Finding Cactus Roots in Polynomial Time*

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Abstract. A cactus is a connected graph in which each edge belongs to at most one cycle. A graph H is a cactus root of a graph G if H is a cactus and G can be obtained from H by adding an edge between any two vertices in H that are of distance 2 in H. We show that it is possible to test in $O(n^4)$ time whether an n-vertex graph G has a cactus root.

1   Introduction

Squares and square roots are well-known concepts in graph theory that have been studied first from a structural perspective [22, 24] but later also from an algorithmic perspective, as we will discuss. The *square* $G = H^2$ of a graph $H = (V_H, E_H)$ is the graph with vertex set $V_G = V_H$, such that any two distinct vertices $u, v \in V_H$ are adjacent in G if and only if u and v are of distance at most 2 in H. A graph H is a *square root* of G if $G = H^2$. It is a straightforward exercise to check that there exist graphs with no square root, graphs with a unique square root as well as graphs with many square roots.

In this paper we consider square roots from an algorithmic point of view. The corresponding recognition problem, which asks whether a given graph admits a square root, is called the SQUARE ROOT problem. Our research is motivated by the result of Motwani and Sudan [21] who proved in 1994 that SQUARE ROOT is NP-complete. Afterwards, SQUARE ROOT was shown to be polynomial-time solvable for various graph classes, such as K_4-free graphs (trivial), planar graphs [18], or more general, any non-trivial minor-closed graph class [23], block graphs [16], line graphs [19], trivially perfect graphs [20], threshold graphs [20], graphs of maximum degree 6 [3], 3-degenerate graphs [11] and (K_r, P_t)-free graphs for any two integers $r, t \geq 1$ [11]. It was also shown that SQUARE ROOT is NP-complete for chordal graphs [13]. We refer to [3, 4, 10] for a number of parameterized complexity results on SQUARE ROOT. The computational hardness of SQUARE ROOT also led to the following natural research question:

Is it possible to test in polynomial time whether a given graph has a square root that belongs to some specified graph class $\mathcal{H}$?

It has been shown that such a polynomial-time algorithm exists if $\mathcal{H}$ is the class of trees [18], proper interval graphs [13], bipartite graphs [12], block graphs [16], strongly chordal split graphs [17], graphs with girth at least g for any fixed $g \geq 6$ [9], ptolemaic graphs [14], 3-sun-free split graphs [14] (see [15] for an extension of the latter result to other subclasses of split graphs). In contrast, NP-completeness of this problem has been shown if $\mathcal{H}$ is the class of split graphs [13], chordal graphs [13], graphs of girth at least 4 [9] or graphs of girth at least 5 [8].

Our Result. We consider the class of all graphs being a cactus as $\mathcal{H}$. A connected graph is a *cactus* if every edge of it is contained in at most one cycle. We give an $O(n^4)$-time algorithm that tests whether an n-vertex graph has a cactus root. Our result is motivated by the nontrivial question whether squares of planar graphs can be recognized in polynomial time. The known result that squares of trees, which form a subclass of the class of cactuses, can be recognized in polynomial time [18] can be seen as a first step in solving this problem. As every cactus is planar, our result could be seen as a second step in solving it. On a side note, cactuses are not a subclass of any of the other aforementioned classes of which the squares can be recognized in polynomial time.

We prove our result by analyzing, in Section 3, the structure of squares of cactuses. In this way we are able to recognize vertices of the input graph G that are cut-vertices in any cactus root (if such a square root exists) together with a set of compulsory edges and a set of forbidden edges of any cactus root of G. In this way we can reduce, in Section 4, the graph G to a number of smaller instances such that G has a cactus root if and only if each of these smaller instances has a cactus root. Showing that each of the smaller instances has bounded treewidth and observing that we can solve the problem in linear time on any graph class of bounded treewidth completes the proof.

We observe that in several variants of the SQUARE ROOT problem where the aim is to find some type of sparse square root [1, 8, 9, 18], such a square root is unique or unique up to isomorphism. This uniqueness can be exploited and as such is very helpful for finding the square root. However, this is not the case for cactus roots: Fig. 1 shows a graph that has two non-isomorphic cactus roots.

In Section 5 we discuss some directions of future work.

2 Preliminaries

We consider only finite undirected graphs without loops and multiple edges. We refer to the textbook of Diestel [7] for any undefined graph terminology.

