Constraint Satisfaction Problems Over The Integers with Successor

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Abstract. A *distance constraint satisfaction problem* is a constraint satisfaction problem (CSP) whose constraint language consists of relations that are first-order definable over $(\mathbb{Z}; \mathrm{succ})$, i.e., over the integers with the successor function. Our main result says that every distance CSP is in P or NP-complete, unless it can be formulated as a finite domain CSP in which case the computational complexity is not known in general.

1 Introduction

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."[4] Leopold Kronecker

A *constraint satisfaction problem* is a computational problem where the input consists of a finite set of variables and a finite set of constraints, and where the question is whether there exists a mapping from the variables to some fixed domain such that all the constraints are satisfied. When the domain is finite, and arbitrary constraints are permitted in the input, the CSP is NP-complete. However, when only constraints for a restricted set of relations are allowed in the input, it might be possible to solve the CSP in polynomial time. The set of relations that is allowed to formulate the constraints in the input is often called the *constraint language*. The question which constraint languages give rise to polynomial-time solvable CSPs has been the topic of intensive research over the past years. It has been conjectured by Feder and Vardi [8] that CSPs for constraint languages over finite domains have a complexity dichotomy: they are in P or NP-complete.

A famous CSP over an infinite domain is *feasibility of linear inequalities over the integers*. It is of great importance in practice and theory of computing, and NP-complete. In order to obtain a systematic understanding of polynomial-time solvable restrictions and variations of this problem, Jonsson and Lööw [13]

* The first author has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement no. 257039).

** The second author was supported by EPSRC grant EP/L005654/1.

[4] *"God made the integers, all the rest is the work of man."* Quoted in *Philosophies of Mathematics*, page 13, by Alexander George, Daniel J. Velleman, Philosophy, 2002.

proposed to study the class of CSPs where the constraint language Γ is definable in *Presburger arithmetic*; that is, it consists of relations that have a first-order definition over $(\mathbb{Z}; \leq, +)$. Equivalently, each relation $R(x_1, \ldots, x_n)$ in Γ can be defined by a disjunction of conjunctions of the atomic formulas of the form $p \leq 0$ where p is a linear polynomial with integer coefficients and variables from $\{x_1, \ldots, x_n\}$. The constraint satisfaction problem for Γ, denoted by $\mathrm{CSP}(\Gamma)$, is the problem of deciding whether a given conjunction of formulas of the form $R(y_1, \ldots, y_n)$, for some n-ary R from Γ, is satisfiable in Γ. By appropriately choosing such a constraint language Γ, a great variety of problems over the integers can be formulated as $\mathrm{CSP}(\Gamma)$. Several constraint languages Γ over the integers are known where the CSP can be solved in polynomial time. However, a complete complexity classification for the CSPs of Jonsson-Lööw languages appears to be a very ambitious goal.

In this paper, we study one of the most basic classes of constraint languages that falls into the framework of Jonsson and Lööw, namely the class of *distance constraint satisfaction problems* [1]. A distance constraint satisfaction problem is a CSP for a constraint language over the integers whose relations have a first-order definition over $(\mathbb{Z}; \mathrm{succ})$ where succ is the successor function. The structure $(\mathbb{Z}; \mathrm{succ})$ has quantifier-elimination, and it is easy to see that a relation is first-order definable over $(\mathbb{Z}; \mathrm{succ})$ if and only if it can be defined by a disjunction of conjunctions of literals of the form $x = \mathrm{succ}^c(y)$ or $x \neq \mathrm{succ}^c(y)$ for $c \in \mathbb{N}$.

It has been shown previously that distance CSPs for constraint languages whose relations have *bounded Gaifman degree* are either NP-complete, or in P, or can also be formulated with a constraint language over a finite domain [1]. The finite Gaifman degree assumption is quite strong; however, here we prove that the same is true even if we drop this assumption. In other words, we show that if the Feder-Vardi dichotomy conjecture for finite domain CSPs is true, then also the class of all distance CSPs exhibits a complexity dichotomy.

Our proof relies on the so-called universal-algebraic approach; this is the first time that this approach has been used for constraint languages that are not finite or countably infinite *ω-categorical.* The central insight of the universal-algebraic approach to constraint satisfaction is that the computational complexity of a CSP is captured by the set of *polymorphisms* of the constraint language. One of the ideas of the present paper is that in order to use polymorphisms when the constraint language is not ω-categorical, we have to pass to the countably saturated model of the integers with successor. The relevance of saturated models for the universal-algebraic approach has already been pointed out in joint work of the authors with Martin Hils [2], but this is the first time that this perspective has been used to perform complexity classification for a large class of concrete computational problems.

