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Some Results on Fixed Points for Monotone Inward Mappings in Geodesic Spaces

Khalid Abed Jassim

Applied mathematics Department, College of Science, University of Anbar, Anbar, Iraq.
E-mail: edw.khalidmathmatic@uoanbarqu.edu.iq .
ORCID ID: https://orcid.org/0000-0002-2712-6821 .

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Abstract:

In this article, the partially ordered relation is constructed in geodesic spaces by betweeness property, A monotone sequence is generated in the domain of monotone inward mapping, a monotone inward contraction mapping is a monotone Caristi inward mapping is proved, the general fixed points for such mapping is discussed and A mutlivalued version of these results is also introduced.

Keywords: Contraction mapping, Fixed points, Geodesic metric spaces, Lower semi-continuous function, Monotone Caristi inward mapping, Multivalued mapping.

Introduction:

Banach fixed point theorem assures the existence and uniqueness of a fixed point for contraction mapping $T: M \longrightarrow M$ in complete metric spaces. In fact, it has many generalization and extention, these extention have been done in many directions. For instance, linear spaces (Banach and Hilbert spaces) or nonlinear spaces (geodesic metric spaces, hyperbolic and CAT(0) spaces). Also generalization includes many types of mappings, namely: weakly contraction mapping, nonexpansive mapping, locally nonexpansive mapping, ect.

In the inward case, when mapping can take values out of its domain, that is, mapping is called non-self mappings, Banach contraction theorem cannot work in this issue Caristi showed[1] when domain of mapping $T: N \longrightarrow M, N \subseteq M$ is closed and convex subset of Banach space and $T: N \longrightarrow M$ is weakly inward mapping with Lipcshitz constant less than one, then$T: Y \longrightarrow X$ has a fixed point in N. This was a very characterized work of caritsi that $T(v)$ is no longer be restricted to stay in N.

A mapping $T: N \longrightarrow M$ is said to be inward if for every $v \in N$ the image $T(v)$ belongs to the set

$$I_{N(v)} = \{u \in X: (1-\lambda)v + \lambda u, \lambda \geq 1$$

The set $I_{N(v)}$is called inward set of N at v and the mapping $T: N \longrightarrow M$is called inward mapping if $T(v)$ belongs to $I_{N(v)}$ also is called weakly inward mapping if $T(v)$ belongs to $\overline{I_{N(v)}}$ closure of $I_{N(v)}$for each $v \in N$.

If $T: N \longrightarrow M$ is an inward mapping then for each $v \in N$ with $v \neq T(v)$ then there exist $u \in N$ and $v \neq u$ which between v and $T(v)$such that $d(v,u) + d(u,T(v) = d(v,T(v)$, the last idea is used to define monotone Caritsi mapping. Moreover, the idea of betweenness in geodesic spaces is used to define the partially ordered relation on geodesic metric spaces.

Lim[2] extended these results to case of multivalued mappings, some results extended to geodesic spaces in multivalued case see[3], the weakly contractive case was discussed in[4]. Recently, the fixed point theory for monotone was initiated by Reurings and Ran[5], the main result discovered was an extension of the Banach's fixed point to metric spaces endowed with a partial ordered, since many mathematicians got interested into the study of monotone Lipschitzain mappings[6].

Banach's proof depends on constructing a Cauchy sequence by iteration $v_{n+1} = T(v_n)$ was called Picard's iteration and proves it converges to the fixed point, while Caristi[1] introduced the lower semi-continuous function

$$\psi: M \longrightarrow [0,\infty) \text{ such that } d\big(v,T(v)\big) \leq \psi(v) - \psi(T(v))$$

Which confirms that the values of function $\psi\,\psi$ are real and bounded below by zero. Moreover its

infimum stands for the fixed point of mapping $T: M \longrightarrow M$.

But the weakness of both theorems was their acting restricted only on self mapping $T: M \longrightarrow M$. Caristi[1] introduced the metrically inward concept to guarantee the existence of fixed point for non-self mapping $.T: N \longrightarrow M$,where $N \subseteq M$, this condition is weaker than Kirk-Assad[7] condition that required T maps ∂N to N (∂N is boundary points of N). In addition, The significant characterization of this theorem found the fixed point for contraction mapping without using iteration to get fixed point.

After that, the inwardness concept extended to multivalued mapping by Hong-Kun[8],for more results on inwardness see for instance [9,10].

Finally, The direction of monotone Lipschitzain mapping was started by Reurings and Ran[11]and the idea to define partially ordered relation on metric space**.** and use monotone Lipschitzain mapping to generate bounded monotone sequence converging to the fixed point was developed by[6, 12–14].

