# New Iterative Method for Solving Nonlinear Equations

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## Abstract:

The aim of this paper is to propose an efficient three steps iterative method for finding the zeros of the nonlinear equation f(x)=0 . Starting with a suitably chosen $x_0$, the method generates a sequence of iterates converging to the root. The convergence analysis is proved to establish its five order of convergence. Several examples are given to illustrate the efficiency of the proposed new method and its comparison with other methods.


**Key words: Nonlinear equation , convergence , three step method .**

## Introduction:

In science and engineering , many of the nonlinear and transcendental problems of the form f(x)=0, are complex in nature . Since it is not always possible to obtain its exact solution by the usual algeric process . The numerical iterative methods are often used to obtain approximate solution of such problems.There are many methods developed on the improvement of quadratically convergent Newton's method . Some modifications of Newton's method have been developed in [1-5].In addition, Hou[6] have proposed and studied method for nonlinear equations with twelfth order convergence. Many iterative methods have been developed using derivative of first order and free from second derivative [7-9],while Yasmin [10] and Ahmed [11] suggested same derivative free iterative methods for finding the zeros of the nonlinear equation based on the central – difference and forward – difference approximations to derivatives. Many two – steps and three steps methods have been proposed by saeed [12] , Rafiullal [13] , Bahgat [14] , Feng [15] and Mir [16] with different order of convergence .

In this paper, three steps iterative method is proposed for solving nonlinear equation. We prove that the new method has order of convergence five. The method and its algorithm is described in section 2. The convergence analysis of the method is discussed in section 3. Finally, in section 4, the method is tested on some numerical examples .

### Derivation of the New Method

Consider iterative method to find a simple root of a nonlinear equation

F(x)=0 … (1)

We assume that α is a simple root of (1) and $\varphi$ is an initial guess sufficiently close to α .

Taking the $1^{st}$ three terms of use Taylor's series expansion of the function f(x) , yields ,

$$f(\varphi) + (x-\varphi) f'(\varphi) + \frac{(x-\varphi)^2}{2!} f''(\varphi) = 0 \quad \dots (2)$$

from (2) one can have

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$$x = \varphi - \frac{f(\varphi)}{f'(\varphi)} \quad \dots (3)$$

and $$x = \varphi - \frac{2f(\varphi)f'(\varphi)}{2(f'(\varphi))^2 - f(\varphi)f''(\varphi)}$$
… (4)

Formulation (3) and (4) allow us to suggest the following three step iterative method for solving the nonlinear eq . (1) .

From eq (3), we can compute the approximate solution $x_{n+1}$ by the following iterative scheme for a given $x_0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \dots (5)$$

Eq.(5) is the well known Newton's method, which has a quadratic convergence [17] .

For a given $x_0$, compute the approximate solution $x_{n+1}$ using eq. (4) ,

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)} \dots (6)$$

Eq.(6) is known a Halley's method, which has cubic convergence [18 , 19] .

Now, eqs.(5) and (6) allow us to suggest the following new three step iterative method for solving eq.(1) .

**Algorithm (1) :**

For a given $x_0$ , compute the approximate solution $x_{n+1}$ by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad \dots (7)$$

$$z_n = y_n - \frac{2f(y_n)f'(y_n)}{2f'^2(y_n) - f(y_n)f''(y_n)} \dots (8)$$

$$x_{n+1} = y_n - \frac{2[f(y_n)+f(z_n)]f'(y_n)}{2f'^2(y_n) - [f(y_n)+f(z_n)]f''(y_n)}$$
… (9)

Algorithm (1) is the main motivation of this paper .

**Analysis of convergence**

In this section we will present the analysis of convergence by giving mathematical proof for the order of convergence of the algorithm defined by eqs . [6-7] is studied .

**Definition (1)** [20] :

Let $\alpha \in R$ , $x_n \in R$ , n=0,1,2, … . Then , the sequence $\{x_n\}$ is said to be converge to α if

$$\lim_{n\to\infty} |x_n - \alpha| = 0$$

If in addition , there exists a constant $c \geq 0$ , an integer $n_0 \geq 0$ , and $p \geq 0$ such that for all $n \geq n_0$ ,

$$|x_{n+1} - \alpha| \leq c\,|x_n - \alpha|^p ,$$

Then $\{x_n\}$ is said to be converge to $\alpha$ with order at least p . If p=2 or 3 , the convergence is said to be 9-quadratic or 9-cubic , respectively .

when $e_n = x_n - \alpha$ is the error in the nth iterate , the relation

$$e_{n+1} = c\, e_n^p + o(e_n^{p+1})$$

is called the error equation .

