ON CERTAIN SUB-SPACE OF X

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ABSTRACT:

The study of properties of space of entire functions of several complex variables was initiated by Kamthan [4] using the topological properties of the space. We have introduced in this paper the sub-space of space of entire functions of several complex variables which is studied by Kamthan.

Keywords: Entire functions; several complex variables

INTRODUCTION:

The space of entire functions over the complex field C was introduced by V.G. Iyer [3] who defined a metric on this space by introducing a real-valued map on it. Spaces of entire functions of several complex variables occupy an important position in view of their vast applications in various branches of mathematics, for instance, the classical analysis, theory of approximation, theory of topological bases etc. Kamthan [4] studied the properties of space of entire functions of several complex variables. Many of eminent mathematician such as Devendra [1], Hazem [2], Kumar [5], Mushtaq [6], Sirivastava [7], and others have contributed richly to space of entire functions of several complex variables.

Let C denote the complex plane, and I be the set of non-negative integers.

We write for $n \in I$

$$C^n = \{(z_1, z_2, \ldots, z_n);\ z_i \in C,\ 1 \le i \le n\}$$

.

$$I^n = \{(p_1, p_2, \ldots, p_n);\ p_i \in I,\ 1 \le i \le n\}$$

C^n and I^n are respectively Banach and metric spaces under the functions

$$\|(z_1, z_2, \ldots, z_n)\| = |z_1| + |z_2| + \ldots + |z_n|$$

$$\|(p_1, p_2, \ldots, p_n)\| = p_1 + p_2 + \ldots + p_n$$

We are concerned here with the space of all entire functions from C^n to C under the usual pointwise addition and scalar multiplication. For the sake of simplicity we consider the case when $n = 2$, though my results can be easily extended to any positive integer n.

Let therefore, X be the space of all entire functions $f : C^2 \to C$, where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n}\, z_1^{\,m} z_2^{\,n},$$

$a_{m,n} \in C$ for $m, n \ge 0$,

and assume that X is equipped with the topology T of uniform convergence on compact in C^n. For more details see Kamthan [4].

1. In this paper X denotes the space of all entire functions as in Kamthan [4]; S denotes the space of all double complex sequences, I^+ is the set of all positive integers, and $I = \{0\} \cup I^+$. Let

$X(\lambda) = \{ f \in X : \lambda \circ f \in X \}$,

where $\lambda = \{ l_{m,n},\ m, n \ge 0,\ m + n \ne 0 \}$ is a fixed element of S such that no coordinate element of λ is zero, and

$$\lambda \circ f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} l_{m,n}\, a_{m,n}\, z_1^{\,m} z_2^{\,n},$$

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where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} \ z_1^{\ m} z_2^n \ .$$

Clearly, $X(\lambda) \neq \varphi$ since every polynomial in z with complex coefficients is in $X(\lambda)$.

It is easy to see that $X(\lambda)$ is a linear sub-space of X.

We now state two theorems, the proofs of which are left to the reader.

Theorem 1.1. $X(\lambda) = X$ if, and only if,

$$\limsup_{|m,n| \to \infty} \left\{ \left| l_{0,0} \right| ; \left| l_{m,n} \right|^{1/(m+n)} , \ m,n \geq 0 \ , \ m+n \neq 0 \right\} < \infty$$

Theorem 1.2. If $\lambda = \{ \left| l_{m,n} \right|, \ m,n \geq 0 \}$, $\mu = \{ \left| k_{m,n} \right|, \ m,n \geq 0 \}$ are any two fixed elements of S, and

$$\left\{ \left| \frac{k_{0,0}}{l_{0,0}} \right| , \left| \frac{k_{m,n}}{l_{m,n}} \right|^{1/(m+n)} , \ m,n \geq 0 \ , \ m+n \neq 0 \right\}$$

is bounded, then $X(\lambda) \subset X(\mu)$.

Remark. The condition stated in Theorem 1.2 is not necessary. For, let λ, μ be such that

$$l_{0,0} = 1, \ l_{m,n} = \frac{1}{m!n!}$$

$$k_{0,0} = 1, \ k_{m,n} = 1$$

Both $\left\{ \left| l_{0,0} \right| , \left| l_{m,n} \right|^{1/(m+n)} \right\}$ and $\left\{ \left| k_{0,0} \right| , \left| k_{m,n} \right|^{1/(m+n)} \right\}$ are bounded sequences, so that, using Theorem 1.1, we note that both $X(\lambda)$ and $X(\mu)$ equal X.

