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EXTENDED ISOGEOMETRIC BOUNDARY ELEMENT METHOD (XIBEM) FOR ACOUSTIC WAVE SCATTERING PROBLEMS

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Abstract. Isogeometric analysis is the concept of using the same functions that describe a geometry in computer-aided design to approximate unknown fields in numerical simulations. This has become a topic of considerable interest to the boundary integral methods community. This paper introduces an eXtended Isogeometric Boundary Element Method (XIBEM), in which isogeometric functions approximating wave potential are enriched using the partition-of-unity method. In this new method, the isogeometric basis is formed from a space of non-uniform rational B-spline (NURBS) functions multiplied by families of plane waves. Using numerical examples, it is shown that this reduces the total number of equations that need to be solved for a given frequency and geometry of problem; this improves the accuracy of and extends the supported frequency range of the boundary element method to include short wave diffraction problems.

1. INTRODUCTION

Meshing is an important aspect of numerical analysis and procedures that reduce the time required, or improve mesh quality, are of interest to both the academic and industrial communities. One such procedure is isogeometric analysis (IGA), introduced by Hughes *et al.* [1]. This is the concept of using the basis functions used to describe a geometry in computer-aided design (CAD) to construct exact geometries for numerical analysis.

Early IGA work concentrated on using the finite element method (FEM) for analysis; this paper focuses on the use of the boundary element method (BEM). The BEM only requires the boundaries of scattering surfaces to be meshed; non-uniform rational B-splines (NURBS), functions commonly used in CAD software, also only describe the boundaries of geometries. This makes IGA and BEM a natural combination. Simpson *et al.* [3] applied this concept to elastostatics, coining the pairing IGABEM; Politis *et al.* [4] applied it to potential flow problems; Takahashi and Matsumoto [5] combined this with the fast multipole method; Scott *et al.* [6] presented some three-dimensional problems using T-splines, an alternative to NURBS.

Like the FEM, conventional BEM schemes require a mesh to be refined as the wavenumber, k, of a problem increases. For a fixed computational resource, this puts a practical limit on the wavelengths that can be considered for a specified geometry. A number of approaches have been developed to increase this limit [7–10]; in this paper, the partition of unity method (PUM) [11] is used. The PUM, in which the character of the wave propagation is included in the approximating function, was first used for wave scattering with the BEM by de la Bourdonnaye [12] under the name 'microlocal discretization'. Perrey-Debain *et al.* [13] showed that the number of degrees of freedom for a given problem could be dramatically reduced using this method.

This paper develops an isogeometric, collocation BEM employing PUM for acoustic wave scattering problems; this is named the eXtended Isogeometric BEM (XIBEM).

2. FORMULATION OF XIBEM FOR THE HELMHOLTZ EQUATION

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain containing a smooth scatterer of boundary $\Gamma := \partial\Omega$. Assuming $e^{-i\omega t}$ time dependence, the wave equation can be reduced to the well-known Helmholtz equation:

$$\Delta\phi(\mathbf{q}) + k^2\phi(\mathbf{q}) = 0, \quad \phi \in \mathbb{C}, \mathbf{q} \in \Omega, \tag{1}$$

where $\Delta(\cdot)$ is the Laplacian operator, $\phi(\mathbf{q})$ is the unknown wave potential at $\mathbf{q}$, and k is wavenumber – related to the wavelength, λ, by $k = 2\pi/\lambda$. The scatterer is impinged by an incident, plane wave,

$$\phi^{\mathrm{I}}(\mathbf{q}) = A^{\mathrm{I}} \exp\left(ik\mathbf{d}^{\mathrm{I}} \cdot \mathbf{q}\right), \quad \left|\mathbf{d}^{\mathrm{I}}\right| = 1, \tag{2}$$

where A^{I} is the incident wave amplitude and $\mathbf{d}^{\mathrm{I}}$ is the direction of propagation.

The conventional boundary integral equation (BIE) for the Helmholtz equation [14] is

$$\frac{1}{2}\phi(\mathbf{p}) = \int_\Gamma \left[\frac{\partial\phi(\mathbf{q})}{\partial n}G(\mathbf{p},\mathbf{q}) - \phi(\mathbf{q})\frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n}\right] \mathrm{d}\Gamma(\mathbf{q}) + \phi^{\mathrm{I}}(\mathbf{p}), \quad \mathbf{p},\mathbf{q} \in \Gamma, \tag{3}$$

where $\mathbf{p}$ is an evaluation point and n is the outward-pointing, unit normal at integration point $\mathbf{q}$. Further, $G(\mathbf{p},\mathbf{q})$ is the fundamental solution (Green's function), representing the field experienced at $\mathbf{q}$ due to a unit source radiating at $\mathbf{p}$ (or *vice versa*).

