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The Approximation of Weighted Hölder Functions by Fourier-Jacobi Polynomials to the Singular Sturm-Liouville Operator

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Abstract:

In this work, a weighted Hölder function that approximates a Jacobi polynomial which solves the second order singular Sturm-Liouville equation is discussed. This is generally equivalent to the Jacobean translations and the moduli of smoothness. This paper aims to focus on improving methods of approximation and finding the upper and lower estimates for the degree of approximation in weighted Hölder spaces by modifying the modulus of continuity and smoothness. Moreover, some properties for the moduli of smoothness with direct and inverse results are considered.

Keywords: Best approximation, Jacobi Polynomials, Moduli of smoothness, Sturm-Liouville equation, Weighted Hölder spaces.

Introduction:

Many problems of differential equations are reduced to orthogonal polynomials or algebraic polynomials, such as Jacobi orthogonal polynomial. There are widely applications to expand the Jacobi polynomials in science and engineering, for instance electrocardiogram data compression and the Gibbs' phenomenon resolution[1-3]. Also, the approximation of weighted Hölder functions and some results on the approximation methods which are defined on weighted Hölder spaces have been considered [4-5]. The significant methods are applied for Fourier-Jacobi operators in periodic weighted Hölder spaces[6-8]. Properties for moduli of smoothness and approximation's degree of functions in weighted Hölder spaces have significant applications for general theory of partial differential equations in non-smoothness domains and estimates of asymptotic[9-10]. Also, numerical applications for modeling of fractional differential equations in applied sciences and engineering to the natural way have been considered in important studies[11-17].

This paper aims to improve the previous results in [18] and that get the functions in the Holder spaces which have best approximation by Jacobi differential operator to a 2^{nd}-order singular Sturm-Liouville type equation. Moreover, estimating the upper and lower bounded for the degree of approximation in weighted Hölder spaces is considered. Also, some properties of the moduli of smoothness that help to get a good approximation on the weighted Hölder space.

Consider the space of all functions which are measurable on $X = [-1,1]$ and weighted. The space is denoted by

$$L_{\varphi,p}(X) = L_{\varphi,p}^{a,b}(X) = L_p(X; \varphi_{a,b}),\ \ 1 \le p \le \infty,$$

which has a finite norm

$$\|f\|_{\varphi,p} = \left(\int_X |f(x)|^p\ \varphi(x)\ dx \right)^{1/p},$$

where the Jacobi-weighted function

$$\varphi = \varphi_{a,b}(x) = (1-x)^a\ (1+x)^b \text{ on } X,$$

with $a, b \in (-1, +\infty)$. In the case $a = b = 0$, the space $L_p(X) = L_p(X\,; \varphi_{0,0})$, is considered with finite norm. Orthonormal polynomials and Sturm-Liouville theory and their applications are considered in significant references[19-20]. They are linearly independent solutions to second-order Sturm-Liouville differential equation

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$$\frac{d}{dx}\left[\varphi^{(a+1,b+1)}(x)\frac{df}{dx}\right]+n(n+a+b+1)$$
$$\times\varphi^{(a,b)}(x)\,f=0,\quad x\in X.\quad \ldots \mathbf{1}$$

The Jacobi polynomial of order n, $P_n^{(a,b)}(x), n=0,1,2,\ldots$ is a solution of **1** and it has been defined in special forms[21]. Let us define $N-$dimensional space $S_N=\text{span}\{x^n : 0\le n\le N-1\}$.

For any $-1<a,b<\infty$, every Jacobi polynomial of $(a,b)-$class is orthogonal under the weighted $L_{\varphi,2}(X)$- inner product

$$\int_X P_m^{(a,b)}P_n^{(a,b)}(1-x)^a(1+x)^b dx=\begin{cases}0, & n\neq m;\\ \mathcal{N}_n^{(a,b)} & n=m.\end{cases}$$

The normalized Jacobi polynomial

$$J_n^{(a,b)}(x)=P_n^{(a,b)}(x)/\mathcal{N}_n^{(a,b)},\quad n=0,1,2,\ldots,$$

where $\mathcal{N}_n^{(a,b)}$ is the normalization constant given as $\left\|P_n^{(a,b)}\right\|=\mathcal{N}_n^{(a,b)}$. Let $f\in L_{\varphi,p}(X), 1\le p\le\infty$, $a,b>0$, which has the Fourier-Jacobi's series as the form

$$f(x)\cong\sum_{k=0}^{\infty}\mathcal{N}_n^{(a,b)}\,\mathcal{F}_k^{(a,b)}(f)\,J_n^{(a,b)}(x),\qquad \ldots \mathbf{2}$$

and the Fourier coefficients has the form

$$\mathcal{F}_k^{(a,b)}(f)=\int_X f(x)J_k^{(a,b)}(x)\varphi(x)dx,\qquad k=0,1,\ldots$$

