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On Finitely Null-additive and Finitely Weakly Null-additive Relative to the σ−ring

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Abstract:

This article introduces the concept of finitely null-additive set function relative to the σ−ring and many properties of this concept have been discussed. Furthermore, to introduce and study the notion of finitely weakly null-additive set function relative to the σ−ring as a generalization of some concepts such as measure, countably additive, finitely additive, countably null-additive, countably weakly null-additive and finitely null-additive. As the first result, it has been proved that every finitely null-additive is a finitely weakly null-additive. Finally, the paper introduces a study of the concept of outer measure as a stronger form of finitely weakly null-additive.

Keywords: Countably weakly null-additive, Measure, Null-additive, σ−field, σ−ring**.**

Introduction:

The theory of measure is an important subject in mathematics. In 2015, Evans et al [1] discussed many details about the measure and proved some important results in measure theory. Let $\mathcal{U}$ be a nonempty set ,and let $\mathcal{P}(\mathcal{U})$ denoted to a power set of a nonempty set $\mathcal{U}$ and the difference of two sets D and B be denoted by $D\backslash B$ and defined as: $D\backslash B = D \cap B^c$. The notion of σ−field was studied by Ash [2] and Mackenzie [3] where a collection $\mathcal{K} \subseteq P(\mathcal{U})$ is called σ−field if and only if $\mathcal{U} \in \mathcal{K}$ and $\mathcal{K}$ is closed under complementation and countable union. Many other authors were interested in studying σ−field to define monotone measure and null-additive[4, 5]. The notion of σ−ring was studied by [6, 7] as a generalization of σ−field, where a collection $\mathcal{K} \subseteq \mathcal{P}(\mathcal{U})$ is called σ−ring if whenever $D_1, D_2, \ldots \in \mathcal{K}$, then $\bigcup_{i=1}^{\infty} D_i \in \mathcal{K}$ and for any $D, B \in \mathcal{K}$, then $D\backslash B \in \mathcal{K}$. It is clear that, every σ−field is σ−ring.The concept of measure was studied by [7, 8, 9] where a measure relative to the σ−ring $\mathcal{K}$ is a set function $\mathfrak{M}: \mathcal{K} \to [0, \infty]$ such that $\mathfrak{M}(\Phi) = 0$ and if $D_1, D_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{K}$, then $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$. The concept of countably additive was studied by [2, 5, 10] where a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is called countably additive relative to the σ−ring $\mathcal{K}$ if whenever $D_1, D_2, \ldots$ are finite or countably infinite collection of disjoint sets in $\mathcal{K}$, then $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$ and $\mathfrak{M}(\Phi) = 0$. If this requirement holds only for the finite collection of disjoint sets in $\mathcal{K}$, then $\mathfrak{M}$ is said to be finitely additive relative to the σ−ring $\mathcal{K}$. The concept of the outer measure studied by [11] where a set function $\mathfrak{M}: \mathcal{P}(\mathcal{U}) \to [0, \infty]$ is called the outer measure if $\mathfrak{M}(\Phi) = 0$ and if $D, B \subseteq \mathcal{U}$ such that $D \subset B$, then $\mathfrak{M}(D) \leq \mathfrak{M}(B)$ and if $D_1, D_2, \ldots$ are subsets of $\mathcal{U}$, then $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) \leq \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$. In 2001 Pap [12] studied the notion of null-additive relative to the σ−ring $\mathcal{K}$ where a null-additive relative to the σ−ring $\mathcal{K}$ is a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ such that whenever E, D are disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(E) = 0$, we have $\mathfrak{M}(E \cup D) = \mathfrak{M}(D)$. In 2002 Pap [13] introduced the countably null-additive on σ−ring, where a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is called countably null-additive relative to the σ−ring $\mathcal{K}$ if whenever $D_1, D_2, \ldots$ are a collection of disjoint sets in $\mathcal{K}$ and $B \in \mathcal{K}$ such that $B \cap D_i = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2, \ldots,$ then $\mathfrak{M}(B \cup \bigcup_{n=1}^{\infty} D_n) = \mathfrak{M}(B)$. Mesiar et al [14] in 2014 introduced the notion of countably weakly null-additive as a generalization of the concept of countably null-additive, where set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is said to be countably weakly null-additive relative to the σ-ring $\mathcal{K}$ if whenever $D_1, D_2, \ldots$ are a collection of disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots$, then $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = 0$.

This paper is a generalization of the concepts of countably null-additive and countably weakly null-additive was introduced, also the concept of finitely null-additive and finitely weakly null-additive were studied respectively.

