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ANTIADJACENCY MATRICES FOR SOME STRONG PRODUCTS OF GRAPHS

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Abstract. Let G be an undirected graphs with no multiple edges. There are many ways to represent a graph, and one of them is in a matrix form, by constructing an antiadjacency matrix. Given a connected graph G with vertex set V consisting of n members, an antiadjacency matrix of the graph G is a matrix B of order $n \times n$ such that if there is an edge that connects vertex v_i to vertex v_j ($v_i \sim v_j$) then the element of i^{th} row and b^{th} column of B is $= 0$, otherwise $= 1$. In this paper we investigate some properties of antiadjacency matrices for some strong product of two graphs. Our results are general forms of the antiadjacency matrix of the strong product of path graphs P_m with P_n for $m, n \geq 3$, and cycle graphs C_m with C_m for $m \geq 3$.

Keywords: antiadjacency matrix, strong product, path graph, cycle graph .

I. INTRODUCTION

In everyday life, some problems can be simplified in the form of mathematical modeling [1]. One of a mathematical modeling that is closely related to our everyday life is a graph theory [2]. Based on the direction, a graph is divided into two types, directed and non-directed graphs [3] . A graph that has at most one side to connect two points and does not have a loop is called a simple graph. These basic definitions of graphs can be found in Chartrand [4] [5]. The development of graph theory is currently associated with many other mathematical subjects, including linear algebra. From these two branches of mathematics, graphs are represented in the form of matrices, known as adjacency matrices and antiadjacency matrices. The elements of the matrix are obtained by looking at the adjacency of the graph, based on the presence or absence of the edge connecting the points on the graph. As we know, the adjacency and distance matrices have been widely studied and applied [6][7][8][9][10]. Beside, several studies on the antiadjacency matrix have been studied, Edwina and Kiki [11] examined the antiadjacency matrix of unions and join of several graphs, and Selvia at al [12] investigated some properties of eigen value of antiadjacency matrices of acyclic graphs. Diwyacitta et al [13] also examined the antiadjacency matrix of directed cyclic graphs with cords.

In this paper we investigate antiadjacency matrices for a graph resulted from operations on graphs, which is a strong product operation.

Before discussing further about the antiadjacency matrix of the strong product graph, the following is given the definition of the antiadjacency matrix and the strong product

Definition 1 *(Bapat [14]) Let $G = (V(G), E(G))$ be a connected graph, with $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{e_1, e_2, ..., e_m\}$. Antiadjacency matrix G, denoted by $B(G)$, is a matrix B of order $n \times n$, with*

$$B = [b_{ij}] = \begin{cases} 0; & \textit{if } e = v_i v_j \in E(G) \\ 1; & \textit{else} \end{cases} .$$

Example 1 Given K_4 as follows

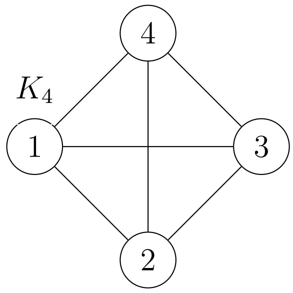

Figure 1. Graph K_4

The antiadjacency matrix of K_4 is

$$B(K_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There are at least three foundamental graph multiplications, i.e, cartesian product, direct product, and strong product. We consider only the last one, that is the strong product.

Definition 2 *(Hammack et al [15]) Let G and H be two graphs. A strong product of G and H , denoted by $G \boxtimes H$ is defined by:*

$$\begin{aligned} V(G \boxtimes H) &= \{(g,h) | g \in V(G) \textit{ and } h \in V(H)\}. \\ E(G \boxtimes H) &= \{((g,h)(g',h')) | g = g' \textit{ and } hh' \in E(H), \\ &\quad \textit{or } h = h' \textit{ and } gg' \in E(G), \\ &\quad \textit{or } gg' \in E(G) \textit{ and } hh' \in E(H)\}. \end{aligned}$$

Example 2 Given two graphs P_4 and P_3. The resulting $P_4 \boxtimes P_3$ is given as follows:

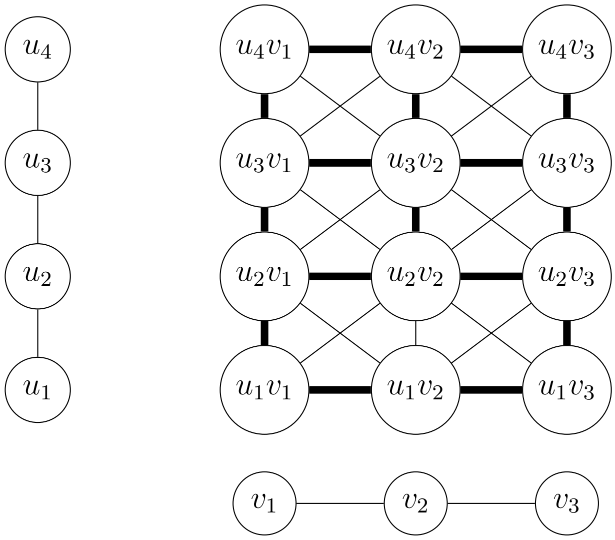

Figure 2. Graph $P_4 \boxtimes P_3$

Based on the operation of graphs, we pose the following question: How the antiadjacency matrix of strong product graph? After some intial investigation, we think it has not a trivial answer, but it perhaps depend on the structure of the graphs
Two special graphs we consider in this work are path and cycle. There are special notations for these two graphs. A path graph G is a graph with its vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and its edges set $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. While if the vertices of G is $V(G) = \{v_1, v_2, ..., v_n\}$ with $n \geq 3$ and its edges set is $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$, then G is called a cycle [16].

II. RESULTS AND DISCUSSION

Before we give a result on the antiadjecency matrix of strong product of paths, we need the following lemma:

Lemma 1 *Let P_m and P_n be two path graphs with $V(P_m) = \{v_1, v_2, ..., v_m\}$ and $V(P_n) = \{u_1, u_2, ..., u_n\}$ where $n, m \geq 3$. There exist submatrices S, Q, R of order $m \times m$ as follow:*

$$S = [s_{ij}] = \begin{cases} 0; & \textit{for } s_{12}, s_{23}, ..., s_{(m-1)m} \\ 0; & \textit{for } s_{21}, s_{32}, ..., s_{m(m-1)} \\ 1; & \textit{for others}. \end{cases}$$

$$Q = [q_{ij}] = \begin{cases} 0; & \textit{for } q_{11}, q_{22}, ..., q_{mm} \\ 0; & for\ q_{12}, q_{23}, ..., q_{(m-1)m} \\ 0; & for\ q_{21}, q_{32}, ..., q_{m(m-1)} \\ 1; & \textit{for other}. \end{cases}$$

$$R = [1_{ij}] = \begin{cases} 1; & \textit{for } i, j = 1, 2, ..., m \end{cases}$$

As a result we have an antiadjacency matrix of $P_m \boxtimes P_n$ of type $n \times n$ as follow:

$$B(P_m \boxtimes P_n) = [b_{ij}] = \begin{cases} S; & for\ b_{11}, b_{22}, ..., b_{nn} \\ Q; & for\ b_{12}, b_{23}, ..., b_{(n-1)n} \\ Q; & for\ b_{21}, b_{32}, ..., b_{n(n-1)} \\ R; & for\ the\ other\ i, j. \end{cases}$$

Proof. Let P_m and P_n be two path graphs, such that $u_i \sim u_{i+1}$, for $i = 1, 2, ..., m-1$ and $v_j \sim v_{j+1}$ for $j = 1, 2, ..., n-1$. From $P_m \boxtimes P_n$, we have $V(P_m \boxtimes P_n) = \{(u_i, v_j)|i = 1, 2, ..., m;\ j = 1, 2, ..., n\}$. The graph $P_m \boxtimes P_n$ can be ilustrated as follows