The expansions of Fourier-Jacobi are approximated to the operator $T_h^{(a,b)}$, where $0<h<\delta<\pi$ on a function $f\in L_{\varphi,p}(X)$ with expansion **2** by the form

$$T_h^{(a,b)}f(x)\cong\sum_{k=0}^{\infty}\mathcal{F}_k^{(a,b)}(f)\mathcal{N}_n^{(a,b)}\;J_n^{(a,b)}(\cos h)$$
$$\times\;J_n^{(a,b)}(x)\qquad \ldots \mathbf{3}$$

where the operator $T_h^{(a,b)}$is positive[18]. If $f\in L_{\varphi,p}(X)$ and $0<h<\delta$, then it is easy to get from **2** and **3**, that

$$\left\|T_h^{(a,b)}f\right\|_{\varphi,p}\le\|f\|_{\varphi,p},\qquad \ldots \mathbf{4}$$

where the restriction $\alpha\ge\beta\ge-1/2$ is guarantee to get the inequality **4** can be satisfied.

Preliminary Results:

Let $f\in L_{\varphi,p}(X), 1<p<\infty$, and $r>0$ then the r-th symmetric difference which is constructed by the Fourier-Jacobi operator $T_h^{(a,b)}$, is defined as

$$\Delta_{r,h}^{(a,b)}=\left(I-T_h^{(a,b)}\right)^{r/2}$$
$$=\sum_{k=0}^{\infty}(-1)^k\binom{\frac{r}{2}}{k}\left(T_h^{(a,b)}\right)^k\quad \ldots \mathbf{5}$$

where I represents the identity operator. Also, the definition of the r-th order weighted modulus of smoothness is defined as

$$w_r(f,\delta)_{\varphi,p}=\sup_{0<h\le\delta}\left\|\Delta_{r,h}^{(a,b)}f\right\|_{\varphi,p},$$

and for $h\ge\delta$, let us define that

$$w_r(f,h)_{\varphi,p}=w_r(f,\delta)_{\varphi,p}.$$

Now, consider the weighted Hölder spaces with respect to the general Jacobi operator $T_h^{(a,b)}$, which is denoted by $C_{\varphi,p}^{r,\mu}(X)$. For any weighted Hölder function $f\in C_{\varphi,p}^{r,\mu}(X)$, $\mu>0$ if $f\in L_{\varphi,p}(X)$ and

$$\|f\|_{C_{\varphi,p}^{r,\mu}(X)}=\|f\|_{\varphi,p}+|f|_{C_{\varphi,p}^{r,\mu}(X)},\qquad \ldots \mathbf{6}$$

is finite, where

$$|f|_{C_{\varphi,p}^{r,\mu}(X)}=\sup_{h>0}\frac{w_r(f,h)_{\varphi,p}}{h^{\mu}}.$$

Moreover, let $0<\mu<r<k<\infty; 1\le p\le\infty$ and for any arbitrary constant $\mathcal{C}$, then the norms of a weighted Hölder function f in weighted Hölder spaces $C_{\varphi,p}^{r,\mu}(X)$ and $C_{\varphi,p}^{k,\mu}(X)$ that are equivalent:

$$\mathcal{C}\,\|f\|_{C_{\varphi,p}^{r,\mu}(X)}\le\|f\|_{C_{\varphi,p}^{r,\mu}(X)}\le\mathcal{C}^{-1}\,\|f\|_{C_{\varphi,p}^{r,\mu}(X)}.\quad \ldots \mathbf{7}$$

Let us consider the modulus of smoothness for the weighted Hölder function f in $C_{\varphi,p}^{r,\mu}(X)$, $0<\mu\le k$, such that

$$w_r^{\mu}(f,t)_{C_{\varphi,p}^{r,\mu}(X)}=\sup_{0<h<t}\frac{w_r(f,h)_{\varphi,p}}{h^{\mu}}.\qquad \ldots \mathbf{8}$$

More properties of the modulus of smoothness on the spaces $C_{\varphi,p}^{r,\mu}(X)$ are in 2018 [5].

