On Higher N-Derivation Of Prime Rings

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Abstract:

The main purpose of this work is to introduce the concept of higher N-derivation and study this concept into 2-torsion free prime ring we proved that:
Let R be a prime ring of char. $\neq 2$, U be a Jordan ideal of R and $D=(d_i)_{i\in n}$ be a higher N-derivation of R, then

$d_n(ur)=\sum_{i+j=n} d_i(u)d_j(r)$, for all $u\in U$, $r\in R$, $n\in N$.

Key words: derivation, Jordan ideal, higher derivation, prime ring.

1. Introduction:

Throughout this paper, R will denote an associative ring with center $Z(R)$, not necessarily with an identity element. A ring R is said to be prime (resp. semiprime) if $xRy=0$ (resp. $xRx=0$) implies either $x=0$ or $y=0$ (resp. $x=0$), [1]. A ring R is 2-torsion free if $2x=0$, for all $x\in R$ implies $x=0$, [2]. A Lie ideal (rsp. Jordan ideal) of R is any additive subgroup U of R with $[u,r]=ur-ru\in U$ (resp. $ur+ru\in U$), for all $u\in U, r\in R$, [2]. A derivation (resp. Jordan derivation) of R is an additive mapping $d:R\to R$ such that $d(ab)=d(a)b+ad(b)$ (resp. $d(a^2)=d(a)a+ad(a)$), for all $a,b\in R$, [3]. For a fixed $a\in R$, define $d:R\to R$ by $d_a(x)=[a,x]$, for all $x\in R$, is called an inner derivation, [4].

Every derivation is obviously a Jordan derivation, but the converse is not true in general. Herstein proved that if R is a prime ring of char. $\neq 2$, then every Jordan derivation of R is a derivation [5].

R . Awtar extended the Herstein's theorem to Lie ideal, [6]. He proved that if U is a Lie ideal of a prime ring of char. $\neq 2$ such that $u^2\in U$, for every $u\in U$, and $d:R\to R$ is an additive mapping such that $d_{|_U}$ is a Jordan derivation of U into R, then $d_{|_U}$ is a derivation of U into R.

On the other hand, N. M. Shammu in [7] extended the Herstein's theorem to Jordan ideals. He proved that if R is a prime ring of char. $\neq 2$, U is a Jordan ideal of R and $d:R\to R$ is an additive mapping satisfying the condition $d(ur+ru)=d(u)r+ud(r)+d(r)u+rd(u)$, for all $u\in U$, $r\in R$, then $d(ur)=d(u)r+ud(r)$, for all $u\in U$, $r\in R$.

Let $D=(d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0=id_R$. D is said to be a higher derivation (resp. Jordan higher derivation) if $d_n(ab)=\sum_{i+j=n} d_i(a)d_j(b)$ (resp. $d_n(a^2)=\sum_{i+j=n} d_i(a)d_j(a)$), for all $a,b\in R, n\in N$, [8]. M, Ferrero and C. Haetinger in [9], extended Hersetine's result to higher derivations, they proved that every Jordan higher

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derivation of 2-torsion free semiprime ring is a higher derivation.

C.Haetinger in [10] extended Awtar's result to higher derivation. Also, A. K. Faraj, C. Haetinger and A. H. Majeed in [2], extended this result to (U,R)-higher derivation.

The main purpose of this work is to introduce the concept of higher N-derivation and study this concept into 2-torsion free prime ring. we extend Nazar's result into higher N-derivation. Throughout this paper N will denote the set of natural numbers including 0 and as usual [x,y] will denote the commutator xy – yx.

2. Preliminaries:

Now we will introduce the definition of higher N- derivations and some basic results which extensively to prove our main results.

Definition (2.1):

Let U be a Jordan ideal of a ring R and $D=(d_i)_{i\in n}$ be a family of additive mappings of R such that $d_0 = id_R$ D is said to be higher N-derivation (HN-D, for short) if for every $n\in N$, we have

$$d_n(ur+ru)=\sum_{i+j=n} d_i(u)d_j(r)+d_i(r)d_j(u)$$

, for all $u\in U$, $r\in R$.