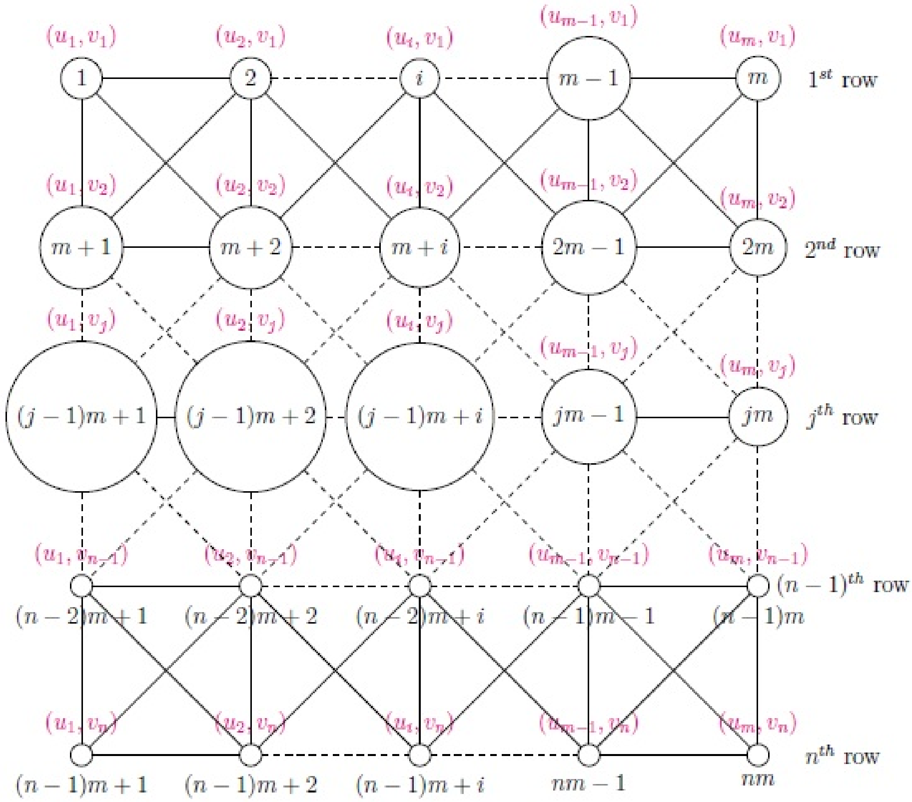

Figure 3. Graph $P_m \boxtimes P_n$

According to Figure 3., we have an antiadjacency submatrix with the following cases:

1. Antiadjacency from the first row, we have the following submatrix

$$S = [s_{ij}]_{m\times m} = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1(m-1)} & s_{1m} \\ s_{21} & s_{22} & \cdots & s_{2(m-1)} & s_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{i1} & s_{i2} & \cdots & s_{i(m-1)} & s_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{(m-1)1} & s_{(m-1)2} & \cdots & s_{(m-1)(m-1)} & s_{(m-1)m} \\ s_{m1} & s_{m2} & \cdots & s_{m(m-1)} & s_{mm} \end{pmatrix}$$

On the strong product, $(u,v)(u',v') \in E(P_m \boxtimes P_n)$ if and only if $uu' \in E(P_m)$ and $v = v'$. Since P_m is a path $u_i \sim u_{i+1}$ and $u_{i+1} \sim u_i$, we have

$$S = [s_{ij}] = \begin{cases} 0; & for\ s_{(i(i+1))} \\ 0; & for\ s_{(i+1)i} \\ 1; & for\ the\ other\ i, j\ . \end{cases}$$

.

