* of a bipartite graph *with respect to* a bipartition (B,W) is the bipartite graph with bipartition (B,W), in which two vertices $u \in B$ and $v \in W$ are adjacent if and only if $uv \notin E$. For instance, the graph $2P_2$ has C_4 as its only bipartite complement, whereas the graph $2P_1$ has $2P_1$ and P_2 as its bipartite complements. For two disjoint vertex subsets X and Y in G, the *bipartite complementation* operation with respect to X and Y acts on G

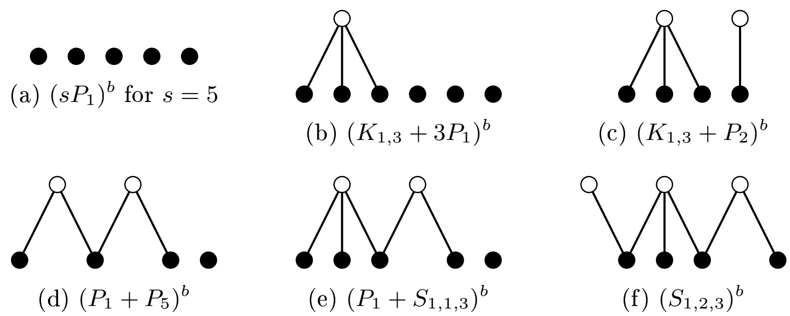


Fig. 3: The labelled bipartite graphs used in Theorems 1 and 3.

by replacing every edge with one end-vertex in X and the other one in Y by a non-edge and vice versa. The *edge subdivision* operation replaces an edge vw in a graph by a new vertex u with edges uv and uw.

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. We will say that a graph operation *preserves* boundedness of clique-width if for every constant k and every graph class $\mathcal{G}$, the graph class $\mathcal{G}_{[k]}$ obtained by performing the operation at most k times on each graph in $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [16].

Fact 2. Bipartite complementation preserves boundedness of clique-width [14].

Fact 3. For a class of graphs $\mathcal{G}$ of *bounded* degree, let $\mathcal{G}'$ be the class of graphs obtained from $\mathcal{G}$ by applying zero or more edge subdivision operations to each graph in $\mathcal{G}$. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [14].

We also use some other elementary results on the clique-width of graphs. In order to do so we need the notion of a *wall*. We do not formally define this notion, but instead refer to Fig. 4, in which three examples of walls of different height are depicted. A *k-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly k times for some constant $k \geq 0$. The next well-known lemma follows from combining Fact 3 with the fact that walls have maximum degree 3 and unbounded clique-width (see e.g. [14]).

Lemma 2. *For every constant k, the class of k-subdivided walls has unbounded clique-width.*

We let $\mathcal{S}$ be the class of graphs each connected component of which is either a subdivided claw $S_{h,i,j}$ for some $1 \leq h \leq i \leq j$ or a path P_r for some $r \geq 1$. This leads to the following lemma, which is well-known and follows from the fact that walls have maximum degree at most 3 and from Lemma 2 by choosing an appropriate value for k (also note that k-subdivided walls are bipartite for all $k \geq 0$).

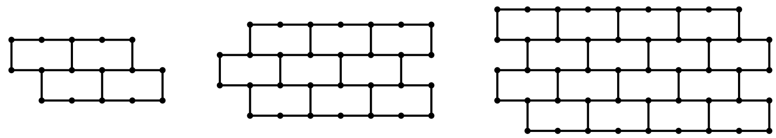


Fig. 4: Walls of height 2, 3, and 4, respectively.

Lemma 3. *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. If $H_i \notin \mathcal{S}$ for $i = 1, \ldots, p$ then the class of $(H_1, \ldots, H_p)$-free bipartite graphs has unbounded clique-width.*

4.2 A Number of Other Lemmas

We start with a lemma which is related to Lemma 1 and which follows immediately from the corresponding definitions.

