Proceedings of the 10th UK Conference on Boundary Integral Methods, University of Brighton, UK, 13-14 July 2015

0 A new enriched dual boundary element method for fracture in anisotropic materials

G. Hattori, J. Trevelyan
School of Engineering & Computing Sciences, Durham University, UK
gabriel.hattori@durham.ac.uk, jon.trevelyan@durham.ac.uk

0.1 Introduction

Fracture mechanics has been studied for over than 50 years, but it is still receiving great attention from the scientific community. One of the reasons is the appearance of a range of new materials, such as composite, piezoelectric, magnetoelectroelastic materials, just to cite a few examples. All these materials have a common characteristic: they present anisotropic behaviour, so material response to a given excitation is also conditioned to the crystal arrangements of this material (for more about crystal arrangements see [1] for instance). Anisotropic materials have been widely employed in the aerospace and automobile industries, wind power generators, sporting goods, and also as piezoelectric sensors and actuators.

Analytical formulations to deal with anisotropic material in fracture mechanics problems are very limited [2, 3]. Hence, fracture mechanics problems are solved using numerical methods, such as the Finite Element Method (FEM) [4] and the Boundary Element Method (BEM) [5]. It has been shown that BEM is more accurate and effective than FEM. However, in the last 15 years, a new paradigm has been established for the FEM, where the partition of unity [6] was applied to describe the discontinuity due to the presence of the crack, where the crack is then modelled through additional degrees of freedom in the elements containing the crack. This approach was named Extended Finite Element Method (X-FEM) [7] and it has been shown that it can provide similar accuracy to the one found in BEM models [8].

In this work we develop an alternative BEM formulation for anisotropic fracture problems using the partition of unity method (PUM). This idea has already been discussed by [9] for isotropic materials, and our objective is to generalise this approach for anisotropic materials. To this end, the PUM is implemented in a dual BEM context. The enrichment functions are derived in terms of the Stroh formalism [10] and further details on the implementation of the X-BEM are discussed.

Furthermore, we compare the results of the X-BEM with those obtained by alternative numerical techniques: dual BEM implemented in combination with discontinuous quarter-point elements [11, 12] and X-FEM with anisotropic enrichment functions [8].

0.2 Constitutive equations

Consider an anisotropic elastic domain Ω, then the static equilibrium equations in the presence of body forces $\mathbf{b}$ are defined as

$$\sigma_{ij,j} + b_i = 0 \tag{0.1}$$

Symmetry holds for the stress and strain tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively,

$$\sigma_{ij} = \sigma_{ji} \tag{0.2}$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \tag{0.3}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{0.4}$$

and u_i represent the displacements.

The linear constitutive equations are given by the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{0.5}$$

where C_{ijkl} define the material constants tensor, satisfying the following symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{0.6}$$

0.3 The Dual Boundary Element Method (DBEM)

The boundary element method (BEM) has been established as a reference numerical method when dealing with linear elastic fracture mechanics problems [5]. BEM is known to be more accurate and robust than domain discretisation methods such as the more popular finite element method (FEM).

The dual BEM (DBEM) is the usual choice when dealing with fracture mechanics problems. Hong and Chen [13] presented the idea of the combined use of a BIE and its derivative. This was used for the first time in a fracture mechanics context when [14] presented the DBEM. It can be summarised by a displacement boundary integral equation (DBIE)

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = \int_\Gamma u^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{0.7}$$

and a traction boundary integral equation (TBIE), obtained by the differentiation of (0.7) and further substitution in (0.5)