In this article, a partially ordered relation is defined on geodesic spaces to enable us to find the minimum value for Caritsi mapping which will be the fixed point. Moreover, the existence of fixed points of inward and monotone mappings is investigated. In particular, monotone Caristi inward mappings are introduced, which are more general than monotone inward contractions. In partially ordered metric space framework, a monotone Caristi inward mapping that has a fixed point is proved. A multivalued version of this result is also discussed in geodesic spaces.

Also, algorithm is introduced analogs to [6,9] to generate monotone sequence in the domain of monotone Caristi weakly inward mapping; furthermore, the image of this sequence under lower semi-continuous is monotone decreasing real sequence bounded below by zero and it has infimum value. The cornerstone of our study that proves the point which is the lower semi-continuous get its infimum becomes a fixed point.

Preliminaries

In this section, a brief discussion on geodesic metric spaces is introduced.

Geodesic space[10,13] is a metric space**.** such that for any pair of distinct points v and u there exists a curve $\alpha: [0, \ell] \longrightarrow M$ connecting v to u such that $\alpha(0) = v$, $\alpha(\ell) = u$ and $d(u, v) = \ell$, α is an isometry such that
$\forall t_1, t_2 \in [0, \ell]$ then $d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2|$

the image of α is denoted by segment $[v, u]$ and is called geodesic segment connecting v to u. For each $t \in [0, \ell]$ assigns a unique point $\alpha(t) \in [v, u]$denoted by v_t and written as $v_t = (1 - t)v \oplus tu$ then $d(v, v_t) = td(v, u)$

Notice, notation $\oplus$ doesn't refer to addition but referes to that, v_t lies in convex geodesic segment $[u, v]$and v,u refers to the endpoints of $[u, v]$. v_t is distinct from $v$and u, also v_t lies between v and u, and it was written by $\in [v, u]$, and $v_t$satisfying this property $d(v, v_t) + d(v_t, u) = d(v, u)$.

Menger[15] called this property by betweenness and he showed this property represents a corner stone for definition of geodesic metric spaces, many authors called this property by (Menger convexity).

In next definition, Menger convexity is used to define the order relation $\preccurlyeq$ on geodesic segment $[u, v]$in geodesic metric space, by definition of geodesic metrics space there exists a mapping $\alpha: [0, \ell] \longrightarrow M$ connecting v to u, the isometry of α implies for every $t_1 < t_2$ hen $\alpha(t_1), \alpha(t_2) \in [u, v]$ and this implies $d(v, \alpha(t_1)) < d(v, \alpha(t_2))$,the partially ordered relation $\preccurlyeq$ is suggested such that if $d(v, \alpha(t_1)) < d(v, \alpha(t_2))$ then $v \preccurlyeq v_{t_1} \preccurlyeq v_{t_2} \preccurlyeq u$, in general $\forall z \in [u, v]$ then $v \preccurlyeq z \preccurlyeq u$.

Def.(1). Let (M, d) be geodesic metric space, for each couple of points v and u define a partially ordered relation on (M, d) such that $\forall z \in [u, v]$, then $v \preccurlyeq z \preccurlyeq u$.

Def.(2). Let (M, d) be a geodesic metric space. (ν_n) (v_n)sequence in M , $(v_n)$is called monotonically increasing (respectively monotonically decreasing) sequence if $((v_n) \preccurlyeq v_{n+1}, \forall n \in \mathbb{N})$ $(resp.\ (v_{n+1}) \preccurlyeq v_n, \forall n \in \mathbb{N})$ and it is called monotone if it either increasing or decreasing.

Def.(3)[9] Let (M, d)be geodesic metric space. A subset $N \subseteq M$ is said to be $\preccurlyeq -closed$ if for any convergent monotone(v_n) in N, its limit belongs to N.

Def.(4)[9] Let (M, d) be geodesic metric space. A subset $N \subseteq M$ is called $\preccurlyeq -complete$ if for each monotone Cauchy sequence(v_n) in$N$, its limit point belongs to N.

Def.(5)[9] Let (M, d)be geodesic metric space. A function $\psi: M \longrightarrow [0, \infty)$ is called $\preccurlyeq -lower\ semi-continuous$ $(\preccurlyeq -lsc)$ if for every sequence (v_n)converges to v in M then, $\psi(v) \lim_{n \to \infty} inf \psi(v_n)$ is gotten.

Next an example of $\preccurlyeq -complete$ subset in geodesic metric space is given.