Lemma 4. *Let G and H be bipartite graphs. Then G is H-free if and only if G is strongly H^ℓ-free for all black-and-white labellings ℓ of H.*

A graph G that contains a graph H as an induced subgraph may be weakly H^ℓ-free for all black-and-white labellings ℓ of H; take for instance the graphs G and H from the proof of Lemma 1. However, we can make the following observation, which also follows directly from the corresponding definitions.

Lemma 5. *Let H be a bipartite graph with a unique black-and-white labelling ℓ (up to isomorphism). Then every bipartite graph G is H-free if and only if it is weakly H^ℓ-free.*

Note that there exist both connected bipartite graphs (for example $H = P_6$) and disconnected bipartite graphs (for example $H = 2P_2$) that satisfy the condition of Lemma 5.

Two black-and-white labellings of a bipartite graph H are said to be *equivalent* if they are isomorphic or opposite to each other; otherwise they are said to be *non-equivalent*. The following lemma follows directly from the definitions.

Lemma 6. *Let ℓ and ℓ^* be two equivalent black-and-white labellings of a bipartite graph H. Then the class of strongly (weakly) H^ℓ-free graphs is equal to the class of strongly (weakly) H^{ℓ^*}-free graphs.*

The following lemma is due to Lozin and Rautenbach [17].

Lemma 7 ([17]). *Let $\{H_1^{\ell_1}, \ldots, H_p^{\ell_p}\}$ be a finite set of labelled bipartite graphs. For $i = 1, \ldots, p$, let F_i denote the bipartite complement of H_i with respect to $(B_{H_i}^{\ell_i}, W_{H_i}^{\ell_i})$. If $H_i \notin \mathcal{S}$ for all $1 \leq i \leq p$ or $F_i \notin \mathcal{S}$ for all $1 \leq i \leq p$, then the class of strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width.*

In the next lemma we demonstrate a list of H-free bipartite classes with unbounded clique-width. It is obtained by combining a known result of Lozin and Voltz [18] with a number of new results.

Lemma 8. *The class of H-free bipartite graphs has unbounded clique-width if* $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3\}$.

Proof. Lozin and Voltz [18] showed that $2P_3$-free bipartite graphs have unbounded clique-width. Let $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$, and let $\{H^{\ell_1}, \dots, H^{\ell_p}\}$ be the set of all non-equivalent labelled bipartite graphs isomorphic to H. For $i = 1, \dots, p$, let F_i denote the bipartite complement of H with respect to $(B_H^{\ell_i}, W_H^{\ell_i})$. We will show that every F_i does not belong to $\mathcal{S}$. Then, by Lemma 7 the class of strongly $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width. Because a bipartite graph is H-free if and only if it is strongly $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$-free (by Lemmas 4 and 6), this means that the class of H-free bipartite graphs has unbounded clique-width.

Suppose $H \in \{2P_1 + 2P_2, 2P_1 + P_4\}$. Let $V_H = \{x_1, \dots, x_6\}$ with $E_H = \{x_1x_2, x_3x_4\}$ if $H = 2P_1 + 2P_2$ and $E_H = \{x_1x_2, x_2x_3, x_3x_4\}$ if $H = 2P_1 + P_4$. Then H has only two non-equivalent black-and-white labellings. We may assume without loss of generality that one of these two labellings colours x_1, x_3, x_5, x_6 black and x_2, x_4 white, whereas the other one colours x_1, x_3, x_5 black and x_2, x_4, x_6 white. Let F_1 and F_2 be the bipartite complements corresponding to the first and second labellings, respectively. The vertices x_2, x_4, x_5, x_6 induce a C_4 in F_1, whereas the vertices x_1, x_4, x_5, x_6 induce a C_4 in F_2. Hence, F_1 and F_2 do not belong to $\mathcal{S}$.