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_\Gamma s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) = N_r\int_\Gamma d^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{0.8}$$

where Γ represents all the boundaries (including crack boundaries) of domain Ω; N_r is the outward unit normal to the boundary at the collocation point $\boldsymbol{\xi}$; c_{ij} is the free term deriving from the Cauchy Principal Value integration of

the strongly singular kernels p^*_{ij}; u^*_{ij} and p^*_{ij} are the displacement and traction fundamental solutions; d^*_{rij} and s^*_{rij} follow from differentiation and substitution into the generalised Hooke's law of u^*_{ij} and p^*_{ij}, respectively. The kernels u^*_{ij} and p^*_{ij} are given by

$$u^*_{ij} = -\frac{1}{\pi}\Re\left\{A_{jm}Q_{mi}\ \ln(z^x_m - z^\xi_m)\right\} \tag{0.9}$$

$$p^*_{ij} = \frac{1}{\pi}\Re\left\{B_{jm}Q_{mi}\ \frac{\mu_m n_1 - n_2}{z^x_m - z^\xi_m}\right\} \tag{0.10}$$

where $\boldsymbol{n} = (n_1, n_2)$ is the unit normal at the observation point; $z^x_m = x_1 + \mu_m x_2$, $z^\xi_m = \xi_1 + \mu_m \xi_2$ are evaluated at the observation and collocation points, respectively; $\Re$ denotes the real part ; $\mathbf{Q} = \mathbf{A}^{-1}(\mathbf{L}^{-1} + \overline{\mathbf{L}}^{-1})^{-1}$ with $\mathbf{L} = i\mathbf{A}\mathbf{B}^{-1}$ and $\overline{L}$ represents the complex conjugate of L.

The matrices $\mathbf{A}$, $\mathbf{B}$ and constant μ_m are parameters from the Stroh formalism and can be obtained through the following eigenvalue problem

$$\left(\begin{array}{c|c} -\mathbf{C_{2ij2}}^{-1}\mathbf{C_{2ij1}} & -\mathbf{C_{2ij2}}^{-1} \\ \hline \mathbf{C_{1ij1}} - \mathbf{C_{2ij1}}^T\mathbf{C_{2ij2}}^{-1}\mathbf{C_{2ij1}} & -\mathbf{C_{2ij1}}^T\mathbf{C_{2ij2}}^{-1} \end{array}\right)\left(\frac{\mathbf{A}_m}{\mathbf{B}_m}\right) = \mu_m\left(\frac{\mathbf{A}_m}{\mathbf{B}_m}\right) \tag{0.11}$$

where there is no summation on the m index.

0.3.1 Extended Boundary Element Method (X-BEM)

The extended boundary element method (X-BEM) was first proposed by Simpson and Trevelyan [9] for fracture mechanics problems in isotropic materials. The main idea is the same as for the X-FEM, to model the asymptotic behaviour of the displacements around the crack tips by introducing new degrees of freedom. The displacements $\mathbf{u}^h(\mathbf{x})$ are thus redefined as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i\in\mathcal{N}} N_i(\mathbf{x})\mathbf{u}_i + \sum_{k\in\mathcal{N}^{\mathcal{CT}}} N_k(\mathbf{x})\sum_\alpha F_\alpha(\mathbf{x})\mathbf{a}^\alpha_k \tag{0.12}$$

where $\mathcal{N}$ and $\mathcal{N}^{\mathcal{CT}}$ are the sets of all nodes and the enriched nodes, respectively, N_i is the standard Lagrangian shape function associated with node i, $\mathbf{u}_i$ is the vector of nodal degrees of freedom, and $\mathbf{a}^\alpha_k$ represents the amplitudes of the enriched basis functions which capture the asymptotic behaviour around the crack tips. In elastic materials, $\mathbf{a}^\alpha_k$ is an 8-component vector for two-dimensional problems, since only two nodal variables (u_1, u_2) and four enrichment functions are needed to describe all the possible deformation states in the vicinity of the crack-tip [8].

In this work, we use the anisotropic enrichment functions obtained by [8] for the X-FEM:

$$F_l(r,\theta) = \sqrt{r}\begin{pmatrix} \Re\{A_{11}B^{-1}_{11}\beta_1 + A_{12}B^{-1}_{21}\beta_2\} \\ \Re\{A_{11}B^{-1}_{12}\beta_1 + A_{12}B^{-1}_{22}\beta_2\} \\ \Re\{A_{21}B^{-1}_{11}\beta_1 + A_{22}B^{-1}_{21}\beta_2\} \\ \Re\{A_{21}B^{-1}_{12}\beta_1 + A_{22}B^{-1}_{22}\beta_2\} \end{pmatrix} \tag{0.13}$$

where $\beta_m = \sqrt{\cos\theta + \mu_m \sin\theta}$, r is the distance between the crack tip and an arbitrary position, θ is the orientation measured from a coordinate system centred at the crack tip.

