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Research Based Teaching in Mathematical Logic

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Introduction: Teaching and Research

According to Humboldt (Humboldt, 1997), the defining responsibility of universities is their role as places of scientific research. Here, scientific knowledge is created, a type of knowledge which is "to be viewed as something not yet completely found, and never to be completely found." Central to the research profession is "with this in view perpetually to aim to find it" (Humboldt, 1997, p. 120; translation by present author).

The competence required for this can only be learned by participation. Oevermann (2005), following Weber (1985), further characterizes this competence as a *social* competence, the *habitus* of a researcher, and interprets the teaching task of universities as follows: that "students act as researchers to become professionals," that is, they learn by participation in a community of researchers.

Healey and Jenkins (2009) stress that research based teaching should apply to every student of a university. Indeed, in a society in which policy and by extension, our further existence on this planet depends on scientifically informed public discourse, there is no getting around universities' responsibility to provide their core service to each student and not just the select few who 'make it' into higher curricular levels such as a PhD-program.

Healey also argues that in practice, students do not regularly participate in research related activities at all. We shall in the following argue that at least in the teaching of mathematics, certain traditions prevent them from doing so, and present a case study in designing lectures to address the issue. I shall begin with a highly exaggerated account of these averse traditions.

The Blackboard Lecture: A Caricature

The epitome of teaching in mathematics is the *classical blackboard lecture* – delivered flawlessly by a superhuman[1] professor who lays down beautiful formulas on the blackboard at the same time as he (of course he is male!) magically brings them to life with explanations. Truths seem to unfold according to their own, innate rhythm. Just as in Euclid's Elements, these truths concern a timeless universe of abstract entities (not, say, a world of diverging strategies and decisions). The student's role, in contrast, is merely to nod to a series of unquestionable definitions, theorems, and proofs.

When the professor is done with his performance, lesser beings take over to drill the student in an exercise class, for application and practice are secondary to the pure, deductive theory.

Lecturing is described in a similar vein in Mazur (1997), where it is also related to very ancient traditions such as a priest's sermon. A remarkably persistent tradition, then – perhaps all this is a re-telling of the story of Zosimo of Panopolis and his student Theosebeia? Upon complaining that she cannot understand the alchemical teachings he intends to impart on her, she is shut down with the words "I gave you what you need to know, and this should be enough for you" (Abt and Fuad, 2012, fol.38b.3-5).

What is wrong with this picture?

So why shouldn't the student proceed from passively listening in lecture, to developing their problem solving capacity in exercise classes, to finally becoming a researcher in the six months or so it takes to write a thesis (if they get that far)?

While a lucky few do take this road, the problem is that most of this curriculum has nothing to do with science in the true sense of *knowledge perpetually in the making*. Firstly, it is not even *about* science in this sense. More crucially, learning *about* science does not seem to impart the competence of being a researcher: According to Mazur (2017) as well as Törnquist (2015), this traditional approach encourages or at least allows bad study habits such as rote learning and a mechanical approach to problem solving. Significant for science in the true sense of *knowledge perpetually in the making* is not possessing knowledge but rather a *type of activity* requiring

[1] This view of academic habitus is starkly contrasted with behavior exemplifying a researcher's attitude by Busch, 2001.

a competence that can only be learned by participating in a community of practice.

Research involves *all* the complementary aspects which are so often described as knowledge, skill, and competence; all of these are all different dimensions of what it means to *act* as a researcher.

In other words, theorems and definitions are not to be viewed as inert book-knowledge, inhabiting an abstract sphere as objects of mathematical knowledge. They are no more and no less than research stopped in its tracks, deposited between strata of pages, but ready to pounce the instant a research mathematician finds a new way to use the tricks, the strategies contained in them. This is one way to interpret the aphorism (attributed in Ulam, 1976, p. 203 to Banach): "Good mathematicians see analogies between theorems or theories; the best see analogies between analogies."

Neither are mathematical problem solving skills to be viewed as the capacity to carry out any one in a certain repertoire of algorithms; rather, we aim for the capacity to solve problems hitherto *unsolved* or even *unknown*.