Finitely Null-additive Relative to the σ−ring

This section, aims to introduce the concept of finitely null-additive relative to the σ−ring and investigated some of its basic properties. Furthermore, the section aims to present the relationships between finitely null-additive and countably null-additive.

Definition 1

A set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is called a finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$, if whenever $D_1, D_2, \ldots D_k$ are a collection of disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ such that $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$, then $\mathfrak{M}(B \cup \bigcup_{n=1}^{k} D_n) = \mathfrak{M}(B)$.

Example 1

Let $\mathcal{U} = \{1, 2, 3\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if\ D = \Phi \text{ or } \{1\} \text{ or } \{2\} \\ 1 & otherwise \end{cases}$$

Then $\mathfrak{M}$ is a finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$.

Example 2

Let $\mathcal{U} = \{1, 2, 3\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if\ D = \Phi \text{ or } \{1\} \text{ or } \{2\} \\ 1 & if\ D = \{3\} \\ 5 & otherwise \end{cases}$$

Assume $D_1 = \{1\}$ and $D_2 = \{2\}$, then D_1, D_2 are disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2$. Consider $B = \{3\}$, then $B \cap D_n = \Phi\ \forall n = 1,2$. Now, since $\mathfrak{M}(B \cup \bigcup_{n=1}^{2} D_n) = \mathfrak{M}(\mathcal{U}) = 5$ and $\mathfrak{M}(B) = 1$,

then $\mathfrak{M}(B \cup \bigcup_{n=1}^{2} D_n) \neq \mathfrak{M}(B)$, thus $\mathfrak{M}$ is not finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$.

The following theorem used mathematical induction to prove that the linear combination of finitely null-additive relative to the σ-ring $\mathcal{K}$ is also finitely null-additive relative to the σ-ring $\mathcal{K}$.

Theorem 1

Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_m: \mathcal{K} \to [-\infty, \infty]$ be a finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$ and $c_j \in [0, \infty)$ for all $j = 1,2,\ldots,m$. If a set function $\sum_{j=1}^{m} c_j \mathfrak{M}_j: \mathcal{K} \to [-\infty, \infty]$ is defined by: $(\sum_{j=1}^{m} c_j \mathfrak{M}_j)(D) = \sum_{j=1}^{m} c_j . \mathfrak{M}_j(D)\ \forall D \epsilon \mathcal{K}$, then $\sum_{j=1}^{m} c_j \mathfrak{M}_j$ is a finitely null-additive relative to the σ-ring $\mathcal{K}$.

Proof:

To prove that the statement is true when $m = 2$, let $D_1, D_2, \ldots, D_k$ be disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ such that $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$,. Then, it is proved that $(\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B \cup \bigcup_{n=1}^{k} D_n) = (\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B)$. Since $\mathfrak{M}_j$ is finitely null-additive relative to the σ-ring $\mathcal{K}$, $j = 1,2$. Then $\mathfrak{M}_j(B \cup \bigcup_{n=1}^{k} D_n) = \mathfrak{M}_j(B)$. Therefore

$$(c_1\mathfrak{M}_1 + c_2\mathfrak{M}_2)(B \cup \bigcup_{n=1}^{k} D_n) = c_1 . \mathfrak{M}_1(B \cup \bigcup_{n=1}^{k} D_n) + c_2 . \mathfrak{M}_2(B \cup \bigcup_{n=1}^{k} D_n)$$
$$= c_1 . \mathfrak{M}_1(B) + c_2 . \mathfrak{M}_2(B)$$
$$= (c_1\mathfrak{M}_1 + c_2\mathfrak{M}_2)(B)$$

Now,

$(\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B \cup \bigcup_{n=1}^{k} D_n) = (\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B)$
where $j = 1,2$

Hence, $(c_1\mathfrak{M}_1 + c_2\mathfrak{M}_2)$ is finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$.

Now, assume that the statement is true when $m = t$ and prove that the statement is true when $m = t + 1$, that is, assume that $\sum_{j=1}^{t} c_j \mathfrak{M}_j$ is finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$ that is $(\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B \cup \bigcup_{n=1}^{k} D_n) = (\sum_{j=1}^{m} c_j \mathfrak{M}_j)(B)$ where $j = 1,2,\ldots,t$, and we prove that $\sum_{j=1}^{t+1} c_j \mathfrak{M}_j$ is finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$. Let $\mathfrak{M}_j$ be a finitely null-additive relative to the σ-ring $\mathcal{K}$ of a set $\mathcal{U}$ and $c_j \in [0, \infty)$ for all $j = 1,2,\ldots,t,t+1$, and let $D_1, D_2, \ldots, D_k$ be disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ such that $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$,. Since $\sum_{j=1}^{t} c_j \mathfrak{M}_j$ is finitely null-additive, then $\sum_{j=1}^{t} c_j \mathfrak{M}_j(B \cup \bigcup_{n=1}^{k} D_n) = (\sum_{j=1}^{t} c_j \mathfrak{M}_j)(B)$. This implies that

