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Approximate Solution of Sub diffusion Bio heat Transfer Equation

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Abstract:

In this paper, author's study sub diffusion bio heat transfer model and developed explicit finite difference scheme for time fractional sub diffusion bio heat transfer equation by using caputo fabrizio fractional derivative. Also discussed conditional stability and convergence of developed scheme. Furthermore numerical solution of time fractional sub diffusion bio heat transfer equation is obtained and it is represented graphically by Python.

Keywords: Bio heat equation, Caputo –Fabrizio fractional derivatives, Fractional Differential Equation, Python, Sub diffusion, Finite difference Method.

Introduction:

In 1948, Harry H. Pennes[1] firstly developed mathematical model for temperature in resting human forearm, this model is called as bio heat transfer equation, given as below:

$$\rho C \frac{\partial Z(x,t)}{\partial t} = k \frac{\partial^2 Z(x,t)}{\partial x^2} + W_b C_b (Z_a - Z) + q_{met} \ , \ x \in [0, L], \ t \in [0, T] \qquad 1$$

Where Z =temperature, t =time, ρ =density, C =specific heat, x =distance, k =thermal conductivity, Z_a = temperature of artillery, W_b =blood perfusion rate, q_{met} =metabolic heat generation in skin tissue, C_b =specific heat of blood.

Researchers are interested to convert above Penne's bio heat equation in terms of fractional partial differential equation[2]. Damor et. al. (2013)[3] developed fractional bio heat model by replacing time derivative by fractional order derivative in Eq.1, as below

$$\rho C \frac{\partial^\alpha Z(x,t)}{\partial t^\alpha} = k \frac{\partial^2 Z(x,t)}{\partial x^2} + W_b C_b (Z_a - Z) + q_{met} \ , \alpha \in (0,1), \ x \in [0, L], \ t \in [0, T] \qquad 2$$

Caputo fractional derivative[4] having singular kernel is used for solving Fractional bio heat model developed by Damor, Ezzat and Ferras. In 2019, Hristov[5] revisited various bio heat model, their non-dimesionalization and fractional approach. Recently many researcher developed fractional bio heat models with new kinds of fractional derivative with non-singular kernel viz. Caputo-Fabrizio fractional derivative, Atangana-Balenau fractional derivative, memory-dependent derivative etc[6-8].

In this scenario, time fractional derivative $\frac{\partial^\alpha Z(x,t)}{\partial t^\alpha}$ in Eq. 2 is replaced by Caputo-Fabrizio fractional derivative of order α, given by H. Yépez-Martínez and J.F. Gómez-Aguilar[9] as follows

Definition 1: Caputo-Fabrizio Fractional Derivative

$$\frac{\partial^\alpha Z(x,t)}{\partial t^\alpha} = \frac{1}{1-\alpha} \int_0^t exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right] \frac{\partial Z(x,t)}{\partial \tau} \, d\tau, \quad \alpha \in (0,1) \qquad 3$$

In regular diffusion, $r^2 \propto D\,t$ where r is mean square displacement, t is time and D is diffusion coefficient. In contrast to regular diffusion anamolus diffusion is given by power law $r^2 \propto D(t)\,t^\alpha$ where α is anamolus diffusion operator and $D(t)$ is time dependent diffusion coefficient[10]. Note that $\alpha = 1$ means regular diffusion, $0 < \alpha < 1$ implies sub diffusion and $\alpha > 1$ gives superdiffusion. Note that time fractional derivative introduce sub diffusion or super diffusion without disturbing properties of density and specific heat of material. To introduce sub diffusion in fractional bio heat transfer equation, thermal conductivity k is changed by $K = k\,\vartheta^{1-\alpha}$ as given by Ferras et.al.[11]; where ϑ has no physical meaning.