Let us emphasise that the anisotropic enrichment functions can also be used for isotropic materials, since this is a degenerated case from anisotropic materials. For more details please refer to reference [8].

The X-BEM formulation is similar to the one used by Simpson and Trevelyan [9] for isotropic materials. The extended DBIE and the TBIE can be restated as

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + \int_{\Gamma_c} p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})F_\alpha(\boldsymbol{x})\mathbf{a}^\alpha_k d\Gamma = \int_\Gamma u^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{0.14}$$

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_\Gamma s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + N_r\int_{\Gamma_c} s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})F_\alpha(\boldsymbol{x})\mathbf{a}^\alpha_k d\Gamma = N_r\int_\Gamma d^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{0.15}$$

where $\Gamma_c = \Gamma_+ \cup \Gamma_-$ stands for the crack surfaces Γ_+ and Γ_-. In this work, only the element containing the crack tip receives the enrichment function. Let us recall that strongly singular and hypersingular terms arise from the integration of the p^*_{ij}, d^*_{rij} and s^*_{rij} kernels and they are regularised in the same way as shown in [15].

The enrichment functions add new degrees of freedom in the BEM formulation, which causes the linear system of equations to become indeterminate. An easy form to overcome this issue is to include additional collocation points. Each new collocation point will provide 2 new equations (in x_1 and x_2 directions). In this work, 3 nodes are enriched in each crack face, which will add $3 \times 4 \times 2 \times 2 = 48$ new degrees of freedom to the problem. Therefore, 24 extra collocation points are necessary to solve the linear system of equations.

0.4 Numerical results

In this section we show the crack opening displacement (COD) for anisotropic materials in two fracture mechanics examples. The numerical results are compared with the results from the extended BEM, the dual BEM with quarter-point elements [11] and the X-FEM.

The BEM formulation presented in [11] has the particularity of modelling only one crack surface, thus obtaining the COD straight from the solution of the linear system of equations.

0.4.1 Double edge crack

Figure 0.1 illustrates a square plate ($h/w = 1$) with two edge cracks of length a under a uniform loading σ. The size of the crack is defined by $a/w = 0.25$. The material is a four graphite-epoxy laminate, where the elastic properties are: $E_1 = 144.8$ GPa, $E_2 = 11.7$ GPa, $G_{12} = 9.66$ GPa and $\nu_{12} = 0.21$. The COD was obtained for two distinct fibre orientation, $\phi = 75°$ and $\phi = 90°$. Results for the X-FEM are obtained using topological and geometrical enrichment, with a fixed area of $r_e/a = 0.2$ and a mesh of 4525 elements. For more information about both adopted enrichment types please refer to [8] for instance.

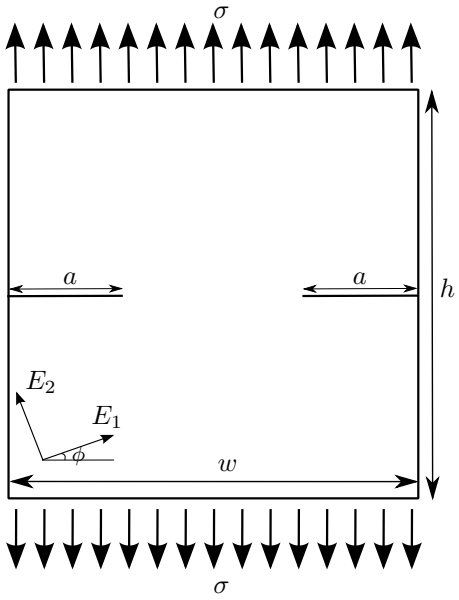

Figure 0.1: Double edge crack problem.