Example(6). Let $M = \mathbb{R}^2$ with the usual Euclidean metric. It is geodesic space because every two points can be joining by a line segment. Define the

partially ordered relation $\preccurlyeq$ on M as $(u,v) \preccurlyeq (r,s)$, if and only if , $u = r$ and $v < s$.

Let $N = \{(u,v) \in \mathbb{R}^2 : u \in [0,1) \; and \; v \in [0,1]\}$

The sequence (u_n, v_n) is convergent in N, because for each $u \in [0,1)$ such that $(u_n = u, \forall n \in \mathbb{N})$ and $(v_n)$is a monotone increasing sequence, so it bounded above by 1 because $v_n \in [0,1]$ and it implies $(v_n \longrightarrow 1)$. Then the sequence $(u_n, v_n) \longrightarrow (u,1)$ and $N \preccurlyeq -complete$, clearly, that N it is not closed in$M$.

Def.(7)[9] Let (M,d) be geodesic metric space. and $\phi \neq N \subseteq M$. A mapping $T: N \longrightarrow M$ is called monotone if $u \preccurlyeq v$ implies $T(u) \preccurlyeq T(v)$.

Def.(8) Let (M,d) be geodesic metric space. and $\phi \neq N \subseteq M$. A mapping $T: N \longrightarrow M$is said to be monotone Caristi inward if $T$is monotone and $\forall u \in N$ such that $T(u) \neq u$ and the geodesic segment $[u, T(u)$ connecting u and $T(u)$ $\exists v \in N$ such that $v \in [u, T(u) \cap N$ and a $\preccurlyeq -lsc \; \psi: N \longrightarrow [0,\infty)$, $d(u,v) \leq \psi(u) - \psi(v)$.

Def.(9)[9] Let (M,d) be geodesic metric space. and $\phi \neq N \subseteq M$. A mapping $T: N \longrightarrow M$ is said to be monotone contraction mapping, that is, there exists $k \in (0,1)$; such that for any $u, v \in N$ with $u \preccurlyeq v$ then $T(u) \preccurlyeq T(v)$ and $d(T(u), T(v) \leq kd(u,v)$.

Remark(10): Let (M,d) be metric space**.** and $T: N \longrightarrow M$be a continuous mapping. Let $\psi: N \longrightarrow [0,\infty)$ be a mapping defined as $\psi(u) = d(u, T(u)$ then ψ is continuous and so it is lower semi-continuous .Furthermore, that monotone contraction mapping is considered as a suitable example of monotone Caristi's inward mapping.

Proof: To prove that, the mapping ψ is continuous. Let a sequence (v_n) be a convergent sequence to a limit point $v \in N$,$(i,e, v_n \longrightarrow v)$, that $\psi(v_n) \longrightarrow \psi(v)$ must explained .

Now,

$$\begin{aligned} |\psi(v_n) - \psi(v)| &= |d(v_n, T(v_n)) - d(v, T(v))| \\ &\leq |d(v_n, v)) + d(v, T(v_n)) - d(v, T(v))| \\ &\leq |d(v_n, v))| + |d(v, T(v_n)) - d(v, T(v))| \end{aligned} \quad (1)$$

It follows From triangle inequality $|d(v, T(v_n)) - d(v, T(v))| \leq |d(T(v_n), T(v))|$ and equation (1) becomes

$$|d(v_n, v))| + |d(v, T(v_n)) - d(v, T(v))| \leq |d(v_n, v))| - |d(T(v_n), T(v))|$$
$$|d(v_n, v))| - k|d(T(v_n), T(v))|$$

But $d(v_n, v) \to 0$, and leads $\psi(v_n) \to \psi(v)$so the mapping ψ is continuous.

Next a lemma shows that the monotone contraction mapping is an example of monotone Caristi's inward mapping.

Lemma(11). Let(M,d) be geodesic metric space. and $N \subseteq M$ and let $T: N \longrightarrow M$ be monotone contraction mapping, then $T: N \longrightarrow M$is monotone Caristi's inward mapping.