Finally our time fractional sub diffusion bio heat transfer equation as follow:

$$\rho C\,\frac{\partial^\alpha Z(x,t)}{\partial t^\alpha} = K\,\frac{\partial^2 Z(x,t)}{\partial x^2} + W_b\,C_b(Z_a - Z) + q_{met}\ ,\alpha \in (0,1),\ x\in[0,L],\ t\in[0,T] \qquad 4$$

Initial condition: $Z(x,0) = Z_a\ ,\quad 0 < x < L \qquad 5$

Boundary conditions: $-K\,\frac{\partial Z(L,t)}{\partial x} = 0\ ,\quad -K\,\frac{\partial Z(0,t)}{\partial x} = q_0\ ,\quad t \geq 0 \qquad 6$

where L is the length of tissue and q_0 is heat flux on the skin surface and $\frac{\partial^\alpha Z(x,t)}{\partial t^\alpha}$ is time fractional derivatives in the sense of Caputo Fabrizio defined in Eq. 3

By using dimensionless variable as follow

$$u = \sqrt{\frac{\rho C}{K}}\ x\ ,\quad v = t\ ,\quad X = (Z - Z_a)\frac{\sqrt{K\rho C}}{q_0}\ ,\quad Q = \frac{q_{met}}{q_0\sqrt{\frac{\rho C}{K}}}\ ,\quad a = \frac{W_b\ C_b}{\rho C}$$

The dimensionless from of time fractional sub diffusion bio heat transfer equation, Eq. 4 to Eq. 6 is:

$$\frac{\partial^\alpha X(u,v)}{\partial v^\alpha} = \frac{\partial^2 X(u,v)}{\partial u^2} - a\,X(u,v) + Q\ ,\alpha \in (0,1),\ u \in \left[0,\sqrt{\frac{\rho C}{K}}\,L\right],\ v\in[0,T] \qquad 7$$

Initial condition: $X(u,0) = 0\ ,\quad 0 < x < \sqrt{\frac{\rho C}{K}}L \qquad 8$

Boundary conditions: $\frac{\partial X\left(\sqrt{\frac{\rho C}{K}}\,L,\ v\right)}{\partial u} = 0\ ,\quad \frac{\partial X(0,v)}{\partial u} = -1\ ,\quad v \geq 0 \qquad 9$

Several methods are developed by researchers to find solution of fractional bio heat equation. Analytical method is discussed by Shih et. al.[12], H. Pandey et.al.[13], H. Patil[14], Takale[15] and Jogdand[16] discussed Explicit and Crank Nicolson[17] Finite difference scheme, Abdulhussein[18] discussed quadratic Spline method for finding solution of fractional bio heat equation. Following explicit finite difference scheme will be discussed, in this paper.

Finite Difference Scheme:

To develop the explicit finite difference scheme for dimensionless sub diffusion bio heat transfer equation, Eq.7 to Eq. 9, define $u_i = i\Delta u, i = 0,1,\cdots,M$ and $v_n = n\Delta v, n = 0,1,\cdots,N$; where $\Delta u = \frac{\left(\sqrt{\frac{\rho C}{K}}\,L\right)}{M}$ and $\Delta v = \frac{T}{N}$. Consider exact solution of dimensionless sub diffusion bio heat transfer equation, Eq.7 at mesh point (u_i, v_n) is $X(u_i, v_n),\ i = 0,1,\cdots,M$ and $v_n = n\Delta v, n = 0,1,\cdots,N$. Also X_i^n be the numerical approximation at point(u_i, v_n). Note that Δu is dimensionless space step size and Δv is dimensionless time step size. Discretization of the space derivatives $\frac{\partial^2 X(u,v)}{\partial u^2}$ by using central difference formula is given by

$$\left(\frac{\partial^2 X}{\partial u^2}\right)_i^n = \frac{X_{i-1}^n - 2X_i^n + X_{i+1}^n}{(\Delta u)^2} + O((\Delta u)^2) \qquad 10$$

Discretization[19] of the Caputo-Fabirizio time fractional derivative of order α as follows

$$\left(\frac{\partial^\alpha X}{\partial v^\alpha}\right)_i^n = \frac{1}{\alpha}\left[\frac{X_i^{n+1} - X_i^n}{\Delta v}\right] d_{\Delta v}^0 + \frac{1}{\alpha}\sum_{j=1}^n \left[\frac{X_i^{n-j+1} - X_i^{n-j}}{\Delta v}\right] d_{\Delta v}^j \qquad 11$$

