An Approximated Solutions for n^{th} Order Linear Delay Integro-Differential Equations of Convolution Type Using B-Spline Functions and Weddle Method

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Abstract:

The paper is devoted to solve *n^{th} order linear delay integro-differential equations of convolution type* (DIDE's-CT) using collocation method with the aid of B-spline functions. A new algorithm with the aid of Matlab language is derived to treat numerically three types (retarded, neutral and mixed) of n^{th} order linear DIDE's-CT using B-spline functions and Weddle rule for calculating the required integrals for these equations. Comparison between approximated and exact results has been given in test examples with suitable graphing for every example for solving three types of linear DIDE's-CT of different orders for conciliated the accuracy of the results of the proposed method.

Key words: n^{th} Order Linear Delay Integro-Differential Equation of Convolution type, Collocation method, B-spline Functions and Weddle method.

Introduction:

One of the most important and applicable subjects of applied mathematics, and in developing modern mathematics is the integral equations. The names of many modern mathematicians notably, Volterra, Fredholm, Cauchy and others are associated with this topic [1].

The name integral equation was introduced by Bois-Reymond in 1888 [2]. However, in 1959 Volterra's book " Theory of Functional and of Integral and Integro-Differential Equations" appeared [1].

The integral and integro-differential equations formulation of physical problems are more elegant and compact than the differential equation formulation, since the boundary conditions can be satisfied and embedded in the integral or integro-differential equation. Also the form of the solution to an integro-differential equation is often more stable for today's extremely fast machine computation. Delay integro-differential equation of convolution type has been developed over twenty years ago where one of its types widely is used in control systems and digital communication systems as, lag-lead compensation and spread spectrum designs [1,3].

In this paper, B-spline functions were employed with collocation method to solve n^{th} order linear (DIDE's-CT) where they are standard representation of smooth geometry in numerical calculations and the required integrals in this method are calculated using Weddle rule as well as Gauss elimination method has been used to solve the resulting equations.

To facilitate the presentation of the material that followed, a brief

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review of some background on the linear DIDE's-CT and their types are given in the following section.

Delay Integro-Differential Equation of Convolution Type (DIDE-CT):

Integro-differential equation (IDE) is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, mechanics, physics, chemistry astronomy, biology, economics, potential theory and electrostatics [3]. IDE is an equation involving one (or more) unknown function $y(t)$ together with both differential and integral operations on y. It means that it is an equation containing derivative of the unknown function $y(t)$, which appears outside the integral sign [1,4].

The delay integro-differential equation is a delay differential equation in which the unknown function $y(t)$ can appear under an integral sign [5]. The main difference between delay differential equation and ordinary differential equation is the kind of initial condition that should be used in delay differential equation differs from ordinary differential equation, so that one should specify in delay differential equations an initial functions on some intervals say $[t_0-\tau,t_0]$ and then try to find the solution for all $t \ge t_0$ [6,7].

When the kernel $k(t,x)$ in integral equation depends only on the difference $t-x$, such a kernel is called a difference kernel and the integral equation with this kind of kernel is called an integral equation of convolution type.

So, the general form of nth order linear delay integro-differential equation of convolution type denoted by (DIDE-CT) is given by:

$$\sum_{i=0}^{n} p_i(t)\frac{d^i y(t)}{dt^i}+\sum_{i=1}^{n} q_i(t)\frac{d^i y(t-\tau_i)}{dt^i}+\sum_{i=0}^{n} r_i(t)y(t-\tau_i)=$$
$$g(t)+\lambda\int_a^{b(t)} k(t-x)\,y(x-\tau)dx \qquad t\in[a,b(t)]$$
… (1)

with initial functions:

$$\left.\begin{array}{l} y(t)=\phi(t) \\ y'(t)=\phi'(t) \\ \quad\vdots \\ y^{(n-1)}(t)=\phi^{(n-1)}(t) \end{array}\right\} \quad for \quad t\le t_0 \,.$$

where $g(t),\ p_i(t),\ q_i(t),\ r_i(t), k(t-x)$ are known functions of t, $y(t)$ is the unknown function, λ is a scalar parameter (in this paper λ =1), a and $b(t)$ are the limits of the integral where a is a constant and $b(t)$ either is given constant or function of t and $\tau, \tau_0, \tau_1, \dots, \tau_n$ are fixed positive numbers. The integral term of eq.(1) can be classified into different kinds according to the limits of integral and the kernel. If the limit $b(t)$ in eq.(1) is constant ($b(t)=b$) then equation (1) is called a delay Fredholm integro-differential equation while if $b(t)=t$ in eq.(1), then eq.(1) is called a delay Volterra integro-differential equation [8,9]

