# Minimal Disconnected Cuts in Planar Graphs*

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**Abstract.** The problem of finding a disconnected cut in a graph is NP-hard in general but polynomial-time solvable on planar graphs. The problem of finding a minimal disconnected cut is also NP-hard but its computational complexity is not known for planar graphs. We show that it is polynomial-time solvable on 3-connected planar graphs but NP-hard for 2-connected planar graphs. Our technique for the first result is based on a structural characterization of minimal disconnected cuts in 3-connected $K_{3,3}$-free-minor graphs and on solving a topological minor problem in the dual. We show that the latter problem can be solved in polynomial-time even on general graphs. In addition we show that the problem of finding a minimal connected cut of size at least 3 is NP-hard for 2-connected apex graphs.

## 1 Introduction

A *cutset* or *cut* in a connected graph is a subset of its vertices whose removal disconnects the graph. The problem Stable Cut is that of testing whether a connected graph has a cut that is an independent set. Le, Mosca, and Müller [12] proved that this problem is NP-complete even for $K_4$-free planar graphs with maximum degree 5. A connected graph $G = (V, E)$ is *k-connected* for some integer $k$ if $|V| \geq k+1$ and every cut of $G$ has size at least $k$. It is not hard to see that if one can solve Stable Cut for 3-connected planar graphs in polynomial-time then one can do so for all planar graphs (in particular the problem is trivial

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if the graph has a cut-vertex or a cut set of two vertices that are non-adjacent). Hence, the problem is NP-complete for 3-connected planar graphs.

Due to the above it is a natural question whether one can relax the condition on the cut to be an independent set. This leads to the following notion. For a connected graph $G = (V, E)$, a subset $U \subseteq V$ is called a *disconnected* cut if $U$ disconnects the graph and the subgraph induced by $U$ is disconnected as well, that is, has at least two (connected) components. This problem is NP-compete in general [13] but polynomial-time solvable on planar graphs [8]. However, the property of the cut being disconnected can be viewed to be somewhat artificial if one considers the 4-vertex path $P_4 = p_1p_2p_3p_4$, which has two disconnected cuts, namely $\{p_1, p_3\}$ and $\{p_2, p_4\}$. Both these cuts contain a vertex, namely $p_1$ and $p_4$, respectively, such that putting this vertex out of the cut and back into the graph keeps the graph disconnected. Therefore, Ito et al. [7] defined the notion of a *minimal disconnected* cut of a connected graph $G = (V, E)$, that is, a disconnected cut $U$ so that $G[(V \setminus U) \cup \{u\}]$ is connected for every $u \in U$ (more generally, we call a cut that satisfies the later condition a *minimal* cut). Here, the graph $G[S]$ denotes the subgraph of $G$ induced by $S \subseteq V(G)$. We note that every vertex of a minimal cut $U$ of a connected graph $G = (V, E)$ is adjacent to every component of $G[V \setminus U]$. See Figure 1 for an example of a planar graph with a minimal disconnected cut.

The corresponding decision problem is defined as follows.

Minimal Disconnected Cut
*Instance:* a connected graph $G$.
*Question:* does G have a minimal disconnected cut?

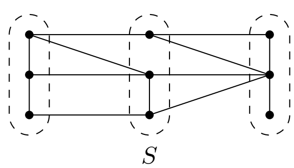


**Fig. 1.** An example of a planar graph with a minimal disconnected cut, namely the set $S$.

Ito et al. [7] showed that Minimal Disconnected Cut is NP-complete. However its computational complexity remains open for planar graphs. It can be seen, via a straightforward reduction, that the problem of deciding whether a graph has a minimal stable cut is NP-complete for any graph class (and thus for the class of planar graphs) for which Stable Cut is NP-complete. Moreover, the problem of deciding whether a graph has a minimal cut (that may be connected or disconnected) is polynomial-time solvable: given a vertex cut $U$ we can remove vertices from $U$ one by one until the remaining vertices in $U$ form a minimal cut.

**Our Results.** As a start we observe that Minimal Disconnected Cut is polynomial-time solvable for outerplanar graphs (as these graphs do not contain

$K_{2,3}$ as a minor, any minimal cut has size at most 2). In Section 2 we prove that MINIMAL DISCONNECTED CUT is also polynomial-time solvable on 3-connected planar graphs. The technique used by Ito et al. [8] for solving DISCONNECTED CUT in polynomial-time was based on the fact that a planar graph either has its treewidth bounded by some constant or else contains a large grid as a minor. However, grids (which are 3-connected planar graphs) do not have minimal disconnected cuts. Hence, we need to use a different approach, which we describe below.

