An implicit enrichment approach in the boundary element method framework for stress intensity factors calculation in anisotropic materials

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Abstract. In this work we explore the use of enrichment functions embedded into the boundary element method (BEM) formulation. The main advantage of this approach is the reduced additional degrees of freedom generated compared to the classic partition of unity approach. The enrichment functions were obtained using the Stroh formalism, a concise formulation which depends only on the material properties. Some numerical examples are provided to show the performance of the proposed approach.

Introduction

The boundary element method (BEM) has been established as a reference discretisation method when dealing with fracture mechanics problems. High accuracy, stability in providing results for the singular stress fields at the crack tips are some of the advantages of the BEM compared to the more commonly used domain discretisation methods such as the finite element method (FEM). Over the last 15 years, FEM has experienced a breakthrough, after the introduction of the partition of unity [1, 2], leading to the so called extended finite element method (X-FEM). Solutions obtained with X-FEM could match the ones found with BEM [3]. Later, the partition of unity has also been applied to BEM for isotropic materials [4].

However, in fracture mechanics, the most important parameter is the stress intensity factor (SIF). Different methods have been used, the most common are the energy approaches using the J-integral [5] or the more general interaction integral . These methods can require as much computational resources as finding the solution of the fracture mechanics problem when using BEM or the extended finite element method (X-FEM). Moreover, dealing with 2D or 3D fracture problems and/or multiple cracks can make SIF calculation cumbersome.

The proposed method in this paper removes the mentioned limitations, since the SIF will now be part of the fracture mechanics problem, extending the work the authors [6] have done for isotropic materials. By using an enrichment similar to the one employed by Benzley [7], it is possible to use enrichment functions that span the asymptotic behaviour at the crack tip for a fully anisotropic material to then include the SIF as part of the solution of the fracture problem. The same enrichment functions obtained in [3] are employed. The Stroh formalism was used in these enrichment functions, which is a powerful mathematical formulation.

The implicit enrichment is embedded into a dual BEM formulation, and some numerical examples are presented to validate the proposed method.

Governing equations

Consider an anisotropic elastic domain Ω, in which the static equilibrium equations in the presence of body forces $\mathbf{b}$ are defined as

$$\sigma_{ij,j} + b_i = 0 \tag{1}$$

Symmetry applies for the stress and strain tensors, i.e.:

$$\sigma_{ij} = \sigma_{ji} \tag{2}$$

$$\varepsilon_{ij} = \varepsilon_{ji} \tag{3}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

and u_i stands for the displacement on the i-direction.

The linear constitutive equations are given by the generalised Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{5}$$

where C_{ijkl} define the material constants tensor, satisfying the following symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{6}$$

that lead to a tensor with only 21 independent components for the 3D case, and 6 components in the 2D case.

Enrichment formulation

Adopting a polar coordinate system (r, θ) with origin at the crack tip, the asymptotic displacement field around a crack-tip in a plane anisotropic domain can be expressed by means of the Stroh formalism [9] as

$$u_j(r, \theta) = \sqrt{\frac{2}{\pi}}\Re\left(K_\alpha A_{jm} B_{m\alpha}^{-1}\sqrt{r\left(\cos\theta + \mu_m\sin\theta\right)}\right) \tag{7}$$

where the summation convention over repeated indices applies; $i, M = 1, 2$; $\alpha = I, II$ is related to the elastic fracture modes; and $\Re(\cdot)$ is the real part of $(\cdot)$; $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\mu}$ are obtained from the following eigenvalue problem

$$\left(\begin{array}{c|c} -\mathbf{C_{22}}^{-1}\mathbf{C_{21}} & -\mathbf{C_{22}}^{-1} \\ \hline \mathbf{C_{11}} - \mathbf{C_{21}}^T\mathbf{C_{22}}^{-1}\mathbf{C_{21}} & -\mathbf{C_{21}}^T\mathbf{C_{22}}^{-1} \end{array}\right)\left(\frac{\mathbf{A}_m}{\mathbf{B}_m}\right) = \mu_m\left(\frac{\mathbf{A}_m}{\mathbf{B}_m}\right) \quad \text{(no sum on } m) \tag{8}$$

with

$$\mathbf{C_{11}} := \mathbf{C}_{1ij1}; \quad \mathbf{C_{21}} := \mathbf{C}_{2ij1}; \quad \mathbf{C_{22}} := \mathbf{C}_{2ij2} \tag{9}$$

