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An adaptive cross approximation (ACA) for the extended boundary element method (XBEM) in anisotropic materials

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ABSTRACT

The extended boundary element method (XBEM) is a modification from the standard BEM, where enrichment functions are embedded into the BEM formulation. The results obtained with this method were seen to be accurate and stable, being especially useful for fracture problems. However, the method suffers from the presence of a linear system containing unsymmetric and fully populated matrices which needs to be solved in order to get the solution of the boundary problem. This can be computationally expensive for problems dealing with multiple cracks for instance. Adaptive cross approximation (ACA) is used to reduce the number of operations necessary to solve the linear system of equations for a fracture problem with an anisotropic material.

Key Words: *Extended boundary element method; Adaptive Cross Approximation; Anisotropic materials, Fracture mechanics*

1. Introduction

The boundary element method (BEM) relies on a mathematical formulation that allows one to model a given domain considering only its boundaries. One of the advantages of this approach is to obtain a reduced mesh due to the boundary only discretisation. However, the linear system of equations is fully populated and non-symmetric. Large problems can become cumbersome to be solved numerically.

For this reason, adaptive cross approximation (ACA) has become a popular technique to accelerate the computational solution time. The idea of the method is to use the smoothness of the operator to approximate the so called admissible blocks, thus accelerating the evaluation of the matrix vector product that lies within each iteration of an iterative solver.

The extended boundary element method (XBEM) has been proposed by [1] for isotropic materials and extended to anisotropic materials by [2]. In this case the additional degrees of freedom correspond to the stress intensity factors (SIF), which are the key parameters in fracture mechanics problems. The advantage of this formulation is that the degrees of freedom do not depend on the number of enriched nodes, as they correspond to the SIF. Moreover, there is no need for post-processing (such as the J-integral) to obtain the SIF, since they are calculated as part of the displacement solution. In this work we tackle for the first time the use of ACA in an XBEM formulation for anisotropic 2D materials.

2. Extended boundary element method (XBEM)

A dual BEM formulation is modified with the enrichment as defined in [2]. Two boundary integral equations (BIE), are necessary to avoid the mathematical degeneration which arises from the coincidence of the crack faces. The displacement boundary integral equation (DBIE) and the traction boundary integral equation (TBIE) are given by [2]

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_{\Gamma} p_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + \int_{\Gamma_c} p_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_{lj}F_{lj}(\boldsymbol{\xi})\mathbf{a}_k^\alpha d\Gamma(\boldsymbol{x}) = \int_{\Gamma} u_{ij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x})$$

(1)

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_{\Gamma} s_{rij}^*(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + N_r\int_{\Gamma_c} s_{rij}^*(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_{lj}F_{lj}(\boldsymbol{\xi})d\Gamma(\boldsymbol{x}) = N_r\int_{\Gamma} d_{rij}^*(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x})$$

(2)

where $\Gamma_c = \Gamma_+ \cup \Gamma_-$ stands for the crack surfaces Γ_+ and Γ_-, N_r is the normal at the observation point $\mathbf{x}$, F_{lj} is the enrichment function and $\tilde{K}_{lj}$ is the additional degree of freedom which stands for the SIF. Let us recall that strongly singular and hypersingular terms arise from the integration of the p_{ij}^*, d_{rij}^* and s_{rij}^* kernels and they have to regularised before numerical integration is possible. More details about the regularisation procedure and the XBEM formulation can be found in [1].

The enrichment function used in Eqs. (1) and (2) are the same as defined in [3] for anisotropic materials using the extended finite element method (XFEM).

3. Adaptive Cross Approximation (ACA)

In this paper the form of ACA used is fully pivoted ACA. While partially pivoted ACA allows for reductions in storage and generation of the system matrix, the subject of this paper is addressing reductions in computations in the solution. If the number of operations required for generation and storage are each proportional to the square of the dimension of the system matrix, the direct solution of the linear system requires a number of operations proportional to its cube. Iterative solvers such as GMRES reduce the complexity but involve an expensive matrix vector product within each iteration.

In order for ACA to produce an accurate low rank approximation, the matrix must be asymptotically smooth. In BEM, this smoothness is only found where the collocation nodes are well separated geometrically from the nodes upon which they are being integrated over. The admissibility of a matrix block is assessed based on geometric considerations, by performing hierarchical clustering [4], before ACA is applied. The whole matrix is recursively subdivided until it satisfies an admissibility criterion or a minimum block size has been reached.

Performing hierarchical clustering on a dual formulated BEM matrix requires separation of the boundary nodes and the crack nodes at all times [5]. Discontinuous elements are used on the crack, with one surface corresponding to the DBIE and the other corresponding to the TBIE. For this reason, when constructing the hierarchical cluster tree, the crack surfaces must be separated from the external boundaries, and both crack surfaces must be separated from each other in collocation, but can be considered as a whole when being integrated over by for a remote collocation point. Thus it is important to place nodes on opposing crack surfaces in different clusters row-wise, but unimportant column-wise in the matrix. This creates asymmetry in the structure of the hierarchical cluster tree.

Constraints other than the admissibility criterion must be put in place to assure that these conditions are met, as although the geometry of crack nodes are usually clearly distinguishable from those of the boundary, this is not always the case. Furthermore, coincident nodes on opposite crack surfaces will be clustered together unless the algorithm is instructed otherwise.

A detailed explanation about ACA applied to BEM can be found in [6].