$$(\sum_{j=1}^{t+1} c_j \mathfrak{M}_j)(B \cup \bigcup_{n=1}^{k} D_n) = (\sum_{j=1}^{t} c_j \mathfrak{M}_j + c_{t+1}\mathfrak{M}_{t+1})(B \cup \bigcup_{n=1}^{k} D_n)$$
$$= \sum_{j=1}^{t} c_j . \mathfrak{M}_j(B \cup \bigcup_{n=1}^{k} D_n) + c_{t+1} . \mathfrak{M}_{t+1}(B \cup \bigcup_{n=1}^{k} D_n)$$
$$= (\sum_{j=1}^{t} c_j \mathfrak{M}_j)(B \cup \bigcup_{n=1}^{k} D_n) + c_{t+1} . \mathfrak{M}_{t+1}(B \cup \bigcup_{n=1}^{k} D_n)$$
$$= (\sum_{j=1}^{t} c_j \mathfrak{M}_j)(B) + c_{t+1} . \mathfrak{M}_{t+1}(B)$$

since $\sum_{j=1}^{t} c_j \mathfrak{M}_j)$ and $\mathfrak{M}_{t+1}$ are finitely null-additive

$= (\sum_{j=1}^{t} c_j \mathfrak{M}_j)(B) + (c_{t+1}\mathfrak{M}_{t+1})(B)$

$= (\sum_{j=1}^{t+1} c_j \mathfrak{M}_j)(B)$

Hence, $\sum_{j=1}^{t+1} c_j \mathfrak{M}_j$ is finitely null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$.
Therefore, $\sum_{j=1}^{m} c_j \mathfrak{M}_j)$ is finitely null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$.

Proposition 1

Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be a countably null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$ and $\mathfrak{M}_1(\Phi) = \mathfrak{M}_2(\Phi) = 0$. Assume $\mathfrak{M}_1$ or $\mathfrak{M}_2$ is finite and define $\mathfrak{M}_1 - \mathfrak{M}_2: \mathcal{K} \to [-\infty, \infty]$ by:
$(\mathfrak{M}_1 - \mathfrak{M}_2)(D) = \mathfrak{M}_1(D) - \mathfrak{M}_2(D), \forall D \in \mathcal{K}$, then $(\mathfrak{M}_1 - \mathfrak{M}_2)$ is a finitely null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$.

Proof:

Let $D_1, D_2, \ldots D_k$ be collection of disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ such that $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$, consider $D_n = \Phi$ for all $n > k$. then
$\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n$ for all $n > k$. Now, since $\mathfrak{M}_i, i = 1,2$ is countably null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$, then $\mathfrak{M}_i(B \cup \bigcup_{n=1}^{\infty} D_n) = \mathfrak{M}_i(B)$. Hence
$(\mathfrak{M}_1 - \mathfrak{M}_2)\left(B \cup \bigcup_{n=1}^{k} D_n\right) = (\mathfrak{M}_1 - \mathfrak{M}_2)(B \cup \bigcup_{n=1}^{\infty} D_n) = \mathfrak{M}_1(B \cup \bigcup_{n=1}^{\infty} D_n) - \mathfrak{M}_2(B \cup \bigcup_{n=1}^{\infty} D_n) = \mathfrak{M}_1(B) - \mathfrak{M}_2(B) = (\mathfrak{M}_1 - \mathfrak{M}_2)(B)$
Therefore $(\mathfrak{M}_1 - \mathfrak{M}_2)$ is a finitely null-additive relative to theσ− ring $\mathcal{K}$.

In the following propositions, the relationships among the countably null-additive, finitely null-additive and null-additive are giving.

Proposition 2

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a countably null-additive relative to the σ−ring $\mathcal{K}$ such that $\mathfrak{M}(\Phi) = 0$. Then $\mathfrak{M}$ is a finitely null-additive relative to the σ−ring $\mathcal{K}$.

Proof:

Let $\mathfrak{M}$ be a countably null-additive relative to the σ−ring $\mathcal{K}$ and let $D_1, D_2, \ldots D_k$ be a collection of disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ such that $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$ and consider $D_n = \Phi$, for all $n > k$, then $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n$ for all $n > k$ and $(D_n) = 0, \forall n$. Hence, $\mathfrak{M}\left(B \cup \bigcup_{n=1}^{k} D_n\right) = \mathfrak{M}(B \cup \bigcup_{n=1}^{\infty} D_n)$

$= \mathfrak{M}(B)$ since $\mathfrak{M}$ is a countably null-additive.
Therefore $\mathfrak{M}$ is a finitely null-additive.