Example (2.2):

Let R be a ring of 2×2 matrices over commutative ring S of char.$\neq 2$.

Let $U=\left\{\begin{pmatrix} x & y \\ y & x\end{pmatrix} : x,y\in S\right\}$. It is clear that U is a Jordan ideal of R let $D=(d_i)_{i\in n}$ be a family of mappings of R into R defined by

$$d_n\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{cases}\begin{pmatrix} o & -b \\ nc & 0\end{pmatrix} & n=1,2 \\ 0 & n\geq 3\end{cases}, \text{ for all}$$

$\begin{pmatrix} \text{a} & \text{b} \\ \text{c} & \text{d}\end{pmatrix}\in R$. Then D is HN-D.

Throughout this work R is a prime ring of char.$\neq 2$ ring, U is a Jordan ideal, $D=(d_i)_{i\in n}$ a HN-D of R and we denote by Φ_n(u,r) the element of R defined by

$$\Phi_n(u,r)=d_n(ur)-\sum_{i+j=n} d_i(u)d_j(r) \quad \text{for}$$

all $u\in U$, $r\in R$, n $\in$N.

Lemma (2.3):

For all $u\in U$, $r\in R$, n $\in$N

$$d_n(uru)=\sum_{i+j+k=n} d_i(u)d_j(r)d_k(u)$$

Proof:

Since $D=(d_i)_{i\in n}$ is a HN-D, then replace r by (2u)r+r(2u) in the definition (2.1),

$$d_n(u((2u)r+r(2u))+((2u)r+r(2u))u)=2\sum_{i+j=n} d_i(u)d_j(ur+ru)+d_i(ur+ru)d_j(u)$$

$$=2\sum_{i+j=n} d_i(u)\sum_{\ell+t=J} d_\ell(u)d_t(r)+d_\ell(r)d_t(u)$$

$$+\sum_{i+j=n}\sum_{p+q=i} d_p(u)d_p(r)+d_p(r)d_q(u)d_j(u)$$

$$=2\sum_{i+\ell+t=n} d_i(u)d_{\ell j}(u)d_t(r)+d_i(u)d_\ell(r)d_t(u)$$

$$+\sum_{P+q+j=n} d_p(u)d_q(r)d_j(u)+d_p(r)d_q(u)d_j(u)$$

....(1)

On the other hand,

$$d_n(u((2u)r+r(2u))+((2u)r+r(2u))u)$$
$$=d_n(u^2r+ru^2)+4d_n(uru)$$

$$=\sum_{i+j=n} d_i(2u^2)d_j(r)+d_i(r)d_j(2u^2)+4d_n(uru)$$

$$= 2\sum_{i+j=n}\sum_{l+t=i} d_i(u)d_l(u)d_t(u)d_j(r) + \sum_{i+j=n} d_i(r)\sum_{p+q=j} d_p(u)d_q(u) + 4d_n(uru)$$

$$= 2(\sum_{\ell+t+j=n} d_\ell(u)d_t(u)d_j(r) + \sum_{i+p+q=n} d_i(r)d_p(u)d_q(u)) + 4d_n(uru)$$

Compare (1) and (2) and since R is 2-torsion free we get the required result.♦

A Linearization of $d_n(uru) = \sum_{i+j+k=n} d_i(u)d_j(r)d_k(u)$ gives

Corollary (2.4):

For all $u, v \in U,\ r \in R, n \in N$

$$d_n(urv + vru) = \sum_{i+j+k=n} d_i(u)d_j(r)d_k(u) + d_i(v)d_j(r)d_k(u)$$

.

Lemma (2.5):

For all u∈U, if u ∈ Z(R), then $d_n(u)$ ∈Z(R) for all n ∈ N.