We first provide a structural characterization of minimal disconnected cuts for the class of 3-connected $K_{3,3}$-free-minor graphs, which contains the class of planar graphs. In particular we show that any minimal disconnected cut of a 3-connected planar graph $G$ has exactly two components and that these components are paths. In order to find such a cut we prove that it suffices to test whether $G$ contains, for some fixed integer $r$, the biclique $K_{2,r}$ as a contraction. We show that $G$ has such a contraction if and only if its dual contains for some fixed $r$ the multigraph $D_r$, which is obtained from the $r$-vertex cycle by replacing each edge by two edges, as a subdivision (see also Figure 2). We then present a characterization of any graph that contains such a subdivision. Next we use this characterization to prove that the corresponding decision problem is polynomial-time solvable even on general graphs.

In Section 3 we give our second result, namely that MINIMAL DISCONNECTED CUT stays NP-complete for the class of 2-connected planar graphs. This proof is based on a reduction from STABLE CUT and as such different from the NP-hardness proof for general graphs [7], the gadget of which contains large cliques. In the same section we show that the problem of finding a minimal *connected* cut of size at least 3 is NP-complete for 2-connected *apex graphs* (graphs that can be made planar by removing one vertex); to the best of our knowledge the computational complexity of this problem has not yet been determined even for general graphs. We note that the problem of finding whether a graph contains a (not necessarily minimal) connected cut of size at most $k$ that separates two given vertices $s$ and $t$ is linear-time FPT when parameterized by $k$ [14].

We finish our paper with some further observations on related problems in Section 4.

**Related Work.** Vertex cuts play an important role in graph connectivity. In the literature various kinds of vertex cuts, besides stable cuts, have been studied extensively and we briefly survey a number of results below.

A cut $U$ of a graph $G = (V, E)$ is a clique cut if $G[U]$ is a clique, a $k$-clique cut if $G[U]$ has a spanning subgraph consisting of $k$ complete graphs; a strict $k$-clique cut if $G[U]$ consists of $k$ components that are complete graphs; and a matching cut if $E_{G[U]}$ is a matching. It follows from a classical result of Tarjan [17] that determining whether a graph has a clique cut is polynomial-time solvable. Whitesides [18] and Cameron et al. [3] proved that the problem of testing whether a graph has a $k$-clique cut is solvable in polynomial time for $k = 1$ and $k = 2$, respectively. Cameron et al. [3] also proved that testing whether a graph has a strict 2-clique cut can be solved in polynomial time. As mentioned the problem of testing whether a graph has a stable cut is NP-complete. This was first shown for general graphs by Chvátal [4]. Also the problem of testing whether a graph has a matching cut is NP-complete. This was shown by Brandstädt et al. [2]. Bonsma [1] proved that this problem is NP-complete even for planar graphs with girth 5 and for planar graphs with maximum degree 4.

The Skew Partition problem is that of testing whether a graph $G = (V, E)$ has a disconnected cut $U$ so that $V \setminus U$ induced a disconnected graph in the complement of $G$. De Figueiredo, Klein and Reed [5] proved that even the list version of this problem, where each vertex has been assigned a list of blocks in which it must be placed, is polynomial-time solvable. Afterwards, Kennedy and Reed [11] gave a faster polynomial-time algorithm for the non-list version.

Finally, for an integer $k \geq 1$, a cut $U$ of a connected graph $G$ is a $k$-cut of $G$ if $G[U]$ contains exactly $k$ components. For $k \geq 1$ and $\ell \geq 2$, a $k$-cut $U$ is a $(k, \ell)$-cut of a graph $G$ if $G[V \setminus U]$ consists of exactly $\ell$ components. Ito et al. [8] proved that testing if a graph has a $k$-cut is solvable in polynomial time for $k = 1$ and NP-complete for every fixed $k \geq 2$. In addition they showed that testing if a graph has a $(k, \ell)$-cut is polynomial-time solvable if $k = 1$, $\ell \geq 2$ and NP-complete otherwise [8]. The same authors show, by using the approach for solving Disconnected Cut on planar graphs, that both problems are polynomial-time solvable on planar graphs.