Proposition 3

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a finitely null-additive relative to the σ−ring $\mathcal{K}$ such that $\mathfrak{M}(\Phi) = 0$. Then $\mathfrak{M}$ is a null-additive relative to the σ−ring $\mathcal{K}$.

Proof:

Let B, C be disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(C) = 0$ and let $\mathfrak{M}$ be a countably null-additive relative to the σ−ring $\mathcal{K}$. Consider $C = D_1$ and $D_n = \Phi$, $\forall n = 2,3,\ldots,k$, then $B \cup C = B \cup \bigcup_{n=1}^{k} D_n$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$. Hence $\mathfrak{M}(B \cup C) = \mathfrak{M}\left(B \cup \bigcup_{n=1}^{k} D_n\right)$
$= \mathfrak{M}(B)$ since $\mathfrak{M}$ is a finitely null-additive and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\ldots,k$.
Therefore $\mathfrak{M}$ is a null-additive.

Definition 2 [1, 2]

Let $D_1, D_2, \ldots$ be subsets of a set $\mathcal{U}$, if $D_1 \subset D_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} D_i = D$, then D_i is called increase to D; and write $D_n \uparrow D$.

Definition 3 [3]

Let $D_1, D_2, \ldots \epsilon \mathcal{K}$ and D= $\bigcup_{n=1}^{\infty} D_n$ such that $D_n \uparrow D$, if $\mathfrak{M}(D_n) \to \mathfrak{M}(D)$, then a set function $\mathfrak{M}$ is called continuous from below at D.

Theorem 2

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a continuous from below at D and $\mathfrak{M}(\Phi) = 0$. Then $\mathfrak{M}$ is a countably null-additive relative to the σ−ring if and only if $\mathfrak{M}$ is a finitely null-additive relative to the σ−ring $\mathcal{K}$.

Proof:

⟹) direct from Proposition 2.

Conversely)

Let $\mathfrak{M}$ be a continuous from below at D and $\mathfrak{M}$ is a finitely null-additive relative to the σ−ring $\mathcal{K}$. Assume that $D_1, D_2, \ldots$ be disjoint sets in $\mathcal{K}$ and $B \epsilon \mathcal{K}$ with $B \cap D_n = \Phi$ and $\mathfrak{M}(D_n) = 0$, $\forall n = 1,2,\ldots$ and let D= $B \cup \bigcup_{n=1}^{\infty} D_n$. If $D_k = B \cup \bigcup_{n=1}^{k} D_n$, then $D_k \uparrow D$, since $\mathfrak{M}$ continuous from below at D, then $\mathfrak{M}(D_k) \to \mathfrak{M}(D)$. But $\mathfrak{M}$ finitely null-additive, then $\mathfrak{M}(D_k) = \mathfrak{M}(B \cup \bigcup_{n=1}^{k} D_n) = \mathfrak{M}(B)$. Thus
$\mathfrak{M}(D) = \lim_{k\to\infty} \mathfrak{M}(D_k)$
$= \lim_{k\to\infty} \mathfrak{M}\left(B \cup \bigcup_{n=1}^{k} D_n\right) = \mathfrak{M}(B)$.
Hence $\mathfrak{M}(B \cup \bigcup_{n=1}^{\infty} D_n) = \mathfrak{M}(B)$, therefore $\mathfrak{M}$ is countably null-additive.

1. Finitely Weakly Null-additive Relative to the σ−ring

This section introduces and studies the concept of finitely weakly null-additive relative to the σ−ring and basic properties of this concept are giving. Furthermore, it presents the relationships between finitely weakly null-additive, finitely null-additive, measure, countably additive, finitely additive, outer measure, countably weakly null-additive and countably null-additive.

Definition 4

A set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is called finitely weakly null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$, if whenever $D_1, D_2, \ldots D_k$ are collection of disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$, then $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = 0$.