Thus $X(\lambda) \subset X(\mu)$ is trivially true. But

$$\left| \frac{k_{m,n}}{l_{m,n}} \right|^{1/(m+n)} = (m!n!)^{1/(m+n)} \to \infty \ \text{as}$$

$m,n \to \infty$, so that

$$\left\{ \left| \frac{k_{0,0}}{l_{0,0}} \right| ; \left| \frac{k_{m,n}}{l_{m,n}} \right|^{1/(m+n)} , \ m,n \geq 0 \ , \ m+n \neq 0 \right\}$$

is bounded.

Theorem 1.3. If either $\left| l_{m,n} \right|^{1/(m+n)}$ or $\left| k_{m,n} \right|^{1/(m+n)}$ tends to infinity, then a necessary and sufficient condition for the relation $X(\lambda) \subset X(\mu)$ to hold is that

$$\left\{ \left| \frac{k_{0,0}}{l_{0,0}} \right| ; \left| \frac{k_{m,n}}{l_{m,n}} \right|^{1/(m+n)} , \ m,n \geq 0 \ , \ m+n \neq 0 \right\}$$

is bounded.

Proof. The sufficiency follows from Theorem 1.2 even without the extra hypothesis. To prove necessity with the extra hypothesis, suppose that

$$\left\{ \left| \frac{k_{0,0}}{l_{0,0}} \right| , \left| \frac{k_{m,n}}{l_{m,n}} \right|^{1/(m+n)} , \ m,n \geq 0 \ , \ m+n \neq 0 \right\}$$

is bounded.

Then this sequences has a sub-sequences, say

$$\left| \frac{k_{m_1,n_1}}{l_{m_1,n_1}} \right|^{1/(m_1+n_1)} , \left| \frac{k_{m_2,n_2}}{l_{m_2,n_2}} \right|^{1/(m_2+n_2)} , \left| \frac{k_{m_3,n_3}}{l_{m_3,n_3}} \right|^{1/(m_3+n_3)} ,\ldots$$

, which tends to infinity.

Define

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} \ z_1^{\ m} z_2^n \quad \text{by}$$

$$a_{m_p,n_q} = \frac{1}{\left| l_{m_p,n_q} \right| + \left| k_{m_p,n_q} \right|} \ , (p,q \in I^+) \ , \ a_{m,n} = 0$$

otherwise.

Then, with the extra hypothesis, $f \in X$. Since $\left| a_{m_p,n_q} \right|^{1/(m_p+n_q)}$ cannot exceed, either

$$\frac{1}{\left| l_{m_p,n_q} \right|^{1/(m_p+n_q)}} \ \text{or} \ \frac{1}{\left| k_{m_p,n_q} \right|^{1/(m_p+n_q)}}$$

, and one of these tends to zero as $m_p \, , \, n_q \to \infty$.

Now

$$\left|l_{m_p,n_q}\ a_{m_p,n_q}\right|^{1/(m_p+n_q)} = \frac{1}{\left[1+\left|\frac{k_{m_p,n_q}}{l_{m_p,n_q}}\right|\right]^{1/(m_p+n_q)}} \le \frac{1}{\left|\frac{k_{m_p,n_q}}{l_{m_p,n_q}}\right|^{1/(m_p+n_q)}},$$

so that

$$\lim_{m_p,n_q\to\infty}\left|l_{m_p,n_q}\ a_{m_p,n_q}\right|^{1/(m_p+n_q)} = 0.$$

Thus $f \in X(\lambda)$. But

$$\left|k_{m_p,n_q}\ a_{m_p,n_q}\right|^{1/(m_p+n_q)} = \frac{1}{\left[\left|\frac{l_{m_p,n_q}}{k_{m_p,n_q}}\right|+1\right]^{1/(m_p+n_q)}} \ge \frac{1}{\left|\frac{l_{m_p,n_q}}{k_{m_p,n_q}}\right|^{1/(m_p+n_q)}+1},$$

so that

$$\lim_{m_p,n_q\to\infty}\left|k_{m_p,n_q}\ a_{m_p,n_q}\right|^{1/(m_p+n_q)} \ge 1,$$

since the sequence

$$\left|\frac{l_{m_1,n_1}}{k_{m_1,n_1}}\right|^{1/(m_1+n_1)}, \left|\frac{l_{m_2,n_2}}{k_{m_2,n_2}}\right|^{1/(m_2+n_2)}, \left|\frac{l_{m_3,n_3}}{k_{m_3,n_3}}\right|^{1/(m_3+n_3)}, \ldots$$

converge to zero. Hence $f \notin X(\mu)$. This show that the condition stated in the theorem is necessary.