The formal definitions of CSPs and distance CSPs can be found in Section 2. The border between distance CSPs in P and NP-complete distance CSPs can be most elegantly stated using the terminology of the mentioned universal-algebraic approach to constraint satisfaction. This is why we first give a brief introduction to this approach in Section 3, and only then give the technical description of our main result in Section 4. Section 5 gives a classification of distance constraint languages that might be of independent interest; this classification is the basis of our classification of the complexity of distance CSPs. Our algorithmic results can be found in Section 6. Finally, we put all the results together to prove our main result in Section 7. We discuss our result and promising future research questions in Section 8.

2 Distance CSPs

Let Γ be a structure with a finite relational signature τ. When R is a relation symbol from τ, we write R^Γ for the relation it denotes in the structure Γ.

A *τ-formula* is a first-order formula built from the relations from τ, and equality. A τ-formula is *primitive positive (pp)* if it is of the form $\exists x_1, \ldots, x_k(\psi_1 \wedge \cdots \wedge \psi_m)$ where each ψ_i is an atomic τ-formula. *Sentences* are formulas without free variables.

Definition 1 ($\mathrm{CSP}(\Gamma)$)**.** *The* constraint satisfaction problem for Γ *is the following computational problem.*
Input: *A primitive positive τ-sentence Φ.*
Question: $\Gamma \models \Phi$*?*

The structure Γ will also be called the *constraint language* of $\mathrm{CSP}(\Gamma)$. A relational structure Γ is a *reduct* of a structure Δ if it has the same domain as Δ and every relation R^Γ of arity k is *first-order definable* over Δ, that is, there exists a first-order formula φ in the signature of Δ with k free variables such that for all elements $u_1, \ldots, u_k$ of Γ we have $R^\Gamma(u_1, \ldots, u_k) \Leftrightarrow \Delta \models \varphi(u_1, \ldots, u_k)$.

We write $(\mathbb{Z}; \mathrm{succ})$ for the structure of the integers with the successor function.

Definition 2 (Distance CSP). *A* distance CSP *is a constraint satisfaction problem where the constraint language is finite and a reduct of* $(\mathbb{Z}; \mathrm{succ})$*.*

It is well-known that $(\mathbb{Z}; \mathrm{succ})$ admits quantifier elimination (this is easy to prove, and can be found explicitly in [9]). Moreover, it is easy to see that every quantifier-free formula is over $(\mathbb{Z}; \mathrm{succ})$ equivalent to a quantifier-free formula in conjunctive normal form (CNF) where every atomic formula is of the form $y = \mathrm{succ}^n(x)$ for $n \in \mathbb{N}$, where $\mathrm{succ}^n(x)$ is defined inductively by $\mathrm{succ}^0(x) = x$, and $\mathrm{succ}^{n+1}(x) = \mathrm{succ}(\mathrm{succ}^n(x))$. We will call formulas of this form *standardized*.

Example 1. We give examples of reducts of $(\mathbb{Z}; \mathrm{succ})$; the relations from those examples will re-appear in later sections.

1. $(\mathbb{Z}; \mathrm{Diff}_S)$, where $\mathrm{Diff}_S := \{(x, y) : x, y \in \mathbb{Z}, y - x \in S\}$ for a finite set $S \subset \mathbb{Z}$.
2. $(\mathbb{Z}; \mathrm{Diff}_{\{2\}}, \{(x, y) : |x - y| \leq 2\})$.
3. $(\mathbb{Z}; F)$ where F is the 4-ary relation $\{(x, y, u, v) : x = \mathrm{succ}(y) \Leftrightarrow u = \mathrm{succ}(v)\}$.
4. $(\mathbb{Z}; \neq, \mathrm{Dist}_i)$ where $\mathrm{Dist}_i := \{(x, y) : |x - y| = i\}$.

The last two examples have unbounded Gaifman degree (see Section 5.1), so they do not fall into the scope of [1]. The following is easy to see.

Proposition 1. *All distance CSPs are in NP.*

3 The Algebraic Approach

The starting point of the universal algebraic approach to analyze the complexity of CSPs is the observation that when a relation R can be defined by a primitive positive formula over Γ, then $\mathrm{CSP}(\Gamma)$ allows to simulate the 'richer' problem $\mathrm{CSP}(\Delta)$ where $\Delta = (\Gamma, R)$ has been obtained from Γ by adding R as another relation. The proof of this fact given by Jeavons, Cohen, and Gyssens [12] works for all structures Γ over finite or over infinite domains. Since we will use this fact very frequently, we will not explicitly refer back to it from now on.

