Fuzzy Subspaces For Fuzzy space of Orderings

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Abstract

The purpose of this paper is to define fuzzy subspaces for fuzzy space of orderings and we prove some results about this definition in which it leads to a lot of new results on fuzzy space of orderings. Also we define the sum and product over such spaces such that: If f = < $a_1,\dots,a_n$ > and g = < $b_1,\dots b_m$>, their sum and product are f + g = < $a_1\dots,a_n$, b_1, …, b_m> and f × g =<$a_1b_1,\dots,a_1b_m,\dots, a_nb_1,\dots, a_n b_m$ >. for all $a_1,\dots,a_n,b_1,\dots,b_m \in G$

Introduction

Let X = (X,A) denoted a space of fuzzy orderings. That is A is a fuzzy subgroup of ablian group G of exponent 2 (i.e. $x^2 = 1$, ∀ x ∈ G) and x is a non empty fuzzy sub set of the character group χ(A) = Hom(A, {1,-1}) satisfying :

1- X is a fuzzy closed subset of χ(A)
2- ∃ an element e ∈ A such that σ(e) = -1 , ∀ σ ∈ X
3- $X^{\perp}$ = { a∈ A / σ(a) = 1 ∀ σ ∈ X } =1
4- If f and g are forms over A and if x ∈D(f ⊕ g) then ∃ y ∈D(f) and z ∈D(g) such that x ∈D < y , z > . [1]

Fuzzy Subspaces

We assume that (X,G) is a finite fuzzy space of orderings.

for $a_1,\dots, a_m \in X$ we can consider all linear combinations

(1) $a = a_1^{e1}\dots a_m^{em}$, $e_1\dots e_m \in \{0,1\}$

in χ(G). we are more interested in linear combinations which are in X. Since a(-1) =1 holds for all a∈X. a necessary condition for a linear combination (1) to be in X is

$e_1 +\dots+ e_m \equiv 1 \pmod 2$.

Let $a_1,\dots,a_m$ be an arbitrary elements of X , and define Y,Z by:

Y ={a ∈ X / a is a linear combination of $a_1,\dots, a_m$ },

Z= {b ∈ G \ a_i (b)= 1 for all i = 1,2,…, m}.

Then Y, Z satisfy the duality condition.

(2) Z = Y , Y = Z ∩ X

The system (Y, G\Z) is referred to as the fuzzy subspace of (X,G) generated by $a_1,\dots, a_m$. More generally, a subspace of (X, G) is any system (Y,G\Z) where Y⊆ X, Z ⊆ G satisfy the duality condition (2).

Conversely suppose we begin with $b_1,\dots, b_m$ ∈G , let f denote the Pfister form <1,b_1>x …x<1,b_m>[2], and let X($b_1,\dots,b_m$) denote {a ∈ X \ a(b_I) = 1 for all I =1,2,…,m}.

Lemma 1

Let Y=X($b_1,\dots,b_m$) and Z= D_f . Then Z satisfy the duality condition(2).

Proof

Let b ∈ D_f , a ∈ X ($b_1,\dots,b_m$) . Then af =2^m =dim f, so a(b) =1.

Thus $D_f \subseteq$ X $(b_1,\dots,b_m)^-$ and, X($b_1,\dots,b_m$) ⊆ D_f ∩ X. Since $b_1,\dots,b_m$∈ D_f, it is clear that D_f ∩ X = X($b_1,\dots,b_m$) . Now let b∈ G satisfy a(b) =1 for all a ∈ X ($b_1,\dots,b_m$). consider the forms af and f.f represents 1, so af represents b.

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comparing signatures at $a \in X$, we see that

$$a(bf) = a(f) = \begin{cases} 2^m & \text{if } a \in X(b_1,\ldots,b_m) \\ 0 & \text{if } a \notin X(b_1,\ldots,b_m) \end{cases}$$

thus $f \equiv af$, so f represents b, thus $X(b_1,\ldots,b_m) \subseteq D_f$.