Where $d_{\Delta v}^j = \left[exp\left[\frac{-\alpha}{1-\alpha}(j-1)\Delta v\right]\right] - \left[exp\left[\frac{-\alpha}{1-\alpha} j\Delta v\right]\right]\ ,\ j = 0,1,2,\cdots,n$

Now, substituting equations, Eq. 10 and Eq. 11 in equation, Eq. 7,

$$\frac{1}{\alpha}\left[\frac{X_i^{n+1} - X_i^n}{\Delta v}\right] d_{\Delta v}^0 + \frac{1}{\alpha}\sum_{j=1}^n \left[\frac{X_i^{n-j+1} - X_i^{n-j}}{\Delta v}\right] d_{\Delta v}^j = \frac{X_{i-1}^n - 2X_i^n + X_{i+1}^n}{(\Delta u)^2} + O((\Delta u)^2) - a\,X_i^n + Q$$

$$X_i^{n+1} = rX_{i-1}^n + (1 - 2r - ap)X_i^n + rX_{i+1}^n - \sum_{j=1}^n h\left[\frac{X_i^{n-j+1} - X_i^{n-j}}{\Delta v}\right] d_{\Delta v}^j + pQ \qquad 12$$

by considering $r = \frac{\alpha\,\Delta v}{d_{\Delta v}^0\,(\Delta u)^2}\ ,p = \frac{\alpha\,\Delta v}{d_{\Delta v}^0}\ ,h = \frac{1}{d_{\Delta v}^0}$.

After simplification, for $n = 0,1,\cdots,N$;

$$\sum_{j=1}^n h\left[\frac{X_i^{n-j+1} - X_i^{n-j}}{\Delta v}\right] d_{\Delta v}^j = h\,d_{\Delta v}^1\,X_i^n - h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]X_i^{n-j} + h\,d_{\Delta v}^n\,X_i^0$$

with this substitution equation, Eq. 12 becomes

$$X_i^{n+1} = rX_{i-1}^n + \left(1 - 2r - ap - h\,d_{\Delta v}^1\right)X_i^n + rX_{i+1}^n + h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]X_i^{n-j} + h\,d_{\Delta v}^n\,X_i^0 + pQ$$

Now, put $n = 0$ in equation, Eq. 12, becomes

$$X_i^1 = rX_{i-1}^0 + (1 - 2r - ap)X_i^0 + rX_{i+1}^0 + pQ$$

Finally, the initial condition Eq. 8 and the boundary conditions Eq.9 are approximated as $X_i^0 = 0$, $i = 0,1,\cdots,M$, and $X_M^n = X_{M-1}^n$; $X_0^n = X_1^n + \Delta u\ ,\ n = 0,1,\cdots,N$.

Finally, the discretized dimensionless sub diffusion bio heat transfer equation along with initial and boundary conditions are as follows,

$X_i^1 = rX_{i-1}^0 + (1-2r-ap)X_i^0 + rX_{i+1}^0 + pQ$ for $n = 0$ 13

$X_i^{n+1} = rX_{i-1}^n + \left(1-2r-ap-h\,d_{\Delta v}^1\right)X_i^n + rX_{i+1}^n + h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]X_i^{n-j} + h\,d_{\Delta v}^n\,X_i^0 + pQ$ for $n \geq 1$ 14

Initial condition: $X_i^0 = 0\,,\quad i = 0,1,\cdots,M;$ 15

Boundary conditions: $X_M^n = X_{M-1}^n\,;\ \ X_0^n = X_1^n + \Delta u\,,\ n = 0,1,\cdots,N.$ 16

where $r = \frac{\alpha\,\Delta v}{d_{\Delta v}^0\,(\Delta u)^2}\,, p = \frac{\alpha\,\Delta v}{d_{\Delta v}^0}\,, h = \frac{1}{d_{\Delta v}^0}$