2. $X(\lambda)$ is endowed with two topologies. One is the metric topology T inherited from X (vide [Kamthan]), its metric d being

$$d(f,g) = \sup\left[\left|a_{0,0}-b_{0,0}\right| . \left|a_{m,n}-b_{m,n}\right|^{1/(m+n)},\ m,n \ge 0,\ m+n \ne 0\right] \quad \ldots (1)$$

where

$$f(z_1,z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\ z_1^m z_2^n ,$$

$$g(z_1,z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{m,n}\ z_1^m z_2^n$$ are

any two elements of $X(\lambda)$. The other is the metric topology T_λ whose metric d_λ is given by

$$d_\lambda(f,g) = \sup\left[\begin{array}{l}\left|l_{0,0}\right| \left|a_{0,0}-b_{0,0}\right|, \left|l_{m,n}\right|^{1/(m+n)} \\ \left|a_{m,n}-b_{m,n}\right|^{1/(m+n)},\quad m,n\ge 0,\ m+n\ne 0\end{array}\right] \quad \ldots\ldots\ldots (2)$$

It is known that X is a complete metric space under its usual topology (vide [4]). We now prove that $(X(\lambda),T_\lambda)$ is complete under a condition to be stated in the following theorem.

Theorem 2.1. $(X(\lambda),T_\lambda)$ is a complete metric space if, and only if

$$\liminf\left\{\left|l_{0,0}\right|, \left|l_{m,n}\right|^{1/(m+n)}\right\} > 0$$

Proof (i) Sufficient. Let $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ be a Cauchy sequence in $(X(\lambda),T_\lambda)$, where

$$f_{p,q}(z_1,z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{p_m,q_n}\ z_1^m z_2^n$$

(Since each $f_{p,q} \in X(\lambda)$, $\lambda \circ f_{p,q} \in X$ for all $p \in I^+$).We then have $d_\lambda(f_{p,q}, f_{i,j}) \to 0$ as $p,i \to \infty$ and $q,j \to \infty$. Hence given an $\varepsilon > 0$, we can find a $p_{0,0} \in I^+$ such that $d_\lambda(f_{p,q}, f_{i,j}) < \varepsilon$ for all $p,i \ge p_0$, $q,j \ge q_0$.

Now

$$d_\lambda(f_{p,q}, f_{i,j}) = \sup\left[\left|l_{0,0}\right| \left|a_{p_0,q_0}-a_{i_0,j_0}\right|, \left|l_{m,n}\right|^{1/(m+n)} \left|a_{p_m,q_n}-a_{i_m,j_n}\right|^{1/(m+n)}\right],$$

so that

$$\left|l_{0,0}\right| \left|a_{p_0,q_0}-a_{i_0,j_0}\right| < \varepsilon,$$

$$\left|l_{m,n}\right|^{1/(m+n)} \left|a_{p_m,q_n}-a_{i_m,j_n}\right|^{1/(m+n)} < \varepsilon$$

for all $p,i \ge p_0$, $q,j \ge q_0$.

Let L be the infimum of the double sequence in the statement of the theorem. Then

$$L\left|a_{p_0,q_0}-a_{i_0,j_0}\right| < \varepsilon,$$

$L\left|a_{p_m,q_n}-a_{i_m,j_n}\right|^{1/(m+n)} < \varepsilon$ for all $p,i \ge p_0$, $q,j \ge q_0$. …(3).

Thus each of the double sequences $\left(a_{p_m,q_n}\right)_{m,n=0}^{\infty}$ is a Cauchy sequence of complex number, so that each of these sequences tends to a limit as $p,q \to \infty$. Let $a_{p_m,q_n} \to a_{m,n}$ for each $m,n \in I$. Using this fact in (3) , we have

$\left|a_{p_0,q_0}-a_{0,0}\right|<\frac{\varepsilon}{L}$,

$\left|a_{p_m,q_n}-a_{m,n}\right|^{1/(m+n)}<\frac{\varepsilon}{L}\quad (m,n\in I^+)$,

for all $p\geq p_0$, $q\geq q_0$ ....(4)

Now for each fixed $p,q\in I^+, \left|a_{p_m,q_n}\right|^{1/(m+n)}\to 0$ as $m,n\to\infty$, since each $f_{p,q}\in X$.