Using the same methodology as in [3], the displacements terms in Eq. (7) can be rearranged into the following set of enrichment functions:

$$\Psi_{lj}(r, \theta) = \left(\begin{array}{cc} \psi_{Ix} & \psi_{IIx} \\ \psi_{Iy} & \psi_{IIy} \end{array}\right) = \sqrt{\frac{2r}{\pi}}\Re\left(\begin{array}{cc} A_{11}B_{11}^{-1}\beta_1 + A_{12}B_{21}^{-1}\beta_2 & A_{11}B_{12}^{-1}\beta_1 + A_{12}B_{22}^{-1}\beta_2 \\ A_{21}B_{11}^{-1}\beta_1 + A_{22}B_{21}^{-1}\beta_2 & A_{21}B_{12}^{-1}\beta_1 + A_{22}B_{22}^{-1}\beta_2 \end{array}\right) \tag{10}$$

where $\beta_i = \sqrt{\cos\theta + \mu_i\sin\theta}$, r is the distance between the crack tip and an arbitrary position, θ is the orientation measured from a coordinate system centred at the crack tip. Note that these enrichment functions are the equivalent of Williams' expansion for the isotropic case [6].

The displacement field can be defined in a similar fashion as [6, 7, 8]

$$u_j = \sum_{a=1}^{M} N^a u_j^a + \tilde{K}_I\psi_{Ij} + \tilde{K}_{II}\psi_{IIj} \tag{11}$$

where N^a represents the shape function for node a, u_j^a is a general coefficient rather than the nodal displacement, M is the number of nodes, $\tilde{K}_I$, $\tilde{K}_{II}$ stand for the mode I and mode II elastic SIF, respectively, and they are now part of the solution vector instead of being calculated after the displacement solution is obtained. For the numerical discretisation of the fracture mechanics problem, the BEM is used.

Boundary Element Method (BEM)

The BEM has been established as a reference when dealing with linear elastic fracture mechanics problems [10]. When dealing with fracture mechanics problems, the dual BEM framework is usually applied. In this case, a new boundary integral equation (BIE) is introduced, in order to avoid the degeneration of the linear system of equations due to the use of the same BIE to model two overlapping surfaces (crack surfaces). The displacement BIE and the traction BIE are defined as

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_{\Gamma} p_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = \int_{\Gamma} u_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{12}$$

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_k \int_{\Gamma} s_{kij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = N_k \int_{\Gamma} d_{kij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{13}$$

where Γ represents the boundaries (including cracks) of the arbitrary elastic domain Ω, N_k is the normal at the observation point, u_{ij}^* and p_{ij}^* are the displacement and traction fundamental solutions, while d_{kij}^* and s_{kij}^* are obtained through derivation and further application of the generalised Hooke's law on the u_{ij}^* and p_{ij}^* kernels, respectively.

Substituting Eq .(11) into Eqs. (12) and (13) yields in

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_{\Gamma} p_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + \int_{\Gamma_c} p_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_l\psi_{lj}(\boldsymbol{\xi})d\Gamma = \int_{\Gamma} u_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{14}$$

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_k \int_{\Gamma} s_{kij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + N_k \int_{\Gamma_c} s_{kij}^*(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_l\psi_{lj}(\boldsymbol{\xi})d\Gamma = N_k \int_{\Gamma} d_{kij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{15}$$

where $\Gamma_c = \Gamma_+ \cup \Gamma_-$ stands for the crack surfaces Γ_+ and Γ_- . Let us remark that strongly singular and hypersingular terms arise from the integration of the p_{ij}^*, d_{rij}^* and s_{rij}^* kernels and they are regularised using the methodology proposed in [11], while the weakly singular terms are handled using Telles transformation [12].