Basic Graph Terminology. We denote the vertex set of a graph G by V_G and the edge set by E_G. The subgraph of G induced by a subset $U \subseteq V_G$ is denoted by $G[U]$. The graph $G - U$ is the graph obtained from G after removing the vertices of U. If $U = \{u\}$, we also write $G - u$. Similarly, we denote the graph obtained from G after deleting a set of edges S (an edge e) by $G - S$ ($G - e$ respectively).

Let G be a graph. A *connected component* of G is a maximal connected subgraph. The *distance* $\mathrm{dist}_G(u, v)$ between a pair of vertices u and v of G is
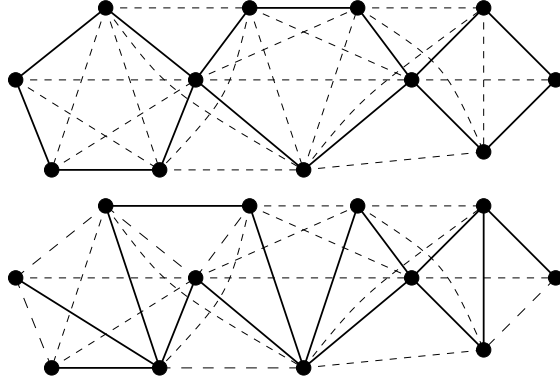
Fig. 1. A graph with non-isomorphic square cactus roots. The edges of the cactus roots are shown by solid lines, whereas the other edges are shown by dashed lines.

the number of edges of a shortest path between them. The diameter $\mathrm{diam}(G)$ of G is the maximum distance between two vertices of G. The *open neighborhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$, and its *closed neighborhood* is defined as $N_G[u] = N_G(u) \cup \{u\}$. Two (adjacent) vertices u, v are said to be *true twins* if $N_G[u] = N_G[v]$. A vertex v is *simplicial* if $N_G[v]$ is a clique, that is, if there is an edge between any two vertices of $N_G[v]$. The *degree* of a vertex $u \in V_G$ is defined as $d_G(u) = |N_G(u)|$. The maximum degree of G is $\Delta(G) = \max\{d_G(v) \mid v \in V_G\}$. A vertex of degree 1 is said to be a *pendant* vertex. If v is a pendant vertex, then we say that the unique edge incident to u is a *pendant* edge.

A vertex u is a *cut vertex* of a connected graph G with at least two vertices if $G - u$ is disconnected. An inclusion-maximal induced subgraph of G that has no cut vertex is called a *block*. Recall that a connected graph G is a cactus if each edge of G is contained in at most one cycle. This implies the following well-known property.

Observation 1 *Each block of a cactus with at least two vertices is either a K_2 (an edge) or a cycle.*

A *tree decomposition* of a graph G is a pair (T, X) where T is a tree and $X = \{X_i \mid i \in V_T\}$ is a collection of subsets (called *bags*) of V_G such that the following three conditions hold:

i) $\bigcup_{i \in V_T} X_i = V_G$,
ii) for each edge $xy \in E_G$, $x, y \in X_i$ for some $i \in V_T$, and
iii) for each $x \in V_G$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V_T\}, T)$ is $\max_{i \in V_T} \{|X_i| - 1\}$. The *treewidth* $\mathbf{tw}(G)$ of a graph G is the minimum width over all tree decompositions of G. If T is restricted to be a path, then we say that (X, T) is a *path decomposition* of G.

Problem Definition. Recall that a graph H is called a cactus root of a graph G if H is a cactus and a square root of G. We consider the following problem:

CACTUS ROOT
Input: a graph G.
Question: is there a cactus H with $H^2 = G$?

We also need to define the following more general variant introduced in [3] for general square roots:

CACTUS ROOT WITH LABELS
Input: a graph G and sets of edges $R, B \subseteq E_G$.
Question: is there a cactus H with $H^2 = G$, $R \subseteq E_H$ and $B \cap E_H = \emptyset$?

By choosing $R = B = \emptyset$ we see that CACTUS ROOT is indeed a special case of CACTUS ROOT WITH LABELS.

3 A Number of Structural Observations and Lemmas

In this section we state three observations and prove seven lemmas. We will use these results, which are all structural, for the design of our $O(n^4)$ time algorithm for CACTUS ROOT presented in Section 4.

The first observation is known and easily follows from the definition of the treewidth.