Proof:

We prove the lemma by indication on n∈N. Since the lemma is true for n=1 ([7], lemma 2.2), we can assume that $d_m(u)$∈Z(R) for all m<n ,m,n ∈ N. Since $u \in Z(R)$ and $D = (d_i)_{i \in n}$ is HN-D, then

$$2d_n(ur) = \sum_{i+j=n} d_i(u)d_j(r) + d_i(r)d_j(u)$$

$$= d_n(u)r + \sum_{i+j=n}^{i<n} d_i(u)d_j(r) + rd_n(u) + \sum_{i+j=n}^{j<n} d_i(r)d_j(u)$$

$$= d_n(u)r + rd_n(u) + 2\sum_{i+j=n}^{i<n} d_i(u)d_j(r).$$

.... (1)

Replace r by vr+rv in equation (1), then

$$2d_n(u(vr+rv)) = d_n(u)(vr+rv) + (vr+rv)d_n(u) + 2\sum_{i+j=n}^{i<j} d_i(u)d_j(vr+rv) \quad \text{...(2)}$$

On the other hand, since $u \in Z(R)$,

$$2d_n(u(vr+rv)) = 2d_n(urv+vru)$$

$$= \sum_{i+j+k=n} d_i(u)d_j(r)d_k(v) + d_i(v)d_j(r)d_k(u)$$

$$= 2(d_n(u)rv + \sum_{i+j+k=n}^{i<n} d_i(u)d_j(r)d_k(v) + vrd_n(u) + \sum_{i+j+k=n}^{k<n} d_i(v)d_j(r)d_k(u)$$

Since $d_m(u) \in Z(R)$, for all m<n then the last equation becomes

$$2d_n(urv+vru) = 2(d_n(u)rv + vrd_n(u) + \sum_{i+j+k=n}^{i<n} d_i(u)d_j(r)d_k(v)) + \sum_{i+j+k=n}^{k<n} d_k(u)d_i(v)d_j(r)$$

.......(3)

Comparing (2) and (3), we have $d_n(u)$ [v,r]=[v,r]$d_n(u)$,i.e [$d_n(u)$,[v,r]]=0 for all v∈U , r∈R, n∈N.

In particular , [$d_n(u)$,[v,w]]=0 for all v,w∈U , r∈R ,n∈N, and this means [$d_n(u)$,[U,U]]=0, so we get [$d_n(u)$,U]=0, and hence U contains a non zero ideal I of R and so

0=[$d_n(u)$,U]=[$d_n(u)$,I R]=[$d_n(u)$,I]R +I [$d_n(u)$,R].

Hence I [$d_n(u)$,R]=0 . since R is prime and I =0 , then [$d_n(u)$,R]=0 and this means

$d_n(u)$ ∈Z(R) for all u∈U, n∈N.♦

Lemma (2.6):

For some u ∈ U and r∈R , if ur = ru then $d_n(ur) = \sum_{i+j=n} d_i(u)d_j(r)$, for all n∈N.

proof:

We prove the lemma by induction on n $\in$ N.

By ([7], lemma 2.3), the lemma is true for n=1, then we can assume that $d_m(ur)=\sum_{i+j=m} d_i(u)d_j(r)$ for all m <n , m,n$\in$N.

Since $D=(d_i)_{i\in n}$ is HN-D, for any v$\in$U

$$d_n(v(ur)+(ur)v)=\sum_{i+j=n} d_i(v)d_j(ur)+d_i(ur)d_j(v)$$

$$=vd_n(ur)+\sum_{i+j=n}^{j<n} d_i(v)d_j(ur)+d_n(ur)v+\sum_{i+j=n}^{i<n} d_i(ur)d_i(v)$$

$$=vd_n(ur)+\sum_{i+j=n}^{j<n} d_i(v)\sum_{p+q=j} d_p(u)d_q(r)+d_n(ur)v$$

$$+\sum_{i+j=n}^{j<n}\sum_{\ell+t=i} d_\ell(u)d_t(r)d_j(v)$$

$$=vd_n(ur)+\sum_{i+p+q=j}^{p+q<n} d_i(v)d_p(u)d_q(r)+d_n(ur)v$$

$$+\sum_{\ell+t+j=n}^{\ell+t<n} d_\ell(u)d_t(r)d_j(v)$$

.....(1)