Polymorphisms are an important tool to study the question of which relations are primitive positive definable in Γ. We say that a function $f\colon D^n \to D$ *preserves* a relation $R \subseteq D^m$ if for all $t_1, \ldots, t_n \in R$ the tuple $f(t_1, \ldots, t_n)$ obtained by applying f componentwise to the tuples $t_1, \ldots, t_n$ is also in R; otherwise, f *violates* R. A *polymorphism* of a relational structure Γ with domain D is a function from D^n to D, for some finite n, which preserves *all* relations of Γ. We write $\mathrm{Pol}(\Gamma)$ for the set of all polymorphisms of Γ. It is clear that a polymorphism of a structure Γ also preserves all relations that are primitive positive definable in Γ; this holds for arbitrary finite and infinite structures Γ. If Γ is finite or ω-categorical [5], then a relation is preserved by all polymorphisms *if and only if* it is primitive positive definable in Γ.

The structures that we consider in this paper will not be ω-categorical; however, following the philosophy in [2], one can refine these universal-algebraic methods to apply them also in our situation. The *(first-order) theory* of a structure Γ, denoted by $\mathrm{Th}(\Gamma)$, is the set of all first-order sentences that are true in Γ. We define some notation to conveniently work with models of $\mathrm{Th}(\Gamma)$ and their reducts.

Definition 3 **($\kappa.\mathbb{Z}$).** *Let κ be a cardinal. We write $\kappa.\mathbb{Z}$ for κ copies of $\mathbb{Z}$ indexed by the elements of κ; formally, $\kappa.\mathbb{Z}$ is the set $\{(a, z) : a \in \kappa, z \in \mathbb{Z}\}$. Then $(\kappa.\mathbb{Z}; \mathrm{succ})$ is the structure where succ denotes the function that maps (a, z) to $(a, z + 1)$.*

It is well-known and easy to see that the models of $\mathrm{Th}(\mathbb{Z}; \mathrm{succ})$ are precisely the structures isomorphic to $(\kappa.\mathbb{Z}; \mathrm{succ})$, for some cardinal κ. When $k \in \mathbb{Z}$ and $u = (a, z) \in \kappa.\mathbb{Z}$, we write $u + k$ for $(a, z + k)$.

Definition 4 **($\kappa.\Gamma$).** *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ with signature τ. Then $\kappa.\Gamma$ denotes the 'corresponding' reduct of $(\kappa.\mathbb{Z}; \mathrm{succ})$ with signature τ. Formally, when $R \in \tau$ and φ_R is a formula that defines R^Γ, then $R^{\kappa.\Gamma}$ is the relation defined by φ_R over $(\kappa.\mathbb{Z}; \mathrm{succ})$.*

We use ω to denote the smallest infinite cardinal throughout the article. Note that $(\omega.\mathbb{Z};\mathrm{succ})$ is isomorphic to the structure $(\mathbb{Q}; x \mapsto x+1)$. In the following, we identify $(\mathbb{Z};\mathrm{succ})$ with the copy of $(\mathbb{Z};\mathrm{succ})$ induced by $0.\mathbb{Z}$ in $(\omega.\mathbb{Z};\mathrm{succ})$. That is, we view $(\mathbb{Z};\mathrm{succ})$ as a substructure of $(\omega.\mathbb{Z};\mathrm{succ})$, and consequently Γ as a substructure of $\omega.\Gamma$ for each reduct Γ of $(\mathbb{Z};\mathrm{succ})$.

A *type* of a structure Δ is a set p of formulas with one free variable x such that $p \cup \mathrm{Th}(\Delta)$ is satisfiable (that is, $\{\varphi(c) : \varphi \in p\} \cup \mathrm{Th}(\Delta)$, for a new constant symbol c, has a model). A τ-structure Γ is ω*-saturated* if for all choices of finitely many constants $c_1, \ldots, c_n$ for elements of Γ, and every type p of $(\Gamma, c_1, \ldots, c_n)$, there exists an element d of Γ such that $(\Gamma, c_1, \ldots, c_n) \models \varphi(d)$ for all $\varphi \in p$. When Γ and Δ are two countable ω-saturated structures with the same first-order theory, then Γ and Δ are isomorphic [11]. Note that $(\omega.\mathbb{Z};\mathrm{succ})$ is ω-saturated. More generally, $\omega.\Gamma$ is ω-saturated for every reduct Γ of $(\mathbb{Z};\mathrm{succ})$.