For compact presentation, only one boundary condition is considered: the case of a perfectly reflecting ("sound-hard") cylinder, expressed as $\partial\phi(\mathbf{q})/\partial n = 0$. A solution to (1) is sought, subject to this boundary condition. The BIE (3) may be then be reformulated as

$$\frac{1}{2}\phi(\mathbf{p}) + \int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n}\phi(\mathbf{q})\,\mathrm{d}\Gamma(\mathbf{q}) = \phi^{\mathrm{I}}(\mathbf{p}). \tag{4}$$

2.1 NURBS

NURBS are used to discretise (4). An important property of B-splines and NURBS is the knot vector, Ξ; this is a sequence of nondecreasing, real numbers (knots). This paper assume the knot vector has the form

$$\Xi = \{\underbrace{0,\ldots,0}_{p+1}, \xi_{p+1}, \ldots, \xi_{s-p+1}, \underbrace{1,\ldots,1}_{p+1}\},$$

where p is the degree of the curve, there are $s+1$ knots, and $\xi_j \leq \xi_{j+1}$ for $j = 0,\ldots,s-1$.

The jth B-spline basis function of pth degree, $N_{j,p}$, is defined for $p = 0$ as

$$N_{j,0}(\xi) = \begin{cases} 1 & \text{if } \xi_j \leq \xi \leq \xi_{j+1} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For $p = 1, 2, 3, \ldots$, it is

$$N_{j,p}(\xi) = \frac{\xi - \xi_j}{\xi_{j+p} - \xi_j}N_{j,p-1}(\xi) + \frac{\xi_{j+p+1} - \xi}{\xi_{j+p+1} - \xi_{j+1}}N_{j+1,p-1}(\xi). \tag{6}$$

The number of basis functions, $J+1$, is related to the degree of the basis and length of the knot vector through $J = s - p - 1$. For NURBS, each basis function is given an individual weighting, w_j; the jth NURBS basis function of pth degree, $R_{j,p}$, is defined as

$$R_{j,p}(\xi) = \frac{N_{j,p}(\xi)w_j}{\sum\limits_{i=0}^{J} N_{i,p}(\xi)w_i}. \tag{7}$$

A NURBS curve is defined using NURBS functions and a set of control points $\mathbf{P}_j$:

$$\mathbf{C}(\xi) = \sum_{j=0}^{J} R_{j,p}(\xi)\mathbf{P}_j. \tag{8}$$

This relationship can provide an analytical geometry given by

$$\Gamma = \{\mathbf{C}(\xi) : \xi \in [0, 1)\}, \tag{9}$$

where $\mathbf{C} : \mathbb{R} \to \mathbb{R}^2$. The mapping between $\mathbf{q} \in \Gamma$ and ξ is unique, hence it is assumed that any function $f(\mathbf{q})$ is equivalent to $f(\xi)$.

2.2 XIBEM

For IGABEM, NURBS are used to represent the boundary, Γ, but also the variation of velocity potential along Γ:

$$\phi(\xi) = \sum_{j=0}^{J} R_{j,p}(\xi)\phi_j, \tag{10}$$

where ϕ_j is the potential associated with each NURBS basis function. XIBEM introduces a linear expansion of plane waves on each NURBS function; (10) is rewritten,

$$\phi(\xi) = \sum_{j=0}^{J} R_{j,p} \sum_{m=0}^{M} A_{jm} \exp\left(ik\mathbf{d}_{jm} \cdot \mathbf{q}\right), \quad |\mathbf{d}_{jm}| = 1, \tag{11}$$

where there are $M + 1$ plane waves related to each NURBS function with prescribed direction of propagation, $\mathbf{d}_{jm} \in \mathbb{R}^2$, and unknown amplitudes, $A_{jm} \in \mathbb{C}$.

Substituting (11) into (4) gives

$$\frac{1}{2}\phi(\mathbf{p}) + \sum_{j=0}^{J}\sum_{m=0}^{M}\int_0^1 \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n} R_j(\xi) \exp(ik\,\mathbf{d}_{jm} \cdot \mathbf{q})|J_\xi|\,\mathrm{d}\xi\, A_{jm} = \phi^{\mathrm{I}}(\mathbf{p}) \tag{12}$$

where $|J_\xi|$ is the Jacobian of the mapping in (9). $\phi(\mathbf{p})$ is also expanded as in (11).