On the other hand , since ur=ru and by using corollary (2.4) then

$$d_n(v(ur)+(ur)v)=d_n(vru+urv)$$

$$=\sum_{i+j+k=n} d_i(v)d_j(r)d_k(u)+d_i(u)d_j(r)d_k(v)$$

$$=v\sum_{j+k=n} d_j(r)d_k(u)+\sum_{i+j+k=n}^{j+k<n} d_i(v)d_j(r)d_k(u)$$

$$+\sum_{i+j=n} d_i(u)d_j(r)(v)+\sum_{i+j+k=n}^{i+j<n} d_i(u)d_j(r)d_k(v)$$

......(2)

Compare (1) and (2) and since $d_m(ur)=\sum_{i+j=m} d_i(u)d_j(r)$, for all m< n, m,n $\in$N then $\Phi_n(u,r)\,v+v\,\Phi_n(r,u)=0$.

.....(3)

Since Φ_n (u,r)= -Φ_n (r,u) for all u$\in$U , r$\in$R, n$\in$N, then equation (3) becomes [Φ_n (u,r) ,v]=0 for all v $\in$ U , n $\in$ N .

Since every Jordan ideal contains a non zero ideal I of R and since R is prime, then Φ_n (u,r) $\in$Z(R).

.....(4)

Since ur= ru , then (2 u^2)r =r (2u^2) .Thus ,

$$\Phi_n(2u^2,r)=d_n(2u^2r)-\sum_{i+j=n} d_i(2u^2)d_j(r)\in Z(R)$$

. ...(5) Also, u(ur) = (ur)u, replace r by ur in equation (4) , then

$$\Phi_n(u,ur)=d_n(uur)-\sum_{i+j=n} d_i(u)d_j(ur)\in Z(R)$$

, so

$$2(d_n(uur)-\sum_{i+j=n} d_i(u)d_j(ur)\in Z(R)$$

........(6)

Subtract equation (6) from equation (5) and since R is prime , then

$$2(\sum_{i+j=n} d_i(u^2)d_j(r)-\sum_{i+j=n} d_i(u)d_j(ur))$$

$$=2(\sum_{\ell+t+j=n} d_\ell(u)d_t(u)d_j(r)-\sum_{i+j=n} d_i(u)d_j(ur))$$

$$=2(u\sum_{t+j=n} d_t(u)d_j(r)+\sum_{\ell+t+j=n}^{t+j<n} d_\ell(u)d_t(u)d_j(r)-ud_n(ur)-\sum_{i+j=n}^{j<n} d_i(u)d_j(ur))$$

$$=2(u\sum_{t+j=n} d_t(u)d_j(r)+\sum_{\ell+t+j=n}^{t+j<n} d_\ell(u)d_t(u)d_j(r)-ud_n(ur)-\sum_{i+p+q=n}^{p+q<n} d_i(u)d_p(u)d_q(r))$$

$= 2u\Phi_n(u,r) \in Z(R)$ and so $u\Phi_n(u,r) \in Z(R)$.

If Φ_n (u,r)$\neq 0$ and Φ_n (u,r)$\in$Z(R) , then by (lemma (1.2, [7])) we get u$\in$Z(R) and by lemma (2.6) we have d_n(u)$\in$Z(R) for all n $\in$ N.

Hence

$$2d_n(ur) = d_n(ur+ru) = \sum_{i+j=n} d_i(u)d_j(r) + d_i(r)d_j(u)$$

$$= 2\sum_{i+j=n} d_i(u)d_j(r)$$

Since R is 2-torsion free, $d_n(ur) = \sum_{i+j=n} d_i(u)d_j(r)$.