It is outright irrational to expect students to evolve the competence to act as a researcher in the last, short phase of a curriculum whose bigger part encourages them to act precisely in an opposed manner. Therefore, I tried to incorporate *activities characteristic of research* into my lecture course.

A Research Based Classroom

From spring 2014 to summer 2017, I shared teaching duties with Asger Törnquist for a course called *Introduction to Mathematical Logic* (IML). Some of its loftier goals are, succinctly, to be able to model and analyze mathematics as a rule-governed formal system, and draw conclusions, for instance, regarding general limitations on what is or isn't provable in such a system.

Experience shows that many students (mostly bachelor students past their first year) find the course challenging; the difficulties are mostly conceptual in nature, as the technical prerequisites are relatively harmless compared to other mathematics courses.

Our intended learning outcomes are aligned, in the sense of Biggs and Tang (2011), with activities in the course as well as with assessment procedures: We require students to solve two problem sets as a mandatory home exercise which will form part of their grade. These consist of problems that are not merely an application of the course material, or given for consolidation or drill. Instead students are asked to find proofs for theorems on further topics, with minimal preparation in lecture. This is, of course, an activity characteristic of research. As is to be expected, performance during oral exams also reflects a much deeper learning (in the sense of Biggs and Tang, 2011) from these mandatory home exercises.

Inspired by my attendance of the *University Pedagogy* course at KU, I decided to center the IML course even more on student activities characteristic of research. In the process, I was supervised by by Camilla Østerberg Rump, Institut for Naturfagens Didaktik, KU, and my postdoc adviser Asger Törnquist.

Case Study 1

One of the conceptually most difficult results taught in the one block long IML course is *Gödel's Completeness Theorem*, which could be described as stating that a mathematical statement is provable by purely logical means if and only if *any* interpretation of that statement (at least in a certain framework) is true.

I chose the following activities in connection to this theorem:

1. Students had to read a set of lecture notes (written by me for an earlier installment of IML) introducing and proving the theorem, before class,
2. Also before class, students had to take an on-line quiz (part of the intention here was to make sure each students did their reading assignment),
3. Finally, students should solve a set of problems in group work, presented to them in class. Their results were then discussed in session.

The on-line quiz was composed of short, conceptual questions, in which students were asked to use definitions from the text to draw simple conclusions. These were intended to require a deep learning approach, or in the words of Sfard (1991), to require students to construct a *mental picture* or *object* relating to the terms and concepts presented in my lecture notes, in preparation for the activities that would later be carried out in class.

The in-class problems required students to provide the some of the essential steps in the proof of the Completeness Theorem. In class, the proof of this theorem was tackled from a somewhat different viewpoint, requiring students to autonomously arrive at solutions different from the ones presented in my lecture notes. Students were therefore required to think creatively, analyze the problem tasks, and apply terms and concepts introduced in my lecture notes in their solution.

Students were asked for some problems to collaborate with their neighbors, for others to work on their own and then to convince a neighbor of their solution (for the positive effects of peer instruction see again Mazur, 2017 and 1997). Finally, I asked them to sum up the main ideas of the proof in their own words (cf. again Törnquist, 2015).

Students carried out several activities typical of mathematical research:

- Reading and understanding a mathematical text,
- Applying definitions to analyze a problem,
- Collaborating in groups to solve a problem without being given a path to its solution,
- Explaining their solution to a colleague.

