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Some New Fixed Point Theorems in Weak Partial Metric Spaces

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Abstract:

The main objective of this work is to introduce and investigate fixed point (F. p) theorems for maps that satisfy contractive conditions $(\psi-\varphi)$in weak partial metric spaces (W.P.M.S), and give some new generalization of the fixed point theorems of Mathews and Heckmann. Our results extend, and unify a multitude of (F. p) theorems and generalize some results in (W.P.M.S). An example is given as an illustration of our results.

Keywords: Coincidence points, Fixed point, Partial metric, Weak partial metric, Weakly Compatible.

Introduction:

A partial metric space (P.M.S.) is a generalization of standard metric space developed by Matthews[1] in 1994 as an extension of standard metric space (M.S), in which self-distance might not be equal to zero. The notion of (P.M.S) plays an important part in the theory of computation. Numerous articles have been published on fixed points for maps satisfying some contractive conditions in (P.M.S)[2-5], and also for generalizing contractions[6]. In 1999, Heckmann[7]developed the notion of (W.P.M.S). which is a generalization of (P.M.S.) by omitting the small self-distance axiom. Some results for mappings in (W.P.M.S) have been obtained [8-11], also many authors proved fixed point (F.P) results for maps satisfying implicit relations[12-15]. The main purpose of this paper is to study fixed point under$(\psi-\varphi)$ contractive conditions in weak partial metric space (W.P.M.S).

Preliminaries

Definition 1 [1]: A (P.M.) on$M\neq\varphi$, is a function $P: M^2\to\Re^2=[0,\infty)$ $\ni$ satisfying the following axioms,

(P_1) $\mu=\eta \Leftrightarrow P(\mu,\mu)=P(\mu,\eta)=P(\eta,\eta)$ (T_0- separation axiom)

(P_2) $P(\mu,\mu)\leq P(\mu,\eta)$, (non-negatively and small self–distance)

(P_3) $P(\mu,\eta)=P(\eta,\mu)$, (symmetry)

(P_4) $P(\mu,\eta)\leq P(\mu,\tau)+P(\tau,\eta)-P(\tau,\tau)$, (triangular inequality)

for all$\mu,\eta,\tau\in M$.Then $(M,P)$is said to be a partial metric space (for short P.M.S).

Remark 1 [1] Clearly $P(\mu,\eta)=0\Rightarrow\mu=\eta$ by using (P_1) and (P_2) But the reverse is false in general For each partial metric P on the set M, the function $d_P: M^2\to\Re^+$is defined by every Pon set M T_0 $-$generates a Topology $\tau(P)$on set M whose base is the collection of open $P-$ball$\{B_P(\mu,r),\mu\in M,r>0\}$, where

$B_P(\mu,r)=\{\eta\in M: P(\mu,\eta)<P(\mu,\mu)+r\}$, For all $\mu\in M$and$r>0$.

Remark 2 [1] If $P$is (P.M.S).on M, then the functions $d_P,d_\omega: M^2\to\Re^+$ given by

$d_P(\mu,\eta)=2P(\mu,\eta)-P(\mu,\mu)-P(\eta,\eta)$

$d_\omega(\mu,\eta)=P(\mu,\eta)-\min\{P(\mu,\mu),P(\eta,\eta)\}$

are ordinary metrics on M. note that d_P , d_ω are equivalent on M.

Definition2 [1]

Let (M,P)be a (P.M.S) then

1- A sequence $\{r_n\}$in(M,P) converges to a point $r\in M$ if and only if $\lim_{n\to\infty}P(r_n,r)=P(r,r)$

2- A sequence $\{r_n\}$in a P.M.S $(M,P)$is called a Cauchy if and only if $\lim_{m,n\to\infty}P(r_m,r_n)$exists (and is finite).

3- If every Cauchy sequence$\{r_n\}$ in M converges, (with respect to the topology $\tau(P)$), to a member $r \in M \ni \lim_{n,m\to\infty} P(r_n, r_m) = P(r,r)$ then(M, P)is complete.