For the dual BEM and the X-BEM a 10 continuous quadratic element per side mesh was used. The crack is discretised with 10 discontinuous elements. The dual BEM presents a quarter-point in the element at the crack tip, while the X-BEM presents an enriched element.

Results are given in Figures 0.2 and 0.3. The COD of the X-FEM models is smaller than the ones of the BEM models. The X-BEM COD is higher at the upper crack face than its equivalent using quarter point. In this case, the X-FEM results can be underestimated compared to the BEM results. Nevertheless, the displacements around the crack tip have the same $\sqrt{r}$ behaviour.

0.4.2 Slanted centred crack

A square plate $h/w = 1$ with a slanted centred crack of length $2a$ under a uniform loading is represented in Figure 0.4, where $\beta = 30°$. The size of the crack is $a/w = 0.5$. The material is a glass-epoxy composite with the following

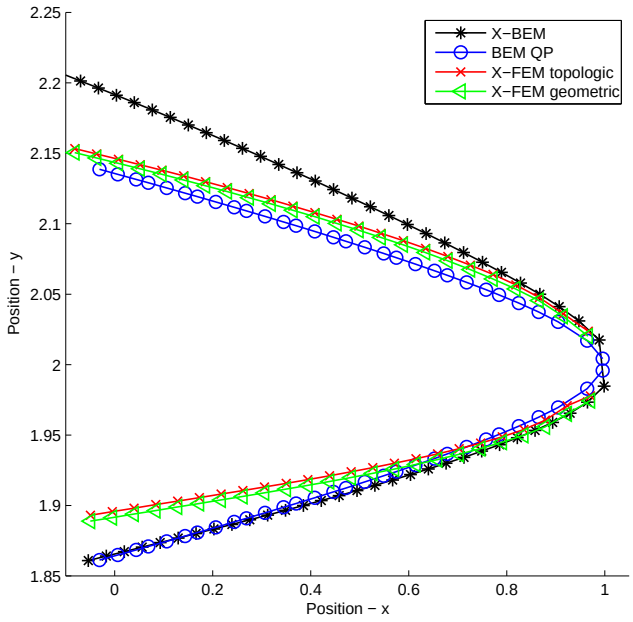

Figure 0.2: COD for the double edge crack - $\theta = 75°$.

elastic properties: $E_1 = 48.26$ GPa, $E_2 = 17.24$ GPa, $G_{12} = 6.89$ GPa and $\nu_{12} = 0.29$. The fibre orientation angle ϕ assumes values of 90° and 120° in this example. The BEM meshes present 6 quadratic elements per side, and 10 discontinuous elements at the crack. The X-FEM mesh contain 2025 elements and the geometrical enrichment uses a fixed area of $r_e/a = 0.2$.

The results are shown in Figures 0.5 and 0.5 for $\phi = 90$ and 120°, respectively. Here we present the results in terms of the u_2 displacement only, since it is difficult to get an accurate COD for the X-FEM model, which can be seen by the deformed shape of the COD. Both CODs from the BEM models result in symmetrical relative displacements. The COD from the extended BEM is considerably larger than the other reference solutions. One of the possible reasons is that the extended BEM model is underestimating the stiffness of the plate, resulting in a larger COD.

0.5 Summary

An enriched boundary element method was proposed in this work. This method uses the benefits of the partition of unit to model in a more general way the asymptotic displacements around the crack tip. Existing anisotropic enrichment functions for the X-FEM were employed at the proposed method. Some numerical examples were evaluated, and the results compared to a well established BEM formulation and the X-FEM . The crack opening displacement was com-