To find the potential on Γ, (12) is collocated at a series of collocation points, $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{Z-1}$, uniformly-spaced on $\xi \in [0, 1)$; $Z = (J + 1)(M + 1)$ is the total number of unknown amplitudes, A_{jm}, that are sought. This yields a square system of linear equations,

$$[(1/2)\mathbf{P} + \mathbf{H}]\{\mathbf{x}\} = \{\mathbf{b}\}, \tag{13}$$

where the (usually sparse) square matrix $\mathbf{P}$ results from interpolations of the plane waves, $\phi(\mathbf{p})$; the right-hand side vector $\{\mathbf{b}\}$ contains the incident wave potentials at the collocation points; and the unknown vector $\{\mathbf{x}\}$ contains the amplitudes, A_{jm}. The square matrix $\mathbf{H}$ is fully populated with integrals,

$$h^z_{jm} = \int_0^1 \frac{\partial G(\mathbf{p}_z, \mathbf{q})}{\partial n} R_j(\xi) \exp(ik\,\mathbf{d}_{jm} \cdot \mathbf{q})|J_\xi|\,\mathrm{d}\xi. \tag{14}$$

3. NUMERICAL RESULTS

For the numerical examples here, the sound-hard boundary condition is applied; there are no singular integrals in this case so no regularisation scheme is required. The integrals of (14) are evaluated using Gauss-Legendre quadrature, subdividing the boundary into cells of approximately $\lambda/4$ in length. To overcome the well-known nonuniqueness problem associated with the BEM, the authors use the CHIEF method [15]; the system is solved using singular value decomposition (SVD). Errors, $\mathcal{E}$, are calculated in a relative L^2 norm sense:

$$\mathcal{E} = \frac{\left\|\Phi - \Phi^{\mathrm{exact}}\right\|_{L^2(\Gamma)}}{\left\|\Phi^{\mathrm{exact}}\right\|_{L^2(\Gamma)}}, \tag{15}$$

where Φ is a vector of potentials along the boundary of the scatterer calculated from the numerical simulation; and Φ^{exact} is a vector of potentials calculated using either an analytical solution or an appropriate converged solution.

3.1 Unit cylinder

Consider a cylinder of radius $a = 1$, centred at $(0, 0)$, being impinged by a unit-amplitude, incident plane wave propagating in the direction $\mathbf{d}^{\mathrm{I}} = (1, 0)$. There is an analytical solution [16].

The quality of solution of conventional, polynomial BEM and IGABEM simulations is compared. Figure 1 shows the errors of solutions by these two methods over a range of ka. The variable τ is defined as the number of degrees of freedom per wavelength of the problem. $\tau \approx 10$ for all simulations; as ka increases, the mesh is refined by increasing the number of elements or, for IGABEM, inserting knots. IGABEM clearly provides a greater accuracy of approximation; this is caused by the integration points being mapped to the analytical surface of the cylinder by the NURBS and the approximation of potential being interpolated by the same functions.

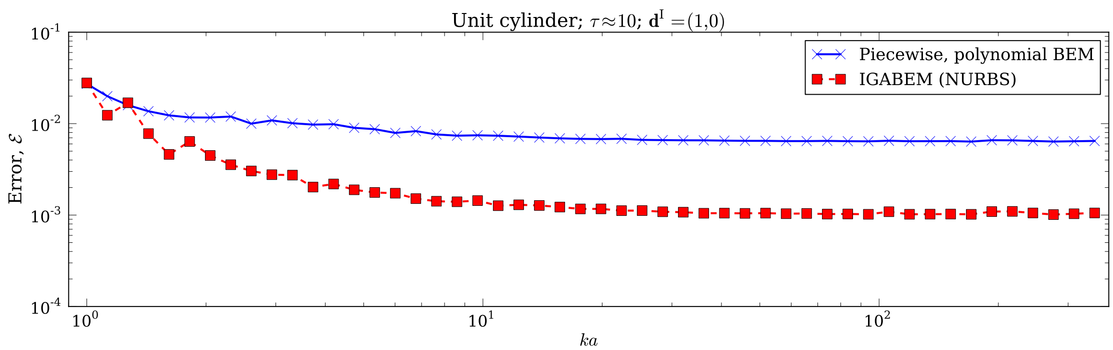

Figure 1: Comparison of errors of conventional, piecewise BEM and IGABEM.

Figure 2 compares the quality of solutions of IGABEM and XIBEM; errors of a piecewise quadratic, partition-of-unity BEM (PU-BEM) [13] are also included. For XIBEM and PU-BEM simulations, $\tau \approx 3$. The IGABEM simulations achieve an accuracy of $\sim 1\%$ while the XIBEM and PU-BEM simulations, with system matrices nine times smaller than the IGABEM, acheive significantly greater accuracy.

The accuracy of XIBEM and PU-BEM are broadly similar. However, for the PU-BEM simulations, geometry points are not located just by the shape functions; instead, collocation and integration points are carefully 'snapped' to the analytical surface. Without this, the accuracy of these simulations is dramatically reduced. A significant benefit of XIBEM is that this mapping is inherent within the NURBS representation.

