

## **COMPARISON OF CHIEF AND BURTON-MILLER APPROACHES IN COLLOCATION PARTITION OF UNITY BEM FOR HELMHOLTZ PROBLEMS**

G.C. DIWAN, J. TREVELYAN and G. COATES

*School of Engineering and Computing Sciences, Durham University, Durham DH1 4JR, UK*  
e-mail: g.c.diwan@durham.ac.uk

**Abstract.** Use of plane wave basis for the numerical solutions of acoustic wave problems using element based methods has become an attractive approach for extending the allowable frequency range for simulations beyond that available using piecewise polynomial elements. The non-uniqueness of the solution at characteristic frequencies resulting from the use of the conventional boundary integral equation is well known. The standard methods of overcoming this problem are the so-called CHIEF method and that of Burton and Miller. The latter method introduces a hypersingular integral which can be treated in several ways. In this paper we present results for Partition of Unity BEM (PUBEM) for Helmholtz problem and compare the performance of CHIEF against a Burton-Miller formulation regularised using the approach of Chen et al.

### **1. INTRODUCTION**

The theory of the boundary element method (BEM) for solving boundary integral equations (BIE) is well established. It is known that the Conventional BIE (CBIE) based on Green's function representations for an exterior acoustic problem results in a non-unique solution at characteristic frequencies for the corresponding interior problem and that this is a purely mathematical phenomenon. There are several methods to handle the non-uniqueness. One of the extensively used method is the so called Combined Helmholtz Integral Equation Formulation (CHIEF) due to Schenck [1], where some extra Helmholtz integral equations evaluated at interior points are added in the original system matrix. Although this results in an over-determined system, the CHIEF method ensures a unique solution at the characteristic frequency. However, one needs to choose the interior points such that they do not lie on the nodal lines of the interior modes of Helmholtz problem thus failing to provide any necessary constraint for uniqueness of the solution. This can therefore introduce uncertainties for complicated geometries at high frequencies as the nodal lines become densely packed in the interior making it difficult to find suitable locations for the placement of interior points. Another method to avoid the non-uniqueness problem is due to Burton and Miller [2]. It was shown in their work that the integral equation resulting from linear combination of the CBIE and its normal derivative at the collocation point always results in a unique solution. The main problem with this method is the evaluation of the hypersingular integral which arises as a result of the differentiation of the CBIE. In the present study, we compare the CHIEF method with one regularized form of the Burton-Miller formulation for acoustic scattering from hard cylinders in two dimensions using PUBEM. The two methods are compared for their accuracy of the solution and efficiency.

### **2. BOUNDARY INTEGRAL EQUATION**

The mathematical formulation for deriving the CBIE from the Helmholtz equation is well established [3]. The CBIE for an acoustic scattering (or radiation) problem governed by the Helmholtz

differential equation is given by

$$c(p)\phi(p) + \int_{\Gamma} \frac{\partial G}{\partial n_q} \phi(q) d\Gamma(q) = \int_{\Gamma} G \frac{\partial \phi(q)}{\partial n_q} d\Gamma(q) + \phi^i(p) \quad (1)$$

where  $p$  is the collocation or source point,  $q$  the field point,  $G$  the free space Green's function for the Helmholtz problem,  $n_q$  and  $n_p$  the normals respectively at points  $q$  and  $p$  pointing away from acoustic domain  $\Omega$ ,  $\phi(q)$  the unknown acoustic potential and  $\phi^i(p)$  the known incident acoustic wave.  $c(p)$  is the free coefficient which depends on the local geometry of  $\Gamma$  at  $p$ . In this study we assume  $\Gamma$  is smooth and take  $c(p) = \frac{1}{2}$ . The normal derivative of (1) at the collocation point  $p$  is given by