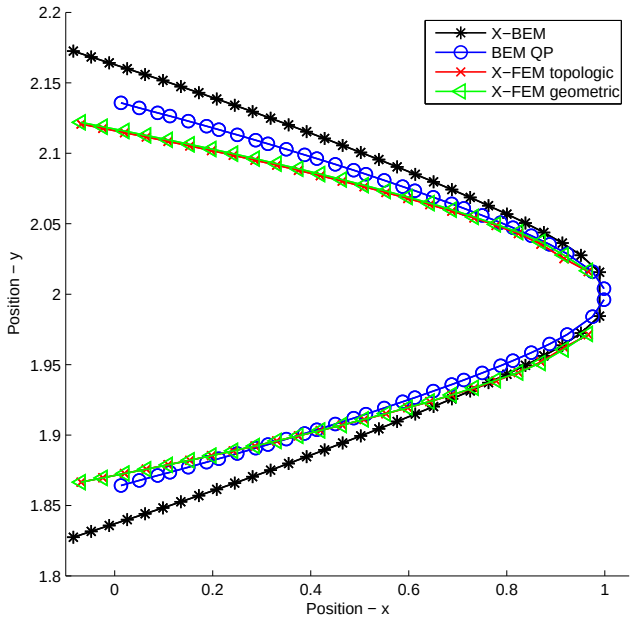

Figure 0.3: COD for the double edge crack - $\theta = 90°$.

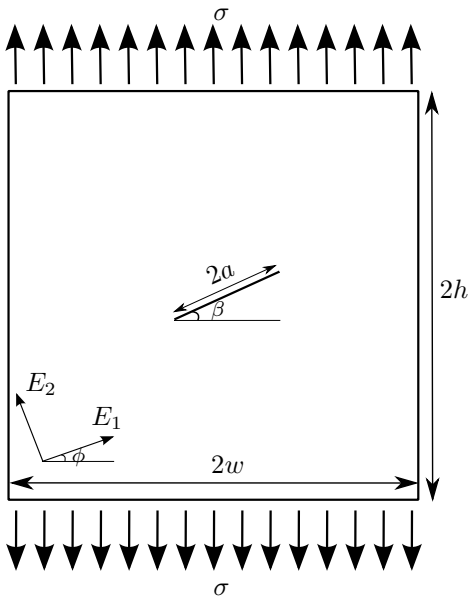

Figure 0.4: Slanted centred crack problem.

pared with the quarter point BEM and the X-FEM. For a flat horizontal edge crack, both BEM models have the same behaviour around the crack tip, and the

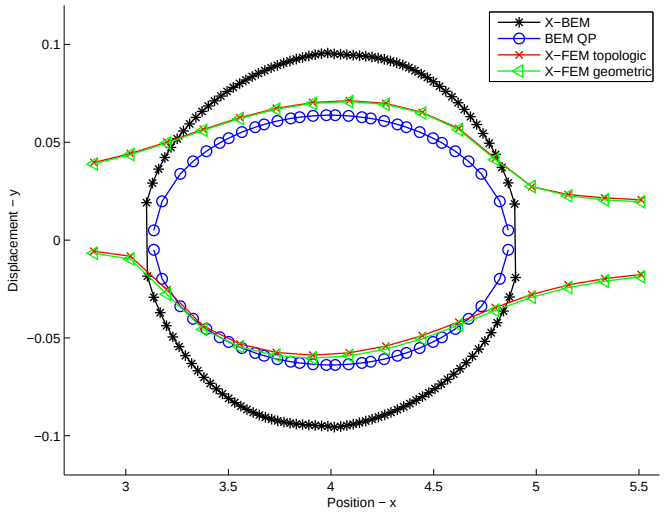

Figure 0.5: COD for the slanted crack problem - $\phi = 90°$.