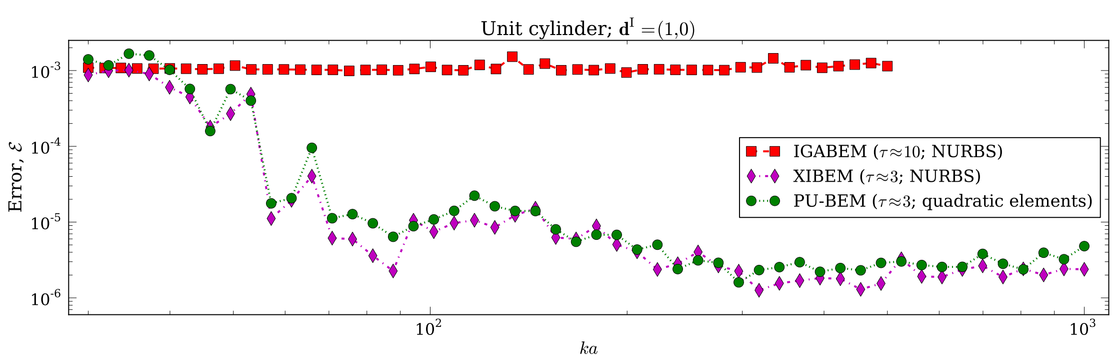

Figure 2: Comparison of errors of IGABEM, XIBEM and PU-BEM.

3.2 Multiple scatterers

A second geometry considers multiple scatterers designed to create internal reflections. Figure 3 displays the geometry used, consisting of two capsules and a cylinder, and illustrates the absolute value of the total wave potential; the angle of incidence, θ^{I}, is $3\pi/4$ radians. The reference solution used to calculate errors, $\mathcal{E}$, is a converged solution obtained using the method of fundamental solutions (MFS) [17].

Figure 4 displays the errors of conventional BEM, IGABEM and XIBEM simulations; the value of τ for each simulation type is noted in the legend. XIBEM approximations are obtained using three times fewers degrees of freedom than used by the other simulations. The IGABEM approximations are clearly more accurate than those of the conventional BEM; futhermore, the XIBEM approximations have smaller errors than both.

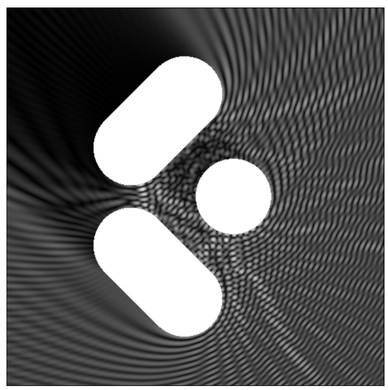

Figure 3: Plot of $\|\phi\|$ illustrating internal reflections of multiple scatterers; $ka = 25$, $\theta^{\mathrm{I}} = 3\pi/4$

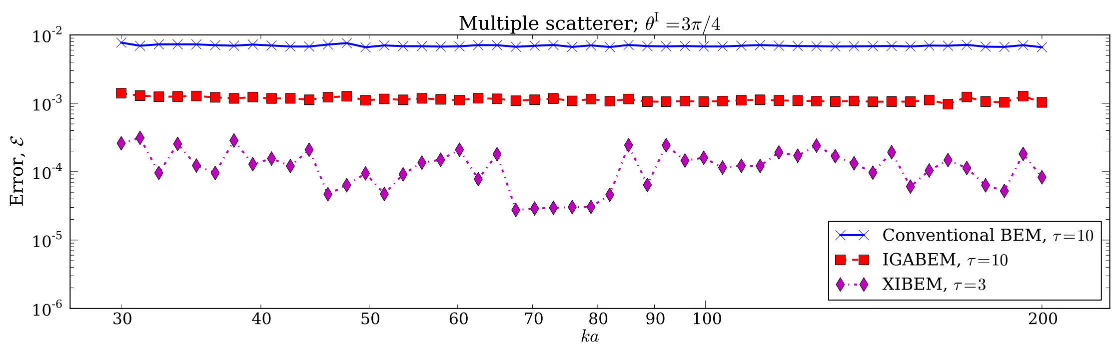

Figure 4: Comparison of errors of conventional, piecewise BEM, IGABEM and XIBEM.

4. CONCLUSIONS

Isogeometric analysis has significant potential for use with boundary elements methods. This paper has demonstrated this for both conventional and enriched methods. XIBEM provides a clear and significant improvement over IGABEM simulations. These improvements are the reduced system size and increased accuracy; this also extends the bandwidth of frequencies over which the method can be considered feasible.

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