Taking $p=p_0$, $q=q_0$ in the second inequality in (4), we have

$\left|a_{p_{0_m},q_{0_n}}-a_{m,n}\right|^{1/(m+n)}<\frac{\varepsilon}{L}$, $(m,n\in I^+)$

Upon simplification, we get

$\left|a_{m,n}\right|^{1/(m+n)}<\left|a_{p_{0_m},q_{0_n}}-a_{m,n}\right|^{1/(m+n)}+\frac{\varepsilon}{L}$

$(m,n\in I^+)$

This proves that $\lim_{m,n\to 0}|a_{mn}|^{1/(m+n)}=0$. This leads us to the fact that

$$f(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}\ z_1^m z_2^n\ \in X.$$

We now show that $f\in(X(\lambda),T_\lambda)$. For this $\lambda\circ f$ must be shown to be in X. And for this $\lim_{m,n\to 0}|l_{mn}\,a_{mn}|^{1/(m+n)}$ must be shown to be zero.

We have already seen that $f_{p,q}\to f$ as $p,q\to\infty$, so that $d_\lambda(f_{p,q},f)<\varepsilon$ for all $p\geq p_0$, $q\geq q_0$.

So $d_\lambda(f_{p_0,q_0},f)<\varepsilon$. Hence

$$|l_{mn}|^{1/(m+n)}\left|a_{p_{0_m},q_{0_n}}-a_{mn}\right|^{1/(m+n)}<\varepsilon$$

i.e.

$$|l_{mn}|^{1/(m+n)}\left[\,|a_{mn}|-\left|a_{p_{0_m},q_{0_n}}\right|\,\right]^{1/(m+n)}<\varepsilon$$

$(m,n\in I^+)$.

Simplification yields

$|l_{mn}|^{1/(m+n)}|a_{mn}|^{1/(m+n)}<$

$|l_{mn}|^{1/(m+n)}\left|a_{p_{0_m},q_{0_n}}\right|^{1/(m+n)}+\varepsilon$.

This proves that $\lim_{m,n\to 0}|l_{mn}\,a_{mn}|^{1/(m+n)}=0$, so that the condition is sufficient for $(X(\lambda),T_\lambda)$ to be complete.

ii) Necessary. Let $\liminf\left\{|l_{0,0}|,|l_{m,n}|^{1/(m+n)},(m,n\in I^+\right\}=0$.

So $\left\{|l_{0,0}|,|l_{m,n}|^{1/(m+n)},(m,n\in I^+\right\}$ contains a sub-sequence, say,

$|l_{m_1,n_1}|^{1/(m_1+n_1)},|l_{m_2,n_2}|^{1/(m_2+n_2)},|l_{m_3,n_3}|^{1/(m_3+n_3)},\ldots$,

which is steadily decreasing and tends to zero.

Consider now the sequence of polynomials $(f_{h,r})_{h,r=1}^{\infty}$, where

$f_{1,1}=z_1^{m_1}z_2^{n_1}$

$f_{2,2}=z_1^{m_1}z_2^{n_1}+z_1^{m_2}z_2^{n_2}$

$f_{3,3}=z_1^{m_1}z_2^{n_1}+z_1^{m_2}z_2^{n_2}+z_1^{m_3}z_2^{n_3}$

:

$f_{h,r}=z_1^{m_1}z_2^{n_1}+z_1^{m_2}z_2^{n_2}+z_1^{m_3}z_2^{n_3}+\ldots+z_1^{m_h}z_2^{n_r}$.

Of course, each $f_{h,r}\in X(\lambda)$

This sequence is a Cauchy sequence in $(X(\lambda),T_\lambda)$, for, let $h,h',r,r'\in I^+$, such that $h'>h$ and $r'>r$. Then

$d_\lambda(f_{h',r'},f_{h,r})=$

$\sup\Big[|l_{m_1,n_1}|^{1/(m_1+n_1)}|1-1|,|l_{m_2,n_2}|^{1/(m_2+n_2)}|1-1|,\ldots,|l_{m_h,n_r}|^{1/(m_h,n_r)}|1-1|,$

$|l_{m_{h+1},n_{r+1}}|^{1/(m_{h+1},n_{r+1})}|1-0|,|l_{m_{h+2},n_{r+2}}|^{1/(m_{h+2},n_{r+2})}|1-0|,|l_{m_{h'},n_{r'}}|^{1/m_{h'},n_{r'})}|1-0|\Big]$

$=|l_{m_{h+1},n_{r+1}}|^{1/(m_{h+1},n_{r+1})}$.