Observation 2 *For a cactus G, $\mathbf{tw}(G) \leq 2$.*

The second observation gives an upper bound for the treewidth of the square of a graph; it follows from the well-known fact that we can transform every tree decomposition (T, X) of a graph G into a tree decomposition of G^2 by adding, to each bag X_i of T, all the neighbors of every vertex from X_i.

Observation 3 *For a graph G, $\mathbf{tw}(G^2) \leq (\mathbf{tw}(G) + 1)\Delta(G) - 1$.*

Let H be a square root of a graph G. We say that H is a *minimal* square root of G if $H^2 = G$ but any proper subgraph of H is not a square root of G. Note that the two cactus roots displayed in Fig. 1 are both minimal. Since any connected subgraph of a cactus is a cactus, we can make the following observation.

Observation 4 *If a graph G has a cactus root, then G has a minimal cactus root.*

A block of a graph G is called a *leaf block* if it contains at most one cut vertex of G. This leads to our first lemma.

Lemma 1. *If a cactus H is a minimal square root of a graph G, then H has no leaf block that is a triangle.*

Proof. Suppose that a cactus H is a minimal square root of G such that a triangle with vertices x, y, z is a leaf block of H. As a leaf block contains at most one cut vertex of H by definition, we may assume that y and z are not cut vertices of H. Let $H' = H - yz$. It is straightforward to verify that $H'^2 = G$, contradicting the minimality of H. $\square$

Suppose that u and v are pendant vertices of a square root H of G and that u and v are adjacent to the same vertex of $H - \{u, v\}$. Then, in G, u and v are simplicial vertices and true twins. We use this observation in the proof of the following lemma.

Lemma 2. *Let H be a minimal cactus root of a graph G. If G contains at least six simplicial vertices that are pairwise true twins, then at least one of these vertices is a pendant vertex of H.*

Proof. Let H be a minimal cactus root of a graph G that contains a set X of six simplicial vertices that are pairwise true twins. The vertices of X cannot all belong to the same block of H, because such a block would be a cycle with at least six vertices (by Observation 1) and any two vertices of this block could not be true twins of G. Hence, there is a cut vertex u of H such that there exist two vertices $x, y \in X$ that are in distinct connected components of $H - u$. Let H' be a connected component of $H - u$ that contains x. If x is not a pendant vertex of H then, by the minimality of H and Lemma 1, there exists a vertex $z \in V_{H'}$ that is adjacent to x and that is at distance 2 from u in H. Then, as every path from y to z in H contains u, we find that $yz \notin E_G$. This is a contradiction since x and y are true twins of G and $xz \in E_G$. We conclude that x is a pendant vertex of H. $\square$

The following definition plays a crucial role in our paper.

Definition 1. *Let u be a cut vertex of a connected graph H. We say that*

(i) u is important *if $H - u$ has three vertices that belong to three distinct connected components of $H - u$ and that are each at distance at least 2 from u in H;*

(ii) u is essential *if $H - u$ has two vertices that belong to two distinct connected components of $H - u$ and that are both at distance at least 2 from u in H.*

Definition 1(i) immediately implies the following lemma.

Lemma 3. *If u is an important cut vertex of a cactus root H of a graph G, then there are three vertices $x, y, z \in N_G(u)$ such that x, y and z are at distance at least 3 from each other in $G - u$.*

Although we have no implication in the opposite direction, we can show the following (which explains why we need the second and weaker part of Definition 1).

Lemma 4. *Let G be a graph with a cactus root H. If $u \in V_G$ has three neighbors x, y and z in G that are at distance at least 3 from each other in $G - u$, then u is an essential cut vertex of H. Moreover, at least two vertices of $\{x, y, z\}$ belong to distinct connected components of $H - u$.*

Proof. Assume that G has a cactus root H. Let $u \in V_G$ be such that u has three neighbors x, y and z in G that are at distance at least 3 from each other in $G - u$. Notice that because x, y and z are at distance at least 3 from each other in $G - u$, these vertices are all at distance 2 from u in H.

For contradiction, assume that u is not a cut vertex of H. Then u has at most two adjacent vertices in H, since H is a cactus (see Observation 1). Then at least two vertices of $\{x, y, z\}$ are adjacent to the same vertex of H (which is one of the two neighbors of u) implying that these two vertices of $\{x, y, z\}$ are adjacent in G and thus in $G - u$; a contradiction. Hence u is a cut vertex of H.