Note

A form f represents $\chi \in G$ if there exist $x_2, \ldots, x_n \in G$ such that $f \equiv \langle x, x_2,\ldots, x_n\rangle$. D_f denotes the set of all elements of G represented by f. [3]

Lemma 2

A form g over G represents $\chi \in G$ modulo $X(b_1,\ldots,b_m)$ if and only if $f \times g$ represents χ modulo X.

Proof

Suppose $g \equiv h \mod X(b_1,\ldots,b_m)$ where h has χ appearing in its diagonal representation. Since f has 1 appearing in its diagonal representation, it follows that χ appears in the diagonal representation of $f \times h$.

Now $ag = ah$ holds for all $a \in X(b_1,\ldots,b_m)$. also $af = 0$ holds, for at $a \notin X(b_1,\ldots,b_m)$. It follows that

$A(f \times g) = afag = afah = a(f \times h)$ holds for all $a \in X$.

Thus $f \times g \equiv f \times h \pmod X$. thus

$f \times g$ represents $\chi \pmod X$.

Conversely, suppose $f \times g$ represents $\chi \pmod X$.

Write $g = \langle y_1,\ldots,y_k\rangle$. Thus $f \times g \equiv y_1f+\ldots+y_kf \pmod X$, there exist $s_1,\ldots, s_k \in D_f$ such that $\langle y_1s_1,\ldots,y_ks_k\rangle$ represents $\chi \pmod X$.

but $\langle y_1 s_1,\ldots, y_k s_k\rangle \equiv \langle y_1,\ldots, y_k\rangle \pmod{X(b_1,\ldots,b_m)}$.

It follows that g represents χ mod $X(b_1,\ldots,b_m)$.

Theorem 1

Let (Y,G\Z) be any fuzzy subspace of (X,G). then (Y,G\Z) is also a fuzzy space of orderings.

Proof

There exist $b_1,\ldots,b_m \in G$ such that $Y=X(b_1,\ldots,b_m)$, and by lemma 1, $Z=D_f$ where f is the Pfister form associated to $b_1,\ldots,b_m$. By lemma 2, suppose g, h are forms over G such that g+h represents $\chi \in G$ modulo Y. thus, by lemma 2 the form $f x (g + h) = (f \times g) + (f \times h)$ represents χ modulo X.

Since there exist y,z represented by $f \times g$ and $f \times h$ respectively (mod X) such that $\langle y,z\rangle$ represents χ (mod X). Thus, by the lemma, y and z are represented by g and h respectively (mod Y), and clearly $\langle y,z\rangle$ represents χ (mod Y).

Definition

An fuzzy ordering $a \in X$ will be called fuzzy Archimedian (in X) if {a} is a component of X.

Definition

We will say two fuzzy orders a, á $\in$ X are fuzzy connected in X denoted by a ~ á if there exist fuzzy orders c, ć $\in$ X {a, á}$\neq$ {c, ć} such that aá = cć.

Notes:

1-If Y is a fuzzy subspace of X and a,á $\in$ Y, then it is conceivable that a, á could be fuzzy connected in X without being fuzzy connected in Y.

2-Let $X = X_1 \cup \ldots \cup X_k$ denote the decomposition of X deter mined by the equivalence relation ~. The classes X_i , $i = 1,\ldots,k$ will be referred to as the fuzzy connected components of X.

Theorem 2

Suppose $X_1,\ldots,X_k$ are the fuzzy connected components of X. Then each

X_I is a fuzzy subspace of X dimension X = $\sum_1^k \dim ensionX i$

Proof

Let Z_i =X_i Then clearly X_i generates $Z_i \cap X$. To show X_i is a fuzzy subspace we must show that $X_i = Z_i \cap X$. This is clear by lemma 2. Let a_{ij} , j= 1,…, n_i be a basis for each X_i , i = 1,…,k; we wish to show that the complete set $\{a_{ij} \setminus i= 1,\ldots,k; j=1,\ldots,n_I\}$ is a basis for X. It is clear this set spans X. If these elements were independent we could find a relation. $\prod_{i,j} a_{ij}^{e_{ij}} = 1$, $e_{ij} \in \{0,1\}$

With not all e_{ij} = 0. Of all such relations pick the one with the minimal number of non-zero e_{ij}. By lemma 2 each a_{ij} appearing with a non-zero exponent is equivalent to every other such a_{ij}. thus ,all such a_{ij} lie in the same fuzzy component. Thus, there exists i such that $e_{rj} = 0$ for $r \neq i$.