Consider $\mathcal{P}_n=\{P_n$: P_n is an algebraic polynomial of degree at most n}, and the degree of approximation for any function $f\in L_{\varphi,p}(X)$ by algebraic polynomials is well-defined as follow :

$$E_n(f)_{\varphi,p}=\inf_{P_n\in\mathcal{P}_{n-1}}\|f-P_n\|_{\varphi,p},\quad n\in N.$$

Where $P_n\in\mathcal{P}_{n-1}$ is an algebraic polynomial of the best approximation of $f\in L_{\varphi,p}(X)$. The following Jacksons type theorem in $L_{\varphi,p}(X)$ is mentioned in [4],

$$\|f-P\|_{\varphi,p}=E_n(f)_{\varphi,p},$$

which is called the Jackson–type form in $L_{\varphi,p}(X)$. Also, $P\in\mathcal{P}_{n-1}$ represents a polynomial of best approximation for a function $f\in C_{\varphi,p}^{r,\mu}(X)$ if

$$\|f-P\|_{C_{\varphi,p}^{r,\mu}(X)}=E_n(f)_{C_{\varphi,p}^{r,\mu}(X)},$$
$$E_n(f)_{C_{\varphi,p}^{r,\mu}(X)}=\inf_{P_n\in\mathcal{P}_{n-1}}\|f-P_n\|_{C_{\varphi,p}^{r,\mu}(X)}, n\in N.$$

Let us consider the differential operator

$$\mathcal{L}=\frac{-1}{\varphi^{(a,b)}}\frac{d}{dx}\left[\varphi^{(a+1,b+1)}\frac{d}{dx}\right],\qquad \ldots \mathbf{9}$$

$$\mathcal{L}P_k^{(a,b)}=\lambda_k\,P_k^{(a,b)};\;\lambda_k=k(k+a+b+1),$$

where the Jacobi polynomials are the eigenfunctions of $\mathcal{L}$. Let us construct the Fourier–Jacobi series $S\in L_{\varphi,p}(X)$ of a function $f\in L_{\varphi,p}(X)$ for $r>0$ such that it has the form

$$S(x)\approx\sum_{j=1}^{\infty}(\lambda_j)^{r/2}\,\mathcal{N}_n^{(a,b)}\,J_j^{(a,b)}(x),$$

where the notation $S=\mathcal{L}^r f$ is considered and $\mathcal{L}^r f$ is called a fractional derivative of the function f with the order r. The generalized $\mathcal{K}_r$-functional term for

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the differential operator $\mathcal{L}^r$ to estimate the results of best approximation for a function f in the weighted space $L_{\varphi,p}(X)$ that has been discussed in [4], where

$$\mathcal{K}_r(f,h)_{\varphi,p} = \inf_{P_n}\{\|f-P_n\|_{\varphi,p} + h^r\|\mathcal{L}^r P_n\|_{\varphi,p}\}.$$

The natural relation between $\mathcal{K}_r$-functional and moduli of smoothness is also observed in [4].

For $f \in L_{\varphi,p}(X), \delta > 0,$ and $r > 0$, then

$$\mathcal{K}_r(f,h)_{\varphi,p} \approx w_r(f,h)_{\varphi,p}, 0 < h < \delta \qquad \ldots \mathbf{10}$$

and for $P \in \mathcal{P}_{n-1}$,

$$h^r \|\mathcal{L}^r P\|_{\varphi,p} \approx w_r(P,h)_{\varphi,p}. \qquad \ldots \mathbf{11}$$

Now, the preliminary results are considered and they will be identified in the next section to prove the main results. The following lemma is introduced in [4]**:**

Lemma 1. Let $n \in N, 0 < h < n^{-1},$ and $r > 0$ then for any algebraic polynomial $P_n \in \mathcal{P}_{n-1}$ and for $f \in L_{\varphi,p}(X)$ such that

$$\left\|\Delta_{r,h}^{(a,b)} P_n\right\|_{\varphi,p} \le C\, w_r(f,h)_{\varphi,p},$$

where C is an independent constant on f, P_n and h.

Proposition 1. Let $n \in N, 0 < h < n^{-1},$ and $r > 0$ then for any algebraic polynomial $P \in \mathcal{P}_{n-1}$ and for $f \in L_{\varphi,p}(X)$ such that

$$w_r(f-P,h)_{\varphi,p} \le w_r(f,h)_{\varphi,p} + w_r(P,h)_{\varphi,p}.$$

Directly, the proof of this proposition is satisfied from combining **4**, **5** and lemma 1. It represents a useful property for the r-th order weighted modulus of smoothness. On the other hand, in the next proposition, some properties on the moduli of smoothness for the weighted Hölder function in $C_{\varphi,p}^{r,\mu}(X)$, will be considered.