**Terminology.** Let $G = (V, E)$ be a connected simple graph. A maximal connected subgraph of $G$ is called a *component* of $G$. Recall that, for a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of $G$ *induced* by $S$, which has vertex set $S$ and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. A vertex $u \in V \setminus S$ is *adjacent* to a set $S \subseteq V \setminus \{u\}$ if $u$ is adjacent to a vertex in $S$. We say that two disjoints sets $S \subset V$ and $T \subset V$ are *adjacent* if $S$ contains a vertex adjacent to $T$, or equivalently if $T$ contains a vertex adjacent to $S$.

Let $G$ be a graph. We define the following operations. The *contraction* of an edge $uv$ removes $u$ and $v$ from $G$, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to $u$ or $v$ in $G$. Unless we explicitly say otherwise we remove all self-loops and multiple edges so that the resulting graph stays simple. The *subdivision* of an edge $uv$ replaces $uv$ by a new vertex $w$ with edges $uw$ and $vw$. Let $u \in V(G)$ be a vertex that has exactly two neighbours $v, w$, and moreover let $v$ and $w$ be non-adjacent. The *vertex dissolution* of $u$ removes $u$ and adds the edge $vw$.

A graph $G$ contains a graph $H$ as a *minor* if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge deletions and edge contractions. We say that $G$ contains $H$ as a *contraction*, denoted by $H \leq_c G$, if $H$ can be obtained from $G$ by a sequence of edge contractions. Finally, $G$ contains $H$ as a *subdivision* if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge deletions and vertex dissolutions, or equivalently, if $G$ contains a subgraph $H'$ that is a *subdivision* of $H$, that is, $H$ can be obtained from $H'$ after applying zero or more vertex dissolutions. We say that a vertex in $H'$ is a *subdivision vertex* if we need to dissolve it in order to obtain $H$; otherwise it is called a *branch vertex* (that is, it corresponds to a vertex of $H$).

For some of our proofs the following global structure is useful. Let $G$ and $H$ be two graphs. An *$H$-witness structure* $\mathcal{W}$ is a vertex partition of a (not necessarily proper) subgraph of $G$ into $|V(H)|$ nonempty sets $W(x)$ called *($H$-witness) bags*, such that

(i) each $W(x)$ induces a connected subgraph of $G$,
(ii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in $G$ if $x$ and $y$ are adjacent in $H$.

In addition, we may require the following additional conditions:

(iii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in $G$ *only* if $x$ and $y$ are adjacent in $H$,
(iv) every vertex of $G$ belongs to some bag.

By contracting all bags to singletons we observe that $H$ is a minor or contraction of $G$ if and only if $G$ has an $H$-witness structure such that conditions (i)-(ii) or (i)-(iv) hold, respectively. We note that $G$ may have more than one $H$-witness structure with respect to the same containment relation.

We denote the complete graph on $k$ vertices by $K_k$ and the complete bipartite graph with bipartition classes of size $k$ and $\ell$, respectively, by $K_{k,\ell}$.

## 2 The Algorithm

We first present a necessary and sufficient condition for a 3-connected $K_{3,3}$-minor-free graph to have a minimal disconnected cut.

**Theorem 1.** *A 3-connected $K_{3,3}$-minor-free graph $G$ has a minimal disconnected cut if and only if $K_{2,r} \leq_c G$ for some $r \geq 2$.*

*Proof.* Let $G = (V, E)$ be a 3-connected graph that has no $K_{3,3}$ as a minor. First suppose that $G$ has a minimal disconnected cut $U$. Let $p$ and $q$ be the number of components of $G[U]$ and $G[V \setminus U]$, respectively. Because $U$ is a disconnected cut, $p \geq 2$ and $q \geq 2$. By definition, every vertex of every component of $G[U]$ is adjacent to all components in $G[V \setminus U]$. Hence, $G$ contains $K_{p,q}$ as a contraction. Because $G$ has no $K_{3,3}$ as a minor, $G$ has no $K_{3,3}$ as a contraction. This means that $p \leq 2$ or $q \leq 2$. Because $p \geq 2$ and $q \geq 2$ holds as well, we find that $K_{2,r} \leq_c G$ for some $r \geq 2$.

Now suppose that $K_{2,r} \leq_c G$ for some $r \geq 2$. Throughout the remainder of the proof we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$. We refer to the bags in a $K_{k,\ell}$-witness structure of $G$ corresponding to the vertices in $X$ and $Y$ as $x$-bags and $y$-bags, respectively. Because $K_{2,r} \leq_c G$, there exists a $K_{2,r}$-witness structure $\mathcal{W}$ of $G$ that satisfies conditions (i)-(iv). Note that $W(x_1) \cup W(x_2)$ is a disconnected cut. However, it may not be minimal.

