S-maximal Submodules

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Abstract:

Throughout this paper R represents a commutative ring with identity and all R-modules M are unitary left R-modules. In this work we introduce the notion of S-maximal submodules as a generalization of the class of maximal submodules, where a proper submodule N of an R-module M is called S-maximal, if whenever W is a semi essential submodule of M with N ⊊ W ⊆ M, implies that W = M. Various properties of an S-maximal submodule are considered, and we investigate some relationships between S-maximal submodules and some others related concepts such as almost maximal submodules and semimaximal submodules. Also, we study the behavior of S-maximal submodules in the class of multiplication modules. Farther more we give S-Jacobson radical of rings and modules. .

Key words: Maximal submodules, S-maximal submodules, Almost maximal submodules, Semimaximal submodules, Semi essential submodules and Jacobson radical of modules.

Introduction:

Throughout this paper R represents a commutative ring with identity and all R-modules M are unitary left R-module, also all R-modules under study contain prime submodules. It is well known that a proper submodule N of an R-module M is called maximal, if whenever W is a submodule of M with N ⊊ W ⊆ M implies that W = M, equivalently, there is no proper submodule of M containing N properly [1].

Inaam and Riyadh in [2] introduced the notion of almost maximal submodules, where a proper submodule N of an R-module M is called almost maximal, if whenever W is an essential submodule of M with N ⊊ W ⊆ M implies that W = M, where a submodule K of M is said to be essential, if for every submodule L of M with K ∩ L = (0) implies that L = (0) [3].

Hatem in [4] gave another generalization for maximal submodules, named semimaximal submodules, where a proper submodule N of an R-module M is called semimaximal, if $\frac{M}{N}$ is a semisimple R-module. Muna in [5] introduced the concept of nearly maximal submodules, where a proper submodule N of an R-module M is called nearly maximal, if whenever a submodule W of M containing N properly implies that W + J(M) = M, where J(M) is the Jacobson radical of M. In this paper, we introduce the concepts of S-maximal submodules as a generalization of maximal submodules, where a proper submodule N of an R-module M is called S-maximal, if whenever W is a semi essential submodule of M with N ⊊ W ⊆ M, implies that W = M, where a submodule K of M is called semi

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essential if $K \cap P \neq (0)$, for every nonzero prime submodule P of M [6]. In section 1, we investigate some main properties of this type of submodules. In section 2, we study the relationships between S-maximal submodules and some other concepts such as almost maximal and semimaximal submodules a. In section 3 we study the behavior of S-maximal submodules in the class of multiplication modules. In section 4, we introduce the S-Jacobson radical of S-maximal submodules.

1. S-maximal submodules

In this section we introduce a class of S-maximal submodules as a generalization of maximal submodules. We give various basic properties for this concept. Firstly we begin by the following definition.

<u>Definition (1.1)</u>: A proper submodule N of an R-module M is called S-maximal, if whenever a semi essential submodule W of M with $N \subsetneq W \subseteq M$, then W = M. Equivalently, there is no proper semi essential submodule of M containing N properly. An ideal I of a ring R is called S-maximal if it is S-maximal R-submodule of R.

<u>Remarks and Examples (1.2):</u>

1. It is clear that every maximal submodule is S-maximal, but the converse is not true in general as the following example shows; The Z-module $M = 2Z \oplus 2Z$ and the submodule $N = 4Z \oplus (0)$ of M is not maximal, since $4Z \oplus (0) \subsetneq 2Z \oplus (0) \subseteq 2Z \oplus 2Z$. While N is an S-maximal, since the only submodule in M containing N properly is $2Z \oplus (0)$, which is not semi essential submodule of M, since there exists a prime submodule $(0) \oplus 2Z$ of M such that $((0) \oplus 2Z) \cap (2Z \oplus (0)) = (0)$.

2. Z is not S-maximal submodule of the Z-module Q, since there exists a submodule $\frac{1}{2}Z$ of Q such that $Z \subseteq \frac{1}{2}Z$, and clearly $\frac{1}{2}Z$ is a semi essential submodule of Q.

3. Not every module has an S-maximal submodule. For example: Z_{p^∞} as Z-module. In fact for each submodule N of Z_{p^∞}, any submodule W of Z_{p^∞} such that $N \subsetneq W \subseteq Z_{p^\infty}$, is an essential submodule of Z_{p^∞} and so a semi essential. That is N is not S-maximal submodule of Z_{p^∞}.

4. If N and W are proper submodules of an R-module M such that $N \subseteq W$. If N is an S-maximal submodule of M, then W is an S-maximal submodule of M.