Lemma 1 [1]

Let(M, P) be any P.M.S. Then

1. A sequence $\{r_n\}$is Cauchy in a P.M.S $\Leftrightarrow$ $\{r_n\}$ is a Cauchy in a metric space(M, d_P),
2. A P.M.S (M, P)is complete$\Leftrightarrow$ (M, d_P) is complete. In addition to that $\lim_{n\to\infty} d_P(r_n, r) = 0 \Leftrightarrow P(r,r) = \lim_{n\to\infty} P(r_n, r) = \lim_{n,m\to\infty} P(r_n, r_m)$.

Definition 3 [7] A weak partial metric space (for short W.P.M.S) on a nonempty set M is a function $\rho: M^2 \to \Re^2$ satisfying the following axioms for all $\mu, \eta, \tau \in M$:

(Wρ_1) $\mu = \eta \Leftrightarrow \rho(\mu,\mu) = \rho(\mu,\eta) = \rho(\eta,\eta)$ (T_0-separation),

(Wρ_2) $\rho(\mu,\eta) = \rho(\eta,\mu)$ (symmetry)

(Wρ_3) $\rho(\mu,\mu) \leq \rho(\mu,\tau) + \rho(\tau,\eta) - \rho(\tau,\tau)$ (modified triangular inequality).

Also, Heckmann[7] showed that if ρ is a(W.P.M.S) on M,$\forall$ $\mu,\eta,\tau \in M$ then the following property is satisfied:

$\rho(\mu,\eta) \geq \frac{\rho(\mu,\mu)+\rho(\eta,\eta)}{2}$ $\forall$ $\mu,\eta,\tau \in M$

It is clear that (P.M.S) implies (WPMS), but the reverse is not true in general [7].

Example 1 [8] Let $M = [0,\infty)$ and $\rho(\mu,\eta) = \frac{(\mu+\eta)}{2}$, then (M, ρ) is a W.P.M.S) space and is not a (P.M.S).

Lemma 2 [8] Let (M, ρ) be a (W.P.M.S). Then

(a) A sequence $\{r_n\}$ is Cauchy sequence in (W.P.M.S). $\Leftrightarrow$ $\{r_n\}$is a Cauchy in(M, d_ω).

(b) A (W.P.M.S) is complete $\Leftrightarrow$ (M, d_ω)is complete. In addition to that $\lim_{n\to\infty} d_\omega(r_n, r) = 0 \Leftrightarrow \rho(r,r) = \lim_{n\to\infty} \rho(r_n, r) = \lim_{n,m\to\infty} \rho(r_n, r_m)$.

Remark 3 Note that(M, d_ω) is a standard(M, S).

Definition 4 [12] The mappings$\alpha$and $\beta: M \to M$on(M, ρ) are called commuting maps if$\forall v \in M, \alpha\beta v = \beta\alpha v$.

Definition 5 [16] Let α and $\beta: M \to M$on be mappings on (M, ρ) if u= $\alpha v = \beta v$ for some $v \in M$, then$v$ is referred to as a coincidence point and $u$is referred as a point of coincidence. The pair(α,β)is weakly compatible (W.C) if $\alpha\beta v = \beta\alpha v$

Remark 4 [17] It is remarked to point out that the definition provided above is taken from the definition in standard metric space (M, d).

Definition 6 [5] The mappings $\alpha$and $\beta: M \to M$on (M, ρ) are called $weak^*$ compatible (w^*, c)If they commute at one of their coincidence points that is if $\exists v \in M$ $\alpha v = \beta v$ then $\alpha\beta v = \beta\alpha v$.

The following example shows that $weak^*$ compatible maps are more general than weakly compatible maps.

Example 2 let $\boldsymbol{\alpha\mu = \frac{\mu^3}{4}}$ and $\beta\mu = \mu^4$ for$\mu \in [0, \frac{1}{4}]$. Then α and β have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.