The DIDE-CT is an important equation in many applications. Convolution can be found in various places in applied mathematics since it plays an important role in heat conduction, wave motion, time series analysis, control systems and digital communication systems [5,6].

DIDE's-CT are classified into three types [10, 11]:-

First type:- Equation (1) is called Retarded type if the derivatives of unknown function appear without difference argument (i.e. the delay comes in y only) and the delay appears in the integrand unknown function (i.e. $\tau \neq 0$).

Second type: Equation (1) is called a neutral type if the highest-order derivative of unknown function appears both with and without difference argument and the delay does not appear in the integrand function (i.e. $\tau = 0$).

Third type:- All other DIDE's-CT in eq.(1) are called mixed types, which are combination of the previous two types.

B-Spline Functions:

The n^th order B-splines as appropriately scaled n^th is divided into difference of truncated power function; these functions have several mathematical definitions [4].

B-spline functions have an explicit function form and are easy to integrate and differentiate [12]. Schoenberg [13] introduced the B-spline in 1949 and B-splines have been applied to geometric modeling since 1970's [4]. According Schoenberg, B-spline means spline basis and the letter B in B-spline stands for basis [4].

Given $t_0, t_1, ..., t_m$ knots $\in$ [0,1] with $t_0 < t_1 < ... < t_m$. Then, a B-spline of degree n is a parametric curve, B: $[0,1] \rightarrow R^n$.

Composed of basis B-spline of degree n: $B(t) = \sum_{i=0}^{m+1} p_i B_{i,n}(t) \qquad t \in [0,1]$.

where the p_i, i=0,1,…,m+1 are called control points or de Boor points.

The B-spline of degree n can be defined using the Cox-de Boor recursion formula as [4,13]:

$$B_{k,o}(t) = \begin{Bmatrix} 1 & if \ t_k \le t < t_{k+1} \\ 0 & otherwise \end{Bmatrix} \ldots (2)$$

$$B_{k,n}(t) = \frac{t - t_k}{t_{k+n} - t_k} B_{k,n-1}(t) + \frac{t_{k+n+1} - t}{t_{k+n+1} - t_{k+1}} B_{k+1,n-1}(t) \quad n \ge 1,\ k \ge 0 \ldots (3)$$

When the knots are equidistant the B-spline is said to be uniform otherwise it is non-uniform [14].

The B-spline can be defined in another way which is [13,15]:

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k} \quad k \ge 0, n \ge 0 \ldots (4)$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

There are (n+1) n^th degree B-spline polynomials for mathematical convenience, we usually set $B_{k,n}(t) = 0 \quad if \quad k < 0 \quad or \quad k > n$.

1. Some Types of B-Spline Functions [4,13,14]:

1.1 The Constant B-spline $B_{k,0}(t)$:

The constant B-spline or B-spline of order 0 is the simplest spline. It is defined at only one knot span and is not even continues on the knots.

$$B_{k,o}(t) = \begin{Bmatrix} 1 & if \ t_k \le t < t_{k+1} \\ 0 & otherwise \end{Bmatrix}$$

1.2 The Linear B-spline $B_{k,1}(t)$:

The linear B-spline or the first order of B-spline is defined on two consecutive knot spans and is continues on the knots.