Now suppose that x, y and z are all in the same connected component H' of $H - u$. Since H is a cactus, we find, by Observation 1, that H' contains at most two vertices that are adjacent to u in H. Again, we obtain that at least two vertices of $\{x, y, z\}$ are adjacent to the same vertex of H; a contradiction. Hence, at least two vertices of $\{x, y, z\}$ belong to distinct connected components of $H - u$. Since x, y and z are at distance 2 from u in H, this implies that u is an essential cut vertex of H. □

We now show that we can recognize edges of a cactus root that are incident to an essential cut vertex.

Lemma 5. *Let u be an essential cut vertex of a cactus root H of a graph G. Then for every $x \in N_G(u)$, it holds that $ux \notin E_H$ if and only if there exists a vertex $y \in N_G(u)$ such that x and y are at distance at least 3 in $G - u$.*

Proof. Let u be an essential cut vertex of a cactus root H of a graph G. Let $x \in N_G(u)$. First suppose that $ux \in E_H$. Let $y \in N_G(u)$. If $uy \in E_H$, then $xy \in E_G$. If $uy \notin E_H$, then there exist a vertex $z \in V_H$ and edges $uz, zy \in E_H$, as $y \in N_G(u)$. As $zy \in E_H$, we find that $zy \in E_G$. As $ux, uz \in E_H$, we also deduce that $xz \in E_G$. In both cases x and y are at distance at most 2 in $G - u$.

Now suppose that $ux \notin E_H$. Then, as $x \in N_G(u)$, we find that x is at distance 2 from u in H. Let H' be the connected component of $H - u$ containing x. Since u is an essential cut vertex of H, $H - u$ has another connected component H'' containing a vertex y at distance 2 from u in H. It remains to observe that $y \in N_G(u)$ and x and y are at distance 3 in $G - u$. □

The next lemma is used to recognize vertices adjacent to an essential cut vertex that belong to the same block of a minimal cactus root.

Lemma 6. *Let H be a minimal cactus root of a graph G. For any $u \in V_H$, two distinct vertices $x, y \in N_H(u)$ are in the same block of H if and only if x and y are in the same connected component of $G' = G - E_{G[N_H(u)]} - u$.*

Proof. Let $x, y \in N_H(u)$. First suppose that x and y are in distinct blocks of H. Then x and y are readily seen to be in distinct connected components of G'. Now suppose that x and y are in the same block C of H. If $xy \in E_G$ then x and y are in the same connected component of G'. Suppose $xy \notin E_G$. Then C is a cycle by Observation 1. If C is not a triangle, then C has a unique (x, y)-path in H (avoiding u) of length at least 2. This path is an (x, y)-path in G' as well.

Hence x and y are in the same connected component of G'. Suppose that C is a triangle.Then $xy \in E_H$. As H is a minimal cactus root, x or y has at least one neighbor $z \neq u$ in H due to Lemma 1. Assume without loss of generality that z is a neighbor of x. Then the edges $xy, xz \in E_H$ imply that $zy \in E_G$. We establish that xzy is an (x, y)-path in G', that is, also in this case x and y are in the same connected component of G'. □

Finally we show how to determine which neighbors in G of an essential cut vertex u of a cactus root H are in the same connected component of $H - u$.

Lemma 7. *Let H be a minimal cactus root of a graph G. For any $u \in V_H$ and $x \in N_H(u)$, a vertex $y \in N_G(u)$ is in the same connected component of $H - u$ as x if and only if either $uy \in E_H$ and y in the same block of H as x, or $uy \notin E_H$ and there is a vertex $z \in N_H(u)$, such that z is in the same block of H as x and $yz \in E_G$.*

Proof. Let $y \in N_G(u)$. First suppose y is in the same connected component of $H - u$ as x. If $uy \in E_H$, then y is in the same block of H as x. Suppose $uy \notin E_H$. As $uy \in E_G$, there is a vertex $z \in N_H(u)$ such that $zy \in E_H$. Then z is in the same block of H as x, as x and y are in the same connected component of $H - u$.