Lemma (2.7):

For all u,v$\in$U, r $\in$ R, m,n$\in$N, if Φ_m (u,v) =0, m<n , m $\in$ N then $[u^2,r]\Phi_n(u^2,r) = 0$ and $\Phi_n(u^2,r)[u^2,r]=0$.

proof:

Let u,v$\in$U such that uv$\in$U and $w = d_n((uv)uv + vu(uv))$ so

$$w = \sum_{i+j+k=n} d_i(uv)d_j(u)d_k(v) + d_i(v)d_j(u)d_k(uv)$$

$$= uv\sum_{j+k=n} d_j(u)d_k(v) + d_n(uv)uv + \sum_{\substack{i+j+k=n\\ i,j+k<n}} d_i(uv)d_j(u)d_k(v) + vud_n(uv)\sum_{i+j=n} d_i(v)d_j(u)$$

$$+ \sum_{\substack{i+j+k=n\\ i,j+k<n}} d_i(v)d_j(u)d_k(uv)$$

On the other hand,

$$w = d_n((uv)^2 + vu^2v)$$

$$= \sum_{i+j=n} d_i(uv)d_j(uv) + \sum_{i+j+k=n} d_i(v)d_j(u^2)d_k(v)$$

$$= \sum_{i+j=n} d_i(uv)d_j(uv) + \sum_{i+\ell+t+k=n} d_i(v)d_\ell(u)d_t(u)d_k(v)$$

$$= d_n(uv)uv + uvd_n(uv) + \sum_{\substack{i+j=n\\ i,j<n}} d_{\ell i}(uv)d_j(uv) + vu\sum_{t+k=n} d_t(u)d_k(v)$$

$$+ \sum_{i+\ell=n} d_i(v)d_\ell(u)uv + \sum_{\substack{i+\ell+t+k=n\\ i+\ell,t+k<n}} d_i(v)d_\ell(u)d_t(u)d_k(v)$$

Compare thr right hand sides of w and since Φ_m (u,v)=0, for all u,v$\in$U, m$\in$n, then

[u,v]Φ_n (u,v)=0 for all u,v$\in$U , n$\in$N.

For any r$\in$R and u$\in$U,the element v= ur+ru satisfies the criterion uv$\in$R, hence by above $[u^2,r]\ \Phi_n(u^2,r)=0$.

In the same way we can prove that $\Phi_n(u^2,r)\ [u^2,r] =0$.♦

If we linearize the result of lemma (2.7) on r we have the following:

Corollary (2.8):

For all u, v$\in$U, r$\in$R, m, n$\in$N, if Φ_m (u,v) =0 m<n, then

(1) $[u^2,r]\ \Phi_n(u^2,s) + [u^2,s]\ \Phi_n(u^2,r)=0$, for all u$\in$U, r,s$\in$R, n$\in$N.

(2) $\Phi_n(u^2,r)\ [u^2,s] + \Phi_n(u^2,s)[u^2,r]=0$, for all u$\in$U, r,s$\in$R, n$\in$N.

Lemma (2.9):

If Φ_m (u,v)=0 for all u,v$\in$U , m<n ,m,n$\in$N , then $\Phi_n(u^2,r)$ =0 , for all u$\in$U, r$\in$R, n$\in$N.

Proof:

Multiply equation (2) of corollary (2.8) from the left by $[u^2,z]$,$z\in R$ and by using equation (1) and (2) of corollary (2.8) we get

$[u^2,r]\ \Phi_n(u^2,s)[u^2,z]+[u^2,z]\ \Phi_n(u^2,s)[u^2,r]=0.$ .....(1)

Replace z by zt in (1) and using Jacobi identities , then

$[u^2,r]\ \Phi_n(u^2,s)\ z[u^2,t]+[u^2,r]\ \Phi_n(u^2,s)[u^2,z]\ t+ z[u^2,t]\ \Phi_n(u^2,s)\ [u^2,r][u^2,z]t\ \Phi_n(u^2,s)[u^2,r]=0.$

By using (1) and corollary (2. 8) ,the last equation becomes

$[[u^2,r]\ \Phi_n(u^2,s),z][u^2,t]+[u^2,z][\Phi_n(u^2,r)[u^2,s],t]=0.$ .....(2)

Replace z by $z[u^2,z]$ in equation (2) ,then

$0=[[u^2,r]\ \Phi_n(u^2,s),z[u^2,z]][u^2,t]+u^2,z[u^2,z]][\Phi_n(u^2,r)[u^2,s]t]$