$$B_{k,1}(t) = \begin{Bmatrix} \dfrac{t - t_k}{t_{k+1} - t_k} & if \ t_k \le t < t_{k+1} \\ \dfrac{t_{k+2} - t}{t_{k+2} - t_{k+1}} & if \ t_{k+1} \le t < t_{k+2} \\ 0 & t \ge t_{k+2} \quad or \quad t < t_k \end{Bmatrix}$$

or $\quad B_{0,1}(t) = 1 - t \quad , \quad B_{1,1}(t) = t$

1.3 Quadratic B-spline $B_{k,2}(t)$:

Quadratic B-spline (or the 2^nd order of B-spline) with uniform knot-vector is a commonly used form of B-spline which is:

$$B_{k,2}(t)=\begin{cases}\dfrac{(t-t_k)^2}{(t_{k+2}-t_k)(t_{k+1}-t_k)} & if\ t_k<t\le t_{k+1}\\[2ex] \dfrac{(t-t_k)(t_{k+2}-t)}{(t_{k+2}-t_k)(t_{k+2}-t_{k+1})}+\dfrac{(t_{k+3}-t)(t-t_{k+1})}{(t_{k+3}-t_{k+1})(t_{k+2}-t_{k+1})} & if\ t_{k+1}<t<t_{k+2}\\[2ex] \dfrac{(t_{k+3}-t)^2}{(t_{k+3}-t_{k+1})(t_{k+3}-t_{k+2})} & if\ t_{k+2}\le t<t_{k+3}\\[2ex] 0 & t\ge t_{k+3}\quad or\quad t\le t_k\end{cases}$$

or $B_{0,2}(t)=(1-t)^2$, $B_{1,2}(t)=2t(1-t)$ *and* $B_{2,2}(t)=t^2$.

1.4 Cubic B-spline $B_{k,3}(t)$:

Cubic B-spline (or the 3rd order of B-spline) with uniform knot-vector is the most commonly used form of B-spline which is:

$$B_{0,3}(t)=(1-t)^3\ ,\quad B_{1,3}(t)=3t(1-t)^2,\quad B_{2,3}(t)=3t^2(1-t)\quad and\quad B_{3,3}(t)=t^3.$$

***2 Some Properties of B-Spline Functions* [12,15]:**

2.1 The Integration property:

For $k=0,1,...,n\ \ and\ \ n\ge 0$:

$$\int_0^1 B_{k,n}(t)=\frac{1}{n+1}$$

2.2 The Differentiation property:

The ith derivative of B-spline polynomials $B_{k,n}(t)$ is given by:

$$\frac{d^iB_{k,n}(t)}{dt^i}=\frac{n!}{(n-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{k+r-i,n-i}(t)$$

.

2.3 The Product property:

For $n,m\ge 0,\ \ i=0,1,...,n\ \ and\ \ j=0,1,...,m$:

$$B_{i,n}(t)*B_{j,m}(t)=\frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}}B_{i+j,n+m}(t)\ .$$

Weddle Method:

Weddle method is one of the basic formula of quadrature approximation methods for integration. Quadrature rule is generic name given to any numerical method for the approximate calculation of definite integral $I[u]$ of the function $u(t)$ over finite integral $[a,b]$ which is [1,3] :

$$I[u]=\int_a^b u(t)dt\qquad a<b.$$

Weddle formula approximates the function on the interval $[t_0,t_6]$ by a curve that possesses through seven points. When it is applied over the interval [a,b], the composite Weddle rule is obtained as [1,4]:

$$\int_a^b f(t)dx=\frac{3H}{10}\begin{bmatrix} f_0+5f_1+f_2+6f_3+f_4+5f_5+\\ 2f_6+5f_7+f_8+6f_9+f_{10}+\\ 5f_{11}+\cdots+2f_{N-6}+5f_{N-5}+f_{N-4}+\\ 6f_{N-3}+f_{N-2}+5f_{N-1}+f_N\end{bmatrix}$$

… (5)

where a, b are the limits of the integral, $H=\frac{(b-a)}{N}$, N is the number of intervals ($[t_0,t_1],[t_1,t_2],\ldots,[t_{N-1},t_N]$) which is the multiple of (6), $f_i=f(t_i)$ $t_0=a$, $t_N=b$ and $t_i=a+iH$ are called the integration nodes which are lying in the interval [a,b] where $i=0,1,\ldots,N$.