To prove the reverse implication, if $uy \in E_H$ and x, y are in the same block of H, then x and y are in the same connected component of $H - u$. Suppose that $uy \notin E_H$ and there is a vertex $z \in N_H(u)$ such that z is in the same block of H as x and $yz \in E_G$. If $yz \in E_H$, then y and z are in the same connected component of $H - u$. If $yz \notin E_H$, then there is a $v \in V_G$ such that $yv, vz \in E_H$. Since $uy \notin E_H$, we obtain $v \neq u$. Therefore, y and z are in the same connected component of $H - u$. Because y and z are in the same connected component of $H - u$ and x, y are in the same block of H, we obtain that x, y are in the same connected component of $H - u$. □

4   The Algorithm

In this section we use the structural results from the previous section to obtain a polynomial-time algorithm for CACTUS ROOT. The main idea is to reduce a given instance of CACTUS ROOT to a set of smaller instances of CACTUS ROOT WITH LABELS, each having bounded treewidth. We therefore need the following two lemmas which show, together with Observations 2 and 3, that we are done if we manage to achieve this goal. The first lemma is due to Bodlaender.

Lemma 8 ([2]). *For any fixed constant k, it is possible to decide in linear time whether the treewidth of a graph is at most k.*

Lemma 9. CACTUS ROOT WITH LABELS *can be solved in time $f(t) \cdot n$ for n-vertex graphs of treewidth at most t.*

Proof. It is not difficult to construct a dynamic programming algorithm for the problem, but for simplicity we give a non-constructive proof based on Courcelle's [5] theorem. By this theorem, it suffices to show that the existence of a cactus root can be expressed in monadic second-order logic.

Let (G, R, B) be an instance of CACTUS ROOT WITH LABELS. We observe that the existence of a cactus H such that $G = H^2$, $R \subseteq E_H$ and $B \cap E_H = \emptyset$ is equivalent to the existence of a subset $X \subseteq E_G$ such that the following four properties hold:

(i) $R \subseteq X$ and $B \cap X = \emptyset$;
(ii) for every $uv \in E_G$, $uv \in X$ or there exists a vertex w such that $uw, wv \in X$;
(iii) for every two distinct edges $uw, vw \in X$, $uv \in E_G$;
(iv) for every $uv \in X$ and for every two (u, v)-paths P_1 and P_2 in G such that $E_{P_1}, E_{P_2} \subseteq X \setminus \{uv\}$, it holds that $P_1 = P_2$.

Each of these properties can be expressed in monadic second-order logic. In particular, with respect to property (iv), expressing that a subgraph P of G is a (u, v)-path in G can be done in monadic second-order logic in a standard way (see, for example, [6]). Hence the lemma follows. □

Now we are ready to prove the main result.

Theorem 1. CACTUS ROOT *can be solved in time $O(n^4)$ for n-vertex graphs.*

Proof. We first give an overview of our algorithm. As we can consider each connected component separately, we may assume without loss of generality that the input graph G is connected. First, we use Lemma 2 to recognize sets of pendant vertices in a (potential) cactus root adjacent to the same vertex that have size at least 7. For each of these sets, we show that it is safe to delete some vertices without changing the answer for the considered instance. After performing this step, we obtain a graph G' such that in any cactus root of G' each vertex is adjacent to at most six pendants. Further, we use Lemmas 3 and 4 to construct a set U of essential cut vertices in a (potential) cactus root such that U contains all important cut vertices. Next, we apply Lemma 5 to recognize which edges incident to the vertices of U are in any cactus root and which edges are not included in any cactus root. We label them *red* and *blue* respectively and obtain an instance of CACTUS ROOT WITH LABELS. Now we can use Lemmas 6 and 7 to determine for each $u \in U$, the partition of the set of vertices of $G - u$ into the sets of vertices of the connected components of $H - u$, where H is a cactus root of G'. This allows us to split G' via the vertices of U as shown in Fig. 2. Due to the presence of labeled edges incident to the vertices of U, we obtain an equivalent instance. Finally, we observe that the obtained graph has bounded treewidth using Observations 2 and 3, so we can use Lemmas 8 and 9 to solve the problem, as we pointed out already.

Now we formally explain the details of our algorithm. Let G be a connected graph. First, we preprocess G using Lemma 2 to reduce the number of pendant vertices adjacent to the same vertex in a (potential) cactus root of G. To do so, we exhaustively apply the following rule.
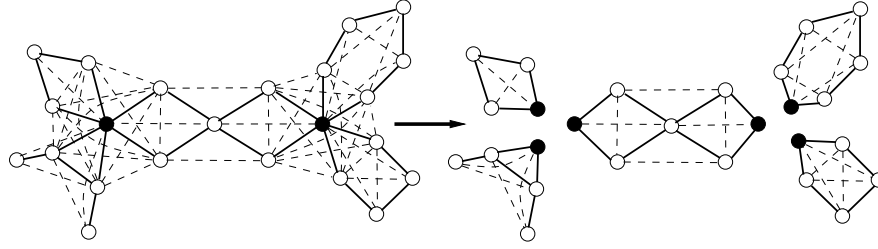
Fig. 2. Splitting of a graph; the vertices of U are black, the edges of a square root are shown by solid lines and the other edges are shown by dashed lines.