Proof:

$T: N \longrightarrow M$is monotone contraction mapping, that is, $\exists k \in (0,1)$ such that for each $u, v \in N$ and $u \preccurlyeq v$ Implies $T(u) \preccurlyeq T(v)$ and $d(T(u), T(v) \leq kd(u,v)$

Let $v \in N$, then there exists a geodesic segment $[v, T(v)]$ joining v and $T(v)$, such that v is comparable with $T(v)$ also $v \preccurlyeq T(v)$, then $d(T(v), T^2(v) \leq kd(v, T(v))$

Hence

$$d(v, T(v)) - k(v, T(v)) \leq d(v, T(v)) - d(T(v), T^2(v)$$
$$(1-k)d(v, T(v) \leq kd(v, T(v)) - d(T(v), T^2(v))$$

Define $\psi: N \longrightarrow [0,\infty)$ by $\psi(v) = \frac{1}{1-k} d(v, T(v))$

Notice, mapping A is contraction so it is continuous and it implies that ψ is continuous (lower semi-continuous) and

$$d(v, T(v)) \leq \left(\frac{1}{1-k}\right)(d(v), T(V)) - d(T(v), T^2(v)) = \psi(v) - \psi(T(v))$$

Def.(12). Let (M,d) be geodesic metric space. and $N \preccurlyeq -closed$ subset of M and $T: N \longrightarrow M$be a mapping. It is called $\preccurlyeq -continuous$ if for any monotone sequence (v_n) converges to $v$in N implies the sequence of images $T(v_n)$ converges to $T(v)$ A(v) in M ($\lim_{n\to\infty} T(v_n) = T(v)$).

Main results

Before discussing the main results, the definition of inward subset is extended into geodesic sense.

Let (M,d)be a geodesic space, N subset of M and $v \in N$, the inward subset of v with respect to N as $I_{N(v)} = \{z \in M: (v,z] \cap N \neq \emptyset\} \cup \{v\}$.
That means any geodesic segment initiated from v to z in M contains at least one point from N except$v$. On another hand that means there exist $t \in (0,1)$ and $v_t \in (v,z]$ such that $d(v,v_t) + d(v_t,z) = d(v,z)$.
And let (M,d)be a geodesic space, N subset of M , $T: N \longrightarrow M$is called inward mapping if $T(v) \in I_{N(v)}$ for each $v \in N$.
When T is a multivalued function, that is, $T: N \to 2^M$, T is called inward mapping if $T(v) \in I_{N(v)}$ for each $v \in N$.
The most advantage of the inward concept that construct a sequence in the domain of a mapping $T: N \longrightarrow M$without need to using Picard iteration.
In this section, the core of our article depends on the idea as mentioned in preliminaries section above, let Mbe $\preccurlyeq -complete$and $N \subseteq M \preccurlyeq -closed$. Let $T: N \longrightarrow M$ be a mapping. a processor is introduced to construct a sequence in the domain of mapping $T: N \longrightarrow M$. Notice, let $v_0 \in N$such that v_0 and $T(v_0)$are comparable $(i.e, v_0 \neq T(v_0)$, then there exists a geodesic segment $[v_0, T(v_0)]$ joining v_0 and $T(v_0)$so that . Moreover, for each $v_1 \in N$ such that $v_0 \neq v_1$ and $v_1 \in [v_0, T(v_0)]$ then $v_0 \preccurlyeq v_1 \preccurlyeq T(v_0)$.
Next, choose v_1 such that is comparable with$T(v_1)$ $T(v_1)$and in the same procedure there exists $v_2 \in [v_1, T(v_1)]$. Also $v_1 \preccurlyeq v_2 \preccurlyeq T(v_1)$ continue in this manner to generate a sequence (v_n) in the domain of$T: N \longrightarrow M$.Next, in the end, the sequence (v_n) is proved converges to a fixed point of mapping $T: N \longrightarrow M$.
Finally, the proof of next theorem depends on this above idea, the image of the sequence under lower semi-continuous defined in the following theorems is a monotone decreasing real sequence and it bounded below by zero. Inwords, it has infimum value at some point $v \in N$ and proves to be a fixed point.

Theorem(13) Let (M,d) be $\preccurlyeq -complete$ geodesic metric space. and let $N \subseteq M$ be $\preccurlyeq -colosed$. Let$T: N \longrightarrow M$ is a monotone contraction. If there exists $v \in M$ comparable to $T(v)$, that is, $v \neq T(v)$ then T has a fixed point, that is, $T^n(z)$ converges to $T(z)$ for any z$\in M$ comparable to $T(z)$.

Proof. Since $T: N \longrightarrow M$ is monotone contraction then it is$\preccurlyeq -continuous$, and let $\psi: M \longrightarrow [0,\infty)$ defined as is $d(v,u) \leq \psi(v) - \psi(u)$, by lemma (11) ψ is a $\preccurlyeq -lsc$.
The existence of $u \in M$ must prove, such that$\psi(u) = \inf_{v \epsilon N} \psi(v)$ and u is a fixed point for T.
Let v_0 in M which is comparable to$T(v_0)$, and by hypothesis assume $v_0 \preccurlyeq T(v_0)$ and let $v_1 = T(v_0)$ then $v_0 \preccurlyeq v_1$ and by monotonically contraction of T then $v_1 = T(v_0) \preccurlyeq T(v_1)$ and $v_2 = T(v_1) \preccurlyeq T^2(v_0)$
And $d(v_n, v_{n+1}) \leq \psi(v_n) - \psi(v_{n+1})$