The Solution of n^{th} Order Linear DIDE-CT Using Collocation Method with B-Spline Functions and Weddle Rule:

Collocation method [16,17] is one of the efficient methods used to solve differential and integro-differential equations without time lag. In this section, collocation method with the aid of B-Spline functions and Weddle rule are candidates to find the approximated solutions for three types of n^{th} order DIDE's-CT as follows:

Recall eq.(1), to solve it the unknown function y(t) is approximated by a set of B-spline functions as:

$$y(t) \cong y_M(t_j) = \sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t_j) \ldots (6)$$

where

$j = 0,1,\ldots,M$, $M > 0$ *and* $c_0, c_1, \ldots, c_M$ are (M+1) unknown coefficients.

By substituting eq.(6) into eq.(1) and by putting $t = t_j$ one gets the following formula:

$$\sum_{i=0}^{n} p_i(t_j)\frac{d^i}{dt^i}\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t_j) + \sum_{i=1}^{n} q_i(t_j)\frac{d^i}{dt^i}\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t_j - \tau_i) + \sum_{i=0}^{n} r_i(t_j)\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t_j - \tau_i)$$
$$= g(t_j) + \int_a^{b(t_j)} k(t_j - x)\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(x - \tau)dx \quad \ldots (7)$$

Hence, by using B-spline's property (3.2.2) for eq.(7) yields:

$$\sum_{i=0}^{n} p_i(t_j)\sum_{\alpha=0}^{M} c_\alpha\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j)\right) +$$
$$\sum_{i=1}^{n} q_i(t_j)\sum_{\alpha=0}^{M} c_\alpha\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j - \tau_i)\right) +$$
$$\sum_{i=0}^{n} r_i(t_j)\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t_j - \tau_i) -$$
$$\int_a^{b(t_j)} k(t_j - x)\sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(x - \tau)dx = g(t_j)$$

then,

$$\sum_{\alpha=0}^{M} c_\alpha\left[\begin{array}{l}\sum_{i=0}^{n} p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j)\right) + \\ \sum_{i=1}^{n} q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j - \tau_i)\right) + \\ \sum_{i=0}^{n} r_i(t_j)B_{\alpha,M}(t_j - \tau_i) - \int_a^{b(t_j)} k(t_j - x)B_{\alpha,M}(x - \tau)dx\end{array}\right] = g(t_j)$$
$\ldots$ (8)

where $j = 0,1,\ldots,M$ *and* $M > 0$.

In Collocation method the unknown coefficients $c_0, c_1, \ldots, c_M$ in eq.(8) are chosen to minimize the residual equation $E_M(t)$ by setting its weighted integral equal to zero, i.e.

$$\int_D w_j E_M(t)dt = 0 \quad j = 0,1,\ldots,M$$
... (9)

where D is a prescribed domain and w_j are weighting functions which are:

$$w_j = \begin{cases}1 & if\ t = t_j \\ 0 & otherwise\end{cases} \ldots (10)$$

where the fixed points $t_j \in D$ *and* $j = 0,1,\ldots,M$ are called collocation points.

By substituting eq.(10) into eq.(9) yields:

$$\int_D w_j E_M(t)dt = E_M(t_j)\int_D w_j dt = 0 \rightarrow$$
$$E_M(t_j) = 0 \quad j = 0,1,\ldots,M \quad \ldots (11)$$

The residual equation $E_M(t_j)$ of DIDE-CT is defined by:

$$E_M(t_j)=\sum_{\alpha=0}^{M}c_\alpha\begin{bmatrix}\sum_{i=0}^{n}p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j)\right)+\\ \sum_{i=1}^{n}q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j-\tau_i)\right)+\\ \sum_{i=0}^{n}r_i(t_j)B_{\alpha,M}(t_j-\tau_i)-\int_a^{b(t_j)}k(t_j-x)B_{\alpha,M}(x-\tau)dx\end{bmatrix}-g(t_j) \qquad \dots (12)$$

By substituting eq.(12) into eq.(11) we get:

$$E_M(t_j)=\sum_{\alpha=0}^{M}c_\alpha\begin{bmatrix}\sum_{i=0}^{n}p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j)\right)+\\ \sum_{i=1}^{n}q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j-\tau_i)\right)+\\ \sum_{i=0}^{n}r_i(t_j)B_{\alpha,M}(t_j-\tau_i)-\int_a^{b(t_j)}k(t_j-x)B_{\alpha,M}(x-\tau)dx\end{bmatrix}-g(t_j)=0$$

$for\ \ j=0,1,\ldots,M$.