Thus our assumed relation has the form $\prod_j a_{ij}^{e_{ij}} = 1$

This would contradict the independence of $a_i ,\ldots,a_{ini}$.

Remark. Let $G_i = G/Z_i$ where $Z_i = X_i$, i= 1,….,k. by the above theorem, the fuzzy injection of G in to $G_1 \times G_2 \times \ldots \times G_k$. is an fuzzy isomorphism. If we identify G with $G_1 \times G_2 \times \ldots \times G_k$ Via this fuzzy isomorphism, we see that Z_i is identified with $\prod_{j \neq i} G_j$ for i= 1,…,k. Also X identified with $U^k Y_i$ where Y_i is obtained from X_I by extending each element by the identity character on $\Pi_{i \neq j} G_j$ Thus, the structure of (X,G) is completely determined by the structure of the fuzzy subspace (X_i ,G_i), i = 1,….,k. we will express this by writing

$$(X,G) = \sum_{i=1}^{k} (Xi, Gi).$$

And will refer to (X,G) as the direct sum of the spaces (X_i ,G_i), $1 \leq i \leq k$.

Theorem 3

Let X be a fuzzy connected space, dimension X ≠ 1. Then there exist

$\propto \in \chi$ (G) , $\propto \neq 1$ such that $\propto X = X$.

Proof

Since dimension X≠1 and X is fuzzy connected there exists a_1, a_2, ∈ X, $a_1 \neq a_2$, $a_1 \sim a_2$ .take $\propto = a_1 a_2$. Then $X_\propto$ has dimension ≥ 3 .of all $\propto \in \chi$ (G) satisfying $\propto \neq 1$, dimension $X_\propto \geq X$, then .

There exist (since X is fuzzy connected) elements $a_1 a_2 \in X$, $a_1 \in X$, $a_2 \notin X_\propto$, $a_1 \sim a_2$. Let **β** = $a_1 a_2$ Then $a_1 \in X_\propto \cap X_\beta$ so by [4, Lemma 4.6] , there exists $\delta \neq 1$, $\delta \in X(G)$ such that $X_\propto \subseteq X_\delta, .X_\beta \subseteq X_\delta$, Since $a_2 \in X_\beta \subseteq X_\delta$ it follows that X_δ Contains $X_\propto$ Properly.

This is α Contradiction and X_δ = X.

Now let T denote the set of all∝∈ X (G) such that α X=X. T is clearly α fuzzy subgroup of X(G) , and will be referred to as the translation group of X. Since we are assuming X is fuzzy Connected, we have , by the above theorem, that T≠1 . if dimension X≠1. let G= T and let X denote the set of all restriction α\G , ∝∈X.

Theorem 4

Let X be a fuzzy connected space and define X,G as above , then (X,G) is a fuzzy space of orderings.

Proof

It is clear that $-1 \in G^1$ X and that is a fuzzy subset of x(G^1) satisfying by definition of fuzzy space of orderings, and by Lemma 4.9 , in [5] , let f,g be forms over G^1 such that f + g represents $X \in G^1$ (mod X^{1}) . We may assume neither f nor g is isotropic (mod X) . There exist y,z $\in$G represented by f,g respectively (mod X) such that <y,z> represents x (mod X) . By the lemma 2, 2 , in [6] , it follows that y, z $\in G^1$ and that f, g, in fact , represent y, z respectively mod X^1. Also it is clear that <y,z> represents x(mod X^1) .

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الفضاءات الجزئية الضبابية من الفضاءات الضبابية الترتيب

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الخلاصة:

الغرض من البحث هو تعريف الفضاءات الجزئية الضبابية من الفضاءات الضبابية الترتيب وبرهان بعض خواص الفضاءات الجزئية وقد تم التوصل إلى بعض النتائج الجديدة حول الفضاءات الضبابية الترتيب .