$$c(p) \frac{\partial \phi(p)}{\partial n_p} + \int_{\Gamma} \frac{\partial^2 G}{\partial n_p \partial n_q} \phi(q) d\Gamma(q) = \int_{\Gamma} \frac{\partial G}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} d\Gamma(q) + \frac{\partial \phi^i(p)}{\partial n_p} \quad (2)$$

and the Combined Hypersingular BIE (CHBIE) due to Burton and Miller [2] is

$$\begin{aligned} c(p)\phi(p) + \alpha c(p) \frac{\partial \phi(p)}{\partial n_p} + \int_{\Gamma} \frac{\partial G}{\partial n_q} \phi(q) d\Gamma(q) + \alpha \int_{\Gamma} \frac{\partial^2 G}{\partial n_p \partial n_q} \phi(q) d\Gamma(q) = \\ \int_{\Gamma} G \frac{\partial \phi(q)}{\partial n_q} d\Gamma(q) + \alpha \int_{\Gamma} \frac{\partial G}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} d\Gamma(q) + \phi^i(p) + \alpha \frac{\partial \phi^i(p)}{\partial n_p} \end{aligned} \quad (3)$$

where  $\alpha$  is a coupling constant most commonly taken as  $i/k$ . In the present study, we analyse the acoustic scattering from sound hard cylinders. A sound hard surface is where the normal derivative of the total acoustic potential vanishes. Therefore, all the terms involving the normal derivative of acoustic potential vanish. As mentioned earlier, the main drawback of (3) is the numerical treatment of the hypersingular integral, i.e. the last integral on the left hand side. Chen et al [4] give the following weakly singular form of the hypersingular integral

$$\begin{aligned} \int_{\Gamma} \frac{\partial^2 G}{\partial n_p \partial n_q} \phi(q) d\Gamma(q) = \int_{\Gamma} \left[ \frac{\partial^2 G}{\partial n_p \partial n_q} - \frac{\partial^2 G_0}{\partial n_p \partial n_q} \right] \phi(q) d\Gamma(q) + \int_{\Gamma} [\phi(q) - \phi(p) - \nabla \phi(p) \cdot (q - p)] \frac{\partial^2 G_0}{\partial n_p \partial n_q} d\Gamma(q) \\ + \int_{\Gamma} \nabla \phi(p) \cdot n_q \frac{\partial G_0}{\partial n_p} d\Gamma(q) - \frac{1}{2} \nabla \phi(p) \cdot n_p \end{aligned} \quad (4)$$

where  $G_0$  is the free space Green's function for the Laplace equation. For the present case of a hard boundary, the last term in the right hand side of (4) vanishes. Consequently, the final equation for this case of a hard boundary can be expanded as

$$\begin{aligned} c(p)\phi(p) + \int_{\Gamma} \frac{\partial G}{\partial n_q} \phi(q) d\Gamma(q) + \alpha \int_{\Gamma} \left[ \frac{\partial^2 G}{\partial n_p \partial n_q} - \frac{\partial^2 G_0}{\partial n_p \partial n_q} \right] \phi(q) d\Gamma(q) + \\ \alpha \int_{\Gamma} [\phi(q) - \phi(p) - \nabla \phi(p) \cdot (q - p)] \frac{\partial^2 G_0}{\partial n_p \partial n_q} d\Gamma(q) + \alpha \int_{\Gamma} \nabla \phi(p) \cdot n_q \frac{\partial G_0}{\partial n_p} d\Gamma(q) = \phi^i(p) + \alpha \frac{\partial \phi^i(p)}{\partial n_p} \end{aligned} \quad (5)$$