Proposition 2. Let $n \in N, 0 < h < n^{-1}, r, k, \sigma > 0,$ and $f \in C_{\varphi,p}^{r,\mu}(X)$ then

$$w_k^{\mu}(f,\sigma)_{C_{\varphi,p}^{r,\mu}(X)} \le 2^{k-r} w_r^{\mu}(f,h)_{C_{\varphi,p}^{r,\mu}(X)}, \quad \ldots \mathbf{12a}$$

where $k > r > 0$and

$$w_k^{\mu}(f,n\sigma)_{C_{\varphi,p}^{r,\mu}(X)} \le Cn^{k-r} w_k^{\mu}(f,h)_{C_{\varphi,p}^{r,\mu}(X)}, \quad \ldots \mathbf{12b}$$

where C is an independent constant of f, σ and n.

Proof. This proposition can be proved later by using **4** and **10**. Let $0 < h \le \sigma$, and $f \in C_{\varphi,p}^{r,\mu}(X)$, and by **5** then

$$\Delta_{k,h}^{(a,b)} f(x) = \left(I - T_h^{(a,b)}\right)^{\frac{k}{2}} f(x) = \sum_{i=0}^{\infty} (-1)^i f(x) \binom{\frac{k}{2}}{i} \left(T_h^{(a,b)}\right)^i$$

and from lemma 1 then

$$\left\|\Delta_{k,h}^{(a,b)} f\right\|_{\varphi,p} \le 2^{k-r} \left\|\Delta_{r,h}^{(a,b)} f\right\|_{\varphi,p} \le C\, 2^{k-r}\, w_r(f,h)_{\varphi,p},$$

$$h^{-\mu} w_k(f,h)_{\varphi,p} \le C\, 2^{k-r} h^{-\mu}\, w_r(f,h)_{\varphi,p}.$$

Thus, the first supremum when $0 < h \le \sigma$, is taken to satisfy **12a** and then when $0 < t \le h$. Similarly, proving the inequality **12b** is satisfied.

Lemma 2. Let $n \in N$, $r > 0$ and $f \in L_{\varphi,p}(X)$ then

$$E_n(f)_{\varphi,p} \le Cw_r(f,n^{-1})_{\varphi,p}$$

where C is an independent constant on n and f. This lemma is proved directly from equation **4**.

Upper Estimate for the Best Approximation:

Depending on the results in sections 1 and 2, the first main result theorem is proved in order to estimate the upper bound for the degree of approximation of the weighted Hölder functions. Fourier-Jacobi operators are applied to the second order singular Sturm-Liouville equation where same strategies and methods which are presented in [19-21].

Theorem 1. Let $0 < \mu < \min(r,k)$ and $f \in C_{\varphi,p}^{r,\mu}(X)$ for an algebraic polynomial $P_n \in \mathcal{P}_{n-1}$ then

$$\|f-P_n\|_{C_{\varphi,p}^{r,\mu}(X)} \le Cw_r^{\mu}(f,t)_{C_{\varphi,p}^{r,\mu}(X)},$$

where C is an independent constant on n and f.

Proof. Assume that $\alpha < \min(r,k)$ and $P_n \in \mathcal{P}_{n-1}$ be any algebraic polynomial and for $f \in L_{\varphi,p}(X)$. By lemma 2 and **6**, let us define

$$\|f-P_n\|_{C_{\varphi,p}^{r,\mu}(X)} = \|f-P_n\|_{\varphi,p} + |f-P_n|_{C_{\varphi,p}^{r,\mu}(X)} \qquad \ldots \mathbf{13}$$

It is enough to estimate the last term of **13**, so

$$|f-P_n|_{C_{\varphi,p}^{r,\mu}} = \sup_{h>0} \frac{w_r(f-P_n,h)_{\varphi,p}}{h^{\mu}} = \sup_{h<1/n} \frac{w_r(f-P_n,h)_{\varphi,p}}{h^{\mu}} + \sup_{h\ge 1/n} \frac{w_r(f-P_n,h)_{\varphi,p}}{h^{\mu}} = I_1 + I_2.$$

By **4**, **8** and Lemma 2 , then

$$I_2 = n^{\mu} w_r(f-P_n,n^{-1})_{\varphi,p} = n^{\mu} \left\|\Delta_{r,n^{-1}}^{(\alpha,\beta)}(f-P_n)\right\|_{\varphi,p} \le \mathcal{C}n^{\mu}\|f-P_n\|_{\varphi,p}$$

$$\le \mathcal{C}n^{\mu} w_r(f,1/n)_{\varphi,p} \le \mathcal{C}n^{\mu} w_r(f,1/n)_{C_{\varphi,p}^{r,\mu}(X)}. \ldots \mathbf{14}$$