Example 3

Let $\mathcal{U} = \{1,2,3\}$ and $\mathcal{K} = \{\Phi, \{1\}, \{2\}, \{1,2\}\}$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by: $\mathfrak{M}(D) = 0$ for all $D \in \mathcal{K}$. Then $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Example 4

Let $\mathcal{U} = \{a,b,c\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if\ D = \{a\} \text{ or } \{b\}\ or\ \{c\} \\ 1 & otherwise \end{cases}$$

Put $D_1 = \{a\}$, $D_2 = \{b\}$ and $D_3 = \{c\}$, then D_1, D_2, D_3 are disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,3$. Now, since $\mathfrak{M}(\bigcup_{n=1}^{3} D_n) = \mathfrak{M}(\mathcal{U}) = 1 \neq 0$, then $\mathfrak{M}$ is not finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Proposition 4

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ and $c \in (0, \infty)$. If a set function $c\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is defined by:

$(c\mathfrak{M})(D) = c\,.\,[\mathfrak{M}(D)]\ \forall D \in \mathcal{K}$, then $c\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Proof:

Let $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ and $(c\mathfrak{M})(D_n) = 0, \forall n = 1,2,\dots,k$. Then $c\,.\,[\mathfrak{M}(D_n)] = 0, \forall n = 1,2,\dots,k$. Since $c > 0$, then $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Now, finitely weakly null-additive and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$, then $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = 0$.

Hence, $(c\mathfrak{M})(\bigcup_{n=1}^{k} D_n) = c\,.\,[\mathfrak{M}(\bigcup_{n=1}^{k} D_n)] = c\,.0 = 0$ and $\mathfrak{M}$ is a finitely weakly null-additive.

Proposition 5

Let $(\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_m): \mathcal{K} \to [-\infty, \infty]$ be a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$. If a set function $\sum_{j=1}^{m} \mathfrak{M}_j: \mathcal{K} \to [-\infty, \infty]$ is defined by: $[\sum_{j=1}^{m} \mathfrak{M}_j](D) = \sum_{j=1}^{m} \mathfrak{M}_j(D)\ \forall D \in \mathcal{K}$, then $\sum_{j=1}^{m} \mathfrak{M}_j$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$.

Proof:

To prove that the statement is true when $m = 2$. Let $D_1, D_2, \dots, D_k$ are disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k,$. Then proved that $[\sum_{j=1}^{m} \mathfrak{M}_j](\bigcup_{n=1}^{k} D_n) = 0$. Since $\mathfrak{M}_j$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$, $j = 1,2$. Then $\mathfrak{M}_j(\bigcup_{n=1}^{k} D_n) = 0$. So,

$$[\mathfrak{M}_1 + \mathfrak{M}_2](\bigcup_{n=1}^{k} D_n) = \mathfrak{M}_1(\bigcup_{n=1}^{k} D_n) + \mathfrak{M}_2(\bigcup_{n=1}^{k} D_n) = 0 + 0 = 0$$

Hence, $(\mathfrak{M}_1 + \mathfrak{M}_2)$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Now, assume that that the statement is true when $m = t$ and it is proved that the statement is true when $m = t + 1$, that is, assume that $\sum_{j=1}^{t} \mathfrak{M}_j$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ that is $[\sum_{j=1}^{m} \mathfrak{M}_j](\bigcup_{n=1}^{k} D_n) = 0$ where $m = t$ and it is proved that $\sum_{j=1}^{t+1} \mathfrak{M}_j$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$. Let $\mathfrak{M}_j$ be a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ for all $j = 1,2,\dots,t,t+1$, and let $D_1, D_2, \dots, D_k$ be disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k,$. Since $\sum_{j=1}^{t} \mathfrak{M}_j$ is finitely weakly null-additive, then

$[\sum_{j=1}^{m} \mathfrak{M}_j](\bigcup_{n=1}^{k} D_n) = 0$. Implies that

$$[\sum_{j=1}^{t+1} \mathfrak{M}_j](\bigcup_{n=1}^{k} D_n) = [\sum_{j=1}^{t} \mathfrak{M}_j + \mathfrak{M}_{t+1}](\bigcup_{n=1}^{k} D_n)$$
$$= \sum_{j=1}^{t} \mathfrak{M}_j(\bigcup_{n=1}^{k} D_n) + \mathfrak{M}_{t+1}(\bigcup_{n=1}^{k} D_n)$$
$$= 0 + 0 = 0$$

since $\sum_{j=1}^{t} \mathfrak{M}_j$) and $\mathfrak{M}_{t+1}$ are finitely weakly null-additive

Hence, $\sum_{j=1}^{t+1} \mathfrak{M}_j$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Therefore $\sum_{j=1}^{m} \mathfrak{M}_j$) is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Theorem 3

Let $(\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_m): \mathcal{K} \to [-\infty, \infty]$ be a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ and $c_j \in (0, \infty)$ for all $j = 1,2,\dots,m$. If a set function $\sum_{j=1}^{m} c_j\mathfrak{M}_j: \mathcal{K} \to [0, \infty]$ is defined by:

$[\sum_{j=1}^{m} c_j\mathfrak{M}_j](D) = \sum_{j=1}^{m} c_j\,.\,\mathfrak{M}_j(D)\ \forall D \in \mathcal{K}$, then $\sum_{j=1}^{m} c_j\mathfrak{M}_j$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$.