Suppose $H = 4P_1 + P_2$. Let $V_H = \{x_1, \dots, x_6\}$ and $E_H = \{x_1x_2\}$. Then H has three non-equivalent black-and-white labellings. We may assume without loss of generality that the first one colours x_1, x_3, x_4, x_5, x_6 black and x_2 white, the second one colours x_1, x_3, x_4, x_5 black and x_2, x_6 white, and the third one colours x_1, x_3, x_4 black and x_2, x_5, x_6 white. Let F_1, F_2, F_3 denote the corresponding bipartite complements. The vertices $x_2, \dots, x_6$ induce a $K_{1,4}$ in F_1. The vertices x_2, x_3, x_4, x_6 induce a C_4 in F_2 and F_3. Hence, none of F_1, F_2, F_3 belongs to $\mathcal{S}$.

Suppose $H = 3P_2$. Let $V_H = \{x_1, \dots, x_6\}$ and $E_H = \{x_1x_2, x_3x_4, x_5x_6\}$. Let ℓ be a black-and-white labelling of H that colours x_1, x_3, x_5 black and x_2, x_4, x_6 white. Then every other labelling ℓ^* of H is isomorphic to ℓ. The bipartite complement of H with respect to (B_H^{ℓ}, W_H^{ℓ}) is isomorphic to C_6, which does not belong to $\mathcal{S}$. □

We will also need the following lemma. We omit the proof due to space restrictions.

Lemma 9. *Let $H \in \mathcal{S}$. Then H is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free if and only if $H = sP_1$ for some integer $s \geq 1$ or H is an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$.*

The last lemma we need before proving the main results of this paper is the following one (we use it several times in the proof of Theorem 3).

Lemma 10. *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has unbounded clique-width in both of the following cases:*

(i) H^ℓ contains a vertex of degree at least 3, or
(ii) H^ℓ contains four independent vertices, not all of the same colour.

Proof. Let b_1 be a black-and-white labelling of $4P_1$ that colours three vertices black and one vertex white. Let b_2 be a black-and-white labelling of $4P_1$ that colours two vertices black and two vertices white. We show below that the class of weakly H^ℓ-free bipartite graphs has unbounded clique-width if $H^\ell \in \{(K_{1,3})^b, (4P_1)^{b_2}, (4P_1)^{b_3}\}$. Then we are done by Lemma 6.

Consider a 1-subdivided wall G' obtained from a wall G. Recall that 1-subdivided walls are bipartite. Moreover, the vertices that were introduced when subdividing every edge of G all have degree 2 and form one class of a bipartition (B, W) of G'. Let this class be B. Then $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, G' is weakly $(K_{1,3})^b$-free. This means that the class of weakly $(K_{1,3})^b$-free graphs contains the class of 1-subdivided walls. As such, it has unbounded clique-width by Lemma 2. The bipartite complement G'' of G' with respect to (B, W) is weakly $(4P_1)^{b_1}$-free, as $(K_{1,3})^b$ is the bipartite complement of $(4P_1)^{b_1}$ and $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, the class of weakly $(4P_1)^{b_1}$-free graphs has unbounded clique-width by Fact 2. The class of weakly $(4P_1)^{b_2}$-free bipartite graphs has unbounded clique-width by Lemma 1 and Theorem 1. □

4.3 The Proof of Theorem 2

Proof. We first deal with the bounded cases. First suppose $H = sP_1$ for some $s \geq 1$. Then every H-free bipartite graph G has at most $s - 1$ vertices in each partition class for every bipartition. This means that the clique-width of G is at most $2s - 2$. Now suppose that $H \in \{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. Then the claim follows from combining Lemma 1 with Theorem 1.

We now deal with the unbounded cases. Suppose $H \neq sP_1$ for some $s \geq 1$ and that H is not an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. Then by Lemma 9, either $H \notin \mathcal{S}$ or, H is not $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free. Hence, the clique-width of the class of H-free bipartite graphs is unbounded by Lemmas 3 and 8, respectively. □

4.4 The Proof of Theorem 3

Proof. We first consider the bounded cases. First suppose $H^\ell = (sP_1)^b$ for some $s \geq 1$ (the $H^\ell = (sP_1)^{\overline{b}}$ case is equivalent). Then every weakly H^ℓ-free bipartite graph has a bipartition (B, W) with $|B| \leq s - 1$. Hence, the clique-width of such a graph is at most $s+1$ (first introduce the vertices of B by using distinct labels, then use two more labels for the vertices of W, introducing them one-by-one).