Hence,

$$\sum_{\alpha=0}^{M}c_\alpha\begin{bmatrix}\sum_{i=0}^{n}p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j)\right)+\\ \sum_{i=1}^{n}q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{\alpha+r-i,M-i}(t_j-\tau_i)\right)+\\ \sum_{i=0}^{n}r_i(t_j)B_{\alpha,M}(t_j-\tau_i)-\int_a^{b(t_j)}k(t_j-x)B_{\alpha,M}(x-\tau)dx\end{bmatrix}=g(t_j) \qquad \dots (13)$$

The values required integrals in eq.(13) are evaluated numerically using Weddle method in eq.(5) as follows:

Let $\psi(t_j,x)=k(t_j-x)\ B_{\alpha,M}(x-\tau)$, then

$$\int_a^{b(t_j)}\psi(t_j,x)dx=Weddle\big(\psi(t_j,x),a,b(t_j),N\big)=$$

$$\frac{3H}{10}\begin{bmatrix}\psi(t_j,a)+5\psi(t_j,x_1)+\psi(t_j,x_2)+6\psi(t_j,x_3)+\\ \cdots+2\psi(t_j,x_{N-6})+5\psi(t_j,x_{N-5})+\psi(t_j,x_{N-4})+\\ 6\psi(t_j,x_{N-3})+\psi(t_j,x_{N-2})+5\psi(t_j,x_{N-1})+\psi(t_j,b(t_j))\end{bmatrix} \qquad \dots (14)$$

where $H=\frac{(b(t_j)-a)}{N}$, $x_i=a+iH$ and $i=0,1,\ldots,N.$

By substituting eq.(14) into eq.(13) we get:

$$\sum_{\alpha=0}^{M} c_\alpha \left[\begin{array}{l} \sum_{i=0}^{n} p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r \binom{i}{r} B_{\alpha+r-i,M-i}(t_j)\right)+ \\ \sum_{i=1}^{n} q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r \binom{i}{r} B_{\alpha+r-i,M-i}(t_j-\tau_i)\right)+ \\ \sum_{i=0}^{n} r_i(t_j) B_{\alpha,M}(t_j-\tau_i) - Weddle\left(\psi(t_j,x),a,b(t_j),N\right) \end{array} \right] = g(t_j) \quad \dots (15)$$

So, by evaluating eq.(15), we have (M+1) simultaneous equations with (M+1) unknown coefficients $c_0, c_1, \dots, c_M$.

Hence, eq.(15) can be written in matrices form as $DC=G$ which they:

$$\left. \begin{array}{l} D = \begin{bmatrix} d_{00} & d_{01} & \dots & d_{0M} \\ d_{10} & d_{11} & \dots & d_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ d_{M0} & d_{M1} & \dots & d_{MM} \end{bmatrix}_{(M+1)\times(M+1)} \\ , C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{bmatrix} \quad and \quad G = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_M) \end{bmatrix} \end{array} \right\} \quad \dots (16)$$

where

$$d_{\alpha,j} = \left[\begin{array}{l} \sum_{i=0}^{n} p_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r \binom{i}{r} B_{\alpha+r-i,M-i}(t_j)\right)+ \\ \sum_{i=1}^{n} q_i(t_j)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r \binom{i}{r} B_{\alpha+r-i,M-i}(t_j-\tau_i)\right)+ \\ \sum_{i=0}^{n} r_i(t_j) B_{\alpha,M}(t_j-\tau_i) - Weddle\left(\psi(t_j,x),a,b(t_j),N\right) \end{array} \right]$$

for $\alpha = 0,1,\dots,M$ and $j = 0,1,\dots,M$.

Then, use Gauss elimination method to find the coefficients $c_\alpha{}'s, \alpha = 0,1,...,M$ which satisfy eq.(6) (the approximate solution $y(t)$ of eq.(1)).