Hence, matrix form of discretized explicit finite difference scheme Eq. 13 to Eq. 16 is as follows

$U^1 = A\,U^0 + B$ for $n = 0$ 17

$U^{n+1} = A_1\,U^n + h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]U^{n-j} + h\,d_{\Delta v}^n\,U^0 + B$ for $n \geq 1$ 18

Initial condition: $X_i^0 = 0\,, i = 0,1,\cdots,M;$ 19

Boundary conditions: $X_M^n = X_{M-1}^n\,;\ \ X_0^n = X_1^n + \Delta u\,,\ n = 0,1,\cdots,N$ 20

where

$U^n = (X_1^n\,,\ X_2^n\,,\ \cdots\cdots\,,\ X_{M-1}^n)^T\,,\quad B = (pQ + r\,\Delta u\,,\ pQ\,,\ pQ\,,\cdots\cdots,\ pQ)$ are the constant matrix, and

$$A = \begin{pmatrix} \Lambda + r & r & & & & & \\ r & \Lambda & r & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & r & \Lambda & r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & r & \Lambda & r \\ & & & & & r & \Lambda + r \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} \Lambda + r - h\,d_{\Delta v}^1 & r & & & & & \\ r & \Lambda - h\,d_{\Delta v}^1 & r & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & r & \Lambda - h\,d_{\Delta v}^1 & r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & r & \Lambda & r \\ & & & & & r & \Lambda + r - h\,d_{\Delta v}^1 \end{pmatrix}$$

where $\Lambda = 1 - 2r - ap$

Stability:

Lemma 1:

The eigenvalues of $M \times M$ tri-diagonal matrix

$$A_1 = \begin{pmatrix} a_1 & a_2 & & & & & \\ a_3 & a_1 & a_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_3 & a_1 & a_2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_3 & a_1 & a_2 \\ & & & & & a_3 & a_1 \end{pmatrix}$$

are given as $\lambda_i = a_1 + 2\sqrt{a_2\ a_3}\cos\left(\frac{i\,\pi}{M+1}\right)$, where a_1, a_2 and a_3 are either real or complex numbers[20].

Theorem 1:

The solution for dimensionless sub diffusion bio heat transfer equations, Eq.7 to Eq.9 developed by explicit finite difference scheme Eq. 13 to Eq. 16 is stable, when

$r \leq \min\left\{\frac{2-ap}{4}, \frac{2-ap-hd_{\Delta v}^1}{4}\right\}$

Proof: For tri-diagonal matrix A, the eigenvalues are given as

$$\lambda_i(A) = (1-2r-ap) + 2\sqrt{(r)\,(r)}\,\cos\left(\frac{i\,\pi}{M}\right)\ ,\text{ for } i = 1,2,\cdots,M$$

$$\therefore\ \lambda_i(A) = (1-2r-ap) + 2\,r\cos\left(\frac{i\,\pi}{M}\right) \leq\ (1-2r-ap) + 2\,r = 1 - ap \leq 1$$

$$\lambda_i(A) = (1-2r-ap) + 2\sqrt{(r)\,(r)}\,\cos\left(\frac{i\,\pi}{M}\right)\ ,\text{ for } i = 1,2,\cdots,M$$

$$\therefore\ \lambda_i(A) = (1-2r-ap) + 2\,r\cos\left(\frac{i\,\pi}{M}\right) \geq\ (1-2r-ap) - 2\,r = 1 - 4r - ap$$

$$\therefore\ \lambda_i(A) \geq -1\ \ if\ \ 1 - 4r - ap \geq -1\ \ \Rightarrow\ r \leq \frac{2-ap}{4}$$