We define the function $-\colon (\kappa.\mathbb{Z})^2 \to (\mathbb{Z} \cup \{\omega\})$ for $x, y \in \kappa.\mathbb{Z}$ by

$$\begin{aligned} x - y := z \in \mathbb{Z} \quad & \text{if } x = \mathrm{succ}^z(y) \text{ for } z \geq 0, \\ & \text{or } y = \mathrm{succ}^{-z}(x) \text{ for } z < 0; \\ x - y := \omega \quad & \text{otherwise.} \end{aligned}$$

When Γ and Δ are two structures with the same relational signature τ, then a *homomorphism* from Γ to Δ is a function from the domain of Γ to the domain of Δ such that for every $R \in \tau$ of arity k we have $R^\Gamma(u_1, \ldots, u_k) \Rightarrow R^\Delta(f(u_1), \ldots, f(u_k))$. It is straightforward to see that if there is a homomorphism from Γ to Δ, and vice versa, then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Delta)$ are the same computational problem.

Lemma 1 (See Lemma 2.1 in [2]). *Let Γ be ω-saturated, let Δ be countable, let $d_1, \ldots, d_k$ be elements of Δ, and let $c_1, \ldots, c_k$ be elements of Γ. Suppose that for all primitive positive formulas φ such that $\Delta \models \varphi(d_1, \ldots, d_k)$ we have $\Gamma \models \varphi(c_1, \ldots, c_k)$. Then there exists a homomorphism from Δ to Γ that maps d_i to c_i for all $i \leq k$.*

An *endomorphism* is a unary polymorphism. To classify the computational complexity of the CSP for all reducts of a structure Γ, it often turns out to be important to study the possible endomorphisms of those reducts first, before studying the polymorphisms, e.g. for the reducts of $(\mathbb{Q}; <)$ in [4] and the reducts of the countably infinite random graph in [6].

We are now in the position to state a general result, Theorem 1, that might explain the importance of ω-saturated models for the universal-algebraic approach. When Γ is a structure, then the *orbit* of a k-tuple $(a_1, \ldots, a_k)$ of elements of Γ is the set $\{(\alpha(a_1), \ldots, \alpha(a_k)) \mid \alpha \in \mathrm{Aut}(\Gamma)\}$.

Theorem 1. *Let Γ be a countable ω-saturated structure, let Δ be a reduct of Γ, and R a relation with a first-order definition in Γ. Then*

- *R has a first-order definition in Δ if and only if R is preserved by the automorphisms of Δ;*

– *R has an existential positive definition in Δ if and only if R is preserved by the endomorphisms of Δ;*
– *if R consists of n orbits of k-tuples in Γ, then R has a primitive positive definition in Δ if and only if R is preserved by all polymorphisms of Δ of arity n.*

4 Statement of Results

The border between NP-complete successor CSPs and successor CSPs in P can be described as follows, modulo the Feder-Vardi dichotomy conjecture. A reduct Γ of $(\mathbb{Z}; \mathrm{succ})$ is *positive* if all relations of Γ have a *positive* first-order definition in $(\mathbb{Z}; \mathrm{succ})$, this is, by a first-order formula without negation. We write $\mathbb{N}$ for the natural numbers including 0, and $\mathbb{N}^+$ for the set of positive natural numbers.

Definition 5. *For $d \in \mathbb{N}^+$, the* d-modular maximum, $\max_d \colon \mathbb{Z}^2 \to \mathbb{Z}$*, is defined by* $\max_d(x, y) := \max(x, y)$ *if* $x = y \mod d$ *and* $\max_d(x, y) := x$ *otherwise. The* d-modular minimum *is defined analogously.*

Note that these two operations are not commutative when $d > 1$.

Theorem 2. *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ with finite signature. Then there exists a structure Δ such that $\mathrm{CSP}(\Delta)$ equals $\mathrm{CSP}(\Gamma)$ and one of the following cases applies.*

1. *Δ has a finite domain, and the CSP for Γ is conjectured to be in P or NP-complete [8].*
2. *Δ is a reduct of $(\mathbb{Z}; \mathrm{succ})$ and preserved by a modular max or modular min. In this case, $\mathrm{CSP}(\Gamma)$ is in P.*
3. *Δ is a reduct of $(\mathbb{Z}; \mathrm{succ})$ such that $\omega.\Delta$ is preserved by an (equivalently, all) isomorphisms between $(\omega.\mathbb{Z}; \mathrm{succ})^2$ and $(\omega.\mathbb{Z}; \mathrm{succ})$. In this case, $\mathrm{CSP}(\Gamma)$ is in P.*
4. *$\mathrm{CSP}(\Gamma)$ is NP-complete.*

5 Definability of Successor

The goal of this section is a proof that the CSPs for reducts of $(\mathbb{Z}; \mathrm{succ})$ fall into four classes. This will allow us to focus in later sections on reducts of $(\mathbb{Z}; \mathrm{succ})$ where succ is pp-definable, where succ is now used to denote the graph of the successor function, that is, $\mathrm{succ} = \{(x, y) \in \mathbb{Z}^2 \mid y = x + 1\}$.