**Theorem (1) :** Let $\alpha \in I$ be a simple zero of $f : I \subseteq R \to R$ for an open interval I which has first and second derivatives . If $x_0$ is sufficiently close to α, then the three – step method defined by algorithm (1) has fifth – order convergence .

**Proof :** Let α be a simple zero of f . since f is sufficiently differentiable , by expanding $f(x_n)$ and $f'(x_n)$ about α , we can get

$f(x_n) = f(\alpha) + (x_n - \alpha) f'(\alpha) + \frac{(x_n-\alpha)^2}{2!} f''(\alpha) + \frac{(x_n-\alpha)^3}{3!} f'''(\alpha) + \ldots$

$f(x_n) = f'(\alpha)\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + \ldots\}$ … (10)

$f'(x_n) = f'(\alpha)\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + \ldots\}$ … (11)

Where $c_k = \frac{1}{k!}\frac{f^{(k)}_{(\alpha)}}{f'_{(\alpha)}}$ , k=1,2,3,….. and $e_n = x_n - \alpha$ .

Now from (10) and (11) we have

$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4) e_n^4 + (8c_2^4 + 10c_2c_4 + 6c_3^2 - 4c_5 - 20c_3c_2^2) e_n^5 + \ldots\}$ ... (12)

Using (12) in (7) , yields

$y_n = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4) e_n^4 + (4c_5 - 10c_2c_4 + 6c_3^2 - 8c_2^4 + 20c_3c_2^2) e_n^5 + \ldots\}$ … (13)

Expanding $f(y_n)$ about $\alpha$ , to obtain

$f(y_n) = f(\alpha) + (y_n - \alpha) f'(\alpha) + \frac{(y_n-\alpha)^2}{2!} f''(\alpha) + \frac{(y_n-\alpha)^3}{3!} f'''(\alpha) + \ldots$ … (14)

Using (13) to get

$f(y_n) = f'(\alpha)\{ c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4) e_n^4 + (4c_5 - 12c_2^4 - 10c_2c_4 + 24c_3c_2^2 - 6c_3^2) e_n^5 + \ldots\}$ … (15)

From (15) and (11) , one can get

$f'(y_n) = f'(\alpha)\{1 + 2c_2^2 e_n^2 + 4(c_2c_3 - c_2^3) e_n^3 + (6c_2c_3 - 11c_3c_2^2 + 8c_2^4) e_n^4 + (8c_2c_3 + 28c_3c_2^3 - 20c_4c_2^2 - 16c_2^5) e_n^5 + \ldots\}$ … (16)

$f''(y_n) = f''(\alpha) + (y_n - \alpha) f'''(\alpha) + \frac{(y_n-\alpha)^2}{2!} f''''(\alpha) + \ldots$

$f''(y_n) = f'(\alpha)\{2c_2 + 6c_2c_3 e_n^2 + 12(c_3^2 - c_3c_2^2) e_n^3 + (24c_3c_2^3 + 18c_4c_3 + 12c_4c_2^2 - 42c_2c_3^2)) e_n^4 + \ldots\}$ … (17)

$\frac{2f(y_n)f'(y_n)}{2f'^2(y_n) - f(y_n)f''(y_n)} = c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4) e_n^4 + (4c_2^4 - 4c_3c_2^2) e_n^5 + \ldots\}$ … (18)

Since $z_n = y_n - \frac{2f(y_n)f'(y_n)}{2f'^2(y_n) - f(y_n)f''(y_n)}$ … (19)

Therefore , using (13) and (18) in (19) , yields

$z_n = \alpha + (-12c_2^4 + 24c_3c_2^2 + 4c_5 - 10c_2c_4 - 6c_3^2) e_n^5$ … (20)

Expanding f ( $z_n$ ) about $\alpha$ , one can get

f ( $z_n$ ) = $f'(\alpha)\{-12c_2^4 + 24c_3c_2^2 + 4c_5 - 10c_2c_4 - 6c_3^2 + 144c_2^9 - 576c_2^7c_3 - 96c_2^5c_5 + 240c_2^6c_4 + 720c_2^5c_3^2 + 192c_2^3c_3c_5 - 480c_2^4c_3c_4 - 288c_2^3c_3^3 + 16c_2c_5^2 - 80c_2^2c_4c_5 - 48c_2c_3^2c_5 + 100c_2^3c_4^2 + 120c_2^2c_3^2c_4 + 72c_2^5c_3^2 + 36c_2c_3^4) e_n^5$ … (21)