Definition 7 [11] A continuous non-decreasing function $\psi: \Re^+ \to \Re^+$with$(\tau) = 0 \Leftrightarrow \tau = 0$,and $\varphi: \Re^+ \to \Re^+$ be a lower semi-continuous with

$\varphi(t) > 0 \forall t > 0$

Main Result:

Theorem 1: Suppose that (M,ρ) are a complete (WPMS), and $\alpha, \beta: M \to M$are mappings such $\alpha M \subseteq \beta M$,

$$\psi(\rho(\alpha\mu, \alpha\eta)) \leq \psi(M_\rho(\mu,\eta) - \varphi\left(M_\rho(\mu,\eta)\right), \quad 1$$

$\forall \mu, \eta \in M$and. Where

$$M_\rho(\mu,\eta) = \max\{\rho(\beta\mu,\beta\eta), \rho(\beta\mu,\alpha\mu), \rho(\beta\eta,\alpha\eta), {}^1\!/_2\,[\rho(\beta\mu,\alpha\eta) + \rho(\beta\eta,\alpha\mu)]\}$$

Then $\alpha$and β possess a point of coincidence, further if $\alpha$and β are(w^*, c).

Then α and β possess a unique common fixed point.

Proof: let $\xi_0$construct the sequences $\{\alpha\xi_n\}$and $\{\xi_n\} \subseteq M$in the following manner.

Since $\alpha M \subseteq \beta M$, choose $\xi_1 \in M$ such that $\beta\xi_1 = \alpha\xi_0$and$\beta\xi_2 = \alpha\xi_1$. Inductively, $\beta\xi_{n+1} = \alpha\xi_n \forall n \geq 0$ is obtained.

If $\alpha\xi_n = \beta\xi_{n+1}$for some $n \in N$, then $\alpha\xi_n = \alpha\xi_{n-1} = \beta\xi_n$, and $\xi_n \in M$ is a coincidence point of α and suppose that $\alpha\xi_n \neq \beta\xi_{n+1} \forall n \geq 0$

By using condition 1, this implies,

$$\psi(\rho(\alpha\xi_n, \alpha\xi_{n-1})) \leq \psi(M_\rho(\xi_{n-1}, \xi_n)) - \varphi\left(M_\rho(\xi_{n-1}, \xi_n)\right) \quad 2$$

Where,

$$M_\rho(\xi_n, \xi_{n-1}) = \max\{\rho(\beta\xi_n, \beta\xi_{n-1}), \rho(\beta\xi_n, \alpha\xi_n), \rho(\beta\xi_{n-1}, \alpha\xi_{n-1}), {}^1\!/_2\,\rho(\beta\xi_n, \alpha\xi_{n-1}) + \rho(\beta\xi_{n-1}, \alpha\xi_n)\}$$
$$= \max\{\rho(\alpha\xi_{n-1}, \alpha\xi_{n-2}), \rho(\alpha\xi_{n-1}, \alpha\xi_n), \rho(\alpha\xi_{n-2}, \alpha\xi_{n-1}), {}^1\!/_2\,\rho(\alpha\xi_{n-1}, \alpha\xi_{n-1}) + \rho(\alpha\xi_{n-2}, \alpha\xi_n)\}$$

By (Wρ_3), it follows that,,

$${}^1\!/_2\,\rho(\alpha\xi_{n-1}, \alpha\xi_{n-1}) + \rho(\alpha\xi_{n-2}, \alpha\xi_n)\} \leq {}^1\!/_2\,\rho(\alpha\xi_{n-1}, \alpha\xi_{n-2}) + \rho(\alpha\xi_{n-1}, \alpha\xi_n)\}$$
$$\leq \max\{\rho(\alpha\xi_{n-1}, \alpha\xi_{n-2}) + \rho(\alpha\xi_{n-1}, \alpha\xi_n)\}$$
$$M_\rho(\xi_{n-1}, \xi_n) = \max\{\rho(\alpha\xi_{n-1}, \alpha\xi_{n-2}) + \rho(\alpha\xi_{n-1}, \alpha\xi_n)\}$$