Then

$$\sum_{n=0}^{m} d(\nu_n, \nu_{n+1}) \leq d(\nu_0, \nu_1) + d(\nu_1,, \nu_2) + ... + d(\nu_m, x_{m+1})$$

$$\leq \psi(\nu_0) - \psi(\nu_1) + \psi(\nu_1) - \psi(\nu_2) + ... + \psi(\nu_m) - \psi(\nu_{m+1})$$

$$\leq \psi(\nu_0) - \psi(\nu_{m+1})$$

Notice that, $inf_{v \in N} \psi(v) \leq \psi(v_m)$ for each $m \in \mathbb{N}$ $m \in$, there for the inequality above become $\leq \psi(v_0) \leq \psi(v_{m+1}) \leq \psi(v_0) - inf_{v \in N} \psi(v)$

then $\lim_{n \to \infty} \sum_{n=0}^{\infty} d(v_n, v_{n+1}) < \infty$ that means the summation is finite.
This implies that (v_n) is a convergent Cauchy sequence $(i.e, v_n \to v)$ in M because it is complete.
Now to prove the mapping T has a fixed point by applying the triangle inequality

$$\begin{aligned} d(v, T(v) &\leq d(v, v_n) + d\big(v_n, T(v)\big) \\ &= d(v, v_n) + d\big(T^{n-1}(v), T(v)\big) \\ &\leq d(v, v_n) + kd(v_{n-1}, v) \end{aligned}$$

Because $(v_n \to v)$ in N then it implies $d(T(v), v) = 0$ and $T(v) = v$
Now and to prove its unique, Let r and s be fixed points with $r \neq s$ then $T(r) = r$ and $T(s) = s$ and $d(r,s) = d(T(r), T(s)) \leq kd(r,s)$

and this contradicts and it concludes $r = s$.
Next theorem explains that Caristi theorem is stronger than Banach contraction theorem; moreover, the contraction condition can be

cancelled when the mapping $T: N \longrightarrow M$is a monotone Caristi inward mapping will proved in next theorem .

Example(14)
Let $M = \mathbb{R}^2$, with Euclidean and let $N = \{(x,0): x \in \mathbb{R}\}$. Define a partially ordered relation on Mas $(u,v) \preccurlyeq (w,z)$if $u \leq w$ and $v \leq z$. Define a mapping $T: N \longrightarrow M$ as following $T\big((x,0)\big) = (kx,0)$, to explain $T$is monotone , let $(u,0) \preccurlyeq (v,0)$ then it leads $u \leq v$, so $T\big((u,0)\big) = (ku,0)$ and $T\big((v,0)\big) = (kv,0)$ and it is clear that $ku \leq kv$ $(ku,0) \preccurlyeq (kv,0)$ also and that$T(u,0) \preccurlyeq (k,0)$ is concluded.
To explain the mapping A is contraction , $d\big(T(u,0), T(v,0)\big) = d\big((ku,0),(kv,0)\big) = kd((u,0),(v,0))$
Therefore, $T$is a contraction mapping.
Notice, $T$is contraction mapping so it is continuous and could define a $\preccurlyeq -lsc$ $\psi: N \to [0.\infty)$ by $\psi(v) = d(v, T(v))$, it follows form remark(13) is continuous and it implies ψ is $\preccurlyeq$-lscand by Caritsi theorem the mapping A has a fixed point .

Theorem (15). Let (M,d) be geodesic metric space. Let N be a $\preccurlyeq -complete$subset of M. Let $T: N \to M$be a monotone Caristi inward mapping. Assume there exists $v \in N$such that $v$and $T(v)$ are comparable. Then T has a fixed point.