Suppose that $W(x_1)$ contains a vertex $u$ that is adjacent to some but not all $y$-bags, i.e., the number of $y$-bags to which $u$ is adjacent is $h$ for some $1 \leq h < r$. Then we move $u$ to a $y$-bag that contains one of its neighbors unless $W(x_1) \cup W(x_2)$

no longer induce a disconnected graph (which will be the case if $u$ is the only vertex in $W(x_1)$). We observe that $G[W(x_1) \setminus \{u\}]$ may be disconnected, namely when $u$ is a cut vertex in $G[W(x_1)]$. We also observe that $u$ together with its adjacent $y$-bags induces a connected subgraph of $G$. Hence, the resulting witness structure $\mathcal{W}'$ is a $K_{q,r'}$-witness structure of $G$ with $q \geq 2$ (as the resulting vertices in $W(x_1) \cup W(x_2)$ still induce a disconnected graph) and $r' = r - (h-1)$. Because $1 \leq h < r$, we find that $2 \leq r' \leq r$. We repeat this rule as long as possible. During this process, $W(x_2)$ does not change, and afterwards, we do the same for $W(x_2)$. Let $\mathcal{W}^*$ denote the resulting witness structure that is a $K_{q^*,r^*}$-witness structure satisfying conditions (i)-(iv) for some $q^* \geq 2$ and $2 \leq r^* \leq r$.

We will now prove the following claim. Afterwards, we are done; due to this claim and because there are at least two $x$-bags and at least two $y$-bags in $\mathcal{W}^*$, the $x$-bags of $\mathcal{W}^*$ form a minimal disconnected cut $U$ of $G$.

*Claim 1. Every vertex of each $x$-bag of $\mathcal{W}^*$ is adjacent to all $y$-bags.*

We prove Claim 1 as follows. First suppose that there exists an $x$-bag of $\mathcal{W}^*$, say $W^*(x_1)$, that contains a vertex $u$ adjacent to some but not to all $y$-bags of $\mathcal{W}^*$, say $u$ is not adjacent to $W^*(y_1)$. By our procedure we would have moved $u$ to an adjacent $y$-bag unless that makes the disconnected cut connected. Hence we find that there are exactly two witness bags $W^*(x_1)$ and $W^*(x_2)$ and that $W^*(x_1) = \{u\}$. In our procedure we only moved vertices from $x$-bags to $y$-bags. This means that $u$ belonged to an $x$-bag of the original witness structure $\mathcal{W}$. This $x$-bag was adjacent to all $y$-bags of $\mathcal{W}$ (as $\mathcal{W}$ was a $K_{2,r}$-witness structure). As we only moved vertices from $x$-bags to $y$-bags, this means that there must still exist a path from $u$ to a vertex in $W^*(y_1)$ that does not use any vertex of $W^*(x_2)$; a contradiction. Hence every $x$-bag of $\mathcal{W}^*$ only contains vertices that are either adjacent to all $y$-bags or to none of them.

Now, in order to obtain a contradiction, suppose that an $x$-bag, say $W^*(x_1)$, contains a vertex $u$ not adjacent to any $y$-bag. Because $G$ is 3-connected, $G$ contains three vertex-disjoints paths $P_1, P_2, P_3$ from $u$ to a vertex in $W^*(y_1)$ (by Menger's Theorem). Each $P_i$ contains a vertex $v_i$ in $W^*(x_1)$ whose successor on $P_i$ is outside $W^*(x_1)$. Hence, by our assumption, $v_i$ has a neighbour in every $y$-bag (including $W^*(y_1)$). Recall that the number of $y$-bags is $r^* \geq 2$. Then the subgraph induced by the vertices from $W^*(y_1)$ and $W^*(y_2)$ together with the vertices on the three paths $P_1, P_2, P_3$ form a $K_{3,3}$-minor of $G$. This is not possible. Hence, every vertex of each $x$-bag of $\mathcal{W}^*$ is adjacent to all $y$-bags. This completes the proof of Claim 1 and thus the proof of Theorem 1. $\square$

By Theorem 1 we may restrict ourselves to finding a $K_{2,r}$-contraction for some $r \geq 2$ in a 3-connected planar graph. Below we state some additional terminology.