If, $M_\rho(\xi_{n-1},\xi_n) = \rho(\alpha\xi_{n-1},\alpha\xi_n)$then by inequality (2) implies,

$$\psi(\rho(\alpha\xi_{n-1},\alpha\xi_n)) \le \psi\rho(\xi_{n-1},\xi_n) - \varphi\rho(\xi_{n-1},\xi_n) \quad 3$$

since $\varphi(t) > 0, \forall t > 0$, and$\psi$ is non-decreasing function. $\psi(\rho(\alpha\xi_{n-1},\alpha\xi_n) < \psi\rho(\alpha\xi_{n-1},\alpha\xi_n)$, which is contradiction.to our assumption,

Hence,$M_\rho(\xi_{n-1},\xi_n) = \rho(\alpha\xi_{n-2},\alpha\xi_{n-1})$ and by use of inequality 2 it yields

$$\psi(\rho(\alpha\xi_{n-1},\alpha\xi_n)) \le \psi\rho(\xi_{n-2},\xi_{n-1}) - \varphi\rho(\xi_{n-2},\xi_{n-1}) \quad 4$$

Since $\varphi(t) > 0, \forall t > 0and\psi$ is non-decreasing function, this implies

$\psi(\rho(\alpha\xi_{n-1},\alpha\xi_n)) \le \psi\rho(\alpha\xi_{n-2},\alpha\xi_{n-1})$.

Therefore, $\{\rho(\alpha\xi_{n-1},\alpha\xi_n)\}$ is a decreasing sequence.

Thus, there exists $\delta \ge 0$such that,

$$\lim_{n\to\infty} \rho(\alpha\xi_{n-1},\alpha\xi_n) = \delta \quad 5$$

Now to show that$\delta = 0$.Suppose $\delta > 0$ Then, making the limit of supremum in $n \to \infty$in the inequality 4, $\psi(\delta) \le \psi(\delta) - \varphi(\delta) < \psi(\delta)$is obtained

Which is a contradiction to our assumption since $\varphi(\delta) > 0$.Therefore $\delta = 0$ and so

$$\lim_{n\to\infty} \rho(\alpha\xi_{n-1},\alpha\xi_n) = 0 \quad 6$$

By weak small self-distance property

$$\rho(\alpha\xi_{n-1},\alpha\xi_n) \ge {}^1/_2 [\rho(\alpha\xi_{n-1},\alpha\xi_{n-1}) + \rho(\alpha\xi_n,\alpha\xi_n)] = 0$$

$$\rho(\alpha\xi_{n-1},\alpha\xi_{n-1}) + \rho(\alpha\xi_n,\alpha\xi_n) = 0 \quad 7$$

Now it can be concluded that $\{\alpha\xi_n\}$is a Cauchy sequence in(M,d_ω). Let us assume otherwise. Then $\exists\varepsilon > 0, \ni$ for each positive integer $j\exists n(j)$and $m(j)$such that $j < m(j) < n(j)$and

$$d_\omega(\alpha\xi_{nj},\alpha\xi_{mj}) \ge \varepsilon \quad 8$$

Pick out nj in such a way that it is the smallest integer with $nj > mj$ satisfying inequality (8). Hence,