Proof. Let $v_0 \in N$ and $v_0 \neq T(v_0)$ such that without loss to generality $v_0 \preccurlyeq T(v)$, it follows from definition**(8)** of monotone Caristi inward mapping, there exists at least one geodesic segment $[v_0, T(v_0)]$ joining $v_0$and $T(v_0)$ such that$[v_0, T(v_0)] \cap N \neq \emptyset$ and a $\preccurlyeq -lsc$ function $\psi: N \to [0.\infty)$ such that $v_1 \in [v_0, T(v_0)] \cap N$ such that $v_1 \neq v_0$ and $v_0 \preccurlyeq v_1 \preccurlyeq T(v_0)$, but T is Caristi monotone then $d(v_0, v_1) = \psi(v_0) - \psi(v_1)$
Now if $v_1 \neq T(v_1)$ and by monotonicity of T then $v_1 \preccurlyeq T(v_1)$ and by hypothesis there exists a geodesic segment $[v_1, T(v_1)] \cap N \neq \emptyset$

$v_2 \in [v_1, T(v_1)] \cap N$ such that $v_2 \neq v_1$ and $v_1 \preccurlyeq v_2 \preccurlyeq T(v_1)$ and $d(v_1, v_2) = \psi(v_1) - \psi(v_2)$

then inductively sequence $(v_n)$is defined in N by the same manner $v_{n+1} \in [v_n, T(v_n)] \cap N$ such that $v_{n+1} \neq v_n$ and $v_n \preccurlyeq v_{n+1} \preccurlyeq T(v_n)$ and $d(v_n, v_{n+1}) = \psi(v_n) - \psi(v_{n+1})$

but $d(v_n, v_{n+1})$ is a positive value, then $\psi(v_n) \geq \psi(v_{n+1})$ and $(\psi(v_n))$ is decreasing real sequence bounded below by zero and it converges to its infimum(bounded below real sequence).
to prove (v_n) is Cauchy sequence

$$d(\nu_n, \nu_m) \leq d(\nu_n, \nu_{n+1}) + d(\nu_{n+1}, \nu_{n+2}) + ... + d(\nu_{m-1}, \nu_m)$$

$$\leq \psi(\nu_n) - \psi(\nu_{n+1}) + \psi(\nu_{n+1}) - \psi(\nu_{n+2}) + + ... + \psi(\nu_{m-1}) - \psi(\nu_m)$$

$$\psi(\nu_n) - \psi(\nu_m)$$

$$\leq \psi(\nu_n) - \inf_{\nu \in N} \psi(\nu)$$

the last equation above is gotten, because $\inf_{v \in N} \psi(v) \leq \psi(v_m), \forall m \in \mathbb{N}$ and get $\lim_{n \to \infty} \sum_{n=0}^{\infty} d(v_n, v_{n+1}) \leq \psi(v_0) - inf_{v \in N} \psi(v) < \infty$
then (v_n) is Cauchy sequence and it has limit point u, $(\lim_{n \to \infty} v_n = u)$ and have $(v_n \preccurlyeq u, \forall n \in \mathbb{N})$ and ψ is $\preccurlyeq -lsc$ it implies $\psi(u) \leq \liminf_{n \to \infty} \psi(v_n)$

Also, the $\psi(u) = \liminf_{n \to \infty} \psi(v_n)$ must proved
Let $m \to \infty$, then get $d(v_n, v_m) = \psi(v_n) - \lim_{m \to \infty} \psi(v_m) \leq \psi(v_n) - \psi(u)$

Now to prove $\psi(u) = inf_{v \in N} \psi(v)$
Let z$\in N$ and u$\preccurlyeq z \preccurlyeq T(u)$ and $inf_{v \in N} \psi(v) < \psi(z)$ then $d(u,z) \leq \psi(u) \leq \psi(z)$
But $(u,z) > 0$, that means $\psi(u) \geq \psi(z)$ and equation **(2)** leads to $\psi(z) < \psi(u) < \liminf_{n \to \infty} \psi(v_n) < \psi(z)$
its contradiction and leads to $\psi(u) = inf_{v \in N} \psi(v)$
To prove T has fixed point y such that $T(u) = u$
Assume $T(u) \neq u$, then $\exists \bar{v} \in N$ and $u \neq \bar{v}$ and $u \preccurlyeq \bar{v} \preccurlyeq T(u)$ then by Properties of $\preccurlyeq -lsc$ of ψ and $d(u, \bar{v}) > 0, d(u, \bar{v}) \leq \psi(u) - \psi(\bar{v})$ this implies $\psi(\bar{v}) < \psi(u)$ but $\psi(u) = inf_{v \in N} \psi(v)$ (2)

but then $\psi(u) = \psi(\bar{v})$, $d(u, \bar{v})$ and $(u = \bar{v})$that show contradiction and concludes $T(u) = u$

In the next theorem, the multi-valued mapping case will be discussed and the core of its proof depends on the idea of theorem **(15)** of the single value above.