Thus, eigenvalues of matrix A, can be write as

$|\lambda_i(A)|\ \leq 1$ for $0 < r \leq \frac{2-ap}{4}$ 21

Also, for tri-diagonal matrix A_1, the eigenvalues are obtained as follow

$$\lambda_i(A_1) = \left(1-2r-ap-h\,d_{\Delta v}^1\right) + 2\sqrt{(r)\,(r)}\,\cos\left(\frac{i\,\pi}{M}\right)\ ,\text{ for } i = 1,2,\cdots,M$$

$$\therefore\ \lambda_i(A_1)\ \leq\ \left(1-2r-ap-h\,d_{\Delta v}^1\right) + 2\,r = 1 - ap - h\,d_{\Delta v}^1 \leq 1$$

$$\lambda_i(A_1) = \left(1-2r-ap-h\,d_{\Delta v}^1\right) + 2\sqrt{(r)\,(r)}\,\cos\left(\frac{i\,\pi}{M}\right)\ ,\text{ for } i = 1,2,\cdots,M$$

$$\therefore\ \lambda_i(A_1)\ \geq\ \left(1-2r-ap-h\,d_{\Delta v}^1\right) - 2\,r = 1 - 4r - ap - h\,d_{\Delta v}^1$$

$$\therefore\ \lambda_i(A_1) \geq -1\ \ if\ \ 1 - 4r - ap - h\,d_{\Delta v}^1 \geq -1 \Rightarrow\ r \leq \frac{2 - ap - h\,d_{\Delta v}^1}{4}$$

Thus, eigenvalues of matrix A, can be write as

$|\lambda_i(A_1)|\ \leq 1$ for $0 < r \leq \frac{2-ap-h\,d_{\Delta v}^1}{4}$ 22

Therefore, from equations, Eq. 21 and Eq. 22, solution of dimensionless sub diffusion bio heat transfer equation, Eq.7 to Eq.9 developed by explicit finite difference scheme Eq.13 to Eq.16 is stable if

$r \le \min\left\{\frac{2-ap}{4}, \frac{2-ap-hd_{\Delta v}^{1}}{4}\right\}$

This proves the theorem.

Convergence:

Lemma 2:

Following conditions are satisfied by the coefficient $d_{\Delta v}^{j}$, $j = 0, 1, 2, \cdots$

(i) $d_{\Delta v}^{j} > 0$ (ii) $d_{\Delta v}^{j} > d_{\Delta v}^{j+1}$

Lemma 3:

If the eigenvalues of A and A_1 are represented by $\lambda_i(A)$ and $\lambda_i(A_1)$ respectively, then

(i) $|\lambda_i(A)| \le 1$, $|\lambda_i(A_1)| \le 1$, $i = 1, 2, 3, \cdots, M$

(ii) $\|(A)\|_\infty \le 1$, $\|(A_1)\|_\infty \le 1$

Theorem 2:

The solution for dimensionless sub diffusion bio heat transfer equation Eq.7 to Eq.9 developed by explicit finite difference scheme Eq.13 to Eq.16 is convergent, if

$$r \le \min\left\{\frac{2-ap}{4}, \frac{2-ap-hd_{\Delta v}^{1}}{4}\right\}$$

Proof: The exact solution of the dimensionless time fractional sub diffusion bio heat transfer equation, Eq.7 to Eq.9 at time level t_n is represented by the vector $\bar{U}^n = (\bar{X}_1^n, \bar{X}_2^n, \bar{X}_3^n, \cdots, \bar{X}_{M-1}^n)^T$ of size $+1$, defined on the region $[0, L] \times [0, T]$. The vector of truncation error is given by $\tau^n = (\tau_1^n, \tau_2^n, \tau_3^n, \cdots, \tau_M^n)^T$ at time level t_n . Then using explicit finite difference scheme Eq.13 to Eq.16, give us

$\tau_i^1 = X_i^1 - rX_{i-1}^0 - (1-2r-ap)X_i^0 + rX_{i+1}^0 - pQ = O(\Delta v + (\Delta u)^2)$ for $n = 0$

$\tau_i^{n+1} = X_i^{n+1} - rX_{i-1}^n - \left(1-2r-ap-h\,d_{\Delta v}^1\right)X_i^n - rX_{i+1}^n - h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]X_i^{n-j} - h\,d_{\Delta v}^n X_i^0 - pQ = O(\Delta v + (\Delta u)^2)$

for $n \ge 1$

Let $U^n = (X_1^n, X_2^n, X_3^n, \cdots, X_{M-1}^n)^T$ be the vector approximate solution of the dimensionless time fractional sub diffusion bio heat transfer equation, Eq.7 to Eq.9 respectively at time level v_n .