The acoustic potential at a point  $\mathbf{x}$  on the boundary  $\Gamma$  using plane wave basis can be approximated as

$$\phi(\mathbf{x}) = \sum_{j=1}^3 N_j \sum_m^{M_j} A_{jm} e^{ik \mathbf{d}_{jm} \cdot \mathbf{x}} \quad \mathbf{x} \in \Gamma \quad (6)$$

where  $N_j$  is the  $j^{\text{th}}$  shape function,  $A_{jm}$  the unknown which can be thought of as the amplitude of the  $m^{\text{th}}$  plane wave with wave number  $k$  associated with node  $j$ . The direction of the  $m^{\text{th}}$  plane wave at node  $j$  is given by unit vector  $\mathbf{d}_{jm}$  and  $\mathbf{x}$  is the location of the point where the

potential  $\phi$  is sought. We consider a three noded element with  $M_j$  as the number of plane waves associated with the  $j^{\text{th}}$  node. In the context of the BEM, the plane wave basis defined in (6) can be used to express the unknown acoustic potential on the boundary  $\Gamma$ . Choosing appropriate locations on the boundary  $\Gamma$  as collocation point  $p$  yields the following set of linear equations from (5)

$$[\mathbf{H}]\{\mathbf{a}\} = \{\mathbf{b}\} \quad (7)$$

where the vector  $\mathbf{a}$  contains the amplitudes of plane waves,  $A_{jm}$ , which can be used to quickly recover the acoustic potential on the boundary  $\Gamma$  using (6).

### 3. NUMERICAL EXAMPLES

This section presents the error analyses for two problems viz. i) plane wave scattering from a single sound hard cylinder and ii) from an array of four cylinders. At the outset, it is convenient to define a parameter  $\tau$  which gives the number of degrees of freedom per wavelength for a given problem, i.e.,

$$\tau = \frac{T}{ka} \quad (8)$$

where  $T$  is the total number of degrees of freedom in the system for one cylinder and  $a$  is the radius of the cylinder. Thus for the problem of scattering from a single cylinder with unit radius,  $\tau = T/k$  where  $T$  will be simply the multiplication of the total number of nodes on the scatterer boundary and number of plane waves per node. It may be noted that the introduction of plane waves into the basis makes the boundary integrals oscillatory in nature and it is very important that these integrals be computed as accurately as possible. We follow the strategy described in [5], namely element subdivision, for evaluating these integrals.

For all the results presented here the parameter  $\tau \approx 3.0$  unless otherwise mentioned. This value has been found to be sufficient to recover solutions with acceptable engineering accuracy of 1% and moderate condition numbers which can be efficiently handled with the Singular Value Decomposition (SVD) algorithm, see [5]. All the results are obtained with 30 integration (Gauss) points per wavelength unless otherwise mentioned. For both the single cylinder and four cylinder examples, we use two 3-noded continuous elements per cylinder along with the trigonometric shape functions presented by Peake et al [6]. For all computations the integration points are placed analytically on the scatterer boundary. We now define the relative  $L^2$  error for the total acoustic potential  $\phi$  on the boundary  $\Gamma$ ,  $E^2(\phi)$  as

$$E^2(\phi) = \frac{\|\phi - \tilde{\phi}\|}{\|\tilde{\phi}\|} \quad (9)$$

where  $\phi$  is the numerically computed solution and  $\tilde{\phi}$  the analytical solution computed using the infinite or approximate series for a given scattering problem. The 2-norm condition number for the matrix  $\mathbf{H}$ ,  $\kappa(\mathbf{H})$  may be defined as

$$\kappa(\mathbf{H}) = \frac{\sigma_{\max}(\mathbf{H})}{\sigma_{\min}(\mathbf{H})} \quad (10)$$

where  $\sigma_{\max}(\mathbf{H})$  and  $\sigma_{\min}(\mathbf{H})$  are respectively the maximum and minimum singular values of the matrix  $\mathbf{H}$  computed using the SVD algorithm. As discussed earlier, the placement of interior collocation points for the CHIEF method might become an issue. For the numerical examples presented in this study, the interior points are placed completely randomly in the interior of the cylinder(s). The number of interior points used here is 20% of the total number of equations in (7) since this has been found to give stable results for the CHIEF method. Also the CHIEF points in the interior of the cylinder(s) are placed such that they are sufficiently away from the boundary.