Now
$\lim_{h,r\to\infty} d_\lambda(f_{h',r'}, f_{h,r}) =$

$\lim_{m_{h+1},n_{r+1}\to\infty} \left|l_{m_{h+1},n_{r+1}}\right|^{1/(m_{h+1}+n_{r+1})} = 0$,

so that $\left(f_{h,r}\right)_{h,r=1}^{\infty}$ is a Cauchy sequence in $(X(\lambda),T_\lambda)$. But $\lim_{h,r\to\infty} f_{h,r}$, if it exists, must be $\sum_{h=1}^{\infty}\sum_{r=1}^{\infty} z_1^{m_h} z_2^{n_r}$, which is clearly not in X. Thus the Cauchy sequence $\left(f_{h,r}\right)_{h,r=1}^{\infty}$ fails to converge to a point of $(X(\lambda),T_\lambda)$, so that , in this case $(X(\lambda),T_\lambda)$ is in complete. Hence the condition is necessary.

3. We have already seen that $X(\lambda)$ can be endowed with two different topologies, viz., T and T_λ. We now state and prove a theorem relating to the comparability of T and T_λ.

Theorem 3.1. T is finer than T_λ if, and only if,

$$\lim\sup\left\{\left|l_{0,0}\right|, \left|l_{m,n}\right|^{1/(m+n)}, (m,n\in I^+)\right\} < \infty$$

.

Proof i) Sufficiency. Let $U<\infty$ be the supremum in the statement of the theorem. To prove that the above condition is sufficient, it is enough to prove that, if $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ is a sequence in $(X(\lambda),T)$ converging to f in $(X(\lambda),T)$, then this sequence converges to f in $(X(\lambda),T_\lambda)$.

Consider now the identity mapping $v: f\to f$ from $(X(\lambda),T)$ to $(X(\lambda),T_\lambda)$. Since it is evidently linear , it is enough that we take $f=\theta$, where θ is the zero-element of X. Let

$$f_{p,q} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{p_m,q_n} z_1^m z_2^n .$$

Since $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ converges to θ in $(X(\lambda),T)$, given an $\varepsilon>0$, there is a $p_0,q_0\in I^+$ such that $d_\lambda(f_{p,q},\theta)<\varepsilon$ for all $p\ge p_0$, $q\ge q_0$
(i.e)

$$\sup\left[\left|a_{p_0,q_0}\right|, \left|a_{p_m,q_n}\right|^{1/(m+n)}, (m,n\in I^+)\right] < \varepsilon$$

for all $p\ge p_0$, $q\ge q_0$.
Now

$$d_\lambda(f_{p,q},\theta) =$$

$$\sup\left[\left|l_{0,0}\right|\left|a_{p_0,q_0}\right|, \left|l_{m,n}\right|^{1/(m+n)}\left|a_{p_m,q_n}\right|^{1/(m+n)}, (m,n\in I^+)\right]$$

$$\le U \sup\left[\left|a_{p_0,q_0}\right|, \left|a_{p_m,q_n}\right|^{1/(m+n)}, (m,n\in I^+)\right]$$

$<U\varepsilon$ for all $p\ge p_0$, $q\ge q_0$.
This shows that $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ converges to θ in $(X(\lambda),T_\lambda)$, so that the condition is sufficient.

ii) Necessity. Let the sequence in the statement of the theorem be unbounded .Then this sequence has a sub-sequence, say,

$$\left|l_{m_1,n_1}\right|^{1/(m_1+n_1)}, \left|l_{m_2,n_2}\right|^{1/(m_2+n_2)}, \left|l_{m_3,n_3}\right|^{1/(m_3+n_3)}, \ldots$$