Using (15),(16) and (21) we have

$2[f(y_n) + f(z_n)] f'(y_n) = 2f'^2(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (7c_2^3 - 7c_2c_3 + 3c_4) e_n^4 + (56c_2^2c_3 - 32c_2^4 + 8c_5 - 20c_2c_4 - 12c_3^2 + 144c_2^9 - 576c_2^7c_3 - 96c_2^5c_5 + 240c_2^6c_4 + 720c_2^5c_3^2 + 192c_2^3c_3c_5 - 480c_2^4c_3c_4 - 288c_2^3c_3^3 + 16c_2c_5^2 -$

$80\ c_2^2 c_4\ c_5$ - $48c_2 c_3^2\ c_5$+100 $c_2^3\ c_4^2$ +120 $c_2^2 c_3^2\ c_4$+72 $c_2^5\ c_3^2$ + 36 $c_2\ c_3^4$ ) $e_n^5$ … (22)

By using (16) we get

$f'^2(y_n)$ = $f'^2(\alpha)\{1 + 4\ c_2^2 e_n^2$ +(8$c_2\ c_3$-8$c_2^3$) $e_n^3$ + (20 $c_2^4$ - 22 $c_2^2\ c_3$ + 12$c_2 c_4$ ) $e_n^4$ + ( 16 $c_2^3\ c_3$ -16 $c_2^5$ ) $e_n^5$ +…} … (23)

Again by (15),(17),(21) and (23) to calculate

$2f'^2(y_n) - [f(y_n) + f(z_n)]\, f''(y_n) =$ $2f'^2(\alpha)\{1 + 3c_2^2\ e_n^2$ + 6($c_2 c_3$-$c_2^3$) $e_n^3$ + (15$c_2^4$ -26$c_2^2 c_3$ +15$c_2 c_4$) $e_n^4$ + (-20 $c_2^3 c_3$ + 8 $c_2^5$ - 12 $c_2 c_3^2$ - 8$c_2\ c_5$+ 20 $c_2^2\ c_4$ +12$c_2 c_3$ -144$c_2^{10}$+576 $c_2^8 c_3$ + 96 $c_2^6 c_5$ - 240 $c_2^7\ c_4$ - 720 $c_2^6\ c_3$ -192 $c_2^4\ c_3\ c_5$ + 480 $c_2^5\ c_3\ c_4$ +288 $c_2^4\ c_3^3$ -16 $c_2^2 c_5^2$ +80$c_2^3 c_4\ c_5$ + 48$c_2^2 c_3^2\ c_5$ -50 $c_2^4\ c_4^2$ - 60 $c_2^3 c_3^2\ c_4$-36 $c_2^6\ c_3^2$ - 36 $c_2^2\ c_3^4$ ) $e_n^5$ +…} … (24)

Dividing (22) by (24)

$\frac{2[f(y_n)+f(z_n)]\ f'(y_n)}{2f'^2(y_n)-[f(y_n)+f(z_n)]\,f''(y_n)}$ = $c_2\ e_n^2$ + 2($c_3$-$c_2^2$) $e_n^3$ + (4$c_2^3$ -7$c_2 c_3$ +3$c_4$) $e_n^4$ + (144 $c_2^2 c_3$ - 25 $c_2^4$ +8 $c_4$ - 20 $c_2 c_4$ – 12 $c_3^2$ + 144 $c_2^9$ – 576 $c_2^7 c_3$ - 96 $c_2^5 c_5$ + 240 $c_2^6\ c_4$ + 720 $c_2^5\ c_3^2$ + 192 $c_2^3\ c_3\ c_5$ - 480 $c_2^4\ c_3\ c_4$ – 288 $c_2^3\ c_3^3$ + 16 $c_2 c_5^2$ - 80 $c_2^2 c_4\ c_5$ - 48$c_2 c_3^2\ c_5$+100 $c_2^3\ c_4^2$ +120 $c_2^2 c_3^2\ c_4$+72 $c_2^5\ c_3^2$ + 36 $c_2\ c_3^4$ ) $e_n^5$ … (25)

$$x_{n+1} = y_n - \frac{2[f(y_n)+f(z_n)]\ f'(y_n)}{2f'^2(y_n)-[f(y_n)+f(z_n)]\,f''(y_n)}$$