$$d_\omega(\alpha\xi_{mj},\alpha\xi_{nj-1}) < \varepsilon \quad 9$$

Now by using the inequalities (8),(9) and the triangular inequality of d_ω

$$\varepsilon \le d_\omega(\alpha\xi_{mj},\alpha\xi_{nj})$$
$$\le d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj},\alpha\xi_{nj-1}) + d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$\le d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{nj}) + 2d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$\le 2\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{mj}) + d_\omega(\alpha\xi_{mj},\alpha\xi_{nj}) + 2d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$\le 3\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{nj-1}) + d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj}) + 2d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$\le 3\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{nj-1}) + 3d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$\le 3\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{mj}) + d_\omega(\alpha\xi_{mj},\alpha\xi_{nj-1}) + 3d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$= 4\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + d_\omega(\alpha\xi_{mj+1},\alpha\xi_{mj-1}) + 3d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$
$$< 4\, d_\omega(\alpha\xi_{mj},\alpha\xi_{mj+1}) + \varepsilon + 3d_\omega(\alpha\xi_{nj-1},\alpha\xi_{nj})$$

Letting $j \to \infty$ yields

$$\operatorname*{Lim}_{j\to\infty} d_\omega(\alpha\xi_{mj},\alpha\xi_{nj}) = \lim_{j\to\infty} d_\omega(\alpha\xi_{mj+1},\alpha\xi_{nj-1}) = \operatorname*{Lim}_{j\to\infty} d_\omega(\alpha\xi_{mj+1},\alpha\xi_{nj}) = \lim_{j\to\infty} d_\omega(\alpha\xi_{mj},\alpha\xi_{nj-1}) = \varepsilon$$

Since $d_\omega(\mu,\eta) = P(\mu,\eta) - \min\{P(\mu,\mu),P(\eta,\eta)\}$for all $\mu,\eta \in M$, then by using inequality (7), $\operatorname*{Lim}_{n\to\infty} \rho(\alpha\xi_n,\alpha\xi_n) = 0$ it concludes that

$$\operatorname*{Lim}_{j\to\infty} \rho(\alpha\xi_{mj},\alpha\xi_{nj}) = \operatorname*{Lim}_{j\to\infty} \rho(\alpha\xi_{mj+1},\alpha\xi_{nj-1}) = \operatorname*{Lim}_{j\to\infty} \rho(\alpha\xi_{mj+1},\alpha\xi_{nj}) = \operatorname*{Lim}_{j\to\infty} \rho(\alpha\xi_{mj},\alpha\xi_{nj-1}) \quad 10$$

Now by using condition (1) to element $\mu = \xi_{mj}$and $\eta = \xi_{nj}$

$$\psi(\rho(\alpha\xi_{mj},\alpha\xi_{nj})) \le \psi(M_\rho(\xi_{mj},\xi_{nj})) - \varphi\left(M_\rho(\xi_{mj},\xi_{nj})\right)$$

$$M_\rho(\xi_{mj},\xi_{nj}) = \max\{\rho(\beta\xi_{mj},\beta\xi_{nj}),\rho(\beta\xi_{mj},\alpha\xi_{mj}),\rho(\beta\xi_{nj},\alpha\xi_{nj}), {}^1/_2\, \rho(\beta\xi_{mj},\alpha\xi_{nj}) + \rho(\beta\xi_{nj},\alpha\xi_{mj})\}$$

Letting $j \to \infty$ and by the property of $\psi$and φ in the above inequality,

$$\psi(\varepsilon) = \psi \max\{\varepsilon,0,0,\varepsilon\} - \varphi\max\{\varepsilon,0,0,\varepsilon\}$$

Hence from condition (1) and (10)$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$, a contradiction to our assumption $\varepsilon > 0$

Thus $\{\alpha\xi_n\}$is a Cauchy sequence in(M,d_ω).then $\operatorname*{Lim}_{n,m\to\infty} d_\omega(\alpha\xi_n,\alpha\xi_m) = 0$ and $\operatorname*{Lim}_{n,m\to\infty} d_\omega\rho(\alpha\xi_n,\alpha\xi_m)$, by completeness of (M,ρ) and lemma 1 (the sequence$\{\alpha\xi_n\}$ converges in (M,d_ω), thus $\exists u \in M$ such that$\operatorname*{Lim}_{n\to\infty} d_\omega\rho(\alpha\xi_n,u) = 0$, and then

