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ON THE GRUNDY BONDAGE NUMBERS OF GRAPHS

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Abstract. For a graph $G = (V, E)$, a sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of G it is called a *dominating sequence* if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \varnothing$. The maximum length of dominating sequences is denoted by $\gamma_{gr}(G)$. We define the Grundy bondage numbers $b_{gr}(G)$ of a graph G to be the cardinality of a smallest set E of edges for which $\gamma_{gr}(G - E) > \gamma_{gr}(G)$. In this paper the exact values of $b_{gr}(G)$ are determined for several classes of graphs.
Keywords: Grundy Domination Number,Grundy Bondage Number.

1. Introduction

In this paper, G is a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For notation and graph theoretical terminology, we generally follow [8]. The order $|V|$ and the size $|E|$ of G is denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ of v is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N_G(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write P_n for the path of order n, C_n for the cycle of order n, K_n for the complete graph of order n and $K_{m,n}$ for complete bipartite graph. Also $K_{1,n}$ is called *star graph* and is denoted by S_n.

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The *cartesian product* of graphs $G = G_1 \times G_2$, are sometimes simply called the graph product of graphs G_1 and G_2 with point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with the point set $V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever ($u_1 = v_1$ and u_2 adjacent v_2) or (u_1 adjacent v_1 and $u_2 = v_2$). The *join* of two graphs G and H is denoted by $G \vee H$ is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. The graph $K_1 \vee C_{n-1}$ is called *wheel graph* and is denoted by W_n.
Let G be a graph of order n and let $H_1, H_2, \cdots, H_n$, be n graphs. The *generalized corona product*, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \cdots, H_n$ and joining the ith vertex of G to every vertex of H_i. This product is denoted by $G \circ \wedge_{i=1}^{n} H_i$. If each H_i is isomorphic to a graph H, then generalized corona product is called the *corona product* of G and H and is denoted by $G \circ H$.

A subset D of $V(G)$ is called a *dominating set* of G if every vertex of G is either in D or adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *γ-set*. For further information about various domination sets in graphs, we refer reader to [9, 10].

Based on the domination number, Grundy domination invariants has been introduced in recent years by some authors [1, 5, 6] and then they continued the study of these concepts in [3, 2, 4, 7].

In [5] the first type of Grundy dominating sequence was introduced. Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. A sequence $S = (v_1, \ldots, v_k)$ is called a *closed neighborhood sequence* if, for each i,

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. Clearly, if $S = (v_1, v_2, \ldots, v_k)$ is a dominating sequence for G, then $k \geq \gamma(G)$. We call the maximum length of a dominating sequence in G the *Grundy domination number* of G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a Grundy dominating sequence of G or γ_{gr}-sequence of G.

The *Grundy bondage number* $b_{gr}(G)$ of a non-empty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with Grundy domination number greater than $\gamma_{gr}(G)$. For empty graph G, we define $b_{gr}(G) = 0$.

In this paper we introduced this concept and in Section 2, we obtain $b_{gr}(G)$ for some families of graphs.

2. Main results

In this section, we compute the Grundy bondage numbers of some special family of graph. First, we state some necessary known results.

Proposition 2.1. *[5] Let n be a positive integer.Then*

i) For $n \geq 3$, $\gamma_{gr}(C_n) = n - 2$, while for $n \geq 2$, $\gamma_{gr}(P_n) = n - 1$.

ii) For $n \geq 1$, we have $\gamma_{gr}(K_n) = 1$, while for complete bipartite graphs $K_{r,s}$ we have $\gamma_{gr}(K_{r,s}) = s$ if $r \leq s$.

iii) If G is the join of G_1 and G_2, Then

$$\gamma_{gr}(G) = max\{\gamma_{gr}(G_1), \gamma_{gr}(G_2)\}.$$

In the following theorem we study some families of graphs with Grundy bondage numbers are equal 1

Theorem 2.1. *Let G be a graph of order $n \geq 4$. If $G \in \{K_n, C_n, W_n, K_2 \times C_n\}$, then $b_{gr}(G) = 1$.*

Proof. We have $\gamma_{gr}(K_n) = 1$, by Proposition 2.1 [ii]. Let $e = xy$. It is not difficult to see that $S = (x, y)$ is a dominating sequence for $K_n - e$. So we conclude that $\gamma_{gr}(K_n - e) > \gamma_{gr}(K_n)$ and thus $b_{gr}(K_n) = 1$.

Now consider the graph C_n. By Proposition 2.1, we have $\gamma_{gr}(C_n) = n - 2$. Consider the edge e from C_n,. Hence $C_n = P_n$ and therefore $\gamma_{gr}(C_n - e) > \gamma_{gr}(C_n)$. Hence, $b_{gr}(C_n) = 1$.