The solution of three types nth order linear ***DIDE's-CT*** using ***C****ollocation* method with ***B-S****pline* functions *and Weddle* method can be summarized by the following algorithm:

DIDECT-CBSW Algorithm :

INPUT

- n : (the order of DIDE-CT).
- N: (the number of intervals of Weddle method)
- M: (the order of B-spline function $B_{k,M}(t)$).
- $t_0, t_1, ..., t_M$: (the (M + 1) collocation points).
- a & $b(t_j)$: (the limits of the integral of DIDE-CT) .
- The function $g(t)$ of DIDE-CT.
- The difference kernel of the DIDE-CT.

OUTPUT

- $c_\alpha{}'s, \alpha = 0,1,...,M$ (the unknown coefficients of eq.(6)).
- $y_M(t)$: (the approximate solution of DIDE-CT)

Step 1: Set $y_M(t) = c_0 B_{0,M}(t) + c_1 B_{1,M}(t) + \dots + c_M B_{M,M}(t)$

Step 2: Define $\psi(t_j, x)$ in eq.(14).

Step3: Compute B-splines $B_{\alpha,M}(t), \alpha = 0,1,\dots,M$ in (step 1) as:

(a) Set $\alpha = 0$

(b) For i = α : M do step (c)

(c) Sum=Sum+ $(-1)^{(i-\alpha)} \binom{M}{i} \binom{i}{\alpha} t^i$

(d) $B_{\alpha,M}(t) =$ Sum

(e) Set $\alpha = \alpha + 1$

(f) If α $=M+1$ then stop and go to (step 4). Else go to step (b)

Step 4: Set $j = 0$

Step 5: Compute eq.(10)

Step 6: Put $j = j+1$

Step 7: If $j = M+1$ then stop and go to (step 8). Else go to (step 5)

Step 8: Find the B-spline functions in (step 5) using eq.(4).

Step 9: Express the $(M+1)$ simultaneous equations in step(5) by matrices form $DC=G$ as eq.(16).

Step 10: Use Gauss elimination method for finding the coefficients c_α's , $\alpha = 0,1,\ldots,M$ which satisfy the solution $y(t)$ in (step 1).

6. Test Examples:

Example (1):

Consider the following 1st order linear Retarded Volterra integro-differential equation of convolution type [11]:

$$\frac{dy(t)}{dt} + t\,y(t-\tfrac{1}{2}) = \frac{1}{2}\left(1 - t + e^{t}\right) + \int_0^t e^{(t-x)}\, y(x-\tfrac{1}{2})\,dx \qquad 0 \le t \le 0.5 \qquad \ldots (17)$$

with initial function:

$$y(t) = e^{t} - \tfrac{1}{2} \qquad -0.5 \le t \le 0 .$$

The exact solution of eq.(17) is: $y(t) = t + \frac{1}{2} \qquad 0 \le t \le 0.5$.

Assume the approximate solution of eq.(17) in the form:

$$y_M(t) = \sum_{\alpha=0}^{M} c_\alpha B_{\alpha,M}(t)$$

When the algorithm (DIDECT-CBSW) is applied, table (1) presents the comparison results between the exact and collocation with B-Spline functions and Weddle method for eq.(17) depending on least square error (L.S.E.) where m=10, h=0.05, $t_j = jh$, $j = 0,1,\ldots,m$.

Table (1) The solution of Ex.(1).

t	*Exact*	*Collocation with B-Splines and Weddle (DIDECT-CBSW) $y_M(t)$*	
		M=1	***M=2***
0	0.5000000	0.5000000	0.5000000
0.05	0.5500000	0.5500000	0.5500000
0.10	0.6000000	0.6000000	0.6000000
0.15	0.6500000	0.6500000	0.6500000
0.20	0.7000000	0.7000000	0.7000000
0.25	0.7500000	0.7500000	0.7500000
0.30	0.8000000	0.8000000	0.8000000
0.35	0.8500000	0.8500000	0.8500000
0.40	0.9000000	0.9000000	0.9000000
0.45	0.9500000	0.9500000	0.9500000
0.50	1.0000000	1.0000000	1.0000000
L.S.E.		**0.0000000**	**0.0000000**

Fig. (1) shows the solution of eq.(17) using DIDECT-CBSW algorithm and the exact solution.