$=z[[u^2,r]\ \Phi_n(u^2,s),[u^2,z]][u^2,t]+[[u^2,r]\ \Phi_n(u^2,s),z][u^2,z][u^2,t]+ z[u^2,[u^2,z]][\Phi_n(u^2,r)[u^2,s],t]+[u^2,z][u^2,z][\Phi_n(u^2,r)[u^2,s],t].$

Substitute equation (2) in the last equation ,

$[[[u^2,r]\ \Phi_n(u^2,s),z],[u^2,z]][u^2,t]=0,$ for all $u\in U$, $r,s,z\in R$, $n\in N$. .....(3)

Replace t by ct in equation (3) and use Jacobi identities, then

$[[u^2,r]\ \Phi_n(u^2,s),z],[u^2,z]](c[u^2,t]+[u^2,c]t)=0.$ .....(4)

In view of equation (3),the second term of equation (4) is zero, then

$[[[u^2,r]\ \Phi_n(u^2,s),z],[u^2,z]]R[u^2,t]=0.$

The primeness of R give us either $[u^2,t]=0$, for all $u\in U$, $t\in R$ or $[[[u^2,r]\ \Phi_n(u^2,s),z],[u^2,z]]=0$, for all $u\in U$, $r,s,z\in R$, $n\in N$.

If $[u^2,t]=0$,then $u^2\in Z(R)$ and by using lemma (2.6) we get $\Phi_n(u^2,r)=0$ for all $u\in U$, $r\in R$, $n\in N$.

Now ,if $[[[u^2,r]\ \Phi_n(u^2,s),z],[u^2,z]]=0$, then

$[[u^2,r]\ \Phi_n(u^2,s),z][u^2,z]=[u^2,z][[u^2,r]\ \Phi_n(u^2,s),z].$ ......(5)

put t=z in equation (2), then

$[u^2,z][\ \Phi_n(u^2,r)[u^2,s],z]+[[u^2,r]\ \Phi_n(u^2,s),z][u^2,z]=0.$

In view of equation (5), the last equation reduce to

$0=[u^2,z][\ \Phi_n(u^2,r)[u^2,s],z]+[u^2,z][[u^2,r]\ \Phi_n(u^2,s),z]$

$=[u^2,z][\Phi_n(u^2,r)[u^2,s],[u^2,r]\ \Phi_n(u^2,s),z]$

$=[u^2,z][\ \Phi_n(u^2,r)[u^2,s]-[u^2,s]\ \Phi_n(u^2,r),z]$

$=[u^2,z][[\ \Phi_n(u^2,r),[u^2,s]],z].$

A linearization of the last equation with respect to z gives

$0=[u^2,z+t][[\ \Phi_n(u^2,r),[u^2,s]],z+t]$

$=[u^2,z][[\ \Phi_n(u^2,r),[u^2,s]],t]+[u^2,t][[\ \Phi_n(u^2,r),[u^2,s]],z].$ .....(6)

Replace t by u^2t in equation (6), then

$[u^2,z]u^2[[\Phi_n(u^2,r),[u^2,s]],t]+u^2[u^2,t][[\ \Phi_n(u^2,r),[u^2,s]],z]=0.$

In view of equation (6),

$[[u^2,z],u^2][[\Phi_n(u^2,r),[u^2,s]],t]=0$ .....(7)

Put t=ct in equation (7),

$0=[[u^2,z],u^2][[\ \Phi_n(u^2,r),[u^2,s]],ct]$

$=[[u^2,z],u^2]c[[\Phi_n(u^2,r),[u^2,s],t]+[[u^2,z],u^2][[\ \Phi_n(u^2,r),[u^2,s],c]t$

$=[[u^2,z],u^2]c[[\Phi_n(u^2,r),[u^2,s],t]$, for all $u\in U$, $r,s,t,z\in R$.

That is,$[[u^2,z],u^2]R[[\Phi_n(u^2,r),[u^2,s],t]=0.$

Since R is prime ,then either $[[u^2,z],u^2]=0$ for all $u\in U$, $z\in R$, or $[[\Phi_n(u^2,r),[u^2,s]],t]=0$, for all $u\in U$, $r,s,t\in R$, $n\in N$.