Let $G = W_n$. Since $W_n = K_1 + C_{n-1}$, by Proposition 2.1, we have

$$\gamma_{gr}(W_n) = max\{\gamma_{gr}(K_1), \gamma_{gr}(C_{n-1})\}.$$

So, $\gamma_{gr}(W_n) = n - 3$. Consider an edge e from C_{n-1}. Then

$$\gamma_{gr}(W_n - e) = \gamma_{gr}(K_1 + P_{n-1}) = n - 2.$$

Thus, $b_{gr}(W_n) = 1$.

Now Consider $K_2 \times P_n$. Let $V(K_2 \times P_n) = \{v_{ij}| \quad 1 \leq i \leq 2, \quad 1 \leq j \leq n\}$. The Grundy domination number of $K_2 \times C_n$ is equal to $2n - 4$. Now consider $K_2 \times C_n - v_{11}v_{1n}$. Hence

$$(v_{11}, v_{21}, v_{12}, v_{22}, \cdots, v_{1n-1})$$

is a Grundy sequences in $K_2 \times C_n - v_{11}v_{1n}$ of size $2n - 3$. Hence $\gamma_{gr}((K_2 \times C_n) - v_{11}v_{1n}) > \gamma_{gr}(K_2 \times C_n)$ and we conclude that $b_{gr}(K_2 \times C_n) = 1$.

□

Theorem 2.2. *Let G be a caterpillar of order $n \geq 2$. Then $b_{gr}(G) = n-1$.*

Proof. Note that for a graph H, we have $\gamma_{gr}(H) = n$ if and only if H is an empty graph. Hence if E_0 is a subset of edge set G, such that $\gamma_{gr}(G - E_0) > \gamma_{gr}(G)$, then $G - E_0$ is an empty graph. Therefore $|E_0| \geq n-1$ and we conclude that $b_{gr}(G) = n-1$.
□

Corollary 2.1. $b_{gr}(P_n) = b_{gr}(S_n) = n-1$.

Proof. The results follows from Theorem 2.2, since paths and stars are caterpillar. □

Theorem 2.3. *Let $2 \leq m \leq n$. Then $b_{gr}(K_{m,n}) \leq n-1$.*

Proof. Let $G = K_{m,n}$ and V_1 and V_2 are two parts of G of sizes m and n, respectively. Suppose that $V_2 = \{w_1, w_2, \cdots, w_n\}$. Consider the arbitrary vertex $v_1 \in V_1$ and edge set $E_0 = \{v_1 w_i | 1 \leq i \leq n\}$. Clearly $K_{m,n} - E_0 = K_1 \bigcup K_{m-1,n}$ and hence $\gamma_{gr}(K_{m,n} - E_0) = n+1$. This implies that $b_{gr}(K_{m,n}) \leq n-1$. □

The following lemma is a useful result for computing $b_{gr}(K_2 \times P_n)$.

Lemma 2.1. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{gr}(G) = n-1$ if and only if G is a caterpillar.*

Proof. We prove by induction on n. For $n = 2$, the result is true. Suppose that result is true for any connected graph of order $n-1$ and G is a connected graph of order $n \geq 3$ with $\gamma_{gr}(G) = n-1$. Let $(v_1, v_2, \cdots, v_{n-2}, v_{n-1})$ be a dominating sequences of G. Hence there exists

$$x \in (N_G[v_{n-1}] \setminus \bigcup_{j=1}^{n-2} N_G[v_j]).$$

Note that $x \neq v_j$ for $1 \leq j \leq n-2$. If $x = v_n$, then v_n is not adjacent to any v_j for $1 \leq j \leq n-2$ and this fact implies that $deg(v_n) = 1$. Hence $(v_1, v_2, \cdots, v_{n-3}, v_{n-2})$ is a dominating sequences for $G - v_n$. The graph $G - v_n$ is a connected graph of order $n-1$ with $\gamma_{gr}(G - v_n) = n-2$. Hence $G - v_n$ is a caterpillar and this fact implies that G is a caterpillar. If $x = v_{n-1}$, then v_{n-1} is not adjacent to any v_j for $1 \leq j \leq n-2$. Since G is connected, we conclude that v_{n-1} is adjacent to v_n and $deg(v_{n-1}) = 1$. By changing the the dominating sequence $(v_1, v_2, \cdots, v_{n-2}, v_{n-1})$ to dominating sequence $(v_1, v_2, \cdots, v_{n-2}, v_n)$ and a same argument the result can be obtained.
The converse of lemma obtained by 2.1. □