Now, in the solution, error vector can be set as $E^n = \bar{X}^n - X^n = (e_1^n, e_2^n, e_3^n, \cdots, e_{M-1}^n)^T$ at time level v_n . Suppose

$$|e_l^n| \le \max_{1 \le i \le M}|e_i^n| = \|E^n\|_\infty , \quad \text{for } l = 1, 2, 3, \cdots$$

and

$$|\tau_l^n| \le \max_{1 \le i \le M}|\tau_i^n| = O(\Delta v + (\Delta u)^2) , \quad \text{for } l = 1, 2, 3, \cdots$$

Exact solution of the equation, Eq.7 to Eq.9, is represented by the vector $\bar{U}^n$. Therefore the equations Eq.17 to Eq.20 are satisfy by $\bar{U}^n$, which is given by

$$\bar{U}^1 = A\,\bar{U}^0 + B + \tau^1 \quad \text{for } n = 0 \qquad 23$$

$$\bar{U}^{n+1} = A_1\,\bar{U}^n + h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]\bar{U}^{n-j} + h\,d_{\Delta v}^n\,\bar{U}^0 + B + \tau^{n+1}$$

for $n \ge 1$ 24

It is possible to prove the convergence of scheme by induction. That is to prove,

$$\|E^m\|_\infty \le K\ O(\Delta v + (\Delta u)^2) , \quad m = 1, 2, 3, \cdots$$

For $m = 1$, from equations, Eq.17 and Eq.23,

$$E^1 = A\,E^0 + \tau^1$$

$$\therefore \quad \|E^1\|_\infty = \|A\,E^0 + \tau^1\|_\infty \le \|E^0\|_\infty + \|\tau^1\|_\infty \le K\ O(\Delta v + (\Delta u)^2)$$

Note that K is not depending on u and v. Hence, result is true for $m = 1$.

Assume that result is true for $m \le n$, therefore

$$\|E^n\|_\infty \le K\ O(\Delta v + (\Delta u)^2)$$

Now, to prove that result is true, for $m = n + 1$. Therefore from equations, Eq.18 and Eq.24, gives

$E^{n+1} = A_1\,E^n + h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]E^{n-j} + h\,d_{\Delta v}^n\,E^0 + B + \tau^{n+1}$

$\therefore \quad \|E^{n+1}\|_\infty \le \|A_1\,E^n\|_\infty + \left\|h\sum_{j=1}^{n-1}\left[d_{\Delta v}^j - d_{\Delta v}^{j+1}\right]E^{n-j}\right\|_\infty + \|h\,d_{\Delta v}^n\,E^0\|_\infty + \|\tau^{n+1}\|_\infty$

$$\|E^{n+1}\|_\infty \le \left[1 + d_{\Delta v}^1 - d_{\Delta v}^n\right]K_1\ O(\Delta v + (\Delta u)^2) + K_2\ O(\Delta v + (\Delta u)^2)$$

Finally, $\|E^{n+1}\|_\infty \le K\ O(\Delta v + (\Delta u)^2)$

where $K = \max\{K_1, K_2\}$ is one of the positive number not depending on u and v .

Hence, by using mathematical induction, for all m,

$$\|E^m\|_\infty \le K\ O(\Delta v + (\Delta u)^2)$$

Therefore, this shows that if

$r \le \min\left\{\frac{2-ap}{4}, \frac{2-ap-hd_{\Delta v}^{1}}{4}\right\}$

Then as $(\Delta u, \Delta v) \to (0, 0)$, the vector U_n converges to $\bar{U}_n$. Hence, this complete the proof.