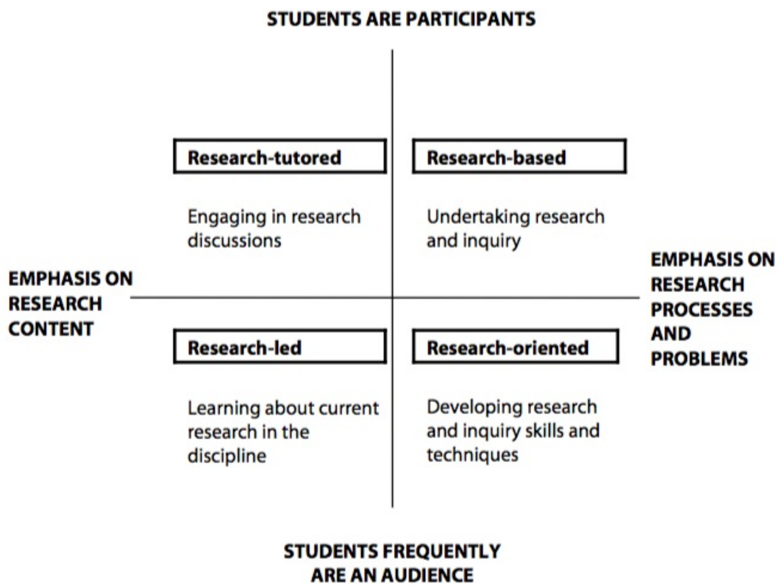

Fig. 2.1. Types of undergraduate inquiry according to Healey and Jenkins (2009), amended from Healey (2005)

According to the taxonomy in Healey and Jenkins (2009), this approach qualifies as "research based" (upper right corner in Figure 2.1) as it focuses on *research processes* and *student activity*. While students may not be performing original research in the sense of producing knowledge which is new to the scientific community, they do construct knowledge that is *new to them* using the same processes that researchers use when creating knowledge. Most significantly, this "transforms their understanding of knowledge and research" (Healey and Jenkins, 2009, p. 9).

For the most part, the experiment went extremely well: Students were very actively engaged, all but one of the problems were solved successfully by a large majority of students, and their written feedback indicates that their experience was also predominantly positive. At the oral exam, when asked about the Completeness Theorem, student answers corroborated the expectation that this approach should lead to deep learning (see Biggs and Tang, 2011, p. 21, which also contains more sources).

Case Study 2

The last part of IML introduces a particular, widely accepted way of formalizing mathematics, namely *set theory*. For this, we introduce a list of *axioms* called ZFC (for 'Zermelo-Fraenkel plus the axiom of Choice').

Students face two difficulties in this topic: Firstly, the formal way of writing axioms; secondly, a few of the axioms are hard to give intuitive meaning to, as reaching operational familiarity with them usually takes a long time.

To address just these difficulties, I took as a starting point the argument by Törnquist (2015) that speaking or writing about mathematics in ordinary, every-day language (rather than in purely technical, mathematically formalized language) helps students overcome conceptual difficulties.

For IML, I first introduced, in a 'classical' lecture style, the axioms of ZFC and an idea called the *iterative concept of set*, often used as in Boolos (1971) to 'justify' the axioms of ZFC. After that,

1. Each group chose a cluster of axioms and worked to come up with a justification of these axioms, based on the *iterative concept of set*.
2. Each group presented their findings to the rest of class.

In the process students are required to improve their 'intuitive grasp' of the ZFC axioms (again, cf. Sfard, 1991). At the same time, on a procedural level, finding an intuitive justification for *why* something might be true, is an indispensable strategy in math research.

Finally and more concretely, students

- Collaborated to investigate a problem without having at the outset a clear a path to its solution,
- Presented their solution to their colleagues,

thus carrying out two further characteristic research activities.

This lecture in IML was extremely successful: Each group found an interesting and compelling justification of 'their' axioms (Figure 2.2, p. 7 shows a diagram drawn by a group justifying the 'Axiom of Union,' found on the top line in the figure), and student feedback was overwhelmingly positive. At the oral exam, performance regarding the ZFC axioms was better than in other years.