Proof:

The result follows from Proposition 4 and Proposition 5.

Proposition 6

Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ and $\mathfrak{M}_1(\Phi) = \mathfrak{M}_2(\Phi) = 0$. Define $\mathfrak{M}_1 - \mathfrak{M}_2: \mathcal{K} \to [-\infty, \infty]$ by:

$(\mathfrak{M}_1 - \mathfrak{M}_2)(D) = \mathfrak{M}_1(D) - \mathfrak{M}_2(D), \forall D \in \mathcal{K}$, then $\mathfrak{M}_1 - \mathfrak{M}_2$ is a finitely null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

Proof:

Let $D_1, D_2, \dots D_k$ are collection of disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Since $\mathfrak{M}_i, i = 1,2$ is finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$, then $\mathfrak{M}_i(\bigcup_{n=1}^{k} D_n) = 0$). Hence

$(\mathfrak{M}_1 - \mathfrak{M}_2)(\bigcup_{n=1}^{k} D_n) = (\mathfrak{M}_1 - \mathfrak{M}_2)(\bigcup_{n=1}^{k} D_n)$
$= \mathfrak{M}_1(\bigcup_{n=1}^{k} D_n) - \mathfrak{M}_2(\bigcup_{n=1}^{k} D_n) = 0$
Therefore $\mathfrak{M}_1 - \mathfrak{M}_2$ is a finitely weakly null-additive relative to the σ− ring $\mathcal{K}$.

Proposition 7

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a finitely null-additive relative to the σ−ring $\mathcal{K}$ such that $\mathfrak{M}(\Phi) = 0$. Then $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$.

Proof:

Let $D_1, D_2, \dots D_k$ be a disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Consider $D_{k+1} = \Phi$ and $C_n = D_{n+1}$ $\forall n = 1,2,\dots,k$ Then $\mathfrak{M}(C_n) = 0$ and

$\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \mathfrak{M}(D_1 \cup \bigcup_{n=1}^{k} C_n)$
$= \mathfrak{M}(D_1)$ since $\mathfrak{M}$ is finitely null-additive
$= 0$

Hence $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$.

Theorem 4

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a continuous from below at D and $\mathfrak{M}(\Phi) = 0$. Then $\mathfrak{M}$ is a countably weakly null-additive relative to the σ−ring if and only if $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$.

Proof:

⟹) direct from Proposition 7.

Conversely)

Let $\mathfrak{M}$ be a continuous from below at D and $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$. Assume that $D_1, D_2, \dots$ be disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0$, $\forall n = 1,2,\dots$ and let D= $\bigcup_{n=1}^{\infty} D_n$. If $D_k = \bigcup_{n=1}^{k} D_n$, then $D_k \uparrow D$, since $\mathfrak{M}$ continuous from below at D, then $\mathfrak{M}(D_k) \to \mathfrak{M}(D)$. But $\mathfrak{M}$ is a finitely weakly null-additive, then $\mathfrak{M}(D_k) = \mathfrak{M}(\bigcup_{n=1}^{k} D_n) = 0$. So, we have,

$\mathfrak{M}(D) = \lim_{k\to\infty} \mathfrak{M}(D_k)$
$= \lim_{k\to\infty} \mathfrak{M}\left(\bigcup_{n=1}^{k} D_n\right) = 0.$

Hence, $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = 0$, therefore $\mathfrak{M}$ is countably weakly null-additive.

Proposition 8

Every finitely additive relative to the σ−ring is a finitely weakly null-additive relative to the σ−ring.

Proof:

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a finitely additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$ and $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Then $\sum_{n=1}^{k} \mathfrak{M}(D_n) = 0$ and $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \sum_{n=1}^{k} \mathfrak{M}(D_n)$ since $\mathfrak{M}$ is finitely additive
$= 0$

Hence, $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$.

The converse of proposition 8 is not true as showing in the following example.

Example 5

Let $\mathcal{U} = \{1,2,3\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if\ D = \Phi \text{ or } \{1\} \\ 1 & otherwise \end{cases}$$

Then $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$, but not finitely additive**.**

Definition 5 [1, 2]

Let $D_1, D_2, \dots$ be subsets of a set $\mathcal{U}$, if $D_1 \supset D_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} D_i = D$, then D_i decrease to D; and write $D_n \downarrow D$.

Definition 6 [3]

Let $D_1, D_2, \dots \in \mathcal{K}$ and $D_n \downarrow \Phi$, if $\mathfrak{M}(D_n) \to 0$ then a set function $\mathfrak{M}$ is called continuous from above at Φ.