The addition of $\tilde{K}_I$ and $\tilde{K}_{II}$ requires two more equations so the linear system of equations can be solved. The additional equations come from a restriction in the crack faces, in order to remove the displacement discontinuity observed at the crack tip. The displacement continuity can be enforced as

$$\sum_{a=1}^{L} N^a u_j^{a\ upper} = \sum_{a=1}^{L} N^a u_j^{a\ lower} \tag{16}$$

where L is the number of nodes used for the crack tip extrapolation. Eq. (16) is applied for both x and y directions, resulting in two different equations per crack tip. Moreover, these shape functions have to be evaluated at the crack tip.

Numerical results

Crack in an infinite anisotropic domain

First, we analyse a crack subject to a uniform loading in an infinite anisotropic domain. This problem has a pure mode I exact solution of $K_I = \sigma_\infty \sqrt{\pi a}$, where σ_∞ represents the applied loading and a is the half-length of the crack. The problem is depicted in Figure 1.

Table 1 shows the results for a crack discretised with 8 discontinuous elements per crack surface, and with the following material constants given in the Voigt notation: $C_{11} = 137.97$ GPa, $C_{12} = 5.78$ GPa, $C_{16} = 20.54$ GPa, $C_{22} = 12.45$ GPa, $C_{26} = 2.30$ GPa and $C_{66} = 12.98$ GPa. A modified version of the J-integral has been used for anisotropic materials, for more details see reference [13].

The extrapolation method consists of using the crack opening displacement (COD) and the crack relative sliding (CRS) to estimate the SIFs. The SIFs are thus given by [14]

$$\begin{pmatrix} K_{II} \\ K_I \end{pmatrix} = \sqrt{\frac{\pi}{8\bar{r}}}(\Re(i\mathbf{A}\mathbf{B}^{-1}))^{-1} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} \tag{17}$$
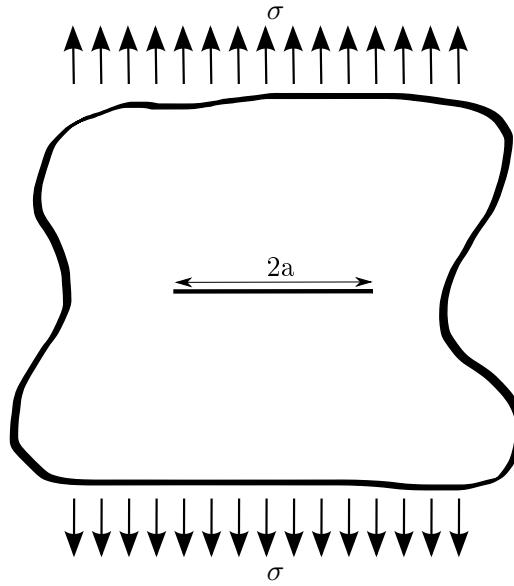
Figure 1: Crack in an infinite domain under uniform loading.

with $\overline{r} = L/6$, and L is the length of the element containing the crack tip.

SIF calculation	$K_I = 1$	Error (%)
Unenriched J-integral	1.0256	2.5656
Unenriched Extrapolation	1.1554	15.5441
Direct SIF	1.0000	0.00115
Enriched J-integral	1.0001	0.0091
Enriched Extrapolation	0.9999	−0.00002

Table 1: Results for the crack in an infinite anisotropic domain.

It is clear that the results obtained with the implicit enrichment are matching the exact solution. It is expected to have higher errors in the SIF extrapolation and the J-integral when no enrichment is used since there is no specific modelling of the asymptotic behaviour at the crack tip in this case.