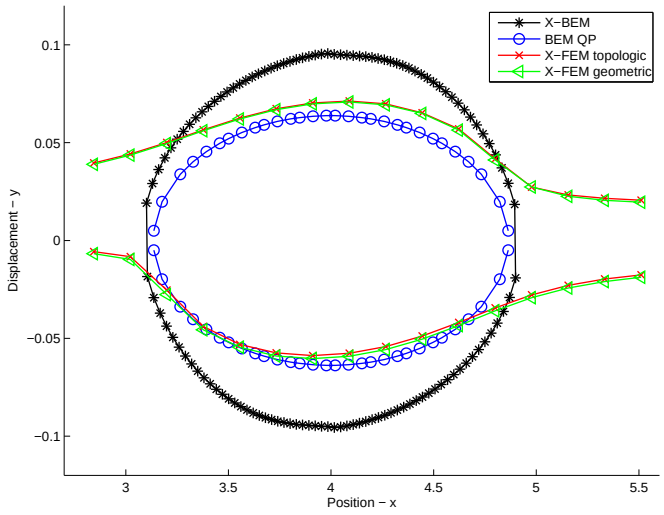

Figure 0.6: COD for the slanted crack problem - $\phi = 120°$.

X-BEM considers a larger COD at the end of the crack. In a slanted crack, the COD in the X-BEM model is larger than both BEM and X-FEM. The X-BEM could be used an alternative to the dual BEM with quarter-point elements.

0.6 Acknowledgments

The first author acknowledges the Faculty of Science, Durham University for his Postdoctoral Research Associate funding.

References

[1] M. J. P. Musgrave. *Crystals Acoustics: Introduction to to the study of elastic waves and vibrations in crystals.* Holden-Day, 1970.

[2] Carloni C. Nobile, L. Fracture analysis for orthotropic cracked plates. *Composite Structures*, 68(3):285–293, 2005.

[3] G. C. Sih, P. C. Paris, and G. R. Irwin. On cracks in rectilinearly anisotropic bodies. *International Journal of Fracture*, 1:189–203, 1965.

[4] T. J. Boone, P. A. Wawrzynek, and A. R. Ingraffea. Finite element modelling of fracture propagation in orthotropic materials. *Engineering Fracture Mechanics*, 26:185–201, 1987.

[5] YJ Liu, S Mukherjee, N Nishimura, M Schanz, W Ye, A Sutradhar, E Pan, NA Dumont, A Frangi, and A Saez. Recent advances and emerging applications of the boundary element method. *Applied Mechanics Reviews*, 64(3):030802, 2011.

[6] I. Babuˇska and J. M. Melenk. The partition of unity method. *International Journal for Numerical Methods in Engineering*, 4:607–632, 1997.

[7] Nicolas Moes, John Dolbow, and T Belytschko. A finite element method for crack growth without remeshing. *Int. J. Numer. Meth. Engng*, 46(1):131–150, 1999.

[8] G Hattori, R Rojas-Díaz, A Sáez, N Sukumar, and F García-Sánchez. New anisotropic crack-tip enrichment functions for the extended finite element method. *Computational Mechanics*, 50(5):591–601, 2012.

[9] Robert Simpson and Jon Trevelyan. A partition of unity enriched dual boundary element method for accurate computations in fracture mechanics. *Computer Methods in Applied Mechanics and Engineering*, 200(1):1–10, 2011.

[10] A. N. Stroh. Dislocation and cracks in anisotropic elasticity. *Philosophical magazine*, 3:625–646, 1958.

[11] F Garcıa, A Sáez, and J Domınguez. Traction boundary elements for cracks in anisotropic solids. *Engineering analysis with boundary elements*, 28(6):667–676, 2004.

[12] A. Sáez, R. Gallego, and J. Domínguez. Hypersingular quarter-point boundary elements for crack problems. *International Journal for Numerical Methods in Engineering*, 38:1681–1701, 1995.

[13] Hong-Ki Hong and Jeng-Tzong Chen. Derivations of integral equations of elasticity. *Journal of Engineering Mechanics*, 114(6):1028–1044, 1988.

[14] A. Portela, M. H. Aliabadi, and D. P. Rooke. The dual bem: effective implementation for crack problems. *International Journal for Numerical Methods in Engineering*, 33:1269–1287, 1992.

[15] F Garcia-Sanchez, Andrés Sáez, and J Dominguez. Anisotropic and piezoelectric materials fracture analysis by bem. *Computers & Structures*, 83(10):804–820, 2005.