Def.(16) Let (M, d) be geodesic metric space. and $\phi \neq N \subseteq M$. A multivalued mapping $T: N \to 2^M$then
T is called monotone if $T(z) \preccurlyeq T(w)$ when z$\preccurlyeq w$, that is $\forall u \in T(z)$, $\exists v \in T(w)$ such that $u \preccurlyeq v$ for all $u, v \in N$ such that . T is called monotone Caristi inward, if T is monotone and there exists a $\psi: N \to [0.\infty)$ is $\preccurlyeq -lsc$, such that for any $x \in N$ $x \notin T(x)$ and , and for each $y \in T(x)$, with $x \preccurlyeq y$ there exists $w \in N$ such that $w \in [x, y]$ with $w \neq x$ and $d(x, w) \leq \psi(x) - \psi(w)$.

Theorem(17). Let (M, d) be geodesic metric space. and $\phi \neq N \subseteq M$ be $\preccurlyeq -complete$. T multivalued mapping $T: N \to 2^M$ is monotone Caristi inward multivalued mapping such that $T(v) \neq \phi$, $\forall v \in N$. Let $v_0 \in N$ such that there exists $u_0 T(v_0)$,it is comparable with v_0 and $v_0 \preccurlyeq u_0$. Then T has a fixed point, that is, $v \in T(v)$

Proof. The idea of proof depends on the procedure, that is, for every $v \in N$ there exists $u \in T(v)$such that $v \preccurlyeq u$, define another mapping $\beta: N \to N$ such that $\beta(v) = u$ and satisfies the conditions $u \in T(v)$ and $v \preccurlyeq u$, by this manner, construct a single value function satisfying condition of the theorem **(15)** of single value above. It has a fixed point such that ;therefore, it's a fixed point for the mapping T.

Let $v_0 \in N$ and $T(v_0) \neq \emptyset$
Define $C_{v_0} = \{u \in T(v_0)\ and\ v_0 \preccurlyeq u\}$ By hypothesis $C_{v_0} C_{v_0}$ is nonempty.

Define $\beta: N \to N$ $\beta(v_0) \in C_{v_0}$such that and this hold by the axiom of choice let $\{U_i: i \in I, U_i \neq \emptyset\ \forall i \in I\}$ and let $f: I \longrightarrow \bigcup_{i \in I} U_i$,, for each $i \in I$, then $f(i) \in U_i$, choose $u_0 \in \beta(v_0)$ such that $v_0 \preccurlyeq u_0$.
if $v_0 \neq u_0$ then there exists geodesic segment $[v_0, u_0]$ joining between v_0 and u_0 in N and there exists $v_1 \in [v_0, u_0]$ such that $v_0 \neq v_1$ and $v_0 \preccurlyeq v_1 \preccurlyeq u_0$
but T is monotone Caristi inward multivalued mapping then there exists $\preccurlyeq -lsc$ function ψ:N→[0.∞) such that $d(v_0, v_1) = \psi(v_0) - \psi(v_1)$ by monotonicity of T then $T(v_0) \preccurlyeq T(v_1)$.
$C_{v_1} = \{u \in T(v_1)\ and\ v_1 \preccurlyeq u\}$ choose $u_1 \in C_{v_1}$ such that $\beta(v_1) = u_1)$ and $u_0 \preccurlyeq u_1$ and if $v_1 \neq u_1$ and if $v_1 \neq u_1$ then there exists geodesic segment $[v_1, u_1]$ joining between v_1 and u_1 there exist $v_2 \in [v_1, u_1]$ such that $v_1 \neq v_2$ and $v_1 \preccurlyeq v_2 \preccurlyeq u_1$
And $d(v_1, v_2) = \psi(v_1) - \psi(v_2)$
by this manner, construct inductively sequence (v_n) such that
1. $(v_n) \in N$
2. $C_{v_n} = \{u \in T(v_n)\ and\ v_n \preccurlyeq u\}$
3. choose $u_n \in C_{v_n}$ such that $u_n = \beta(v_n)$, and $v_n \preccurlyeq u_n$
4. if $v_n \neq u_n$ then there exists geodesic segment $[v_n, u_n]$ joining v_n and u_n, and $v_n \neq v_{n+1}$ and $v_{n+1} \in [v_n, u_n]$ and $d(v_n, v_{n+1}) = \psi(v_n) - \psi(v_{n+1})$
Now the mapping β is monotone Caristi inward mapping satisfying the required conditions of Theorem**(15)** for single-valued valued above then it has a fixed point and the existence of a fixed point of β is equivalent to existence the fixed point of T $(i.e, v \in T(v))$, because $\beta(v_n) \in T(v_n), \forall n \in \mathbb{N}$ notice, the mapping β has a fixed point v ;therefore, $v = \beta(v) \in T(v)$, and it leads $v = \in T(v)$ therefore T has a fixed point.