Notice that, if $[[u^2,z],u^2]=0$ then by (theorem 1.4, [7]) we get $u^2\in Z(R)$ and by lemma (2.6), $\Phi_n(u^2,r)=0$ for all $u\in U$, $r\in R$, $n\in N$.

Now, if $[[\Phi_n(u^2,r),[u^2,s]],t]=0$, then

$[\Phi_n(u^2,r),[u^2,s]]= \Phi_n(u^2,r)[u^2,s]-[u^2,s]\Phi_n(u^2,r)\in Z(R)$.

Put $\propto=\Phi_n(u^2,r)[u^2,s]$ and $\beta=[u^2,s]\Phi_n(u^2,r)$.

Now we must show that $\propto-\beta=0$.

Trivially we have $\propto^2=0$ and $\beta^2=0$.So $(\propto-\beta)^3=\beta\propto\beta-\propto\beta\propto$.

Since $[\Phi_n(u^2,r),[u^2,s]]\in Z(R)$, then $[u^2,s][\Phi_n(u^2,r),[u^2,s]]=[\Phi_n(u^2,r),[u^2,s][u^2,s]$.

By expanding and using the corollary (2.8) and lemma (2.7) we get

$-[u^2,s][u^2,s]\Phi_n(u^2,r)=\Phi_n(u^2,r)[u^2,s][u^2,s]$. …..(8)

Also,from $[\Phi_n(u^2,r),[u^2,s]]\in Z(R)$ we have

$\Phi_n(u^2,r)[\Phi_n(u^2,r),[u^2,s]]=\Phi_n(u^2,r),[u^2,s]\Phi_n(u^2,r)$.

Then by expanding and using corollary (2.8) and lemma (2.7) ,we get

$\Phi_n(u^2,r)\Phi_n(u^2,r)[u^2,s]=-[u^2,s]\Phi_n(u^2,r)\Phi_n(u^2,r)$. ……(9)

Now, $\propto\beta=\Phi_n(u^2,r)[u^2,s][u^2,s]\Phi_n(u^2,r)$ and by equation (8) we get

$\propto\beta=-[u^2,s][u^2,s]\Phi_n(u^2,r)\Phi_n(u^2,r)$.

By applying equation (9) on the last equation,

$\propto\beta=[u^2,s]\Phi_n(u^2,r)\Phi_n(u^2,r)[u^2,s]=\beta\propto$. So ,$(\propto-\beta)^3=0$.

Since R is prime and $\propto-\beta\in Z(R)$, then $\propto-\beta=0$.

That is ,$[\Phi_n(u^2,r),[u^2,s]]=0$, for all $u\in U$, $r,s\in R$, $n\in N$. ……(10)

Replace s by st in equation (10) and by Jacobi identities and by using equation (10) itself we have

$[u^2,s][\Phi_n(u^2,r),t]+[\Phi_n(u^2,r),s][u^2,t]=0$, for all $u\in U$, $r,s,t\in R$, $n\in N$.

Put $s=[u^2,s]$ in the last equation ,then

$[u^2,[u^2,s]][\Phi_n(u^2,r),t]+[\Phi_n(u^2,r),[u^2,s]][u^2,t]=0$.

Again by equation (10) ,then $[u^2,[u^2,s]][\Phi_n(u^2,r),t]=0$.

So, $[u^2,[u^2,s]]R[\Phi_n(u^2,r),t]=0$. Since R is prime either $u^2\in Z(R)$ which gives us

$\Phi_n(u^2,r)=0$ for all $u\in U$, $r\in R$, $n\in N$, or $[\Phi_n(u^2,r),t]=0$, for all $u\in U$, $r\in R$, $n\in N$ and this means $\Phi_n(u^2,r)\in Z(R)$.

By lemma (2.7), $\Phi_n(u^2,r)[u^2,r]=0$,for all $u\in U$, $r\in R$, $n\in N$.