,
which is strictly increasing and tends to infinity. So the sequence

$$\left|l_{m_1,n_1}\right|^{-1/(m_1+n_1)}, \left|l_{m_2,n_2}\right|^{-1/(m_2+n_2)}, \left|l_{m_3,n_3}\right|^{-1/(m_3+n_3)}, \ldots$$

converges to zero.
Now consider the sequence $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ of polynomials

$$f_{1,1} = \frac{1}{\left|l_{m_1,n_1}\right|^{1/(m_1+n_1)}} z_1^{m_1} z_2^{n_1}$$

$$f_{2,2} = \frac{1}{\left|l_{m_2,n_2}\right|^{1/(m_2+n_2)}} z_1^{m_2} z_2^{n_2} \quad :$$

$$f_{p,q} = \frac{1}{\left|l_{m_p,n_q}\right|^{1/(m_p+n_q)}} z_1^{m_p} z_2^{n_q} .$$

Each of these polynomials is an element of $X(\lambda)$.

Now $d_\lambda(f_{p,q},\theta)=\dfrac{1}{\left|l_{m_p,n_q}\right|^{1/(m_p+n_q)}}$.

Hence

$$\lim_{p,q\to\infty} d_\lambda(f_{p,q},\theta)=$$

$$\lim_{m_p,n_q\to\infty}\frac{1}{\left|l_{m_p,n_q}\right|^{1/(m_p+n_q)}}=0,$$

so that $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ converges to θ in $(X(\lambda),T_\lambda)$. On the other hand, $d_\lambda(f_{p,q},\theta)=1$, so that $\lim_{p,q\to\infty} d_\lambda(f_{p,q},\theta)=1$. Thus $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ fails to converge to θ in $(X(\lambda),T_\lambda)$. This shows that v is discontinuous at θ in $(X(\lambda),T_\lambda)$, so that the condition is necessary.

4. Lastly we determine the form of a continuous linear functional on the complete metric space $(X(\lambda),T_\lambda)$.

Theorem 4.1. Every continuous linear functional ϕ on the complete metric space $(X(\lambda),T_\lambda)$ is of the form

$$\phi(f)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\, c_{m,n},$$

where

i) $f(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\, z_1^m z_2^n$ is any one element of $X(\lambda)$;

ii) $\left(c_{m,n}\right)_{m,n=0}^{\infty}$ is a chosen sequence of complex number such that

$$\left\{\left|\frac{c_{0,0}}{l_{0,0}}\right|,\left|\frac{c_{m,n}}{l_{m,n}}\right|^{1/(m+n)}\quad m,n\ge 0, m+n\ne 0,(m,n\in I^+)\right\}$$

is bounded.

Before we proceed with the proof of this theorem, we shall state and prove a lemma.

Lemma 4.2. A necessary and sufficient condition that the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}\, c_{m,n}$ should converge for every sequence $f=\left(f_{m,n}\right)_{m,n=0}^{\infty}$ in the complete metric space $(X(\lambda),T_\lambda)$ is that

$$\left\{\left|\frac{c_{0,0}}{l_{0,0}}\right|,\left|\frac{c_{m,n}}{l_{m,n}}\right|^{1/(m+n)}\quad m,n\ge 0, m+n\ne 0,(m,n\in I^+)\right\}$$

…(5)

should be bounded.

Proof i) Sufficiency. Let (5) be bounded . Then we can find an $M>0$ such that

$$\left|\frac{c_{0,0}}{l_{0,0}}\right|\le M,\ \left|\frac{c_{m,n}}{l_{m,n}}\right|^{1/(m+n)}\le M.$$

Let $f\in X(\lambda)$.

Since $\lim_{m,n\to 0}\left|l_{m,n}\, a_{m,n}\right|^{1/(m+n)}=0$, we can find an $m_\circ, n_\circ \in I^+$ such that $\left|l_{m,n}\, a_{m,n}\right|<\left(\frac{1}{2M}\right)^{(m+n)}$ for all $m>m_\circ$, $n>n_\circ$.

Hence

$$\left|a_{m,n}\, c_{m,n}\right|=\left|a_{m,n}\right|\left|l_{m,n}\right|\left|\frac{c_{m,n}}{l_{m,n}}\right|<\frac{1}{2^{(m+n)}}$$

for all $m>m_\circ$, $n>n_\circ$.

Hence $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}\, c_{m,n}$ converges for the f chosen.

ii) Necessity. Now let (5) be unbounded. We shall show that there is a sequence in $X(\lambda)$ the corresponding series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\, c_{m,n}$ of which does not converge.