Also, from proposition 1 then I_1 should be estimated as follows:

$$I_1 = \sup_{0<h<1/n} \frac{w_r(f-P_n,h)_{\varphi,p}}{h^{\mu}} \le \sup_{0<h<1/n} \left(\frac{w_r(f,h)_{\varphi,p}}{h^{\mu}} + \frac{w_r(P_n,h)_{\varphi,p}}{h^{\mu}}\right) \le n^{\mu} w_r(f,n^{-1})_{C_{\varphi,p}^{r,\mu}(X)} + I_3 \quad \ldots \mathbf{15}$$

To estimate I_3 in **15**, depending on the results of both lemma 1 and lemma 2,

$$I_3 = \sup_{0<h<1/n} \frac{w_r(P_n,h)_{\varphi,p}}{h^{\mu}} \le \mathcal{C}n^{-1} w_r(P_n,n^{-1})_{\varphi,p} = \mathcal{C}n^{\mu} w_r(f-f+P_n,n^{-1})_{\varphi,p},$$

Thus the estimation of I_3 becomes

$$I_3 \le \mathcal{C}n^{\mu}\left(\|f-P_n\|_{\varphi,p} + w_r(f,n^{-1})_{\varphi,p}\right) \le \mathcal{C}n^{\mu} w_r(f,n^{-1})_{\varphi,p} \le \mathcal{C}n^{\mu} w_r^{\mu}(f,n^{-1})_{C_{\varphi,p}^{r,\mu}(X)} \ldots \mathbf{16}$$

therefore, to combine equations**14** $-$ **16**, then proof of theorem 1 is done. ■

Lower Estimate for the Best Approximation:

The second result that is a lower estimate for the best approximation should be proved based on the auxiliary results in sections 1 and 2. Also, the lower bound for the degree of approximation to weighted Hölder functions is estimated.

Theorem 2. Let $0 < \mu < \min(r, k)$ and $f \in C^{r,\mu}_{\varphi,p}(X)$ then

$$w^{\mu}_{r}(f,t)_{C^{r,\mu}_{\varphi,p}(X)} \le C\, n^{\mu-k} \sum_{\rho=0}^{m} 2^{\rho(r-\mu)} E_{2^{\rho}}(f)_{C^{r,\mu}_{\varphi,p}(X)}$$

where C is an independent constant on n and f.

Proof. Let $P_n \in \mathcal{P}_{n-1}, n \in N$ be the polynomial of the best approximation of $f \in C^{r,\mu}_{\varphi,p}(X)$ in the case $\mu < \min(r, k)$ and for any $m \in N \cup \{0\}$ such that $2^{m-1} \le n < 2^{m}$. Suppose that $\sum_{\rho=m}^{\infty} \rho^{n-1} E_{\rho}(f)_{\varphi,p}$ is finite. Then let us consider

$$f \cong P_{2^m} + \sum_{\rho=m}^{\infty}(P_{2^{\rho+1}} - P_{2^{\rho}}). \qquad \ldots\mathbf{17}$$

From **17** and proposition 1, then

$$w^{\mu}_{r}(f,1/n)_{C^{r,\mu}_{\varphi,p}(X)} \le w^{\mu}_{r}(f - P_{2^{m+1}},1/n)_{C^{r,\mu}_{\varphi,p}(X)} + w^{\mu}_{r}(P_{2^{m+1}},1/n)_{C^{r,\mu}_{\varphi,p}(X)}. \qquad \ldots\mathbf{18}$$

Then by applying the Holder inequality and **7**, the best approximation is

$$E_n(f)_{C^{r,\mu}_{\varphi,p}(X)} = \inf_{P_n \in \mathcal{P}_{n-1}} \|f - P_n\|_{C^{r,\mu}_{\varphi,p}(X)}$$
$$= \inf_{P_n \in \mathcal{P}_{n-1}} \left\{\|f - P_n\|_{\varphi,p} + |f - P_n|_{C^{r,\mu}_{\varphi,p}(X)}\right\}$$
$$\ge \inf_{P_n \in \mathcal{P}_{n-1}} \|f - P_n\|_{\varphi,p} + \inf_{P_n \in \mathcal{P}_{n-1}} |f - P_n|_{C^{r,\mu}_{\varphi,p}(X)}.$$