Before considering the case $H^\ell = (P_1 + P_5)^b$, we first consider the case where $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$. We first assume that $H = P_2 + P_4$ or $H = P_6$. Then

$H \subseteq_i S_{1,2,3}$, which implies that that the class of H-free bipartite graphs has bounded clique-width by Theorem 2. All black-and-white labellings of $P_2 + P_4$ are isomorphic. Similarly, all black-and-white labellings of P_6 are isomorphic. Hence, the class of H-free bipartite graphs coincides with the class of weakly H^ℓ-free graphs by Lemma 5. We therefore conclude that the latter class also has bounded clique-width.

Now let $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$, but $H \notin \{P_2 + P_4, P_6\}$. Note that $P_2 + P_4$ and P_6 have a unique labelling b (up to isomorphism). If H^ℓ is not a labelled induced subgraph of one of $\{(P_2 + P_4)^b, P_6^b\}$ then H must have two non-equivalent black-and-white labellings. Since H is a linear forest, it must have at least two components with an odd number of vertices. Therefore $H \in \{2P_1, 3P_1, P_1 + P_3, 2P_1 + P_2\}$. However, in all these cases, for every labelling ℓ of H, $H^\ell \subseteq_{li} P_6^b$ or $H^\ell \subseteq_{li} (P_2 + P_4)^b$. Therefore, if $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$ then for every labelling ℓ of H, the weakly H^ℓ-free bipartite graphs are a subclass of either the P_6-free or $(P_2 + P_4)$-free bipartite graphs. In particular, this holds for $H^\ell = (P_1 + 2P_2)^b$ (we need this observation for the following case).

Finally, suppose $H^\ell = (P_1 + P_5)^b$. Let G be a weakly H^ℓ-free bipartite graph. Then G has a labelling ℓ^* such that H^ℓ is not a labelled induced subgraph of $(B_G^{\ell^*}, W_G^{\ell^*}, E_G)$. If $|B_G^{\ell^*}|$ is even, then we delete a vertex of $B_G^{\ell^*}$. We may do this by Fact 1. Hence $|B_G^{\ell^*}|$ may be assumed to be odd. Let X be the subset of $W_G^{\ell^*}$ that consists of all vertices that are adjacent to less than half of the vertices of $B_G^{\ell^*}$. We apply the bipartite complementation between X and $B_G^{\ell^*}$. We may do this by Fact 2. Let G_1 be the resulting bipartite graph, with bipartition classes $B_{G_1}^{\ell^*} = B_G^{\ell^*}$ and $W_{G_1}^{\ell^*} = W_G^{\ell^*}$.

Suppose $B_{G_1}^{\ell^*}$ contains three vertices b_1, b_2, b_3 and $W_{G_1}^{\ell^*}$ contains two vertices w_1, w_2 such that $G_1^{\ell^*}[b_1, b_2, b_3, w_1, w_2]$ is isomorphic to $(P_1 + 2P_2)^b$. By construction and because $|B_{G_1}^{\ell^*}| = |B_G^{\ell^*}|$ is odd, w_1 and w_2 have at least one common neighbour $b_4 \in B_{G_1}^{\ell^*}$. Then $G_1^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is isomorphic to $(P_1 + P_5)^b$. However, then $G^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is also isomorphic to $(P_1 + P_5)^b$ (irrespective of whether w_1 or w_2 belong to X), which is a contradiction. We conclude that G_1 is weakly $(P_1 + 2P_2)^b$-free. As observed above, this means that G_1 has bounded clique-width. Hence G has bounded clique-width.