$\rho(u,u) = \lim_{n\to\infty} \rho(\alpha\xi_n, u) = \lim_{n,m\to\infty} \rho(\alpha\xi_n, \alpha\xi_m) = 0$
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Also, the subsequences $\{\alpha\xi_{nj}\}$ and $\{\alpha\xi_{mj}\}$ are convergent to u,$\exists\omega \in M \ni u = \alpha\omega$

Now to show that $\beta\omega = \alpha\omega = u$,

$$\rho(\beta\omega,\alpha\omega) \leq \rho(\beta\omega,\alpha\xi_{n+1}) + \rho(\alpha\xi_{n+1},\alpha\omega) - \rho(\alpha\xi_{n+1},\alpha\xi_{n+1}) \leq \rho(\beta\omega,\alpha\xi_{n+1}) + \rho(\alpha\xi_{n+1},\alpha\omega)$$

$$\psi(\rho(\alpha\xi_{mj+1},\alpha\omega)) \leq \psi(M_\rho(\xi_{mj+1},\omega)) - \varphi\left(M_\rho(\xi_{mj+1},\omega)\right)$$

Where,

$$M_\rho(\xi_{mj+1},\omega) = \max\{\rho(\beta\xi_{mj},\beta\omega), \rho(\beta\xi_{mj+1},\alpha\xi_{mj+1}), \rho(\beta\omega,\alpha\omega), {}^1/_2\, \rho(\beta\xi_{mj+1},\alpha\omega) + \rho(\beta\omega,\alpha\xi_{mj+1})\}$$
$$= \max\{\rho(\alpha\xi_{mj},\beta\omega), \rho(\beta\omega,\alpha\omega), \rho(\alpha\xi_{mj},\alpha\xi_{mj+1})\, {}^1/_2\, \rho(\alpha\xi_{mj},\beta\omega) + \rho(\alpha\omega,\alpha\xi_{mj+1})$$

Letting $j\to\infty$ it can be concluded that,

$$M_\rho(\xi_{mj+1},\omega) = \max\{\rho(\beta\omega,\alpha\omega), \rho(u,u)\} \quad \rho(\beta\omega,\alpha\omega)$$

Therefore, as $j\to\infty$ condition (1) reduce to

$$\psi(\rho(\alpha\xi_{mj+1},\omega)) = \psi(\rho(\beta\omega,\alpha\omega)) \leq \psi(\rho(\beta\omega,\alpha\omega)) - \varphi(\rho(\beta\omega,\alpha\omega))$$
$$\psi(\rho(\beta\omega,\alpha\omega)) < \varphi(\rho(\beta\omega,\alpha\omega))$$

since $\varphi(t) > 0$if $t > 0$, and$\psi$is non- increasing this implies,

$$(\rho(\beta\omega,\alpha\omega)) < (\rho(\beta\omega,\alpha\omega))$$

So $(\rho(\beta\omega,\alpha\omega)) = 0 \Rightarrow \alpha\omega = \beta\omega$

Thus, it follows that $\alpha\omega = \beta\omega = u, u$ is a point of coincidence. If the mapping $\alpha$and $\beta$are $weak^*$ compatible , then $\beta(\alpha\omega) = \alpha(\beta\omega) = u$ since, thus $\beta u = \alpha u = u$, i.e. $u$is a common fixed point of α and β , the uniqueness of common fixed point of α and β, follows from condition 1 . If not assume that$\exists$ another fixed point $\varpi \ni \beta u = \alpha u = u$ and $\beta\varpi = \alpha\varpi = \varpi$.