Choose $n = 2^{m+1}$, and from **6** and **8** then

$$E_{2^{m+1}}(f)_{C^{r,\mu}_{\varphi,p}(X)} \ge \inf_{P_{2^{m+1}} \in \mathcal{P}_{n-1}} \left\{ |f - P_{2^{m+1}}|_{C^{r,\mu}_{\varphi,p}(X)} \right\}$$
$$= |f - P_{2^{m+1}}|_{C^{r,\mu}_{\varphi,p}(X)}$$
$$= \sup_{h<1/n} w_r(f - P_{2^{m+1}},h)_{\varphi,p}/h^{\mu} + \sup_{h\ge 1/n} w_r(f - P_{2^{m+1}},h)_{\varphi,p}/h^{\mu}$$
$$= w^{\mu}_{r}(f - P_{2^{m+1}},1/n)_{C^{r,\mu}_{\varphi,p}(X)} + \sup_{h\ge 1/n} \frac{w_r(f - P_{2^{m+1}},h)_{\varphi,p}}{h^{\mu}}.$$

Then

$$w^{\mu}_{r}\left(f - P_{2^{m+1}},\frac{1}{n}\right)_{\varphi,p} \le |f - P_{2^{m+1}}|_{C^{r,\mu}_{\varphi,p}(X)} \le E_{2^{m+1}}(f)_{C^{r,\mu}_{\varphi,p}(X)}. \qquad \ldots\mathbf{19}$$

By lemma 2 , **7**, **10** and **11** then

$$\|\mathcal{L}^r P_{2^{m+1}}\|_{\varphi,p} \le \|\mathcal{L}^r P_2 - \mathcal{L}^r P_1\|_{\varphi,p} + \sum_{\rho=0}^{m} \|\mathcal{L}^r P_{2^{\rho+1}} - \mathcal{L}^r P_{2^{\rho}}\|_{\varphi,p}$$
$$\le C w_r(P_2 - P_1, 1)_{\varphi,p} + C\sum_{\rho=0}^{m} 2^{\rho r} \times w_r\left(P_{2^{\rho+1}} - P_{2^{\rho}}, 2^{-(\rho+1)}\right)_{\varphi,p}$$
$$\le C\|P_2 - P_1\|_{C^{r,\mu}_{\varphi,p}(X)} + C\sum_{\rho=0}^{m} 2^{(\rho r-\mu)} \times \|P_{2^{\rho+1}} - P_{2^{\rho}}\|_{C^{r,\mu}_{\varphi,p}(X)}$$
$$\le CE_1(f)_{C^{r,\mu}_{\varphi,p}(X)} + C\sum_{\rho=0}^{m} 2^{(\rho r-\mu)} E_{2^{\rho}}(f)_{C^{r,\mu}_{\varphi,p}(X)} \qquad \ldots\mathbf{20}$$

Now, from **20** then

$$w^{\mu}_{r}(P_{2^{m+1}}, n^{-1})_{C^{r,\mu}_{\varphi,p}(X)} = \sup_{h<1/n} \frac{w_r(P_{2^{m+1}},h)_{\varphi,p}}{h^{\mu}},$$
$$\le \mathcal{C} \sup_{0<h\le 1/n} \frac{\mathcal{K}_r(P_{2^{m+1}},h)_{\varphi,p}}{h^{\mu}} \le \mathcal{C} n^{\mu-r} \|\mathcal{L}^r P_{2^{m+1}}\|_{\varphi,p}$$
$$\le \mathcal{C} n^{\mu-r} E_1(f)_{C^{r,\mu}_{\varphi,p}} + \sum_{\rho=0}^{m} 2^{(\rho r-\mu)} E_{2^{\rho}}(f)_{C^{r,\mu}_{\varphi,p}} \qquad \ldots\mathbf{21}$$

Choose m such that $2^{m} \le n < 2^{m+1}$. Then the required estimation follows from $\mathbf{18 - 19}$ and **20**, then proof of theorem 1 is done. ■

In the next theorem, proof of the sharp lower estimate will be considered for the degree of approximation in $C^{r,\mu}_{\varphi,p}(X)$:

Theorem 3. Let $0 < \mu < r$, $\nu \ge \mu$ and $f \in C^{r,\mu}_{\varphi,p}(X)$. Then the following two parts are equivalent :

(i) $w^{\mu}_{r}(f, n^{-1})_{C^{r,\mu}_{\varphi,p}(X)} \le C_1 E_n(f)_{C^{r,\mu}_{\varphi,p}(X)}$,

(ii) $w^{\mu}_{\nu}(f, h)_{C^{r,\mu}_{\varphi,p}(X)} \le C_2 w^{\mu}_{k}(f, h)_{C^{r,\mu}_{\varphi,p}(X)}$,

where $h > 0$, for some $k > \nu$ and existing positive constants C_1 and C_2.