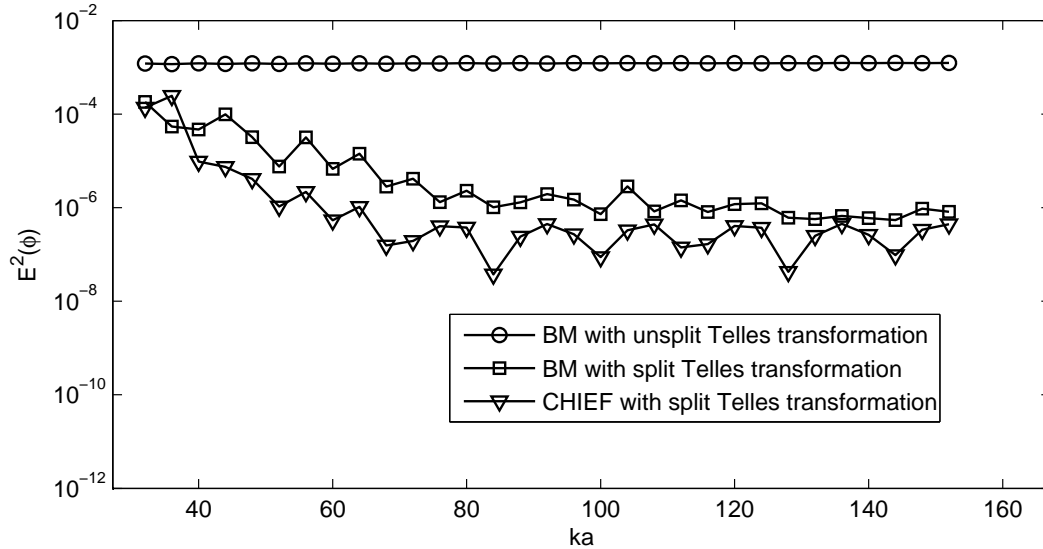


Figure 1:  $E^2(\phi)$  for the single cylinder problem

### 3.1 Scattering from a single sound hard cylinder

We first investigate the performance of CHIEF and Burton-Miller methods for the classical problem of plane wave scattering from an acoustically hard cylinder of infinite extent. We use the infinite series in [7] to compute the analytical solution for the scattered potential,  $\phi^s$ , on the surface of a hard cylinder. The total acoustic potential  $\phi$  can be computed by simply performing a complex addition of incident wave to the scattered potential obtained using the infinite series from [7], i.e.,  $\phi = \phi^i + \phi^s$ . The relative  $L^2$  error for the total acoustic potential is then computed using (9). Fig. 1 shows the relative  $L^2$  error,  $E^2(\phi)$  for CHIEF and Burton-Miller methods. As seen from Fig.1, CHIEF provides better accuracy compared to Burton-Miller results. Note that when the weak singularity in (5) is handled with the Telles scheme without splitting the interval containing the singularity, the Burton-Miller formulation gives poorer results. Despite the regularization, the integrals in the (5) are slowly converging. As is evident, to achieve a comparable accuracy to that of the CHIEF method, the integrals in the regularized Burton-Miller formulation needs to use Telles transformation with splitting the interval towards left and the right of the singularity.

### 3.2 Scattering from an array of four cylinders

The scattering from a multi-cylinder array presents a more challenging case as it involves multiple reflections from individual cylinders which ultimately forms the total acoustic field. The recursive multiple reflections make this problem an ideal candidate to test the efficacy of PUBEM to obtain an accurate solution. We consider a setting of four unit radius sound hard cylinders of infinite extent with their centres placed at  $(-2,-2)$ ,  $(2,-2)$ ,  $(2,2)$  and  $(-2,2)$  in a two dimensional homogeneous unbounded acoustic medium (air). Let this array then be impinged by a unit amplitude plane wave with wavenumber  $k$  at an angle of  $\theta^I = 45^\circ$  with the horizontal. We use the formula proposed by Linton and Evans [8] (eq. 2.15) to compare our PUBEM solution for the total acoustic potential on the surface of each cylinder. The formula proposed by Linton and Evans is based on the addition theorem that combines the separable solutions of Helmholtz equation, see [9] for details. The addition theorems can be efficiently used to compute the solution but the infinite series has to be truncated in practice. Theoretically of course, an infinite sum should result in a converged solution. However, when solving even the truncated system of linear equations, the addition of extra terms in the series can make the matrix formed