2. Antiadjacency submatrix from first row and second row

$$Q = [q_{ij}]_{m\times m} = \begin{pmatrix} q_{1(m+1)} & q_{1(m+2)} & \cdots & q_{1(2m-1)} & q_{1(2m)} \\ q_{2(m+1)} & q_{2(m+2)} & \cdots & q_{2(2m-1)} & q_{2(2m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i(m+1)} & q_{i(m+2)} & \cdots & q_{i(2m-1)} & q_{i(2m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{(m-1)(m+1)} & q_{(m-1)(m+2)} & \cdots & q_{(m-1)(2m-1)} & q_{(m-1)2m} \\ q_{m1} & q_{m2} & \cdots & q_{m(m-1)} & q_{mm} \end{pmatrix}$$

On the strong product $(u,v)(u',v') \in E(P_m \boxtimes P_n)$ if $u = u'$ and $vv' \in E(P_n)$ or $uu \in E(P_m)$ and $vv' \in E(P_n)$. Because P_m and P_n are path, so $(u_i, v_j) \in V(P_m \boxtimes P_n)$ will be adjacent with (u_{i+1}, v_j), (u_i, v_{j+1}), and (u_{i+1}, v_{j+1}). Therefore, the entry of Q as follows

$$Q = [q_{ij}] = \begin{cases} 0; & for\ q_{11}, q_{22}, ..., q_{mm} \\ 0; & for\ q_{12}, q_{23}, ..., q_{(m-1)m} \\ 0; & for\ q_{21}, q_{32}, ..., q_{m(m-1)} \\ 1; & for\ the\ others\ i, j. \end{cases}$$

3. Antiadjacency of first row with 3^{rd} row, Based on Figure 3. above, there is no edge that connect first row and 3^{rd} row, so antiadjacency submatrix will be given as follows

$$R = [r_{ij}] = \left\{ \begin{array}{ll} 1; & for\ i = 1, 2, ..., m\ j = 1, 2, ..., n \end{array} \right.$$

In the same way, note that relation among each columns from Figure 3.. Submatrix S is obtained from $1 = 2 = \cdots = m$, submatrix Q is obtained from row $1 \sim 2 = 2 \sim 3 = 3 \sim 4 = \cdots = m - 1 \sim m$ and submatrix R is obtained from $i - th$ row with i+2, i+3 rows and so on, so we get antiadjacency matrix of $P_m \boxtimes P_n$ is as follows

$$B(P_m \boxtimes P_n) = [b_{ij}] = \begin{cases} S; & for\ b_{11}, b_{22}, ..., b_{nn} \\ Q; & for\ b_{12}, b_{23}, ..., b_{(n-1)n} \\ Q; & for\ b_{21}, b_{32}, ..., b_{n(n-1)} \\ R; & for\ the\ others\ i, j \end{cases}$$

□

From the above result we have the following corollary:

Corollary 1 *Let P_m and P_n be two path graphs. The antiadjacency matrix of $P_m \boxtimes P_n$ of type $n \times n$ can be contructed as follow:*

$$B(P_m \boxtimes P_n) = [b_{ij}] = \begin{cases} S; & for\ b_{11}, b_{22}, ..., b_{nn} \\ Q; & for\ b_{12}, b_{23}, ..., b_{(n-1)n} \\ Q; & for\ b_{21}, b_{32}, ..., b_{n(n-1)} \\ R; & for\ the\ other\ i, j. \end{cases}$$

2.1. Antiadjacency Matrix of Strong Product of Cycles

It is known that a cycle graph C_m is a graph such that every vertex has degree 2. We are interested in examining the antiadjacency matrix of strong product of cycle graphs of the same order.

Lemma 2 *Let C_m be cycle graph, where $V(C_m) = \{v_1, v_2, ..., v_m\}$ and $E(C_m) = \{v_i v_{1+1} \cup v_1 v_m | i = 1, 2, ..., m-1,\ m \geq 3\}$, there will be submatrices S, Q, R of order $m \times m$ such that:*

$$S = [s_{ij}] = \begin{cases} 0; & for\ s_{1m}, s_{12}, s_{23}, ..., s_{(m-1)m} \\ 0; & for\ s_{21}, s_{32}, ..., s_{m(m-1)}, s_{m1} \\ 1; & for\ the\ others\ i, j. \end{cases}$$

$$Q = [q_{ij}] = \begin{cases} 0; & for\ q_{1m}, q_{11}, q_{22}, ..., q_{mm} \\ 0; & for\ q_{12}, q_{23}, ..., q_{(m-1)m} \\ 0; & for\ q_{21}, q_{32}, ..., q_{m(m-1)}, q_{m1} \\ 1; & for\ the\ others\ i, j. \end{cases}$$

$$R = [1_{ij}] = \left\{ \begin{array}{ll} 1; & for\ i, j = 1, 2, ..., m \end{array} \right.$$