Pendants reduction. If G has a set X of simplicial true twins of size at least 7, then delete an arbitrary $u \in X$ from G.

The following claim shows that this rule is safe.

Claim A. *If $G' = G - u$ is obtained from G by the application of* **Pendant reduction**, *then G has a cactus root if and only if G' has a cactus root.*

We prove Claim A as follows. Suppose that H is a minimal cactus root of G. By Lemma 2, H has a pendant vertex $u \in X$. It is easy to verify that $H' = H - u$ is a cactus root of G'. Assume now that H' is a minimal cactus root of G'. By Lemma 2, H has a pendant vertex $w \in X \setminus \{u\}$, since the vertices of $X \setminus \{u\}$ are simplicial true twins of G' and $|X \setminus \{u\}| \geq 6$. Let v be the unique neighbor of w in H'. We construct H from H' by adding u and making it adjacent to v. It is readily seen that H is a cactus root of G. This completes the proof of Claim A.

For simplicity, we call the graph obtained by exhaustive application of the pendants rule G again. The following property is important for us.

Claim B. *Every cactus root of G has at most six pendant vertices adjacent to the same vertex.*

Now we construct an instance of CACTUS ROOT WITH LABELS together with a set U of cut vertices of a (potential) cactus root.

Labeling. Set $U = \emptyset$, $R = \emptyset$ and $B = \emptyset$. For each $u \in V_G$ such that there are three distinct vertices $x, y, z \in N_G(u)$ that are at distance at least 3 from each other in $G - u$ do the following:

 (i) set $U = U \cup \{u\}$,
 (ii) set $B' = \{uv \in E_G \mid \exists w \in N_G(u) \text{ s.t. } \text{dist}_{G-u}(v, w) \geq 3\}$,
(iii) set $R' = \{uv \mid v \in N_G(u)\} \setminus B'$,
(iv) set $R = R \cup R'$ and $B = B \cup B'$,
 (v) if $R \cap B \neq \emptyset$, then return a no-answer and stop.

Lemmas 3–5 immediately imply the following claim.

Claim C. *If G has a cactus root, then* **Labeling** *does not stop in Step (v), and if H is a minimal cactus root of G, then $R \subseteq E_H$ and $B \cap E_H = \emptyset$. Moreover,*

every vertex $u \in U$ is an essential cut vertex of any cactus root of G, and any important cut vertex u of any cactus root of G is contained in U.

For each $u \in U$, let $R(u) = \{v \in N_G(u) \mid uv \in R\}$ and $B(u) = N_G(u) \setminus R(u)$ and construct a partition $P(u) = \{S_1, S_2, \ldots, S_{k(u)}\}$ of $N_G(u)$ as follows.

Partition. For each $u \in U$,

(i) put $x, y \in R(u)$ in the same set of $P(u)$ if and only if x and y are in the same connected component of $G' = G - E_{G[R(u)]} - u$,

(ii) for each $x \in R(u)$, put $y \in B(y)$ in the same set with x if $xy \in E_G$,

(iii) if at least one of the following holds, then return a no-answer and stop:

- $P(u)$ is not a partition of $N_G(u)$,
- there is a set of $P(u)$ with at least three vertices of $R(u)$,
- there is a vertex of $B(u)$ that is not in a set of $P(u)$ with a vertex of $R(u)$,
- there are distinct $S, S' \in P(u)$ such that for some $x \in S$ and $y \in S'$, $xy \in R$,
- there are distinct $S, S' \in P(u)$ such that for some $x \in S$ and $y \in S'$, $xy \in E_G$ but $ux \notin R$ or $uy \notin R$,
- there are distinct $S, S' \in P(u)$ such that for some $x \in S$ and $y \in S'$, $xy \notin E_G$ but $ux \in R$ and $uy \in R$,
- the graph $G - E_{G[R(u)]} - u$ has a path connecting vertices of distinct sets of $P(u)$.

By Lemmas 6, 7 and Claim C, we have the following.