Test Problem:

In this Paper, Parameter values[3] are considered as follow:

$$\rho = 1050\,Kg\,m^{-3}, C = 4180\,Kg^{-1}C, W_b = 0.5\,Kg\,m^{-3}, C_b = 3770\,J\,Kg^{-1}\,C,$$

$$K = 0.5\,Wm^{-1}C, q_0 = 5000\,Kg\,m^{-2}, q_{met} = 368.1\,W\,m^{-3}, L = 0.02m$$

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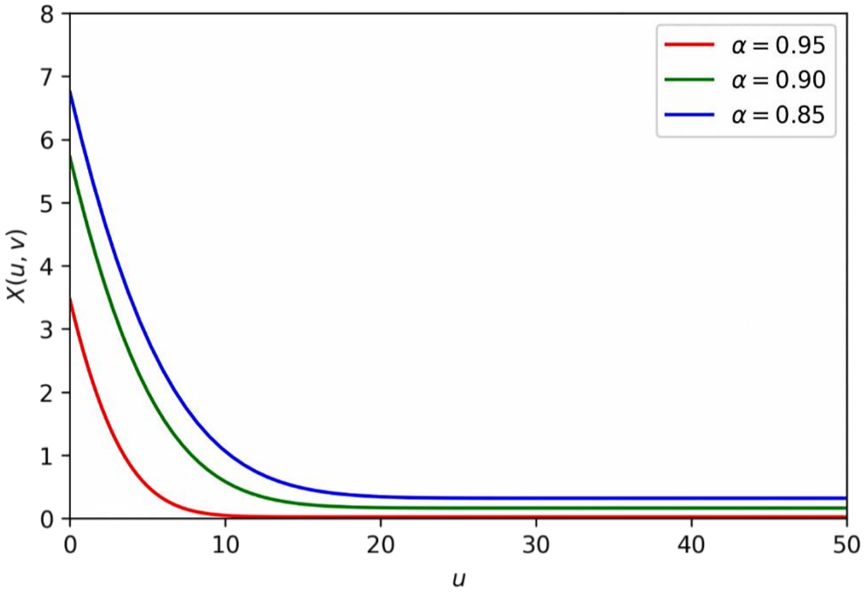

Figure 1. Approximate solution of dimensionless sub diffusion bio heat transfer equation for $\alpha = 0.95, 0.90, 0.85$

Fig. 1 shows that dimensionless temperature distribution over the dimensionless distance for various values of α . Note that result obtained by our Python code is same as analytical solution provided by Shih[12] and numerical solution obtained by Damor[3].

Conclusion:

Explicit finite difference scheme for dimensionless time fractional sub diffusion bio heat transfer equation is developed. Condition for stability and convergence of the developed scheme is obtained. Approximate graphical solution is obtained by using Python code.

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- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given the permission for re-publication attached with the manuscript.
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Authors' Contributions Statement:

J. S proposed the idea and developed method. K. T collected parameter values and designed graph. B. S discussed convergence and stability. All authors read manuscript carefully and approve final manuscript.

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الحل التقريبي لمعادلة نقل الحرارة الحيوية للانتشار الفرعي

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الخلاصة:
في هذه الورقة، ندرس نموذج نقل الحرارة الحيوية الفرعي ونطور مخطط فرق محدود صريح لمعادلة نقل الحرارة الحيوية للانتشار الجزئي الجزئي للوقت باستخدام مشتق كابوتو فابريزيو الكسري. ناقشنا الاستقرار المشروط والتقارب من مخطط المتقدمة. علاوة على ذلك ، يتم الحصول على حل رقمي لمعادلة نقل الحرارة الحيوية الجزئية الجزئية ويتم تمثيلها بيانيا بواسطة بايثون.

الكلمات المفتاحية: معادلة الحرارة الحيوية، مشتقات كابوتو-فابريزيو ، المعادلة التفاضلية الجزئية ، بايثون ، الانتشار الفرعي ، طريقة الفروق المحدودة.