Recall that $D_n$ is the multigraph obtained from the cycle on $n \geq 3$ vertices by doubling its edges. We let $D_2$ be the multigraph that has two vertices with four edges between them. The *dual* graph $G_d$ of a plane graph $G$ has a vertex for each face of $G$, and there exist $k$ edges between two vertices $u$ and $v$ in $G_d$ if and only if the two corresponding faces share $k$ edges in $G$. Note that the dual of a graph may be a multigraph. As 3-connected planar graphs have a unique embedding

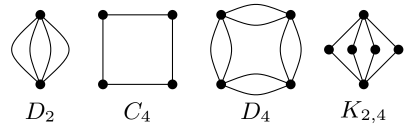

**Fig. 2.** The graphs $D_2$, $C_4$, $D_2$, $K_{2,4}$. Note that the dual of $C_4 = K_{2,2}$ is $D_2$, that $D_4$ is obtained from $C_4$ by duplicating each edge and that $D_4$ is the dual of $K_{2,4}$.

(see e.g. Lemma 2.5.1, p.39 of [16]) we can speak of the dual of a 3-connected planar graph. We need the following lemma. Its proof, which we omit, follows from using a result from [9].

**Lemma 1.** *Let $G$ be a 3-connected planar graph. Then $G$ contains $K_{2,r}$ as a contraction for some $r \geq 2$ if and only if the dual of $G$ contains $D_r$ as a subdivision.*

By Lemma 1 it suffices to check if the dual of the 3-connected planar input graph contains $D_r$ as a subdivision for some $r \geq 2$. We show how to solve this problem in polynomial time for general graphs. In order to do so we need the next lemma which gives a necessary condition for a graph $G$ to be a yes-instance of this problem. In its proof we use the following notation. For a path $P = v_1 v_2 \dots v_p$, we write $v_i P v_j$ to denote the subpath $v_i v_{i+1} \dots v_j$ or $v_j P v_i$ if we want to emphasize that the subpath is to be traversed from $v_j$ to $v_i$.

**Lemma 2.** *Let $v, w$ be two distinct vertices of a multigraph $G$ such that there exist four edge-disjoint $v$-$w$-paths in $G$. Then $G$ contains a subdivision of $D_r$ for some $r \geq 2$.*

*Proof.* We prove the lemma by induction on $|V(G)| + |E(G)|$. Then we can assume that $G$ is the union of the four edge-disjoint $v$-$w$-paths. Let us call these paths $P_1, P_2, P_3$, and $P_4$. If these four paths are vertex-disjoint (apart from $v$ and $w$) then they form a subdivision of $D_2$. Hence, we may assume that there exists at least one vertex of $G$ not equal to $v$ or $w$ that belongs to more than one of the four paths.

First suppose that there exists a vertex $u$ that belongs to all four paths $P_1$, $P_2$, $P_3$ and $P_4$. Let $G'$ be the graph consisting of the vertices and edges of the four subpaths $vP_1u$, $vP_2u$, $vP_3u$ and $vP_4u$. As $G'$ does not contain $w$, it holds that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, $G'$, and thus $G$, contains a subdivision of $D_r$ for some $r \geq 2$.

Now suppose that there exists a vertex $u$ that belong to only three of the four paths, say to $P_1$, $P_2$, and $P_3$. Let $G'$ be the graph that consists of the vertices and edges of the four paths $uP_1w$, $uP_2w$, $uP_3w$ and $uP_1vP_4w$. As $G'$ does not contain an edge of $vP_2u$ we find that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, $G'$, and thus $G$, contains a subdivision of $D_r$ for some $r \geq 2$.

From now on assume that every inner vertex of every path $P_i$ $(i = 1, \dots, 4)$ belongs to at most one other path $P_j$ $(j \neq i)$. We say that two different paths

$P_i$ and $P_j$ *cross* in a vertex $u$ if $u$ is an inner vertex of both $P_i$ and $P_j$. Suppose $P_i$ and $P_j$ cross in some other vertex $u'$ as well. Then we say that $u$ is *crossed before* $u'$ by $P_i$ and $P_j$ if $u$ is an inner vertex of both $vP_iu'$ and $vP_ju'$.

We now prove the following claim.

*Claim 1. If $P_i$ and $P_j$ ($i \neq j$) cross in both $u$ and $u'$ then we may assume without loss of generality that either $u$ is crossed before $u'$ or $u'$ is crossed before $u$.*

We prove Claim 1 as follows. Suppose that $u$ is not crossed before $u'$ by $P_i$ and $P_j$ and similarly that $u'$ is not crossed before $u$ by $P_i$ and $P_j$. Then we may assume without loss of generality that $u$ is an inner vertex of $vP_iu'$ and that $u'$ is an inner vertex of $vP_ju$. See Figure 3 for an example of this situation. However, in that case we can replace $P_i$ and $P_j$ by the paths $vP_iuP_jw$ and $vP_ju'P_iw$. These two paths together with the two unused original paths form a subgraph $G'$ of $G$ with fewer edges than $G$ (as for instance no edge on $uP_iu'$ belongs to $G'$). We apply the induction hypothesis on $G'$. This completes the proof of Claim 1.