So, if for some u and r, $\Phi_n(u^2,r)=0$, since R is prime then $[u^2,r]=0$, so by lemma (2.6), $\Phi_n(u^2,r)=0$ for all $u\in U$, $r\in R$, $n\in N$.♦

Lemma (2.10), [7]:

For any $t\in R$,if $tv^2+v^2t=0$ for all $v\in U$, then t=0.

Now,we reach to the main theorem

Theorem (2.11):

Let R be a prime ring of char.$\neq$ 2, U be a Jordan ideal of R and $D=(d_i)_{i\in N}$ be a HN-D of R ,such that $\Phi_m(u,v)=0$ for all $u,v\in U$, m<n, $m,n\in N$, then

$d_n(u,r)=\sum_{i+j=n}d_i(u)d_j(r)$, for all $u\in U$, $r\in R$, $n\in N$.

proof:

We prove by induction on $n\in N$. By [7], the theorem is true for n=1, then we can assume that

$d_m(ur)=\sum_{i+j=m}d_i(u)d_j(r)$, for all $u\in U$, $r\in R$, m<n , $m,n\in N$.

Since $D=(d_i)_{i\in N}$ is HN-D, then

$$d_n(u(ur)+(ur)u)=\sum_{i+j=n}d_i(u)d_j(ur)+d_i(ur)d_j(u)$$

$$=ud_n(ur)+\sum_{i+j=n}^{j<n}d_i(u)d_j(ur)+d_n(ur)u+\sum_{i+j=n}^{i<n}d_i(ur)d_j(u)$$

Since $\Phi_m(u,r)=0$, for all $u\in U$, $r\in R$, m<n, then

$$d_n(u(ur)+(ur)u)=ud_n(ur)+\sum_{i+l+t=n}^{l+t<n} d_i(u)d_l(u)d_t(r)+d_n(ur)u+\sum_{p+q+j=n}^{p+q<n} d_p(u)d_q(r)d_j(u) \text{.....} \quad (1)$$

On the other hand, by using lemma (2.9) and lemma (2.3) we have

$$d_n(u(ur)+(ur)u)=d_n(u^2r)+d_n(uru)$$

$$=\sum_{\ell+t+j=n} d_\ell(u)d_t(u)d_j(r)+\sum_{i+j+k=n} d_i(u)d_j(r)d_k(u)$$

$$=u\sum_{t+j=n} d_t(u)d_j(r)+\sum_{\ell+t+j=n}^{t+j<n} d_\ell(u)d_t(u)d_j(r)$$

$$+\sum_{i+j=n} d_i(u)d_j(r)u+\sum_{i+j+k=n}^{i+j<n} d_i(u)d_j(r)d_k(u)$$

. ...(2)

By comparing (1) and (2) we have

$u\Phi_n(u,r)+\Phi_n(u,r)u=0$,for all $u\in U$, $r\in R$, $n\in N$.

A linearization of the last equation with respect to u gives

$v\Phi_n(u,r)+u\Phi_n(v,r)+\Phi_n(u,r)v+\Phi_n(v,r)u=0$, for all $u,v\in U, r\in R$, $n\in N$. …...(3)

Replace v by $2v^2$ in equation (3) and by lemma (2.9) then

$2(v^2\Phi_n(u,r)+\Phi_n(u,r)v^2)=0$.

Since R is 2-torsion free and by using lemma (2.10),

we have $\Phi_n(u,r)=0$ for all $u\in U$, $r\in R$, $n\in N$. i.e, $d_n(ur)=\sum_{i+j=n} d_i(u)d_j(r)$.

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حول اشتقاق N العالي للحلقات الاوليه

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الخلاصه :
الهدف الرئيسئ للبحث هو اعطاء مفهوم اشتقاق- N العالي والقيام بدراسة هذا المفهوم بالحلقات الاوليه طليقة الالتواء من النمط 2 وقد برهنا
اذا كانت R حلقه اوليه ذات مميز $\neq 2$ و U هو مثالي جوردان للحلقه R و $D=(d_i)_{i\in n}$ هو اشتقاق- N العالي للحلقه R فان $d_n(ur)=\sum_{i+j=n} d_i(u)d_j(r)$ لكل r∈R , u∈U ،n∈N.