Proof. Let C_m is a cycle graph, so $u_i \sim u_{i+1}$ for $i = 1, 2, ..., m-1$ and $u_1 \sim u_m$. For $V(C_m \boxtimes C_m) = \{(u_i, u_j)|i, j = 1, 2, ..., m-1\}$. The graph $C_m \boxtimes C_m$ can be described as follows:

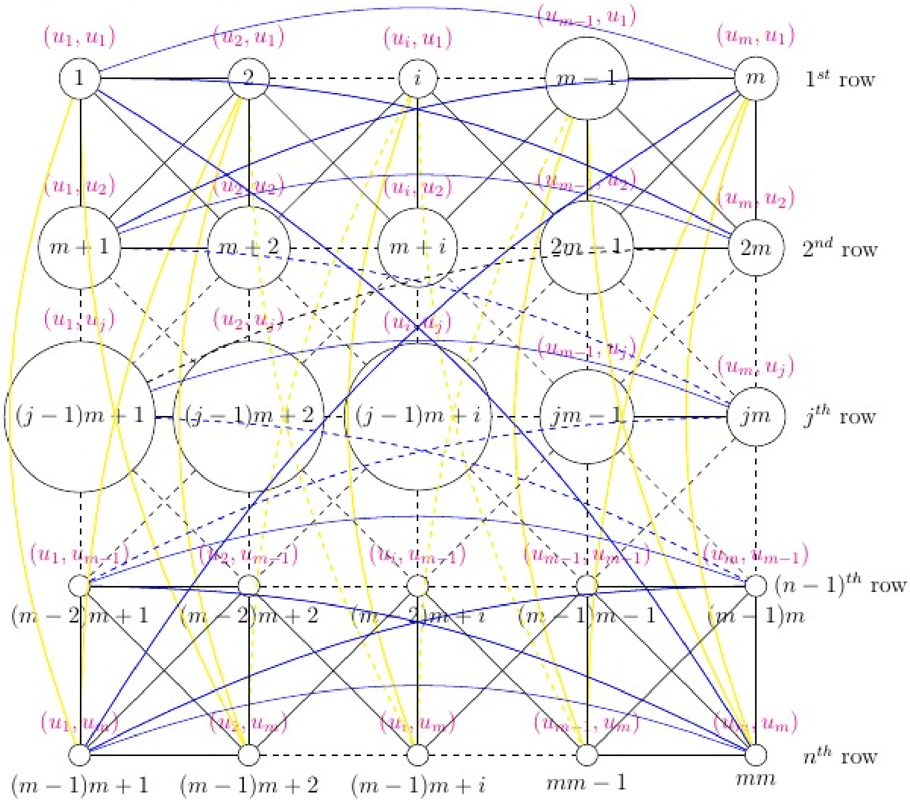

Figure 4. Graph $C_m \boxtimes C_m$

Based on Figure 4. above, we get antiadjacency submatrix by reviewing some cases below:

1. Antiadjacency of first row, we get submatrix as follows

$$P = [p_{ij}]_{m\times m} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1(m-1)} & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2(m-1)} & p_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1} & p_{i2} & \cdots & p_{i(m-1)} & p_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{(m-1)1} & p_{(m-1)2} & \cdots & p_{(m-1)(m-1)} & p_{(m-1)m} \\ p_{m1} & p_{m2} & \cdots & p_{m(m-1)} & p_{mm} \end{pmatrix}$$

Note, on strong product, $(u, v)(u', v') \in E(C_m \boxtimes C_m)$ if and only if $uu' \in E(C_m)$ and $v = v'$. Just because C_m is cycle graph, so $u_i \sim u_{i+1},\ u_{i+1} \sim u_i$, and $u_1 \sim u_m$ so

$$S = [s_{ij}] = \begin{cases} 0; & for\ s_{i(i+1)} \\ 0; & for\ s_{(i+1)i} \\ 0; & for\ s_{1m}\ and\ s_{m1} \\ 1; & for\ the\ others\ i, j. \end{cases}$$

Then, we will get the same submatrix for second row, third row and so on.