Theorem 3. *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ with finite signature. Then $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Delta)$ where Δ is one of the following:*

1. *a finite structure;*
2. *a reduct of $(\mathbb{Z}; =)$;*
3. *a reduct of $(\mathbb{Z}; F)$ where Dist_k is pp-definable for all $k \geq 1$ (see Example 1);*
4. *a reduct of $(\mathbb{Z}; \mathrm{succ})$ where succ is pp-definable.*

The proof of this result requires some effort and spreads over the following subsections. Before we go into this, we explain the significance of the four classes for the CSP.

It is easy to see that there exists a structure Δ with a finite domain such that $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Delta)$ if and only if Γ has an endomorphism with finite range. So we will assume in the following that this is not the case.

The CSPs for reducts of $(\mathbb{Z}; =)$ have been studied in [3]; they are either in P or NP-complete. Hence, we are also done if there exists a reduct Δ of $(\mathbb{Z}; =)$ such that $\mathrm{CSP}(\Delta) = \mathrm{CSP}(\Gamma)$. Several equivalent characterizations of those reducts Γ will be given in Section 5.2. This is essential for proving Theorem 3.

When Γ is a reduct of $(\mathbb{Z}; \mathrm{succ})$ where for all $k \geq 1$ the relation Dist_k is pp-definable, then $\mathrm{CSP}(\Gamma)$ is NP-complete; this is a consequence of the following proposition from [1].

Proposition 2 (Proposition 26 in [1]). *Suppose that the relations* Dist_1 *and* Dist_5 *are pp-definable in* Γ*. Then* $\mathrm{CSP}(\Gamma)$ *is NP-hard.*

The previous paragraphs explain why Theorem 3 indeed reduces the complexity classification of CSPs for finite-signature reducts Γ of $(\mathbb{Z}; \mathrm{succ})$ to the case where succ is pp-definable in Γ.

5.1 Degrees

We consider three notions of *degree* for relations R that are first-order definable in $(\mathbb{Z}; \mathrm{succ})$:

- For $x \in \mathbb{Z}$, we consider the number of $y \in \mathbb{Z}$ that appear together with x in a tuple from R; this number is the same for all $x \in \mathbb{Z}$, and called the *Gaifman-degree* of R (it is the degree of the Gaifman graph of $(\mathbb{Z}; R)$).
- The *distance degree* of R is the supremum of d such that there are $x, y \in \mathbb{Z}$ that occur together in a tuple of R and $|x - y| = d$.
- The *quantifier-elimination-degree (qe-degree)* of R is the minimal q so that there is a quantifier-free definition of R containing no nesting of succ that is greater than q.

The degree of a reduct of $(\mathbb{Z}; \mathrm{succ})$ is the supremum of the degrees of its relations, for any of the three notions of degree. The paper [1] considered reducts of $(\mathbb{Z}; \mathrm{succ})$ with finite Gaifman-degree. Note that the Gaifman-degree is finite if and only if the distance degree is finite. In this paper, qe-degree will play the central role, as any reduct of $(\mathbb{Z}; \mathrm{succ})$ with finite relational signature clearly has finite qe-degree. We call a binary relation *trivial* if it is pp-definable over $(\mathbb{Z}; \mathrm{succ})$, and *non-trivial* otherwise.

5.2 Petrus

The following theorem is the rock upon which we build our church.

Theorem 4 (Petrus). *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ with finite relational signature and without an endomorphism of finite range. Then the following are equivalent:*