4. Results

The results are presented for a two dimensional anisotropic plate containing 12 internal cracks. The problem is discretised with 2000 nodes for the external boundaries and 300 nodes per crack. Continuous elements are employed on the external boundaries and discontinuous elements are used on the crack faces. There are 9248 degrees of freedom (DOF) in total, of which 48 DOFs correspond to the SIFs of each crack (mode I and II). The material properties are given by: $C_{11} = 117.97$ GPa, $C_{12} = 14.19$ GPa, $C_{16} = 35.43$ GPa, $C_{22} = 15.64$ GPa, $C_{26} = 7.49$ GPa, $C_{66} = 21.38$ GPa. Figure 1 represents the matrix structure resulting from the hierarchical clustering of the problem.

Table 1 illustrates the savings of solving the linear system of equations using ACA. The solution error is defined as

$$\text{error} = \frac{||\mathbf{x}_{ACA} - \mathbf{x}||_F}{||\mathbf{x}||_F} \tag{3}$$

where $\mathbf{x}_{ACA}$ is the displacement solution when the linear system of equations was approximated with ACA, $\mathbf{x}$ is the displacement solution obtained by solving the full system using a direct solver and $||(.)||_F$
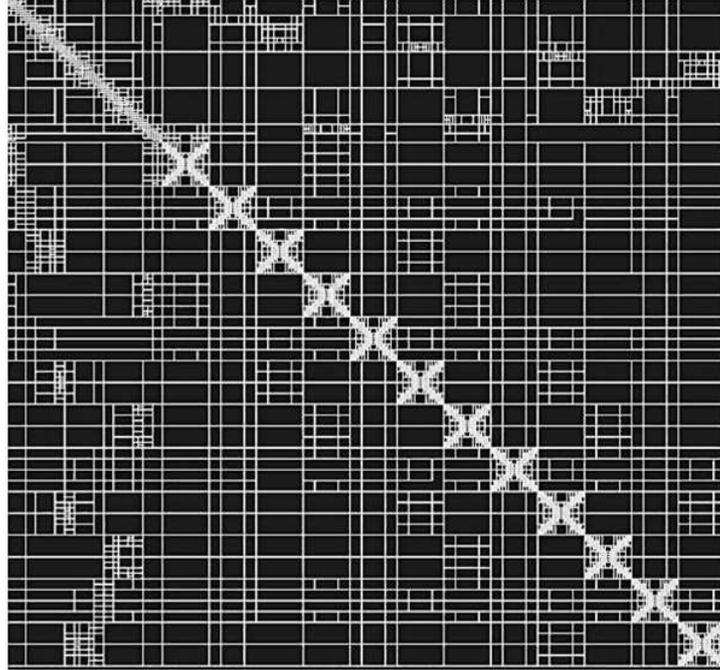
Figure 1: : ACA generated matrix partitioning.

stands for the Frobenius norm. The parameter ε_C is the threshold error of the ACA approximation using the Frobenius norm.

Table 1: Operational savings data

ε_C	Solution error		Operations	Saving(%)
	Total (%)	SIFs (%)		
1×10^{-4}	0.310	0.130	6.562×10^6	92.3
1×10^{-5}	0.022	0.018	7.499×10^6	91.2
1×10^{-6}	1.27×10^{-5}	1.26×10^{-5}	8.449×10^6	90.1

The data given as 'Operations' is the number of computations required to perform a matrix vector product using the approximated system matrix. This is the total sum of $O(r(N+M))$ for every low rank block combined with $O(NM)$ for every full rank block, where r represents the rank and N and M represent the rows and columns, respectively. 'Saving' stands for the gain in number of computations required to perform the full rank matrix-vector product.

The results show that reductions in ε_C, as expected, improves the accuracy of the eventual solution. ACA is successfully applied to all parts of the matrix, with exception of the last 48 rows and columns. For the latter, the columns are partitioned into 50×48 blocks and then approximated using ACA. For the former, these are extra rows necessary to balance the system of equations as shown in [1], and contain the so called tying equations. These rows are sparsely populated, therefore are not considered for ACA. Applying these configurations produces an accurate solution with computational savings in excess of 90% per iteration and low errors for both displacement field and SIFs.

In Table 1 the error for the Frobenius norm is the same for boundary displacements and the SIFs in the solution vector. However, it is known that the terms of the linear system of equations associated to the enriched terms can be of very different order of magnitude from the other terms. Moreover, it might be speculated that the accuracy of these terms is strongly influential over the accuracy of the computed SIFs. In this case, it is worthwhile, then, to consider lower error in the ACA approximation for the sub-blocks

containing terms related to the SIF and allowing a higher error in approximating all other sub-blocks. Table 2 analyses this issue considering two errors for the Frobenius norm, ε_C and ε_{SIF}. One can verify that the error in SIFs is strongly governed by ε_C, and less so by ε_{SIF} for the SIFs reduce slightly even for high accuracy, nevertheless the number of operations increases.

Table 2: Operational savings data for different levels of error

ε_C	ε_{SIF}	Solution error		Operations	Saving(%)
		Total (%)	SIFs (%)		
1×10^{-3}	1×10^{-4}	2.995	1.407	5.515×10^6	93.55
	1×10^{-8}	2.995	1.404	5.627×10^6	92.42
1×10^{-4}	1×10^{-4}	0.306	0.127	6.507×10^6	92.39
	1×10^{-8}	0.306	0.125	6.618×10^6	92.26
1×10^{-5}	1×10^{-4}	0.023	0.017	7.471×10^6	91.26
	1×10^{-8}	0.022	0.018	7.583×10^6	91.13

5. Conclusions

In this work we applied ACA with the XBEM for solving an anisotropic fracture problem for the first time in literature. ACA is employed as a method to accelerate the solution of the system of equations obtained from the XBEM formulation. The achieved saving times are more than 90% of efficiency compared to solving the system using regular Gauss elimination. We are looking forward to extend this formulation to deal with 3D problems.

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