Since (5) is unbounded, we can find a steadily increasing sequence $\left(m_p\, n_q\right)_{p,q=1}^{\infty}$ in I^+, such that

$$\left|\frac{c_{m_p,n_q}}{l_{m_p,n_q}}\right|\ge P^{m_p+n_q}\ (P\in I).$$

Consider $\left(a_{m,n}\right)_{m,n=0}^{\infty}$ such that $a_{m,n}=0$ if $m\neq m_p$, $n\neq n_q$,

$$a_{m_p,n_q}=\frac{1}{\left|l_{m_p+n_q}\right|}\cdot\frac{1}{P^{m_p+n_q}}.$$

Clearly $\left|a_{m,n}\right|=0$, $\left|l_{m,n}\,a_{m,n}\right|=0$ for all $m\neq m_p$, $n\neq n_q$. Let

$$\inf\left\{\left|l_{0,0}\right|,\left|l_{m,n}\right|^{1/(m+n)}\ m,n\geq 0, m+n\neq 0,(m,n\in I^+)\right\}=L>0$$

(That $L>0$ exists follows from Theorem (2.1)). Then $\left|l_{m_p,n_q}\right|^{1/(m_p,n_q)}\geq L$, so that

$$\left|l_{m_p,n_q}\right|^{-1/(m_p+n_q)}\geq L^{-1}.$$

Thus

$$\lim_{m_p,n_q\to\infty}\left|a_{m_p,n_q}\right|^{1/(m_p+n_q)}=\lim_{m_p,n_q\to\infty}\frac{1}{\left|l_{m_p,n_q}\right|^{1/(m_p+n_q)}}\cdot\frac{1}{P}\leq\lim_{m_p,n_q\to\infty}\frac{1}{L_P}=0$$

.

Also $\left|l_{m_p,n_q}\,a_{m_p,n_q}\right|^{1/(m_p+n_q)}=\frac{1}{P}$, so that

$$\lim_{m_p,n_q\to 0}\left|l_{m_p,n_q}\,a_{m_p,n_q}\right|^{1/(m_p,n_q)}=0.$$

Thus $\left(a_{m,n}\right)_{m,n=0}^{\infty}$ represents an element of $X(\lambda)$. However, $\left|a_{m_p,n_q}\,c_{m_p,n_q}\right|\geq 1$, so that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}\,c_{m,n}$ fails to converges. Hence the condition is necessary .

Proof of Theorem 4.1. Let $f\in X(\lambda)$ and ϕ a continuous linear functional on $(X(\lambda),T_\lambda)$. Let

$$f_{m,n}=z_1^{m_1}\ z_2^{n_1},\ \phi(f_{m,n})=c_{m,n}.$$

Then

$$\phi(f_{m,n})=\lim_{m,n\to\infty}\left[\phi(a_{0,0}\,f_{0,0}+a_{1,1}\,f_{1,1}+\ldots+a_{m,n}\,f_{m,n})\right]$$

$$=\lim_{m,n\to\infty}\left[(a_{0,0}\,\phi(f_{0,0})+a_{1,1}\,\phi(f_{1,1})+\ldots+a_{m,n}\,\phi(f_{m,n})\right]$$

$$=\lim_{m,n\to\infty}\left[(a_{0,0}\ c_{0,0}+a_{1,1}\ c_{1,1}+\ldots+a_{m,n}\ c_{m,n}\right]$$

.

Thus, for every $f\in X(\lambda)$, $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}\,c_{m,n}$ converges , and $\phi(f)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}\,c_{m,n}$. Since $f\in X(\lambda)$, we use Lemma (4.2) to note that

$$\left\{\frac{c_{0,0}}{l_{0,0}},\left|\frac{c_{m,n}}{l_{m,n}}\right|^{1/(m+n)}\ m,n\geq 0, m+n\neq 0,(m,n\in I^+)\right\}$$

is bounded

Conversely, let this sequence be bounded. Then by Lemma (4.2) , $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}\,c_{m,n}$ converges for every $f\in X(\lambda)$. So

$$\phi(f)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}\,c_{m,n},\ f\in X(\lambda)$$

is functional on $X(\lambda)$. It is clearly linear on $X(\lambda)$. We now show that it is continuous on $(X(\lambda),T_\lambda)$. For this it is enough to show that , if $\left(f_{p,q}\right)_{p,q=1}^{\infty}$ is a double sequence in $(X(\lambda),T_\lambda)$ converging to θ , then $\left[\phi\left(f_{p,q}\right)\right]_{p,q=1}^{\infty}$ converges to zero. Let

$$f_{p,q}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{p_m,q_n}\ z_1^m\ z_2^n\ .$$