Theorem 5

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is finitely additive relative to the σ−ring $\mathcal{K}$, if $\mathfrak{M}$ is continuous from above at Φ, then it's countably weakly null-additive relative to the σ−ring.

Proof:

Let $D_1, D_2, \dots$ be disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots$ and let D= $\bigcup_{n=1}^{\infty} D_n$. If $D_k = \bigcup_{n=1}^{k} D_n$. Now, D = $D_k \cup D_k^{c}$ = $D_k \cup (D \cap D_k^{c}) = D_k \cup (D \backslash D_k)$. Since $\mathfrak{M}$ is finitely additive, then $\mathfrak{M}(D) = \mathfrak{M}(D_k) + \mathfrak{M}(D \backslash D_k)$, but $(D \backslash D_k) \downarrow \Phi$ and $\mathfrak{M}$ is continuous from above at Φ, thus $\mathfrak{M}(D \backslash D_n) \to 0$ and hence $\mathfrak{M}(D_k) \to \mathfrak{M}(D)$. Since $\mathfrak{M}$ is finitely additive, then from proposition 8 implies that $\mathfrak{M}$ is finitely weakly null-additive. Hence $\mathfrak{M}(D_k) = \mathfrak{M}(\bigcup_{n=1}^{k} D_n) = 0$.

but $\mathfrak{M}(D) = \lim_{k\to\infty} \mathfrak{M}(D_k) = \lim_{k\to\infty} \mathfrak{M}\left(\bigcup_{n=1}^{k} D_n\right) = 0)$, so $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = 0$, therefore $\mathfrak{M}$ is countably weakly null-additive relative to the σ−ring.

The converse of above theorem is not true as showing in the following example.

Example 6

Let $\mathcal{U} = \{a, b, c\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if\ D = \Phi \text{ or } \{b\} \\ 1 & otherwise \end{cases}$$

Then $\mathfrak{M}$ is a countably weakly null-additive relative to the σ−ring $\mathcal{K}$ of a set $\mathcal{U}$, but not finitely additive**.**

Proposition 9

Let $\mathcal{K}$ be a σ−ring of a set $\mathcal{U}$ and $\mathfrak{M}: \mathcal{K} \to [0, \infty]$ be a set function relative to the σ−ring $\mathcal{K}$.

1- If $\mathfrak{M}$ is measure, then $\mathfrak{M}$ is finitely weakly null-additive.

2- If $\mathfrak{M}$ is an outer measure, then $\mathfrak{M}$ is finitely weakly null-additive.

Proof:

1- Let $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Then $\sum_{n=1}^{k} \mathfrak{M}(D_n) = 0$. Hence $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \sum_{n=1}^{k} \mathfrak{M}(D_n)$ since $\mathfrak{M}$ is measure $= 0$

Therefore, $\mathfrak{M}$ is finitely weakly null-additive.

2- Let $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ and $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Consider $D_n = \Phi, \forall n > k$, then $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n \; \forall n > k$ and $\mathfrak{M}(D_n) = 0, \forall n$. Hence $\sum_{n=1}^{\infty} \mathfrak{M}(D_n) = 0$. Since $\mathfrak{M}$ is an outer measure, then $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) \le \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$. Therefore,

$\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) \le \sum_{n=1}^{\infty} \mathfrak{M}(D_n) = 0$.

Now, let $D_n = \Phi, \forall n > k$ since $D_n \subseteq \bigcup_{n=1}^{\infty} D_n \; \forall n = 1,2,\dots$ and $\mathfrak{M}$ is an outer measure, then $\mathfrak{M}(D_n) \le \mathfrak{M}(\bigcup_{n=1}^{\infty} D_n)$. But $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n \; \forall n > k$ and $\mathfrak{M}(D_n) = 0, \forall n$. Thus $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) \ge 0$. Hence $\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = 0$. Therefore, $\mathfrak{M}$ is finitely weakly null-additive.

Proposition 10

Every countably additive relative to the σ–ring is a finitely weakly null-additive relative to the σ–ring.

Proof:

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a countably additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ and $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Consider $D_n = \Phi, \forall n > k$, then $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n \; \forall n > k$ and $\mathfrak{M}(D_n) = 0, \forall n$. Hence, $\sum_{n=1}^{\infty} \mathfrak{M}(D_n) = 0$ and

$\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \mathfrak{M}(\bigcup_{n=1}^{\infty} D_n)$
$= \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$ since $\mathfrak{M}$ is countably additive
$= 0$

Therefore, $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

The converse of proposition 10 is not true as showing in the following example.