2. Antiadjacency from the first row to the second row, we will get submatrix as follows

$$Q = [q_{ij}]_{m\times m} = \begin{pmatrix} q_{1(m+1)} & q_{1(m+2)} & \cdots & q_{1(2m-1)} & q_{1(2m)} \\ q_{2(m+1)} & q_{2(m+2)} & \cdots & q_{2(2m-1)} & q_{2(2m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i(m+1)} & q_{i(m+2)} & \cdots & q_{i(2m-1)} & q_{i(2m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{(m-1)(m+1)} & q_{(m-1)(m+2)} & \cdots & q_{(m-1)(2m-1)} & q_{(m-1)2m} \\ q_{m1} & q_{m2} & \cdots & q_{m(m-1)} & q_{mm} \end{pmatrix}$$

As we know that on strong product $(u, v)(u', v') \in E(C_m \boxtimes C_m)$ if $u = u'$ and $vv' \in E(C_m)$ or $uu \in E(C_m)$ and $vv' \in E(C_m)$. Thus C_m is cycle, so $(u_i, v_j) \in V(C_m \boxtimes C_m)$ will be adjacent with $(u_{i+1}, v_j), (u_i, v_{j+1}), (u_{i+1}, v_{j+1})$ and $(u_i, v_1) \sim (u_{i+1}, v_m),\ (u_1, v_m) \sim (u_{i+1}, v_1), (u_1, v_i) \sim u_m, v_{i+1},\ (u_m, v_i) \sim (u_1, v_{i+1})$. Thus the entry of Q is as follows

$$Q = [q_{ij}] = \begin{cases} 0; & for\ q_{11}, q_{22}, ..., q_{mm} \\ 0; & for\ q_{12}, q_{23}, ..., q_{(m-1)m}, q_{1m} \\ 0; & for\ q_{21}, q_{32}, ..., q_{m(m-1)}, q_{m1} \\ 1; & for\ the\ others\ i, j. \end{cases}$$

3. Antiadjacency of the first row with the third row, based on Figure 4. above, there is no edge that connect the first row and the third row, so antiadjacency submatrix will be given as follows

$$R = [r_{ij}] = \left\{ 1; \quad for\ i, j = 1, 2, ..., m \right.$$

In the same way, noted that relation among each columns from Figure 4., and submatrix S will be obtained from $1^{st} = 2^{nd} = \cdots = m^{th}$ row, submatrix Q will be obtained from antiadjacency of $1 \sim 2 = 2 \sim 3 = 3 \sim 4 = \cdots = m-1 \sim m = 1 \sim m$ row and submatrix R is obtained from i^{th} row with $(i+2)^{th}$, $(i+3)^{th}$ rows and so on. □

Corollary 2 *The antiadjacency matrix of $C_m \boxtimes C_m$ of order $m \times m$ is given by:*

$$B(C_m \boxtimes C_m) = [b_{ij}] = \begin{cases} S; & for\ b_{11}, b_{22}, ..., b_{mm} \\ Q; & for\ b_{1m}, b_{12}, b_{23}, ..., b_{(m-1)m} \\ Q; & for\ b_{m1}, b_{21}, b_{32}, ..., b_{m(m-1)} \\ R; & for\ the\ others\ i, j. \end{cases}$$

III. CONCLUSIONS

We have investigated antiadjency matrices for strong products between two paths P_m and P_n where $m, n \geq 3$, and between cycles C_m and C_m where $m \geq 3$. The strong product graph of path and cycle always forms a symmetry matrix that consist of four antiadjacency matrix patterns.

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