Then,

$$\psi(\rho(u,\varpi)) = \psi\left(M_\rho(\alpha u,\alpha\varpi)\right) \leq \psi(M_\rho(u,\varpi)) - \varphi(M_\rho u,\varpi)$$

Where,

$$M_\rho(u,\varpi) = \max\{\rho(\beta u,\beta\varpi), \rho(\beta u,\alpha u), \rho(\beta\varpi,\alpha\varpi), {}^1/_2\, [\rho(\beta u,\alpha\varpi) + \rho(\beta\varpi,\alpha u)]\}$$
$$= \max\{\rho(u,\varpi), 0, 0, {}^1/_2\, [\rho(u,\varpi) + \rho(\varpi,u)]\}$$

So,

$$\psi(\rho(u,\varpi)) = \psi(\rho(\alpha u,\alpha\varpi)) \leq \psi(\rho(u,\varpi)) - \varphi(\rho(u,\varpi)) < \psi(\rho(u,\varpi)).$$

This implies $u = \varpi$.

Corollary 1 Let (M,ρ) be a (C. WPMS). Assume that $\alpha,\beta: M \to M$ are mappings $\ni \psi(\rho(\alpha\mu,\alpha\eta)) \leq \theta\max\{\rho(\mu,\eta), \rho(\mu,\alpha\mu), \rho(\eta,\beta\eta), \frac{1}{2}[\rho(\mu,\beta\eta) + \rho(\eta,\alpha\mu)]\}$ 12

For all $\mu,\eta \in M$and. Where, $\theta: \Re^+ \to \Re^+$is continuous and $\alpha(\tau) < \tau\ \forall\tau > 0$

Then α andβ have a point of coincidence, if moreover α and β are $weak^*$ compatible.

Then α and β possess common fixed point.

Proof: Take $\psi(\tau) = \tau$and $\varphi(t) = t - \theta(t)$. Then by Theorem 1, implies the condition 12.

Corollary 2 Let (M,ρ) be a (C. WPMS). Suppose that $\alpha: M \to M$ be a mapping such that

$$\psi(\rho(\alpha\mu,\alpha\eta)) \leq \psi(M_\rho(\mu,\eta) - \varphi\left(M_\rho(\mu,\eta)\right),$$
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For all $\mu,\eta \in M$,, where

$$M_\rho(\mu,\eta) = \max\{\rho(\mu,\eta), \rho(\mu,\alpha\mu), \rho(\eta,\alpha\eta), {}^1/_2\, [\rho(\mu,\alpha\eta) + \rho(\eta,\alpha\mu)]\}$$

Where $\psi$and $\varphi$are altering distance function and. Then αhas a unique fixed point.

Proof: Take $\alpha = \beta$. Then by Theorem 1, implies the condition 13 . αpossess a unique fixed point.

Corollary 3 Let (M,ρ) be a (C. WPMS). Suppose that $\alpha: M \to M$ be a map $\ni$

$$\rho(\alpha\mu,\alpha\eta) \leq \delta[(\mu,\eta) + \rho(\mu,\alpha\mu) + \rho(\eta,\alpha\eta)]$$
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For all $\mu,\eta \in M$, $0 \leq \delta \leq \frac{1}{3}$ then α possess a unique F.P.

Proof: Take $\psi(\tau) = \tau, \varphi(t) = (1 - 3\delta)t$ and $\alpha = \beta$ Then by Theorem 1, α possess a unique F.P.

Corollary 4 Let (M,ρ) be a (C. WPMS).

Suppose that $\alpha: M \to M$ be a mapping such that

$$\rho(\alpha\mu,\alpha\eta) \leq k[(\mu,\eta), \rho(\mu,\alpha\mu), \rho(\eta,\alpha\eta)],$$
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For all $\mu,\eta \in M, k \in [0,1)$, then α possess a unique fixed point.

Proof: Take $\psi(\tau) = \tau, \varphi(t) = (1 - k)t$ for $k \in [0,1)$ and $\alpha = \beta$, then by theorem 1, α possess a unique F.P.

An example is given to illustrate our main result.

Example 3 let $M = [0,1]$ and$\rho: M^2 \to \Re^+, \rho(r,s) = {}^1/_2\,(r + s)$, then$(M,\rho)$ is a WPMS. Then, $d_\omega(\mu,\eta) = {}^1/_2\,|\mu - \eta|$. Therefore,$(M, d_\omega)$ is a complete. By lemma 2(M,ρ) is a (C. WPMS).