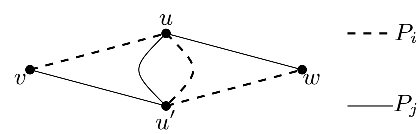


**Fig. 3.** The paths $P_i$ and $P_j$ where $u$ is not crossed before $u'$ by $P_i$ and $P_j$ and similarly $u'$ is not crossed before $u$ by $P_j$ and $P_i$. Note that the paths $P_i$ and $P_j$ may have more common vertices, but for clarify this is not been shown.

We need Claim 1 to prove the following claim, which is crucial for our proof.

*Claim 2. We may assume without loss of generality that there exists a vertex $u \notin \{v, w\}$ that is on two paths $P_i$ and $P_j$ ($i \neq j$) so that every inner vertex of $vP_iu$ and $vP_ju$ has degree 2 in $G$.*

We prove Claim 2 as follows. By our assumption there exists at least one vertex in $G$ that is on two paths. Let $s \notin \{v, w\}$ be such a vertex, say $s$ belongs to $P_1$ and $P_2$. Assume without los of generality that every inner vertex of $vP_1s$ has degree 2. Then, by Claim 1, we find that $P_1$ and $P_2$ do not cross in an inner vertex of $vP_2s$.

If every inner vertex of $vP_1s$ and $vP_2s$ has degree 2 in $G$ then the claim has been proven. Suppose otherwise, namely that there exists an inner vertex $s'$ of $vP_1s$ or $vP_2s$ whose degree in $G$ is larger than 2, say $s'$ belongs to $vP_2s$. As $P_1$ does not cross $vP_2s$, we find that $s'$ must belong to $P_3$ or to $P_4$. Choose $s'$ in such a way that every inner vertex of $vP_2s'$ has degree 2 in $G$. Assume without loss of generality that $s'$ belongs to $P_3$.

If every inner vertex of $vP_3s'$ has degree 2 then the claim has been proven (as every inner vertex of $vP_2s'$ has degree 2 as well). Suppose otherwise, namely that there exists an inner vertex $s''$ of $vP_3s'$ whose degree in $G$ is larger than 2.

Choose $s''$ in such a way that every inner vertex of $vP_3s''$ has degree 2 in $G$. By Claim 1, no inner vertex of $vP_3s'$ belongs to $P_2$, so $s''$ does not lie on $P_2$. This means that $s''$ belongs either to $P_1$ or to $P_4$.

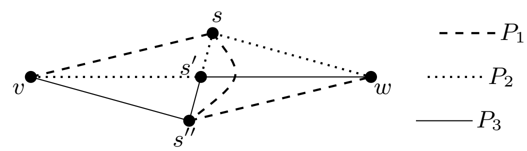


**Fig. 4.** The paths $P_1$, $P_2$ and $P_3$ where $s$ belongs to $P_1$ and $P_2$, $s'$ belongs to $vP_2s$ and $P_3$ and $s''$ belongs to $vP_3s'$ and $P_1$.

Suppose $s''$ belongs to $P_1$. See Figure 4 for an example of this situation. As every inner vertex of $vP_1s$ has degree 2, we find that $s$ is an inner vertex of $vP_1s''$. However, we can now replace $P_1$, $P_2$ and $P_3$ by the three paths $vP_1sP_2w$, $vP_2s'P_3w$ and $vP_3s''P_1w$. These three paths form, together with $P_4$, a subgraph of $G$ with fewer edges than $G$ (for instance, no edge of $sP_1s''$ belongs to $G'$). We can apply the induction hypothesis on this subgraph. Hence we may assume that $s''$ does not belong to $P_1$.

From the above we conclude that $s''$ belongs to $P_4$. See Figure 5 for an example of this situation. We consider the paths $vP_3s''$ and $vP_4s''$. If every inner vertex of $vP_4s''$ has degree 2 in $G$ then we have proven Claim 2 (recall that every inner vertex of $vP_3s''$ has degree 2 in $G$ as well). Suppose otherwise, namely that there exists an inner vertex $t$ of $vP_4s''$ whose degree in $G$ is larger than 2. Choose $t$ in such a way that every inner vertex of $vP_4t$ has degree 2 in $G$. By Claim 1 we find that $t$ is not on $P_3$. If $t$ is on $P_2$ we can use a similar replacement of three paths by three new paths as before that enables us to apply the induction hypothesis. Hence, we find that $t$ belongs to $P_1$.