1. *there exists a reduct Δ of $(\mathbb{Z}; =)$ such that $\mathrm{CSP}(\Delta)$ equals $\mathrm{CSP}(\Gamma)$;*
2. *$\omega.\Gamma$ has an endomorphism whose range induces a structure isomorphic to a reduct of $(\mathbb{Z}; =)$;*
3. *for all ℓ greater than the qe-degree of Γ, there exists $e \in \mathrm{End}(\Gamma)$ so that the range of e is included in $\{\ell z \mid z \in \mathbb{Z}\}$;*
4. *for all $t \geq 1$, there is an $e \in \mathrm{End}(\Gamma)$, $z \in \mathbb{Z}$, such that $|e(z+t) - e(z)| > t$;*
5. *for all $t \geq 1$, there is an $e \in \mathrm{End}(\omega.\Gamma)$, $z \in \omega.\mathbb{Z}$, such that $|e(z+t)-e(z)| > t$;*
6. *all binary relations with a primitive positive definition in Γ are either the equality relation or have unbounded distance degree;*
7. *for all distinct $z_1, z_2 \in \mathbb{Z}$ there is a homomorphism $h \colon \Gamma \to \omega.\Gamma$ such that $h(z_1) - h(z_2) = \omega$;*
8. *for all distinct $z_1, z_2 \in \mathbb{Z}$ there is an $e \in \mathrm{End}(\omega.\Gamma)$ such that $e(z_1) - e(z_2) = \omega$, and for all $x, y \in \omega.\mathbb{Z}$ with $x - y = \omega$ we have $e(x) - e(y) = \omega$;*
9. *there exists an $e \in \mathrm{End}(\omega.\Gamma)$ with infinite range such that $e(x) - e(y) = \omega$ or $e(x) = e(y)$ for any two distinct $x, y \in \omega.\Gamma$.*

We would like to mention that the finite-signature assumption in the statement of Theorem 4 is necessary.

Example 2. Consider the reduct $\Gamma := (\mathbb{Z}; I_1, I_2, \dots)$ of $(\mathbb{Z}; \mathrm{succ})$ where $I_i := \{(x,y) : x \neq \mathrm{succ}^i(y)\}$. Then the endomorphisms of Γ are precisely the automorphisms of $(\mathbb{Z}; \mathrm{succ})$, and hence Γ does not satisfy items (3) and (4), but it does satisfy the remaining items.

5.3 Boundedness and Rank

Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ without a finite-range endomorphism. Theorem 4 (Petrus) characterized the "degenerate case" when $\mathrm{CSP}(\Gamma)$ is the CSP for a reduct of $(\mathbb{Z}; =)$. For such Γ, as we have mentioned before, the complexity of the CSP has already been classified. In the following we will therefore assume that the equivalent items of Theorem 4, and in particular, item (5), do *not* apply. To make the best use of those findings, we introduce the following terminology.

Definition 6. *Let $k \in \mathbb{N}^+, c \in \mathbb{N}$. A function $e \colon \kappa_1.\mathbb{Z} \to \kappa_2.\mathbb{Z}$ is* (k,c)-bounded *if for all $u \in \kappa_1.\mathbb{Z}$ we have $|e(u+k) - e(u)| \leq c$.*

We say that e is *tightly-k-bounded* if it is (k,k)-bounded, and *k-bounded* if it is (k,c)-bounded for some $c \in \mathbb{N}$. We say that $\kappa.\Gamma$ is *(k,c)-bounded* if all its endomorphisms are; similarly, $\kappa.\Gamma$ is *tightly-k-bounded* if all its endomorphisms are. We call the smallest $t \in \mathbb{N}^+$ such that $\kappa.\Gamma$ is tightly-t-bounded the *tight rank* of $\kappa.\Gamma$. Similarly, we call the smallest $r \in \mathbb{N}^+$ such that $\kappa.\Gamma$ is r-bounded the *rank* of $\kappa.\Gamma$. The negation of item (5) in Theorem 4 says that there exists a $t \in \mathbb{N}^+$ such that $\omega.\Gamma$ is tightly-t-bounded. Clearly, being tightly-t-bounded implies being t-bounded. Hence, the negation of item (5) in Theorem 4 also implies that $\omega.\Gamma$ has finite rank $r \leq t$.

Example 3. There are rank one reducts of $(\mathbb{Z}; \mathrm{succ})$ which do have non-injective endomorphisms, but no finite-range endomorphisms. Consider the second structure in the Example 1:

$$\Gamma := (\mathbb{Z}; \mathrm{Diff}_{\{2\}}, \{(x, y) : |x - y| \leq 2\}) \ .$$

Note that Γ has rank one: as e preserves the relation $\{(x, y) : |x - y| \leq 2\})$ we have $|e(x+1) - e(x)| \leq 2$. Also note that Γ has the non-injective endomorphism e defined by $e(x) = x$ for even x, and $e(x) = x + 1$ for odd x.

These two notions of rank are the key to generalize the results from [1] about reducts of $(\mathbb{Z}; \mathrm{succ})$ with finite distance degree to general finite-signature reducts.