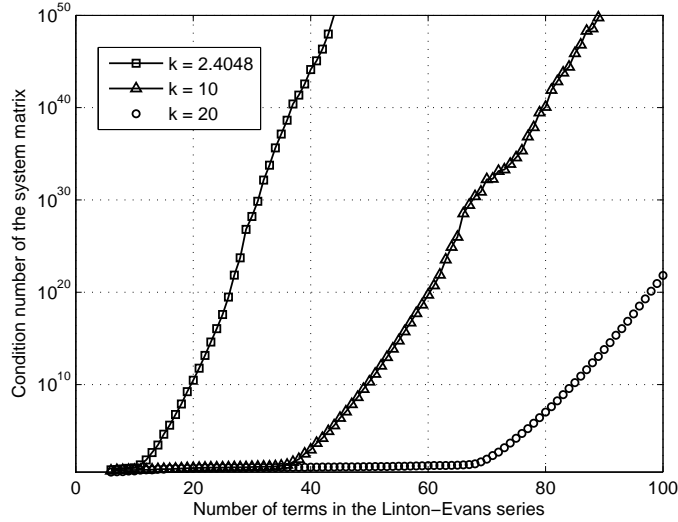


Figure 2: Stability of Linton-Evans series, eq. (2.15) in [8]

	Cyl. 1		Cyl. 2		Cyl. 3		Cyl. 4	
	CHIEF	BM	CHIEF	BM	CHIEF	BM	CHIEF	BM
$\tau \approx 3.0$	5.98E-04	1.03	2.15E-04	2.71	2.78E-04	6.26	1.67E-04	2.01
$\tau \approx 3.5$	8.67E-06	4.06E-05	1.28E-05	9.39E-05	7.6E-06	3.29E-05	516E-06	3.82E-05
$\tau \approx 3.9$	2.01E-07	4.07E-06	1.87E-07	4.01E-06	3.46E-07	6.32E-06	1.92E-07	5.17E-06

Table 1: PUBEM results -  $E^2(\phi)$  for scattering from four cylinder array for  $k = 36.9171$  and  $\theta^I = 45^\circ$ , 100 terms in Linton-Evans series

using (2.15) in [8] highly ill-conditioned. Fig.2 shows the dependence of the condition number of the system matrix formed from (2.15) in [8] on the number of terms included in the series. Note that  $k = 2.4048$  is an irregular wavenumber (first zero of the first kind Bessel function,  $J_0$ ). Clearly the reason for such significantly high condition numbers is the wide spread of eigenvalues with the growing number of terms in the series.

We use  $M$  terms in the Linton-Evans series, thus, a system of linear equations of size  $N_c(2M + 1)$  is formed where  $N_c$  is the number of cylinders (4 in the present case). We use a linear least squares solver with QR factorisation to solve this system of linear equations using suitable routines from the LAPACK library and obtain the total acoustic potential on each cylinder surface. This solution is considered as the reference solution and used to compute the relative  $L^2$  error (see (9)) for our PUBEM solution with the CHIEF and Burton-Miller methods. For the error analysis of the four cylinder problem, we consider two cases of the wavenumber, namely,  $k = 36.9171$  and  $k = 150$ . It may be noted that both  $k = 36.9171$  and  $k = 150$  are irregular wavenumbers. The  $L^2$  error results shown in Tables 1-2 are obtained using two continuous elements per cylinder with trigonometric shape functions as before. All the results are obtained with 30 integration points per wavelength.