Reflection

I gave students several opportunities to provide oral or written feedback (handwritten, or anonymously via the learning platform) after both experiments. The majority of written feedback strongly indicates the positive effect of research related activities on student learning. Here are some of their written answers to my question "how did this lecture help you learn?":

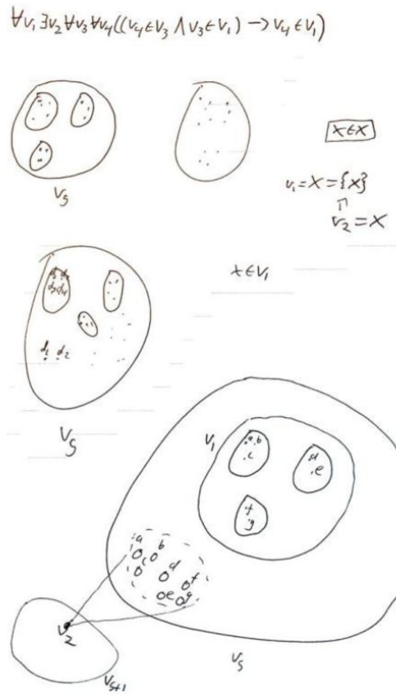

Fig. 2.2. Justifying the Union Axiom

- "It was very helpful to be ‘forced’ to understand each step in a complicated proof"
- "It broke with the pattern of passively listening"
- "It gives something to try and explain"
- "There is more than one way to prove things"
- "My concentration is better in group assignments"
- "[Justifying the axioms] helped me understand how you can argue from the intuitive picture"

One student indicated in their written feedback that they had further and autonomously studied a generalized version of the Completeness Theorem for uncountable languages in Enderton (2001), indicating a very deep learning approach.

On the other hand, centering student activity can put additional pressures on students. Such additional factors are the more dynamic structure of communications, as well as stress caused by competition or fear of not ‘performing’ as hoped – and indeed one student indicated a strong frustration with their self-perceived inability to find sufficiently many correct answers.

Understanding difficult mathematics and creatively finding solutions both take *time* and perhaps sometimes a bit of quiet, resources that become scarcer as more of students time is spent on managing the more dynamic setting of group interactions.

On a more abstract level, being unable to solve a problem *is itself a research activity* (unfortunately, a frequent one!). According to Weber (1985, pp. 524 – 55), it is part of the necessary attitude of a researcher to pursue her work at the same time with cool systematicity, a prerequisite for being open to critical dissonance, as well as to be emotionally invested in this work and driven by enthusiasm. Only both together can entice the crucial idea – but research is not churned out mechanically as in a factory, and "[i]deas occur to us when they please, not when it pleases us." This is eminently illustrates how acquiring the researcher’s *habitus* is only possible through active participation.

This situation is quite opposite to that of being tested: In a graded test, being unable to solve a problem is usually not experienced as an unavoidable part of the process, but simply an undesirable end to it. And yet many a lecture that puts students activity in the center (e.g., one described in Mazur, 1997) can end up resembling a multiple choice test similar to the one used in grading, and it is perhaps easy for the student to conflate these activities – that of being tested and that of being given an opportunity to act as a researcher. This indicates that it is vital to be very clear about the distinction and frame research activities accordingly.

While many of activities which I list above are easily judged to be characteristic of carrying out research, especially in the face of the previous discussion, it seems less obvious when one is ‘solving a problem without being given a clear path to its solution’ and what a good problem task for such an activity might be.

The criterion I used was that the solution or the method of obtaining it could be expected to to be previously unknown to the student, as well as centrally relevant to deeper learning goals, and that the student should experience a sense of agency (rather than following some sort of recipe). Such questions can still be simple, and a more complicated problem can easily be broken down into parts where each partial problem retains this *research-like* character. For instance, if one can expect a student to make a novel realization from following a particular algorithm, this *can* be a research problem, but such cases will be rare and I completely avoided such problem tasks in this case study.

Sometimes, problem task will in practice turn out not to meet these requirements, as indeed happened with one problem in this case study. There is necessarily an empirical element to finding problems that meet the above criteria; a repertoire of such problems must be built over time, keeping the ones that work well, and modifying the problem set if some do not.

Where Do We Go From Here

I have argued that in-class problem sessions, with the necessary preparations, are capable of allowing students to act as researchers in activities such as utilizing literature, autonomously solving a problem, collaborating, explaining, and so on. I have also argued that test-like or fast-paced formats can stand in the way of real participation in these activities.