<u>Proof (4):</u> Suppose that W is not S-maximal submodule of M, then there exists a semi essential submodule U of M such that $W \subsetneq U \subseteq M$. This implies that $N \subsetneq U \subseteq M$, that is N is not S-maximal which is a contradiction.

5. If U and V are proper submodules of an R-module M such that $U \cap V$ is an S-maximal submodule of M, then both of U and V are S-maximal submodules of M.

<u>Proof (5):</u> Follows directly from (4).

The converse of (5) is not true in general as we see in the following example:

The submodules (2) and (3) of the Z-module Z are S-maximal submodules, but $(2) \cap (3) = (6)$, and (6) is not S-maximal submodule of Z since $(6) \subsetneq (3) \subseteq Z$, and (3) is a semi essential submodule of Z.

6. Let M be an R-module and let N and K be submodules of M. If N and K are S-maximal submodules of M, then N+K is an S-maximal submodule of M.

<u>Proof (6):</u> The result follows by (4).

7. Let M be an R-module and let N and K be proper submodules of M, such that $N \subsetneq K$. If N is an S-maximal submodule of K and K is an S-maximal of M, then N is not necessary S-maximal submodule of M. For example: Consider the Z-module $M=Z_{24}$ and the submodules $K = (\bar{2})$ and $N = (\bar{4})$ of Z_{24}. Note that N is an S-maximal of K and K is S-maximal submodule of M, but N is not S-maximal submodule of M.

8. Let M be an R-module, and let A be an S-maximal submodule of M. If B is a submodule of M such that $B \simeq A$, then it is not necessary that B is an S-maximal submodule of M. For example: Consider the Z-module Z, the submodule 2Z is an S-maximal in Z, and $2Z \simeq Z$, but Z is not S-maximal submodule of Z. In fact any S-maximal submodule must be a proper in any R-module.

9. Every nonzero F-regular module has an S-maximal submodule. In fact every nonzero F-regular module has a maximal submodule [7] and the result follows from (1), where an R-module M is called F-regular if every submodule of M is pure [7].

Recall that a prime radical of an R-module M, is the intersection of all prime submodules of M, and denoted by rad (M) [8]. We have the following proposition.

Proposition (1.3): Let M_1 and M_2 be R-modules and let $N \subsetneq M_1$, assume that $f: M_1 \longrightarrow M_2$ be an epimorphism such that $\ker(f) \subseteq \text{rad}(M_1) \subseteq N$. If N is an S-maximal submodule of M_1, then f(N) is S-maximal submodule of M_2.

Proof: Since $\ker(f) \subseteq \text{rad}(M_1)$, and $\text{rad}M_1 \subsetneq M_1$, then we can show that $f(N) \subsetneq M_2$. In fact if $(N) = M_2 = f(M_1)$, since $N \neq M_1$, so there exists $m \in M_1$ such that $m \notin N$. Now $y = f(m) \in f(N)$, this implies that $f(m) = f(n)$ for some $n \in N$, and hence $m-n \in \ker(f) \subseteq N$. Therefore $m-n= n_1$ for some $n_1 \in N$, that is $m=n+n_1 \in N$ which is a contradiction, since $N \neq M_1$. Now, If f(N) is not S-maximal submodule of M_2, then there exists a semi essential submodule W of M_2 such that $f(N) \subsetneq W \subseteq M_2$. This implies that $f^{-1}(f(N) \subsetneq f^{-1}(W) \subseteq f^{-1}(M_2)$. But f is an epimorphism and $\ker f \subseteq \text{rad}(M_1) \subseteq N$, then $N \subsetneq f^{-1}(W) \subseteq M_1$. Since W is a semi essential submodule of M_2 and $\ker f \subseteq \text{rad}(M_1)$, so by [6] $f^{-1}(W)$ is a semi essential of M_1, that is N is not S-maximal submodule of M_1 which is a contradiction.

Corollary (1.4): Let N be an S-maximal submodule of M and let $K \subseteq N$. If $K \subseteq \text{rad}(M)$, then $\frac{N}{K}$ is an S-maximal submodule of $\frac{M}{K}$.

Corollary (1.5): If N is an S-maximal submodule of M such that $\text{rad}(M) \subseteq N$, then $\frac{N}{\text{rad}(M)}$ is S-maximal submodule of $\frac{M}{\text{rad}(M)}$.

Recall that a nonzero R-module M is called semi uniform, if each nonzero submodule of M is semi essential of M. A ring R is called semi-uniform if each nonzero ideal of R is a semi essential [6].

It's clear that every proper ideal is contained in an S-maximal ideal, then we have the following.

Remark (1.6): Let M be a semi uniform R-module and let N be a proper submodule of M. Then N is an S-maximal submodule if and only if N is maximal submodule.