Proof. Assume that the condition (ii) is satisfied. From proposition 2, the two inequalities **12a-12b** are satisfied, then

$$w^{\mu}_{\nu}(f, n\sigma)_{C^{r,\mu}_{\varphi,p}(X)} \le CC_2 n^{\nu-\mu} w^{\mu}_{k}(f,\sigma)_{C^{r,\mu}_{\varphi,p}(X)} \qquad \ldots\mathbf{22}$$

for $n \in N$, $\sigma > 0$, therefore,

$$w^{\mu}_{\nu}(f, \lambda\sigma)_{C^{r,\mu}_{\varphi,p}(X)} \le CC_2 (1+\lambda)^{\nu-\mu} w^{\mu}_{k}(f,\sigma)_{C^{r,\mu}_{\varphi,p}(X)}.$$

Now, to prove the following inequality

$$n^{\mu-k} \sum_{\rho=0}^{n} \rho^{-(\mu-k+1)} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}} \le CC_2 w^{\mu}_{k}(f, 1/n)_{C^{r,\mu}_{\varphi,p}} \qquad \ldots\mathbf{23}$$

Definitely, from results of theorem 1 and the form **22**, the following estimate

$E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}(X)} \le w^{\mu}_{k}(f, 1/\rho)_{C^{r,\mu}_{\varphi,p}(X)}$ for $\rho = 0,1,\ldots,n$.

is valid. Then

$$n^{\mu-k} \sum_{\rho=0}^{n} \rho^{-(\mu-k+1)} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}}$$
$$\le CC_2 n^{\mu-k} w^{\mu}_{k}(f, 1/n)_{C^{r,\mu}_{\varphi,p}} \sum_{\rho=0}^{n} \rho^{-(\mu-k+1)}$$
$$\le CC_2 w^{\mu}_{k}(f, 1/n)_{C^{r,\mu}_{\varphi,p}}.$$

Now, from results of theorem 2 and the form **23**,then for all $m, n \in N$, with $d = nm$

$$w^{\mu}_{\nu}(f, 1/d)_{C^{r,\mu}_{\varphi,p}} \le Cd^{\mu-\nu} \sum_{\rho=1}^{d} \rho^{-(\mu-\nu+1)} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}}$$
$$\le Cd^{\mu-\nu} \sum_{\rho=1}^{n} \rho^{-(\mu-\nu+1)} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}(X)} + Cd^{\mu-\nu} \sum_{\rho=n+1}^{d} \rho^{-(\mu-\nu+1)} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}(X)}$$
$$\le C\, C_2 m^{\mu-\nu} w^{\mu}_{k}\left(f, \tfrac{1}{n}\right)_{C^{r,\mu}_{\varphi,p}(X)} + Cd^{\mu-\nu} \times \sum_{\rho=n+1}^{d} \rho^{\nu-\mu-1} E_{\rho}(f)_{C^{r,\mu}_{\varphi,p}(X)},$$

which implies that

$$d^{\mu-k} w_k^{\mu}\left(f,\frac{1}{n}\right)_{C_{\varphi,p}^{r,\mu}} - CC_2 m^{\mu-k} w_k^{\mu}\left(f,\frac{1}{n}\right)_{C_{\varphi,p}^{r,\mu}} \le \sum_{\rho=n+1}^{d} C\rho^{-(\mu-k+1)} E_\rho(f)_{C_{\varphi,p}^{r,\mu}(X)}$$

from proposition 2, **22** and by the monotonicity property of $E_n(f)_{C_{\varphi,p}^{r,\mu}(X)}$, then

$$\left(C^{-1}n^{\mu-k} - C_2\right) m^{\mu-k} w_k^{\mu}\left(f,\frac{1}{n}\right)_{C_{\varphi,p}^{r,\mu}(X)} \le E_n(f)_{C_{\varphi,p}^{r,\mu}(X)} \sum_{\rho=n+1}^{mn} \rho^{k-\mu-1}.$$

Therefore,

$$E_n(f)_{C_{\varphi,p}^{r,\mu}(X)} \ge C w_k^{\mu}(f,1/n)_{C_{\varphi,p}^{r,\mu}(X)}. \qquad \ldots \mathbf{24}$$

where, $C = C(k,\mu,C_2)$ is a positive constant. From **24** and (ii); (i) is satisfied. On the other hand, the opposite direction comes immediately from theorem 1. Moreover, the degree of approximation has lower bound which is a weighted modulus of smoothness is considered. ■

Applications:

Consider the Jacobi polynomial of $(a,b)-$ class of order n,

$$P_n^{(a,b)} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n+a}{i}\binom{n+b}{n-i}(x-1)^{n-i}(x+1)^n$$

with its norm,

$$\mathcal{N}_n^{(a,b)} = \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{(a+b+2n+1)n!\,\Gamma(a+b+n+1)},$$

which satisfies the differential equation **1**. Let us consider the case $a = b = 0$

$$P_n^{(0,0)} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}(x-1)^{n-i}(x+1)^n,$$

$$\mathcal{N}_n^{(0,0)} = \frac{2(n+1)}{(2n+1)n!},$$

Then the normalized Jacobi polynomial is valid

$$J_n^{(0,0)}(x) = \frac{(2n+1)n!}{2^{n+1}(n+1)}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}(x-1)^{n-i}(x+1)^n$$

Let $f \in L_{\varphi,p}[-1,1], 1 < p \le +\infty,\ r = 2$ and let us consider the operator $T_h^{(0,0)}, 0 < h < n^{-1}$, of a function f with expansion form **3**, and it is proved in **5** that

$$\left\|f - T_h^{(0,0)}(f)\right\|_{\varphi,p} \asymp \inf_s\Big\{\|f-s\|_{\varphi,p} + n^{-2}\left\|\frac{d}{dx}\left((x-1)(x+1)\frac{d}{dx}\right)s\right\|_{\varphi,p}\Big\},$$

by **10-11** and **6**, then the following estimation is valid

$$\left\|f - T_h^{(0,0)}(f)\right\|_{C_{\varphi,p}^{2,\mu}([-1,1])} \asymp w_2(f,n^{-1})_{\varphi,p} \qquad \ldots \mathbf{25}$$

With $0 < \mu \le 2$ and to estimate the modulus of smoothness $w_2(f,n^{-1})_{\varphi,p}$ by the following

$$w_2(f,n^{-1})_{\varphi,p} = \sup_{0<h\le n^{-1}}\left\|\Delta_{2,h}^{(0,0)} f\right\|_{\varphi,p},$$

where

$$\Delta_{2,h}^{(0,0)} f(x) = f(x) - T_h^{(0,0)} f(x)$$

Thus, from **8** and **25**, the following estimation is valid,

$$\left\|f - T_n^{(0,0)}(f)\right\|_{C_{\varphi,p}^{2,\mu}([-1,1])} \asymp w_2(f,n^{-1})_{C_{\varphi,p}^{2,\mu}([-1,1])} \asymp \mathcal{O}(n^{-\eta}) \qquad \ldots \mathbf{26}$$

as $n \to +\infty$. Therefore, for $0 < \mu \le 2$, $0 < \eta < 2$, and by applying the theorems 1, 2 and 3; the inequality **26** as $n \uparrow +\infty$ is equivalent to the estimates,

$$E_n(f)_p = \mathcal{O}(n^{-(\eta+\mu)}) \text{ and }$$
$$E_n(f)_{C_{\varphi,p}^{2,\mu}([-1,1])} = \mathcal{O}(n^{-\eta}), \text{ as } n \to +\infty.$$

Conclusion:

The estimation of a good approximation to the Hölder functions with respect to the second order singular Sturm-Liouville operators has been developed. To satisfy a strong convergent, a special form of the weighted moduli of smoothness is used to apply for Fourier-Jacobi operators in expanding periodic Hölder spaces. In theorem 1, The upper estimate for the degree of approximation of weighted Hölder functions has considered. Also, the lower bound for the degree of approximation in both theorem 2 and theorem 3, are estimated. Moreover, some properties of weighted modulus of smoothness on Hölder spaces in two propositions 1,and 2, have been proved. Finally, the application is shown to satisfy the estimates of the degree of the best approximation.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Thi-Qar.

Authors' contributions statement:

H. A. ., A. H. K. and Kh. F. Al O. contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript. All authors discussed the results to the final manuscript.

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تقريب دوال هولدر المرجحة باستخدام متعددات حدود فوريه- جاكوبي لمؤثر ستورم- ليوفيل المنفرد

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الخلاصة:
تم مناقشة دوال هولدر المرجحة التي تحقق تقارب متعددات حدود جاكوبي لمعادلة ستورم- ليوفيل المنفردة من الدرجة الثانية. هذا يتوافق مع تحويلات جاكوبي المعممة ومعايير النعومة. يهدف هذا البحث ويركز على تحسين طرق التقريب وإيجاد أفضل تقريب على هذا النوع من الفضاءات عن طريق تحسين معايير النعومة. علاوة على ذلك، يتم النظر في بعض خواص معايير النعومة والقيود العليا والسفلى لدرجة تقريب الدالة.

الكلمات المفتاحية: أفضل تقريب، متعددات حدود جاكوبي، معيار النعومة، معادلة ستورم- ليوفيل، فضاءات هولدر المرجح.