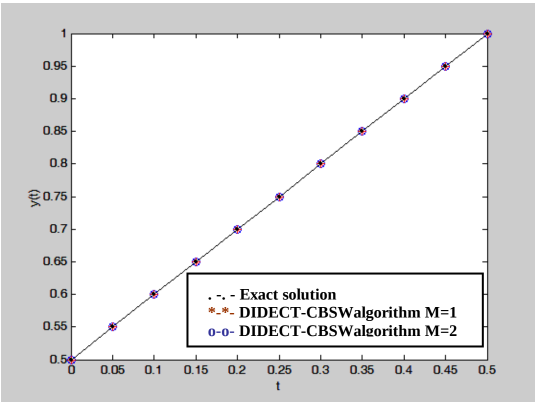

Fig.(1) The comparison between the exact and DIDECT-CBSW algorithm for eq.(17) in Ex.(1).

Example (2):

Consider the following second order neutral Volterra integro-differential equation of convolution type [10]:

$$\frac{d^2y(t-1)}{dt^2}+\frac{dy(t-0.5)}{dt}=\left(-6\sin t-t^3+3t^2+9t-\frac{21}{4}\right)+\int_0^t \sin(t-x)\,y(x)dx \qquad 0\le t\le 1 \quad \dots (18)$$

with initial functions :

$$y(t)=t^3 \qquad t\le 0$$

$$y'(t)=3t^2 \qquad t\le 0$$

The exact solution of eq.(18) is:

$$y(t)=t^3 \qquad 0\le t\le 1 \ .$$

Assume the approximate solution of eq.(18) in the form:

$$y_4(t)=\sum_{\alpha=0}^{4} c_\alpha B_{\alpha,4}(t)$$

When the algorithm (DIDECT-CBSW) is applied, table (2) presents the comparison between the exact and approximated solutions for eq.(18) using collocation with B-spline functions and Weddle method for m=10, h=0.1, $t_j = jh$, $j=0,1,...,m$ with least square error (L.S.E.).

Table (2) The solution of Ex.(2).

t	***Exact***	***(DIDECT-CBSW) algorithm $y_M(t)$***
0	0.0000000	0.0000000
0.1	0.0010000	0.0010000
0.2	0.0080000	0.0080000
0.3	0.0270000	0.0270000
0.4	0.0640000	0.0640000
0.5	0.1250000	0.1250000
0.6	0.2160000	0.2160000
0.7	0.3430000	0.3430000
0.8	0.5120000	0.5120000
0.9	0.7290000	0.7290000
1	1.0000000	1.0000000
L.S.E.		**0.0000000**

Fig. (2) shows the solution of eq.(18) by using DIDECT-CBSW algorithm and the exact solution.

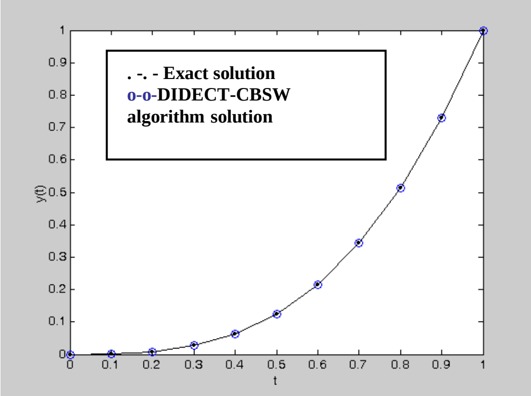

Fig.(2) The comparison between the exact and DIDECT-CBSW solution for eq.(18) in Ex.(2)

Example (3):

Consider the following third order mixed Fredholm integro-differential equation of convolution type [6]:

$$\frac{d^3y(t-1)}{dt^3}-y(t)=\left(-t^4+\frac{119}{5}t-\frac{719}{30}\right)+\int_0^1 (t-x)y(x-1)dx \quad 0\le t\le 1 \quad \dots (19)$$

with initial functions :

$$\left.\begin{array}{l} y(t)=t^4 \\ y'(t)=4t^3 \\ y''(t)=12t^2 \end{array}\right\} \quad -1\le t\le 0 .$$

The exact solution of eq.(19) is:

$$y(t)=t^4 \qquad 0\le t\le 1$$

Assume the approximate solution of eq.(19) in the form:

$$y_5(t)=\sum_{\alpha=0}^{5} c_\alpha B_{\alpha,5}(t)$$

When the algorithm (DIDECT-CBSW) is applied, table (3) presents the comparison between the exact and approximate solutions of eq.(19) using collocation with B-spline functions and Weddle method for m=10, h=0.1, $t_j = jh$, $j = 0,1,\ldots,m$ depending on least square error (L.S.E.).