Since

$$\left\{\frac{c_{0,0}}{l_{0,0}},\left|\frac{c_{m,n}}{l_{m,n}}\right|^{1/(m+n)}\ m,n\geq 0, m+n\neq 0,(m,n\in I^+)\right\}$$

is bounded,

then there exists an $M>0$ such that

$$\left|\frac{c_{0,0}}{l_{0,0}}\right|\leq M,\qquad\left|\frac{c_{m,n}}{l_{m,n}}\right|\leq M^{(m+n)},$$

$m,n\geq 0, m+n\neq 0,(m,n\in I^+)$.

Let $\varepsilon < \frac{1}{2}$ be given. Let $\eta = \frac{\varepsilon}{M}$.

Since $d_\lambda(f_{p,q},\theta) \to 0$ as $p,q \to \infty$, we can find a $p_\circ, q_\circ \in I^+$ such that $d_\lambda(f_{p,q},\theta) < \eta$ for all $p > p_0$, $q > q_0$. Now

$$d_\lambda(f_{p,q},\theta) = \sup\left[|l_{0,0}||a_{p_0,q_0}|, |l_{m,n}|^{1/(m+n)} |a_{p_m,q_n}|^{1/(m+n)}, (m,n \in I^+) \right].$$

So, for all $p > p_0$, $q > q_0$,

$|l_{0,0}||a_{p_0,q_0}| < \eta$,

$|l_{m,n}|^{1/(m+n)} |a_{p_m,q_n}|^{1/(m+n)} < \eta$.

So

$$|\phi(f_{p,q})| = \left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{p_m,q_n} c_{m,n} \right|$$

$$\le \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{p_m,q_n} c_{m,n}|$$

$$= |a_{p_0,q_0} c_{0,0}| + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{p_m,q_n}| \, |c_{m,n}|.$$

Thus, for $p > p_0$, $q > q_0$

$$|\phi(f_{p,q})| < \frac{\eta}{|l_{0,0}|}.M\, |l_{0,0}| + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\eta^{m+n}}{|l_{m,n}|}.M^{m+n} |l_{m,n}|$$

$$= \varepsilon + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varepsilon^{m+n} = \varepsilon \left[1 + \frac{1}{1-\varepsilon} \cdot \frac{1}{1-\varepsilon} \right]$$

$$< \varepsilon \left[1 + \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{2}} \right] = 5\varepsilon .$$

This proves that $[\phi(f_{p,q})]_{p,q=1}^{\infty}$ converges to zero.

REFERENCES

1. Devendra Kumar and G.S.Srivastava, *Applications to interpolation and approximation of entire functions*, Ganita, **2001,**52, 95-115.

2. Hazem Shaba Behnam and G.S.Srivastava , *Space of analytic functions of two complex variables*, Demonstratio Mathematica., **2000**, 33 , 65-74.

3. Iyer, V.G., *On the space of integral functions (I)*, J. Indian Math. Soc. (New Series), **1948,** 12, 13-30.

4. Kamthan, P.K., *A study on the space of entire functions of several complex variables*, Yoko. Math. J., **1973,** 21,11-20.

5. Kumar, D. and Balbir Singh, Fourier *expansions of entire functions of two complex variables, Bulletin of institute of mathematics Academia Sinica,* **2004,**32,45-59.

6. Mushtaq Shakir , *On Fourier expansions of integral functions of two complex variables having zero order*, Al-Mustansiriya Journal of Science, **2008,** 19, 52-65.

7. Srivastava, G.S. and Mushtaq Shakir , *Bornological properties of the space of entire functions represented by Dirichlet series*, Soochow Journal of mathematics, **2005** ,31, 213-224.

حول فضاء جزئي معين من X

مشتاق شاكر الشيباني*

* قسم الرياضيات – كلية العلوم – الجامعة المستنصرية

الخلاصة:

باستعمال الخواص التوبولوجية Kamthan اول من درس خواص الدوال الكلية لمتغيرات معقدة ولأكثر من متغير واحد , في بحثنا هذا بنينا وأنتجنا فضاء جزئي من فضاء الدوال الكلية لمتغيرات معقدة ولأكثر من متغير واحد.