For future implementations, I therefore suggest and intend to more often and more casually integrate such problem sessions into the lecture. Integrating problem sessions without *first* having to redesign an entire sequence of lectures (as is the case with an in-class quiz, for example) has the advantage of allowing more frequent and flexible implementation, with the same or greater positive effect on learning depth. Eventually, the course will be redesigned in its entirety through this integration.

Numerous additional activities can be suggested that integrate research or research-like activities from the later curriculum into a lecture course or the corresponding exercise class. For example, to require students to:

- Discuss or present material that has been assigned for autonomous study, such as a reading assignment
- Themselves select material for autonomous study, for example from a pre-compiled list, or to compile a bibliography
- Write short expositions of topics assigned or selected for autonomous study or based on in-class activities

An excellent way of integrating such activities into the course in a manner that aligns intended learning outcome, activities and assessment could be a mandatory or even graded *portfolio* documenting students' research activities.

Here is one of *my own* learning outcomes from the process this paper reports on: Written student feedback as well as writing down my own reflections are an immensely useful tool for developing my teaching methods. Therefore, implementations of the above ideas should be documented using both. In this manner, results can form a basis for reflections that in turn lead to further experimentation and new implementation – teaching competence perpetually in the making.

Finally, my personal experience strongly indicates that students success in the later curriculum (especially writing a thesis) correlates very strongly to the amount and quality of participation in research activities that they have had up to that point. Therefore, it is highly desirable to increase this participation.

Designing lectures that include research activities for students is important since such a large portion of the early curriculum is currently spent in lectures; but other ways may be even more effective and should be pursued, such as seminars in which students prepare lectures, or formats in which students prepare written works autonomously.

Ultimately and optimally, this transformation process can and should be carried to the institutional level, affecting curriculum design so that research activities permeate the early as well as the late curriculum, and are required for every student.

References

Abt, T., & Fuad, S. (2012). The book of pictures by Zosimos of Panopolis. Zurich.

Biggs, J., & Tang, C. (2011). *Teaching for quality learning at university: What the student does*. McGraw-Hill Education (UK).

Boolos, G. (1971). The iterative conception of set. *The Journal of philosophy*, *68*(8), 215–231.

Busch, H. (2001). Når Superman smider skjorten. *Kvant*, *2001*(3).

Enderton, H., & Enderton, H. B. (2001). *A mathematical introduction to logic*. Elsevier.

Healey, M. (2005). Linking research and teaching exploring disciplinary spaces and the role of inquiry-based learning. *Reshaping the university: New relationships between research, scholarship and teaching*, 67–78.

Healey, M., & Jenkins, A. (2009). Developing students as researchers. In S. Haslett & H. Rowlands (Eds.), *Linking research and teaching in higher education* (Vol. 1, pp. 7–11). Proceedings of the Newport NEXUS Conference.

Humboldt, W. (1997). Über die innere und äußere organisation der höheren wissenschaftlichen anstalten in berlin. In C. Menze (Ed.), *Bildung und sprache* (pp. 118–126). Schöningh, Paderborn.

Jenkins, A., & Healey, M. (2009). *Developing undergraduate research and inquiry*. York: Higher Education Academy.

Mazur, E. (1997). Peer instruction: Getting students to think in class. In *AIP conference proceedings* (Vol. 399, pp. 981–988). AIP.

Mazur, E. (2017). Peer instruction. In *Peer instruction* (pp. 9–19). Springer.

Oevermann, U. (2005). Wissenschaft als beruf. die professionalisierung wissenschaftlichen handelns und die gegenwärtige universitätsentwicklung. *die hochschule*, *1*(2005), 15–51.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(2), 1–36.

Törnquist, A. (2015). Promoting deep learning in mathematics through conceptual exercises. *Improving University Science Teaching and Learning*, *8*(1–2), 155–166.

Ulam, S. (1976). Adventures of a mathematician (new york: Scribner's). *quoted by Steve J. Heims, op. cit*, 117–118.

Weber, M. (1985). Wissenschaft als berufung. In H. von Johannes Winckelmann (Ed.), *Gesammelte aufsätze zur wissenschaftslehre* (Vol. 6, pp. 524–555). J. C. B. Mohr (Paul Siebeck).