Define α and $\beta: M \to M$ such that $\alpha\mu = {}^{\mu}/_{3}$ and$\beta\mu = \mu$. Let$\psi(t) = t$, $\varphi(t) = {}^{2}/_{3t}$. Then for all $\mu, \eta \in M$, it yields,

$$\psi\rho(\alpha\mu, \alpha\eta) = \psi\rho({}^{\mu}/_{3}, {}^{\eta}/_{3})$$
$$= \psi\left({}^{1}/_{2}\left(({}^{\mu + \eta}/_{3}\right)\right)$$
$$= {}^{1}/_{3}\,\rho(\mu, \eta)$$

$$\leq \rho(\mu, \eta) - {}^{2}/_{3}\,\rho(\mu, \eta)$$
$$= \psi(M_{\rho}(\mu, \eta) - \varphi(M_{\rho}(\mu, \eta))$$
$$= {}^{1}/_{3}\,\rho(\mu, \eta)$$

$$M_{\rho}(\mu, \eta) = \max\left\{\begin{matrix}\rho(\beta\mu, \beta\eta), \rho(\beta\mu, \alpha\mu), \rho(\beta\eta, \alpha\eta), \\ {}^{1}/_{2}\,[\rho(\beta\mu, \alpha\eta) + \rho(\beta\eta, \alpha\mu)]\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix}\rho(\mu, \eta), \rho(\mu, {}^{\mu}/_{3}), \rho(\eta, {}^{\eta}/_{3}), \\ {}^{1}/_{2}\,[\rho(\mu, {}^{\eta}/_{3}) + \rho(\eta, {}^{\mu}/_{3})]\end{matrix}\right\}$$

$$= \max\left\{\frac{(\mu + \eta)}{2}, \frac{2\mu}{3}, \frac{2\eta}{3}, \frac{(\mu + \eta)}{3}\right\}$$
$$= \frac{(\mu + \eta)}{2} = \rho(\mu, \eta)$$

Therefore, all conditions of Theorem 1 are satisfied. Since $\alpha 0 = \beta 0$, and α, β are $weak^{*}$ compatible then α and β possess a unique common F.P $\alpha 0 = \beta 0 = 0$.

It can be remarked that α and β are single valued maps and for multivalued maps see [18].

Conclusions:

In this paper, the theorems of coincidence and fixed point for two maps satisfying a generalized contractive condition 1 in a weak partial metric space are proven as a generalized of partial metric space and the standard metric space in the sense that the self-distance of any point need not equal to zero.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah.

Authors' contributions statement:

A. M. H. was in charge of developing the idea of fixed point theorem in weak partial metric space which is a generalization of partial metric space. A. T. H. verified the analytical procedures used in the research and she proved an example to support the results. Both of the authors discussed the findings and contributed to the final draft of the paper.

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حول مبرهنات النقطة الصامدة في الفضاءات المترية الجزئية الضعيفة

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الخلاصة:

الهدف الرئيسي لهذا البحث هو استعراض ودراسة بعض النقاط الصامدة للدوال التي تحقق الشرط $(\phi-\psi)$ في الفضاءات المترية الجزئية الضعيفة وتم إعطاء بعض التعميمات الجديدة لمبرهنات النقطة الصامدة لكل من ماثيوس وهيكمان. ان اهم النتائج التي حصلنا عليها هي توسيع وتوحيد العديد من النتائج في مبرهنات النقطة الصامدة وتعميم بعض النتائج الحديثة في الفضاء المتري الجزئي الضعيف كما اعطينا مثال لتوضيح نتائجنا.

الكلمات المفتاحية: النقاط المتطابقة. المتري الجزئي، النقطة الصامدة، المتري الجزئي الضعيف, التوافق الضعيف.