Example(18)
Let $M = \mathbb{R}$ with usual metric and ordered relation $\leq$, Let $[a, b] = N \subseteq M$and a multi valued mapping $T: N \longrightarrow CB(M)$ defined as , where$CB(M)$ is a family of closed and bounded subsets in $\mathbb{R}$, $T(x) = [x - 1, x + 1]$. Clear that T is monotone, because for each $x, y \in [a, b]$ and $x \leq y$then there exists a $x_1 \in T(x)$ and $y_1 \in T(y)$ such that $x_1 \leq y_1$.
Notice, N is closed and bounded then for each $x \in N$, there exists $z \in T(x)$.

Now define $\psi(x) = \min\{d(x, z): z \in T(x)\}$, it follows form remark**(10)** that ψ is continuous so it $\preccurlyeq -lsc$ it easily to see that the mapping T has infinite fixed points because $x \in [x - 1, x + 1]$and it concludes $x \in T(x)$

Conclusion:

In conclusion, Betweenness property in geodesic metric space is a powerful tool that enables us to produce algorithm analogs to Picard iteration to generate sequence converges to a fixed point. Furthermore, the same manner is used to find a fixed point for nonespansive mapping.

Author's declaration:

- Conflicts of Interest: None.

- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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References:

1. Caristi J, kirk W. Fixed poit theorem for mapping satisfying inward condition. The, Trans. O F Soc. Am. Math. 1976; 215:241–252.
2. Lim T-C. Note a fixed point theorem for weakly inward multivalued contractions. J. Math. Anal. Appl. 2000; 247:323–327.
3. Alisawi K, Salwa S A. Some results on weakly inward mapping in geodesic metric spaces. Al-Qadisiyah J. Pure Sci. 2020; 25(3):109–118.
4. Al-Thagafi M, Naseer S. Fixed point theorems for generalized metrically inward maps. Nonlinear Anal. Theory, Methods Appl. 2010; 73(1):31–36.
5. Ran ACM, Reurings MCB. A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. 2004; 132(5):1435–1443.
6. Bachar M, Khamsi MA. Recent contributions to fixed point theory of monotone mappings. J. Fixed Point Theory Appl. 2017; 19(3):1953–1976.
7. Kirk W A A. Fixed point theorems for set-valued mappings of contractive type. Pacific J. Math. 1972; 43:553–562.
8. Xu HK. Multivalued nonexpansive mappings in Banach spaces. Nonlinear Anal. Theory, Methods Appl. 2001; 43(6):693–706.
9. Alfuraidan MR, Jorquera ED, Khamsi MA. Fixed point theorems for monotone Caristi inward mappings. Numer Funct Anal Optim. 2018 Jul 27;39(10):1092-101.
10. Piatek B. The behavior of fixed point free nonexpansive mappings in geodesic spaces. J. Math. Anal. Appl. 2017; 445(1):1071–1083.
11. Ran A. A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. 2004; 132(5):1435–1443.
12. Saejung S. Remarks on endpoints of multivalued mappings on geodesic spaces. Fixed Point Theory Appl. 2016; 52:1–12.
13. Kifayat U, Junaid A, Nabil M. On Noor iterative process for multivalued nonexpansive mappings with application. Int. J. Math. Anal. 2019; 13(6):293–304.
14. Ansari AH, Došenovic T, Radenovic S, Saleem N, Šešum-Cavic V, Vujakovic J. C-class functions on some fixed point results in ordered partial metric spaces via admissible mappings. Novi Sad J. Math. 2019; 49(1):101–116.
15. Papadopoulos A. Metric spaces, convexity and nonpositive curvature. European Mathematical Society 2005.

بعض النتائج حول النقاط الصامدة لدالة كارتسي الرتيبة المتجهة الى الداخل في الفضاءات المترية الجيوديسية

خالد عبد جاسم

قسم الرياضيات التطبيقة، كلية العلوم، جامعة الانبار، الانبار،العراق.

الخلاصة:

في هذه الدراسة:تم انشاء متتابعة رتيبة في مجال دالة كارتسي الرتيبة المتجهة الى الداخل باستخدام الخاصية البينية التي تتمتع بها الفضاءات المترية الجيوديسية واثبات ان هذه المتتابعة تقترب الى النقطة الصامدة لتلك الدالة باستخدام نظرية كارتسي للنقطة الصامدة.

الكلمات المفتاحية: الدالة الانكماشية،النقاط الصامدة، الفضاءات المترية الجيوديسية، النقطة الصامدة، الدالة شبة المستمرة الدنيا، دالة كارتسي الرتيبة المتجهة للداخل، الدوال متعددة القيم.