Remark. All reducts of $(\mathbb{Z}; \mathrm{succ})$ are *strongly minimal* (see [11][14]), another important concept from model theory. Our notion of rank resembles the notion of *dimension* in this context. However, the two notions are different. Consider for instance the structure

$$\left(\mathbb{Z}; \mathrm{succ}^2, \neq, \{(x, y) : x \neq \mathrm{succ}^3(y)\}\right) \ .$$

This structure has dimension one, since the algebraic closure of any of its elements is all of $\mathbb{Z}$. However, the rank of this structure is two and not one.

In order to understand the relations pp-definable in a reduct of $(\omega.\mathbb{Z}, \mathrm{succ})$ with finite rank, we start with the structures which have rank 1, and then show how to factor structures with higher rank to structures of rank 1.

Theorem 5. *Let Γ be a finite-signature reduct of $(\mathbb{Z}; \mathrm{succ})$ so that $\omega.\Gamma$ has rank one. Then $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Delta)$ where Δ is one of the following:*

1. *a finite structure;*
2. *a reduct of $(\mathbb{Z}; F)$ where Dist_k is pp-definable for all $k \geq 1$ (see Example 1);*
3. *a reduct of $(\mathbb{Z}; \mathrm{succ})$ where succ is pp-definable.*

Definition 7. *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ and $k \in \mathbb{N}^+$. Then we write Γ/k for the substructure of Γ induced by the set $\{z \in \mathbb{Z} : z = 0 \bmod k\}$.*

For instance, in Example 3 the structure $\Gamma/2$ is isomorphic to

$$(\mathbb{Z}; \mathrm{succ}, \{(x, y) : |x - y| \leq 1\}) \ .$$

Proposition 3. *Let Γ be a reduct of $(\mathbb{Z}; \mathrm{succ})$ such that $\omega.\Gamma$ has rank $r \in \mathbb{N}$. Then Γ/r has the same CSP as Γ, and is isomorphic to a reduct Δ of $(\mathbb{Z}; \mathrm{succ})$ such that $\omega.\Delta$ has rank one.*

Theorem 3 can now be proved using a combination of Proposition 3, Theorem 5, and Theorem 4.

6 Algorithms

We treat items 2 and 3 in Theorem 2. Let si be any isomorphism between $(\omega.\mathbb{Z}, \text{succ})^2$ and $(\omega.\mathbb{Z}, \text{succ})$. A standardized formula is *Horn* if all its clauses have at most one *positive literal*, i.e., a literal of the form $x = \text{succ}^p(y)$.

Proposition 4. *Let Γ be a reduct of $(\mathbb{Z}; \text{succ})$. If $\omega.\Gamma$ is preserved by* si *then every relation of Γ has a quantifier-free Horn definition over $(\mathbb{Z}; \text{succ})$. In this case,* $\text{CSP}(\Gamma)$ *is in P.*

The key algorithmic result here is that satisfiability of Horn formulas can be decided as follows: when the positive unit clauses imply that a literal in the input is false (this can be checked in polynomial time), remove this literal. Repeat this step. If we derive an empty clause in this way, there is no satisfying assignment. Otherwise, we are finally in a situation in which every literal is satisfied by a solution to the positive clauses. Using the assumption that si is a polymorphism of $\omega.\Gamma$, we obtain a satisfying assignment for all clauses in the input.

Theorem 6. *Let Γ be a finite-signature reduct of $(\mathbb{Z}; \text{succ})$ preserved by $\max_d$ or $\min_d$ for some $d \in \mathbb{N}$. Then* $\text{CSP}(\Gamma)$ *is in P.*

We describe two ideas for the proof of Theorem 6. The first is to reduce $\text{CSP}(\Gamma)$ to $\text{CSP}(\Gamma/d)$. We prove that Γ/d is preserved by max or min. The second idea is to solve $\text{CSP}(\Gamma/d)$ using the (still polynomial-time) *uniform version* of the *arc-consistency procedure*, where both the instance and the (finite) template are given in the input. It suffices to work with templates that are finite substructure of Γ/d whose size is linear in the size of the instance of $\text{CSP}(\Gamma/d)$.

7 The Classification

In this section we prove Theorem 2. By Theorem 3, we are essentially left with the task to classify the CSP for finite-signature *expansions* of $(\mathbb{Z}; \text{succ})$, i.e., reducts of $(\mathbb{Z}; \text{succ})$ which have succ among their relations.