$x_{n+1}$ = $\alpha$ + (4 $c_5$ + 10 $c_2 c_4$ + 6$c_3^2$ + 17 $c_2^4$ - 24 $c_3 c_2^2$ - 8$c_4$ - 144$c_2^9$ + 576 $c_2^7 c_3$ + 96 $c_2^5 c_5$ - 240 $c_2^6 c_4$ - 720 $c_2^5\ c_3^2$ - 192 $c_2^3 c_3 c_5$ + 480 $c_2^4\ c_3\ c_4$ + 288 $c_2^3 c_3^3$ - 16 $c_2 c_5^2$ + 80 $c_2^2 c_4\ c_5$ + 48$c_2 c_3^2\ c_5$ - 100 $c_2^3\ c_4^2$ - 120 $c_2^2 c_3^2\ c_4$ - 72 $c_2^5\ c_3^2$ - 36 $c_2\ c_3^4$ ) $e_n^5$ + 0( $e_n^6$ )

$e_{n+1} = x_{n+1} - \alpha$

$e_{n+1}$ = (4 $c_5$ + 10 $c_2 c_4$ + 6$c_3^2$ + 17 $c_2^4$ - 24 $c_3 c_2^2$ - 8$c_4$ - 144$c_2^9$ + 576 $c_2^7 c_3$ + 96 $c_2^5 c_5$ - 240 $c_2^6 c_4$ - 720 $c_2^5\ c_3^2$ - 192 $c_2^3 c_3 c_5$ + 480 $c_2^4\ c_3\ c_4$ + 288 $c_2^3 c_3^3$ - 16 $c_2 c_5^2$ + 80 $c_2^2 c_4\ c_5$ + 48$c_2 c_3^2\ c_5$ - 100 $c_2^3\ c_4^2$ - 120 $c_2^2 c_3^2\ c_4$ - 72 $c_2^5\ c_3^2$ - 36 $c_2\ c_3^4$ ) $e_n^5$ + 0( $e_n^6$ )

Which implies that the three step method eqs. (7 – 9) has fifth order convergence .

## Numerical Results

Some examples are presented to illustrate the efficiency of the new three step method . The results are compared with the Newton method and two step Halley's method [18].The stopping criteria which is used for computer program is

$|x_{n+1} - x_n| < \varepsilon$ and $|f(x_{n+1})| < \varepsilon$ .

where $\varepsilon = 10^{-14}$ .

The test examples are

$f_1$(x) = $x^3 + 4x^2 - 10$

$f_2$(x) = $x^2 - e^x$ - 3x + 2

$f_3$(x) =sin x

$f_4$(x) =$x^3 - e^{-x}$

The number of iterations to approximate the zeros with different initial guess $x_0$ , and the approximate zero are displayed in Table 1 .

**Table 1: Results and comparisons**

| Function F(x) | $x_0$ | i NM | i T.S.H.M | i New method | Root |
|---|---|---|---|---|---|
| (1) $x^3 + 4x^2 - 10$ | 1<br>2 | 6<br>5 | 3<br>3 | 2<br>3 | 1.36523001341410<br>1.36523001341410 |
| (2) $x^2 - e^x - 3x + 2$ | 2<br>3 | 5<br>6 | 4<br>3 | 2<br>3 | 0.25753028543977<br>0.25753028543977 |
| (3) sin x | 1.5 | 5 | 5 | 3 | -12.56637061435917 |
| (4) $x^3 - e^{-x}$ | 0.473<br>1<br>1.073 | 6<br>6<br>6 | 3<br>4<br>4 | 3<br>3<br>3 | 0.77288295914921<br>0.77288295914921 |

NM :- Newton's method .
T.S.H.M :- Two step Halley's method .
i :- Number of iterations to approximate the root to 14 decimal places .
we Notice that from table (1) that the new three step method converges with equal or less number of iterations than the other methods .

## Conclusions:

It was noted that the new method is comparable with the well known existing methods and in many cases gives better results .
Our results can be considered as an improvement of the previously known results in the literature .

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# طريقة تكرارية جديدة لحل معادلات لاخطية

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**الخلاصة :**

الهدف من هذا البحث هو اقتراح طريقة تكرارية كفؤة ذات ثلاث خطوات لأيجاد الجذور للمعادلة اللاخطية f(x)=0 . بأختيار قيمة بدائية ملائمة لـ $x_0$ ، يكون الاقتراب للجذر مضمون بهذه الطريقة . اعطيت بعض الامثلة لتوضيح كفاءة الطريقة المقترحة الجديدة وقورنت مع طريقة Halleys̉ method .