Table (3) The solution of Ex.(3).

t	*Exact*	*B-Spline and Weddle (DIDECT-BSB) y(t)*
0	0.0000000	0.0000000
0.1	0.0001000	0.0001000
0.2	0.0016000	0.0016000
0.3	0.0081000	0.0081000
0.4	0.0256000	0.0256000
0.5	0.0625000	0.0625000
0.6	0.1296000	0.1296000
0.7	0.2401000	0.2401000
0.8	0.4096000	0.4096000
0.9	0.6561000	0.6561000
1	1.0000000	1.0000000
L.S.E.		**0.0000000**

Fig.(3) shows the solution of eq.(19) by using DIDECT-CBSW algorithm and the exact solution.

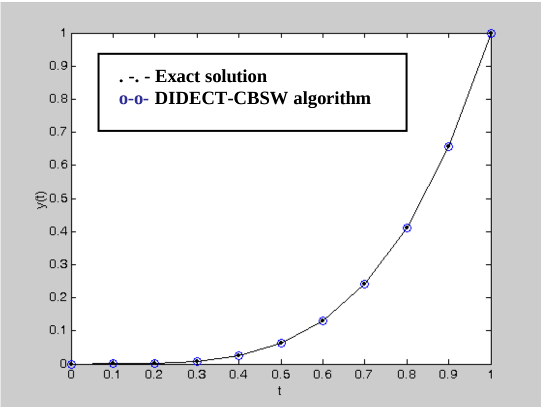

Fig.(3) The comparison between the exact and DIDECT-CBSW solution for eq.(19) in Ex.(3)

Conclusions:

Collocation method with the aid of B-Spline functions and Weddle method have been presented to find the approximated solutions for n^{th} order retarded, neutral and mixed linear DIDE's-CT. The results show a marked improvement in the least square error (L.S.E.). From solving three test examples, the following points are drawn:

1. Collocation method with B-spline functions and Weddle method give qualified way for solving 1^{st} order linear DIDE's-CT as well as n^{th} order linear DIDE-CT
2. The good approximation of Weddle method depends on the size of *H*, if *H* is decreased then the number of nodes increases and the L.S.E. approaches to zero where this gives the advantage in numerical computation.
3. The good approximation solution of DIDECT-CBSW algorithm depends on the number *M* of B-spline functions where as *M* increased, the error term approaches to zero.

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الحلول التقريبية للمعادلات التفاضلية- التكاملية التباطؤية الألتفافية الخطية من الرتبة n باستخدام الدوال الثلمة التوصيلية و طريقة ويدل

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الخلاصة:

البحث مكرس لحل المعادلات التكاملية التفاضلية التباطؤية الألتفافية الخطية من الرتبة n باستخدام طريقة التجميع مع الدوال الثلمة التوصيلية. حيث تم اشتقاق خوارزمية جديدة تمت برمجتها بلغة (Matlab) لمعالجة ثلاثة أنواع من المعادلات التفاضلية- التكاملية التباطؤية الألتفافية الخطية و المتضمنة (التراجعية ، المتعادلة والمختلطة) من الرتبة n باستخدام طريقة التجميع مع الدوال الثلمة التوصيلية مع استخدام قاعدة ويدل لحساب التكاملات للطريقة المقترحة حيث من الممكن ملاحظة كفاءة الطريقة و دقة الحسابات فيها. كما تمت مقارنة النتائج التقريبية و الحقيقية لثلاثة أنواع من الرتب المختلفة لهذه المعادلات من خلال الأمثلة و الرسوم وقد تم الحصول على نتائج دقيقة.