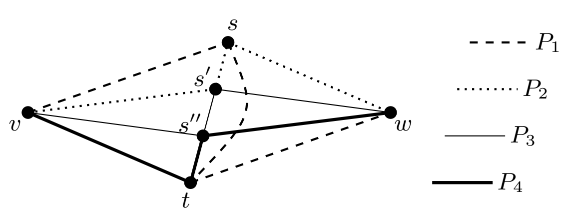


**Fig. 5.** The paths $P_1$, $P_2$, $P_3$ and $P_4$ where $s$ belongs to $P_1$ and $P_2$, $s'$ belongs to $vP_2s$ and $P_3$, $s''$ belongs to $vP_3s'$ and $P_4$ and $t$ belongs to $vP_4s''$ and $P_1$.

As every inner vertex of $vP_1s$ has degree 2 in $G$ we find that $s$ is an inner vertex of $vP_1t$. Then we take the four paths $vP_1sP_2w$, $vP_2s'P_3w$, $vP_3s''P_4w$ and $vP_4tP_1w$. These four paths form a subgraph $G'$ of $G$ with fewer edges than $G$

(as for instance $G'$ contains no edge from $sP_1t$). We can apply the induction hypothesis on $G'$. Hence we may assume that such a vertex $t$ cannot exist. Thus we have found the desired vertex and subpaths, namely $s''$ with subpaths $vP_3s''$ and $vP_4s''$. This completes the proof of Claim 2.

By Claim 2 we may assume without loss of generality that there exists a vertex $u$ that belongs to $P_1$ and $P_2$ such that every inner vertex of $vP_1u$ and $vP_2u$ has degree 2. Let $G^*$ be the graph obtained from $G$ by contracting all edges of $vP_1u$ and $vP_2u$ (recall that we remove loops and multiple edges). Let $u^*$ be the new vertex to which all the edges were contracted. Notice that there are four edge-disjoint $u^*$-$w$-paths in $G^*$. Then, by the induction hypothesis, $G^*$ contains a subdivision $H$ of $D_r$ for some $r \geq 2$. If $u^*$ does not belong to $H$, then $G$ contains $H$ as well and we would have proven the lemma. Assume that $u^*$ belongs to $H$.

First suppose that $u^*$ is a subdivision vertex of $H$ in $G^*$. Let $u^*$ have neighbours $s_1$ and $s_2$ in $H$. Take a shortest path $Q$ from $s_1$ to $s_2$ in the subgraph of $G$ induced by $s_1$, $s_2$ and the vertices of $vP_1u$ and $vP_2u$. This results in a graph $H'$, which is a subgraph of $G$ and which is a subdivision of $D_r$ as well.

Now suppose that $u^*$ is a branch vertex of $H$ in $G^*$, say $u^*$ corresponds to $z \in V(D_r)$. Note that any vertex in $D_r$ has one neighbour if $r = 2$ and two neighbours if $r \geq 3$. We let $s$ and $t$ be the two branch vertices of $H$ that correspond to the neighbours of $z$ in $D_r$ (note that $s = t$ if $r = 2$). Let $s_1$ and $s_2$ be the neighbours of $u^*$ on the two paths from $u^*$ to $s$, respectively, in $H$. Similarly, let $t_1$ and $t_2$ be the neighbours of $u^*$ on the two paths from $u^*$ to $t$, respectively, in $H$. Note that, as $G$ is a multigraph, it is possible that $s_1 = s_2 = s$ and $t_1 = t_2 = t$.

Recall that every internal vertex on $vP_1u$ and on $vP_2u$ has degree 2 in $G$. As $u$ is an inner vertex of $P_1$ and $P_2$ but not of $P_3$ and $P_4$, it has degree 4 in $G$. As $G$ is the union of $P_1$, $P_2$, $P_3$ and $P_4$, we find that $v$ has degree 4 as well. Then, after uncontracting $u^*$, we have without loss of generality one of the following two situations in $G$. First, $u$ is adjacent to $s_1$ and $s_2$ and $v$ is adjacent to $t_1$ and $t_2$. In that case $u$ and $v$ become branch vertices of a subdivision of $D_{r+1}$ in $G$ (to which the internal vertices on the paths $uP_1v$ and $uP_2v$ belong as well, namely as subdivision vertices). Second, $u$ is adjacent to $s_1$ and $t_1$, whereas $v$ is adjacent to $s_2$ and $t_2$. Then $u$ and $v$ become subdivision vertices of a subdivision of $D_r$ in $G$ (and we do not use the internal vertices on the paths $uP_1v$ and $uP_2v$). This completes the proof of the lemma. □

Lemma 2 gives us the following result.