Theorem 2.4. *Let $n \geq 2$. Then $b_{gr}(K_2 \times P_n) = n-1$.*

Proof. Let $V(K_2 \times P_n) = \{v_{ij} | \quad 1 \leq i \leq 2, \quad 1 \leq j \leq n\}$. We know that $\gamma_{gr}(K_2 \times P_n) = 2n - 2$ [2]. Consider the set $E_0 = \{v_{1i}v_{2i} | \quad 1 \leq i \leq n-1\}$. Clearly $E_0 \subseteq E(K_2 \times P_n)$ and $K_2 \times P_n - E_0 = P_{2n}$. Hence $\gamma_{gr}(K_2 \times P_n - E_0) = 2n - 1$. Thus $b_{gr}(K_2 \times P_n) \leq n - 1$. On the other hand, if $E_0 \subseteq E(K_2 \times P_n)$ such that $\gamma_{gr}(K_2 \times P_n - E_0) = 2n - 1$, then $(K_2 \times P_n) - E_0$ is a forest such that all components except one are a single vertex. Hence $|E_0| \geq n - 1$ and we conclude that $b_{gr}(K_2 \times P_n) = n - 1$. □

An additional variant of the Grundy domination number was introduced in [1]. Let G be a graph without isolated vertices. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a $Z-$*sequence* if for each i,

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

Then the Z-Grundy domination number $\gamma_{gr}^Z(G)$ of the graph G is the length of a longest Z-sequence.

The following results are known

Proposition 2.2. *[5, 1] For $n \geq 3$, $\gamma_{gr}(C_n) = \gamma_{gr}^Z(C_n) = n - 2$, while for $n \geq 2$, $\gamma_{gr}(P_n) = \gamma_{gr}^Z(P_n) = n - 1$.*

Theorem 2.5. *[11] Let G and $H_1, H_2, \ldots, H_n$ be $n+1$ graphs with without isolated vertices. Then*

$$\gamma_{gr}(G \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}(H_i) + \gamma_{gr}^Z(G).$$

Theorem 2.6. *Let G and $H_1, H_2, \ldots, H_n$ be $n + 1$ graphs with without isolated vertices. If $G = C_n$ or $H_1 = C_n$, then $b_{gr}(G \circ \wedge_{i=1}^n H_i) = 1$.*

Proof. Suppose that $G = C_n$ and consider an edge e from G. Hence $G - e = P_n$ and therefor by Proposition 2.2 and Theorem 2.5

$$\gamma_{gr}(G \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}(H_i) + n - 2 < \gamma_{gr}(G - e \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}(H_i) + n - 1.$$

Thus $b_{gr}(G \circ \wedge_{i=1}^n H_i) = 1$. □

REFERENCES

1. B. Brešar, Cs. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Z. Tuza and M. Vizer: *Grundy dominating sequences and zero forcing sets*, Discrete Optim., **26** (2017), 66-77.
2. B. Brešar, C. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Z. Tuza and M. Vizer: *Dominating sequences in grid-like and toroidal graphs*, Electron. J. Combin., **23** (2016), P4.34 (19 pages).
3. B. Brešar, T. Gologranc and T. Kos: *Dominating sequences under atomic changes with applications in Sierpinski and interval graphs*, Appl. Anal. Discrete Math., **10** (2016), 518-531.
4. B. Brešar, Kos and Terros: *Grundy domination and zero forcing in Kneser graphs*, Ars Math. Contemp., **17** (2019), 419-430.
5. B. Brešar, T. Gologranc, M. Milanič, D. F. Rall and R. Rizzi: *Dominating sequences in graphs*, Discrete Math., **336** (2014), 22-36.
6. B. Brešar, M. A. Henning and D. F. Rall: *Total dominating sequences in graphs*, Discrete Math., **339** (2016) 1165-1676.
7. B. Brešar, T. Kos, G. Nasini and P. Torres: *Total dominating sequences in trees, split graphs, and under modular decomposition*, Discrete Optim., **28** (2018), 16-30.
8. G. Chartrand and L. Lesniak: *Graphs and digraphs*, Third Edition, CRC Press,(1996).
9. T. W. Haynes, S. Hedetniemi and P. Slater: *Fundamentals of Domination in Graphs*, CRC Press, (1998).
10. M. A. Henning and A. Yeo: *Total domination in graphs*, (Springer Monographs in Mathematics.) ISBN-13: 987-1461465249 (2013).
11. S. M. Moosavi Majd and H. R. Maimani: *Grundy domination sequences in generalized corona products of graphs*, Facta Universitatis Ser: Math. Inform., Vol. **35**, No 4 (2020) 1231–1237.