Claim D. *If G has a cactus root, then* **Partition** *does not stop in Step (iii), and if H is a minimal cactus root of G, then*

(i) $R \subseteq E_H$ and $B \cap E_H = \emptyset$,

(ii) every important cut vertex u of H is in U,

(iii) for any $u \in U$, $x, y \in N_G(u)$ are in the same connected component of $H - u$ if and only if x and y are in the same set of $P(u)$.

Now we split the instance (G, R, B) of CACTUS ROOT WITH LABELS into several instances of the problem.

Splitting. For each $u \in U$, let $P(u) = \{S_1, \ldots, S_k\}$ and do the following:

(i) delete u and introduce k new vertices $u_1, \ldots, u_k$,

(ii) for each $i \in \{1, \ldots, k\}$, make u_i adjacent to all vertices of S_i,

(iii) for each $i \in \{1, \ldots, k\}$ and $v \in S_i$, if $uv \in R$, then replace uv by $u_i v$ in R, and if $uv \in B$, then replace uv by $u_i v$ in B,

(iv) for each $i, j \in \{1, \ldots, k\}$, $i \neq j$, delete the edges xy with $x \in S_i$ and $y \in S_j$,

(v) for each $i \in \{1, \ldots, k\}$ and $v \in S_i$, update $P(v)$ by replacing v by v_i in the sets and deleting the vertices of $N_G(u) \setminus S_i$ from the sets.

Let $G_1, \ldots, G_r$ be the connected components of the obtained graph. For $i \in \{1, \ldots, r\}$, let $R_i = R \cap E_{G_i}$ and $B_i = B \cap E_{G_i}$. By Claims B and D, we establish the following crucial claim.

Claim E. *The input graph G has a cactus root if and only if (G_i, R_i, B_i) is a yes-instance of* CACTUS ROOT WITH LABELS *for each $i \in \{1, \ldots, r\}$. Moreover, if (G_i, R_i, B_i) is a yes-instance, then G_i has a cactus root H with $R_i \subseteq E_H$ and $B_i \cap E_H = \emptyset$ such that every cut vertex of H belongs to at most eight blocks and to at most two blocks not being a K_2.*

By Claim E, if G has a cactus root, then $\Delta(G_i) \le 10$ for $i \in \{1, \ldots, k\}$. By Observations 2 and 3, we obtain that $\mathbf{tw}(G_i) \le 29$ in this case. We use Lemma 8 to check whether this holds for each $i \in \{1, \ldots, r\}$. If the algorithm reports that $\mathbf{tw}(G_i) \ge 30$ for some $i \in \{1, \ldots, r\}$, then we return a no-answer and stop. Otherwise, we solve CACTUS ROOT WITH LABELS for each instance (G_i, R_i, B_i) using Lemma 9 for $i \in \{1, \ldots, r\}$.

It remains to evaluate the running time of our algorithm. We can find all simplicial vertices and sort them into the equivalence classes with the true twin relation in time $O(n^3)$. This implies that the exhaustive application of the **Pendant reduction** rule can be done in time $O(n^3)$. For each vertex $u \in V_G$, we can compute the distances between the vertices of $G - u$ in time $O(n^3)$. Hence, the **Labeling** step can be done in time $O(n^4)$. For each $u \in U$ the sets $R(u)$ and $B(u)$ can be constructed in time $O(n^2)$. For each $u \in U$, we can construct $G' = G - E_{G[R(u)]}$ and find the connected components of G' in time $O(n^2)$. It follows, that the **Partition** step can be done in time $O(n^3)$. The **Splitting** step takes $O(n^3)$ time. The algorithm in Lemma 8 runs in $O(n)$ time. We conclude that the total running time is $O(n^4)$. $\qquad\qquad\square$

5   Conclusions

We proved that the problem of testing whether a graph has a cactus root is $O(n^4)$-time solvable. In fact, our algorithm can be modified to find a cactus root in the same time (if it exists).

We recall that every cactus is planar and that the problem of settling the complexity of recognizing squares of planar graphs is open. We also recall that a cactus is a connected graph, in which each block is either a cycle or an edge. This leads to the following (known) generalization: a *cactus block graph* is a connected graph, in which each block is a cycle or a complete graph. Can we decide in polynomial time whether a given graph has a square root that is a cactus block graph? In order to answer this question, we need new arguments as our current proof for cactus roots does not carry over.

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