We have used  $M = 100$  for  $k = 36.9171$  and  $M = 200$  for  $k = 150$ , in the Linton-Evans series. The condition number of the coefficient matrix for the Linton-Evans series for  $k = 36.9171$  with 100 terms was 14.28 and that for  $k = 150$  with 200 terms was 16.29. It can be noted from Tables 1-2 that the accuracy of both CHIEF and regularised Burton-Miller methods improves with more plane waves per node i.e. by increasing the value of the parameter  $\tau$ .

	Cyl. 1		Cyl. 2		Cyl. 3		Cyl. 4	
	CHIEF	BM	CHIEF	BM	CHIEF	BM	CHIEF	BM
$\tau \approx 2.2$	8.26	15.26	12.53	27.67	42.43	98.36	15.68	37.90
$\tau \approx 2.6$	6.73E-05	3.0E-03	7.30E-05	2.6E-03	7.23E-05	4.8E-03	7.98E-05	7.0E-03
$\tau \approx 3.0$	6.48E-05	6.47E-05	6.40E-05	6.39E-05	6.68E-05	6.70E-05	6.40E-05	6.46E-05

Table 2: PUBEM results -  $E^2(\phi)$  for scattering from four cylinder array for  $k = 150$  and  $\theta^I = 45^\circ$ , 200 terms in Linton-Evans series.

## 4 CONCLUSION

We have presented a plane wave enriched BEM formulation of the regularised Burton-Miller equations for the exterior acoustic scattering problem in two dimensions. The error analyses presented for the classical single and the multiple scattering problems show that the CHIEF method outperforms Burton-Miller method by at least 1 order of magnitude for the problems considered in this paper. The Burton-Miller method can prove competitive despite the difficult and slowly converging integrals if suitable coordinate transformations are implemented. The CHIEF method may be preferred over the Burton-Miller formulation, at least for simpler geometries and moderate wavenumbers ( $k < 200$ ) as the former does not have the problem of hypersingular integrals and provided that a sufficient number of interior collocation points are chosen that ensure the linear independence of the coefficient matrix  $\mathbf{H}$ .

## Acknowledgements

The first author gratefully acknowledges generous financial support from Durham University.

## REFERENCES

1. H.A. Schenck (1968) Improved Integral Formulation for Acoustic Radiation Problems, *Journal of the Acoustical Society of America*, **44**, pp.41-58.
2. A.J. Burton and G.F. Miller (1971) The application of integral equation methods to the numerical solution of some exterior boundary-value problems, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **323**, pp.201-210.
3. R.D. Ciskowski and C.A. Brebbia (1991) *Boundary Element Methods in Acoustics*, CMP and Elsevier Applied Science.
4. K. Chen, J. Cheng and P.J. Harris (2009) A new study of the Burton and Miller method for the solution of a 3D Helmholtz problem, *IMA Journal of Applied Mathematics*, **74**(2), pp.163-177.
5. E. Perry-Debain, O. Laghrouche, P. Bettess and J. Trevelyan (2004) Plane-wave basis finite elements and boundary elements for three-dimensional wave scattering, *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, **362**, pp.561-577.
6. M.J. Peake, J. Trevelyan and G. Coates (2012) Novel basis functions for the partition of unity boundary element method for Helmholtz problems, *International Journal for Numerical Methods in Engineering*, **93**, pp.905-918.
7. J.J. Bowman, T.B.A. Senior and P.L.E. Uslenghi (1970) *Electromagnetic and Acoustic Scattering by Simple Shapes*, North-Holland Pub. Co.

8. C.M. Linton and D.V. Evans (1990) The interaction of waves with arrays of vertical circular cylinders, *Journal of Fluid Mechanics*, **215**, pp.549-569.
9. P.A. Martin (1985) Integral-equation methods for multiple-scattering problems I. Acoustics, *The Quarterly Journal of Mechanics and Applied Mathematics*, **38**, pp.105-118.