Theorem 7. *Let Γ be a first-order expansion of $(\mathbb{Z}; \text{succ})$. Then at least one of the following is true:*

1. *Γ is positive and preserved by $\max_d$ or $\min_d$ for some $d \in \mathbb{N}$,*
2. *Γ is non-positive and $\omega.\Gamma$ is preserved by* si*,*
3. $\text{CSP}(\Gamma)$ *is NP-hard.*

To show this theorem, we first prove the following lemma. A standardized formula over the signature of $(\mathbb{Z}; \text{succ})$ in DNF is called *reduced* when every formula obtained by removing literals or clauses is not equivalent over $(\mathbb{Z}; \text{succ})$. It is clear that every quantifier-free formula is equivalent to a reduced formula.

Lemma 2. *For a first-order expansion Γ of $(\omega.\mathbb{Z}; \text{succ})$, are equivalent:*

1. *every reduced DNF that defines a relation of Γ is positive,*
2. *Γ has an endomorphism that violates the binary relation given by $|x-y| = \omega$,*
3. *Γ does not pp-define a non-trivial binary relation of infinite distance degree.*

Using Lemma 2, we treat positive and non-positive expansions Γ of $(\mathbb{Z}; \mathrm{succ})$ separately. In the non-positive case, we first show that when $\omega.\Gamma$ omits si as a polymorphism, then there exists a non-trivial binary relation with finite distance degree with a pp-definition in Γ. Together with the non-trivial binary relation of infinite distance degree from Lemma 2, one can then prove hardness of $\mathrm{CSP}(\Gamma)$ by a reduction from CSPs for finite undirected graphs G, using the classic result that $\mathrm{CSP}(G)$ is hard if G contains an odd cycle [10].

To treat the positive case, we make essential use of results and techniques that have been developed for reducts with finite distance degree in [1], based on the following lemma.

Lemma 3. *Let Γ be a positive first-order expansion of $(\mathbb{Z}; \mathrm{succ})$ that does not admit a modular max or modular min polymorphism. Then there is a non-trivial finite binary relation pp-definable in Γ.*

One of the concepts needed in the proof of Lemma 3 above and Proposition 5 below is the notion of *decomposability.* A relation R of arity n is r-*decomposable* if $R(x_1, \ldots, x_n)$ is equivalent to $\bigwedge_J \exists_{j \notin J} x_j . R(x_1, \ldots, x_n)$ where J ranges over all the r-element subsets of $\{1, \ldots, n\}$.

Definition 8. *A d-*progression *is a set of the form $[a, b \mid d] := \{a, a+d, a+2d, \ldots, b\}$, for $a \leq b$ with $b - a$ divisible by d.*

One can show that if there is a non-trivial finite binary relation R pp-definable in Γ, and $\{b - a \in \mathbb{Z} \mid (a, b) \in R\}$ is not a d-progression for any $d \geq 1$, then $\mathrm{CSP}(\Gamma)$ is NP-hard. By considering Γ/d instead of Γ, we can reduce to the case $d = 1$. In order to prove Theorem 7, it thus suffices to show the following.

Proposition 5. *Let Γ be a positive first-order expansion of $(\mathbb{Z}; \mathrm{succ})$, and $S \subset \mathbb{Z}$ a 1-progression, $|S| > 1$, such that Diff_S is pp-definable in Γ. Then Γ is preserved by max or min; or $\mathrm{CSP}(\Gamma)$ is NP-hard.*

In the proof of this proposition we use known results about finite domain CSPs. More specifically, we apply these results to substructures Δ of $(\Gamma, 0)$ induced by $\{-n, \ldots, n\}$. All singleton unary relations are pp definable in Δ. Then it is known that $\mathrm{CSP}(\Delta)$ is NP-hard, or Δ has a so-called *weak near unanimity* polymorphism of arity $k \geq 2$ (combining a result from [7] with a result from [15]). We show that in our situation, such polymorphisms must generate min or max on $\{-n, \ldots, n\}$, which then implies that also Γ is preserved by min or max.

8 Discussion

The structure $(\mathbb{Z}; \mathrm{succ})$ is among the simplest structures that is not ω-categorical. Note that $(\mathbb{Z}; \mathrm{succ})$ and its reducts are uncountably categorical and ω-stable. They are also *automatic* in the sense of algorithmic model theory.

We want to stress that the difficulties we had to overcome when classifying reducts of $(\mathbb{Z};\mathrm{succ})$ will be present in classifications of reducts of richer structures, such as $(\mathbb{Z};\mathrm{succ},\leq)$ (which has the same reducts as $(\mathbb{Z};<)$), $(\mathbb{Z};+)$, or even $(\mathbb{Z};+,\leq)$, i.e., Presburger arithmetic, and we view it as an interesting question which of our techniques might generalise to such more general contexts.

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