Example 7

Let $\mathcal{U} = \{1, 2, 3\}$ and $\mathcal{K} = P(\mathcal{U})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if \; D = \Phi \text{ or } \{1\} \\ 5 & otherwise \end{cases}$$

Then $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$, but not countably additive**.**

In the end of this section we give the relation between finitely weakly null-additive and countably weakly null-additive in following proposition.

Proposition 11

Let $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ be a set function relative to the σ–ring $\mathcal{K}$ such that $\mathfrak{M}(\Phi) = 0$. Then every countably weakly null-additive relative to the σ–ring is a finitely weakly null-additive relative to the σ–ring.

Proof:

Let $\mathfrak{M}$ be a countably weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$ and $D_1, D_2, \dots D_k$ be a collection of disjoint sets in $\mathcal{K}$ such that $\mathfrak{M}(D_n) = 0, \forall n = 1,2,\dots,k$. Consider $D_n = \Phi, \forall n > k$, then $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{k} D_n \; \forall n > k$ and $\mathfrak{M}(D_n) = 0, \forall n$. Hence $\sum_{n=1}^{\infty} \mathfrak{M}(D_n) = 0$. Which implies that to,

$\mathfrak{M}(\bigcup_{n=1}^{k} D_n) = \mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = 0$ since $\mathfrak{M}$ is countably weakly null-additive

Therefore, $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$.

The converse of proposition 11 is not true as shown in the following example.

Example 8

Let $\mathcal{U} = \mathbb{N}$ and $\mathcal{K} = P(\mathbb{N})$. Define a set function $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ by:

$$\mathfrak{M}(D) = \begin{cases} 0 & if \; D = \Phi \text{ or } D = \{n\}, \; n \in \mathbb{N} \\ 1 & otherwise \end{cases}$$

Then $\mathfrak{M}$ is a finitely weakly null-additive relative to the σ–ring $\mathcal{K}$ of a set $\mathcal{U}$, but not countably weakly null-additive**.**

Conclusions:

In this article, the concepts of finitely null-additive and finitely weakly null-additive have been introduced as a generalization of countably null-additive and countably weakly null-additive respectively and some properties of these concepts have been discussed such as the linear combination of finitely null-additive relative to the σ–ring $\mathcal{K}$ is a finitely null-additive. Every countably null-additive relative to the σ–ring $\mathcal{K}$ is a finitely null-additive. Every finitely null-additive relative to the σ–ring $\mathcal{K}$ is a null-additive. If $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is continuous from below at D and $\mathfrak{M}(\Phi) = 0$, then $\mathfrak{M}$ is a countably null-additive if and only if $\mathfrak{M}$ is a finitely null-additive. Every finitely null-additive relative to the σ–ring $\mathcal{K}$ is a finitely weakly null-additive. If $\mathfrak{M}: \mathcal{K} \to [-\infty, \infty]$ is continuous from below at D and $\mathfrak{M}(\Phi) = 0$, then $\mathfrak{M}$ is a countably weakly null-additive if and only if $\mathfrak{M}$ is a finitely weakly null-additive. Every countably additive is a finitely weakly null-additive. Every countably weakly null-additive is a finitely weakly null-additive.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Tikrit University.

Authors' contributions statement:

H. H. E., conceived of the presented and plan idea of this work and introduced the definition of finitely null-additive and finitely weakly null-additive. S. H. A., developed the theory and performed the computations and the results of this paper. I. S. A., introduced the examples and conclusion of this work. All authors written the introduction and references and discussed the results and contributed to the final manuscript.

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حول المضافات الفارغة المنتهية و المضافات الفارغة الضعيفة المنتهية نسبةً الى الحلقة من النمط σ–

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الخلاصة:

يهدف هذا البحث الى تقديم مفهوم المضافات الفارغة المنتهية نسبةً الى الحلقة من النمط σ– ومناقشة العديد من الخصائص لهذا المفهوم. اضافةً الى ذلك قدمنا ودرسنا مفهوم المضافات الفارغة الضعيفة المنتهية نسبةً الى الحلقة من النمط σ– وهي أعم من بعض المفاهيم كالقياس والمضافات المعدودة والمضافات المنتهية والمضافات الفارغة المعدودة والمضافات الفارغة الضعيفة المعدودة والمضافات الفارغة المنتهية و على هذا الاساس برهنا ان كل المضاف الفارغ المنتهي يؤدي الى المضاف الفارغ الضعيف المنتهي . واخيراً درسنا مفهوم القياس الخارجي إذ يكون اقوى من مفهوم المضافات الفارغة الضعيفة المنتهية.

الكلمات المفتاحية: المضاف الفارغ الضعيف المعدود، القياس، المضاف الفارغ، الحقل من النمط σ– ، الحلقة من النمط σ–