**Theorem 2.** *It is possible to find in $O(mn^2)$ time whether a graph $G$ with $n$ vertices and $m$ edges contains $D_r$ as a subdivision for some $r \geq 2$.*

*Proof.* Let $G$ be a graph with $m$ edges. We check for every pair of vertices $s$ and $t$ whether $G$ contains four edge-disjoint paths between them. We can do this via a standard reduction to the maximum flow problem. Replace each edge $uv$ by the arcs $(u, v)$ and $(v, u)$. Give each arc capacity 1. Introduce a new vertex $s'$ and an arc $(s', s)$ with capacity 4. Also introduce a new vertex $t'$ and an arc

$(t, t')$ with capacity 4. Check if there exists an $(s', t')$-flow of value 4 by using the Ford-Fulkerson algorithm. As the maximum value of an $(s', t')$-flow is at most 4, this costs $O(m)$ time per pair, so $O(mn^2)$ time in total.

If there exists a pair $s, t$ in $G$ with four edge-disjoint paths between them then $G$ has a subdivision of $D_r$, for some $r \geq 2$, by Lemma 2. If not then we find that $G$ has no subdivision of any $D_r$ ($r \geq 2$) as any subdivision of $D_r$ immediately yields four edge-disjoint paths between two vertices and our algorithm would have detected this. □

We are now ready to state our main result.

**Theorem 3.** Minimal Disconnected Cut *can be solved in $O(n^3)$ time on 3-connected planar graphs with $n$ vertices.*

*Proof.* Let $G$ be a 3-connected planar graph with $n$ vertices. By Theorem 1 it suffices to check whether $K_{2,r} \leq_c G$ for some $r \geq 2$. By Lemma 1, the latter is equivalent to checking whether the dual of $G$, which we denote by $G^*$, contains $D_r$ as a subdivision for some $r \geq 2$. To find $G^*$ we first embed $G$ in the plane using the linear-time algorithm from Mohar [15]. As the number of edges in a planar graph is linear in the number of vertices, $G^*$ has $O(n)$ vertices and $O(n)$ edges and can be constructed in $O(n)$ time. We are left to apply Theorem 2. □

## 3 Hardness

We show the following result, which complements Theorem 3. We omit its proof.

**Theorem 4.** Minimal Disconnected Cut *is* NP*-complete for the class of 2-connected planar graphs.*

A cut $S$ in a graph $G$ is a *minimal connected* cut if $G[S]$ is connected and for all $u \in S$ we have that $G[(V \setminus S) \cup \{u\}]$ is connected. We call the problem of testing whether a graph a minimal connected cut of size at least $k$ the Minimal Connected Cut($k$) problem. By modifying the proof of Theorem 4 we obtain the following result (proof omitted).

**Theorem 5.** Minimal Connected Cut(3) *is* NP*-complete even for the class of 2-connected apex graphs.*

We cannot use the reduction in the proof of Theorem 5 to get NP-hardness for Minimal Connected Cut(1), the reason being that the gadget graph constructed in our omitted proof contains minimal disconnected cuts of size 2.

## 4 Conclusions

We proved that Minimal Disconnected Cut is NP-complete for 2-connected planar graphs and polynomial-time solve for planar graphs that are 3-connected. Our proof technique for the latter result was based on translating the problem to a dual problem, namely the existence of a subdivision of $D_r$ for *some* $r$, for which we obtained a polynomial-time algorithm even for general graphs. One can also solve the problem of determining whether a graph contains $D_r$ as a subdivision for some *fixed* integer $r$ by using the algorithm of Grohe, Kawarabayashi, Marx, and Wollan [6] which tests in cubic time, for any fixed graph $H$, whether a graph contains $H$ as a subdivision. However, when $r$ is part of the input we can show the following result via a reduction from Hamilton Cycle (proof omitted).

**Theorem 6.** *The problem of deciding whether a graph contains the graph $D_r$ as a subdivision is* NP*-complete if $r$ is part of the input.*

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