Edge crack in a square composite plate

Next a square plate ($h/w = 1$) with an edge crack ($a/w = 0.5$) subject to a uniform loading is presented. The plate is a symmetric angle ply composite laminate consisting of four graphite-epoxy laminae. Figure 2 illustrates the problem.

The material properties of the plate are given as: $E_1 = 144.8$ GPa, $E_2 = 11.7$ GPa, $G_{12} = 9.66$ GPa and $\nu_{12} = 0.21$. The fibre orientation of the plate is rotated from $\theta = 0°$ to $\theta = 90°$. Results are given in Figure 3 and are compared with the BEM formulation from reference [14]. The BEM mesh consists of 8 discontinuous elements for the external boundaries, plus 8 discontinuous elements for each crack surface.

The error of the extrapolation method for the unenriched case compared to the reference [14] is over 16 %. One can verify that the direct SIF approach, the enriched extrapolation and J-integral as well as the unenriched J-integral present excellent agreement with the reference solution.

Conclusions

An implicit enrichment framework covering anisotropic materials has been presented in this work. The SIFs have been introduced as additional degrees of freedom straight into the dual BEM formulation, so when the displacement solution is obtained, so are the SIFs. This technique can save precious computational time especially when dealing with a large number of cracks. Moreover, only 2 new
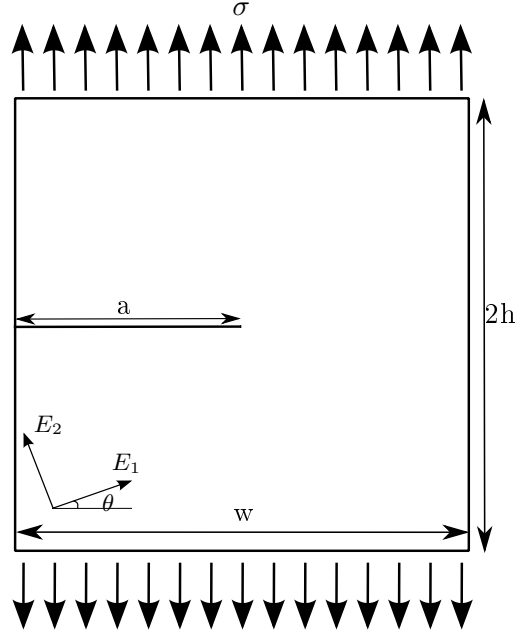
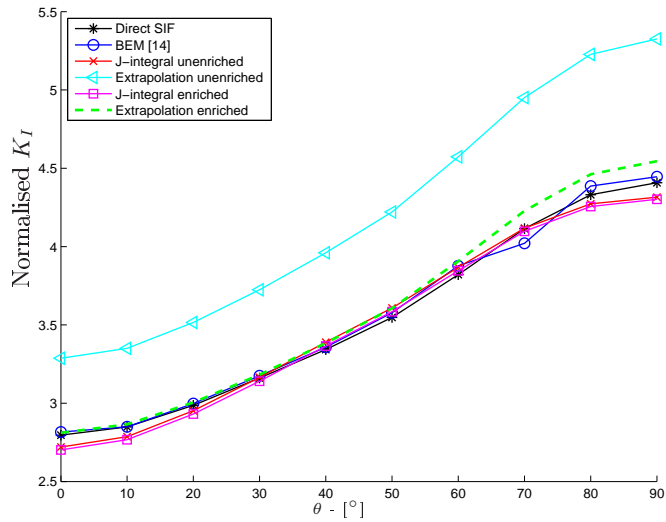Figure 2: Edge crack plate under a uniform load.

Figure 3: Results for the normalised mode I in an anisotropic composite plate.

degrees of freedom are introduced per crack tip, compared to the partition of unity where every enriched node means additional degrees of freedom. The numerical examples show excellent agreement with exact and reference solutions.

Acknowledgements

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