We now consider the unbounded cases. Let H^ℓ be a labelled bipartite graph that is not isomorphic to one of the (bounded) cases considered already. Suppose that H contains a cycle or an induced subgraph isomorphic to $2P_3$. Then the class of weakly H^ℓ-free graphs has unbounded clique-width by combining Lemma 1 with Theorem 2. Suppose that H contains a vertex of degree at least 3. Then the class of weakly H^ℓ-free bipartite graphs has unbounded clique-width by Lemma 10(i). It remains to consider the case when $H = sP_1 + tP_2 + P_r$ for some constants $1 \le r \le 6$, $s \ge 0$ and $t \ge 0$, where $\max\{s, t\} \ge 1$ (as H is not an induced subgraph of P_6).

Suppose $5 \le r \le 6$. Assume without loss of generality that three vertices of the copy of P_r in H^ℓ are coloured black. If $r = 6$ or $t \ge 1$ or some copy P_1 in H^ℓ is coloured white, or two copies of P_1 in H^ℓ are coloured black, then we can apply Lemma 10(ii). Hence, $H^\ell = (P_1 + P_5)^b$, which is not possible by assumption.

Suppose $r = 4$. If two vertices in the induced subgraph of H^ℓ isomorphic to $sP_1 + tP_2$ have the same colour then we can apply Lemma 10(ii). Hence we may assume that $s \leq 2$ and $t \leq 1$, and moreover that $s = 0$ if $t = 1$. Also we would have $H \subseteq_i P_2 + P_4$ if $s = 0$ and $t = 1$ or if $s = 1$ and $t = 0$. Hence, it remains to consider the case $s = 2$ and $t = 0$, such that one copy of P_1 is coloured black and the other one white. In that case, we may apply Lemma 10(ii).

Suppose $r = 3$. Assume without loss of generality that the two vertices of the copy of P_3 in H^ℓ are coloured black. Recall that $s \geq 1$ or $t \geq 1$. If $t \geq 2$, then we can apply Lemma 10(ii). Suppose $t = 1$. Then $s = 0$ otherwise H^ℓ would contain an induced $4P_1$ in which not all the vertices are the same colour, in which case we could apply Lemma 10(ii). However, this means that H is an induced subgraph of $P_2 + P_4$. Now suppose $t = 0$. Then $s \geq 2$, as otherwise H is an induced subgraph of $P_2 + P_4$. If $s \geq 3$ then H^ℓ contains an induced $4P_1$ in which not all the vertices are the same colour, in which case we apply Lemma 10(ii). Hence, $s = 2$ and both copies are coloured black (otherwise we apply Lemma 10(ii)). However, in this case H^ℓ is a labelled induced subgraph of $(P_1 + P_5)^b$, which is not possible by assumption.

Finally suppose that $r \leq 2$. Then we may write $H = sP_1 + tP_2$ instead. We must have $s + t \geq 4$ or $t \geq 3$, otherwise H would be an induced subgraph of $P_2 + P_4$ or P_6. If $t = 0$ then since $H^\ell \neq (sP_1)^b$ and $H^\ell \neq (sP_1)^{\overline{b}}$ we can find four copies of P_1 in H that are not all of the same colour and apply Lemma 10(ii). If $t \geq 1, s + t \geq 4$, we can also find four copies of P_1 that are not all of the same colour and apply Lemma 10(ii). Finally, suppose $s = 0, t = 3$. In this case we combine Lemmas 1 and 8. This completes the proof. $\square$

5 Conclusions

We have completely determined those bipartite graphs H for which the class of H-free bipartite graphs has bounded clique-width. We also characterized exactly those labelled bipartite graphs H for which the class of weakly H-free bipartite graphs has bounded clique-width. These results complement the known characterization of Lozin and Volz [18] for strongly H-free bipartite graphs. A natural direction for further research would be to characterize, for each of the three notions of H-freeness, the clique-width of classes of $\mathcal{H}$-free bipartite graphs when $\mathcal{H}$ is a set containing at least 2 graphs. In a follow-up paper [12], we apply our results for H-free bipartite graphs to determine classes of (H_1, H_2)-free (general) graphs of bounded and unbounded clique-width.

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