Definition (1.7): An R-module M is called S-semisimple, if M has no proper semi essential submodule of M. That is if a submodule N is a semi essential submodules of M, then N = M.

It is clear that every S-semisimple module is a semisimple module.

Examples (1.8):

1. $(\bar{3})$ in the Z-module Z_{30} is an S-semisimple module since $(\bar{3})$ has no proper semi-essential submodule.

2. Z_{12} as Z-module is not S-semisimple module.

Proposition (1.9): Let M be an R-module. Then the zero submodule of M is an S-maximal submodule if and only if M is an S-semisimple module.

Proof: **⇒)** If the zero submodule (0) is S-maximal, then M has no proper semi essential submodule, that is M is an S-semisimple module.

⇐) Since M is an S-semisimple R-module, then M has no proper semi essential submodule, which implies that every submodule of M is an S-maximal. Thus (0) is an S-maximal of M.

Corollary(1.10): Let M be an R-module, then the following statements are equivalent:

1. M is an S-semisimple module.

2. (0) is an S-maximal submodule.

3. Every proper submodule of M is an S-maximal submodule.

Proof: **(1) ⇔ (2)** By Prop (1.10).

(2) ⇒ (3) By Rem and Ex (1.2)(4).

(3) ⇒ (2) It is obvious.

Proposition (1.11): If N is an S-maximal submodule of an R-module M and I is an ideal of R, If $(N:_M I)$ is a proper submodule of M, then $(N:_M I)$ is an S-maximal submodule of M.

Proof: Since $(N:_M I) \subsetneq M$ and $N \subseteq (N:_M I)$,so by Rem and Ex (1.2)(4), we get $(N:_M I)$ is an S-maximal submodule.

Note that sometimes$(N:_M I)$ = M, for example: If M is a multiplication module, then any submodule N of M can be written as the form N=IM, hence $(N:_M I)$ = M.

The converse is not true for example. The Z-module $M=Z_{12}$, and the ideal I=2Z of Z, N = $\{\bar{0}, \bar{4}, \bar{8}\}$ is not S-maximal submodule of Z_{12}, while $(N:_M I)$ = $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ is an S-maximal submodule of M.

Definition (1.12): An R-module M is called SM-module, if every proper submodule of M is an S-maximal. And a ring R is called SM-ring if every proper ideal of R is an S-maximal ideal.

Examples (1.13):

1. Both of Z-module Z_6 and Z-module Z_{10} are SM- modules.

2. Z as Z-module is not SM-module, since the submodule (6) of Z is not S-maximal submodule. In fact a nonzero submodule (n) of Z is S-maximal if and only if (n) prime submodule of Z.

3. Every S-semisimple module is an SM-module.

Proof (3): It follows from Cor (1.10).

Recall that an R-module M is called fully prime, if every proper submodule of M is prime [9]. In the following theorem we prove under some conditions, the direct sum of two SM-modules is an SM-module, before that we need to give the following lemma.

Lemma (1.14): If M is a fully prime R-module, then every nonzero semi essential submodule of M is an essential submodule of M.

Lemma (1.15): Let $M = M_1 \oplus M_2$ be a fully prime R-module where M_1 and M_2 are submodules of M, and let $(0) \neq K_1 \subseteq M_1 \subseteq M$ and $(0) \neq K_2 \subseteq M_2 \subseteq M$, then $K_1 \oplus K_2$ is a semi essential submodule of $M_1 \oplus M_2$ if and only if

K_1 is a semi essential submodule of M_1 and K_2 is a semi essential submodule of M_2.

Proof: **⇒)** Since M is a fully prime module, then by Lemma (1.14) $K_1 \oplus K_2$ is an essential submodule of $M_1 \oplus M_2$, and by [8], K_1 is an essential submodule of M_1 and K_2 is an essential submodule of M_2. But every essential submodule is a semi essential, so we get the result.

⇐) Also by using [8].

Theorem (1.16): Let M be a fully prime R-module, and $M=M_1\oplus M_2$, where M_1 and M_2 be are modules, and let $annM_1 + annM_2 = R$. If M_1 and M_2 are SM-modules, then M is an SM-module.

Proof: Let N be a proper submodule of M, and let K be a submodule of M such that $N \subseteq K \subseteq M$ where K is a semi essential submodule of M. Since $annM_1 + annM_2 = R$, then $N=N_1\oplus N_2$ for some submodules N_1 of M_1 and N_2 of M_2, also $K= K_1\oplus K_2$ for some submodules K_1 of M_1 and K_2 of M_2 [10]. There are three cases: (1) Both of N_1 and N_2 are proper submodules of M_1 and M_2 respectively (2) N_1 is a proper submodule of M_1 and $N_2 = M_2$ (3) N_2 is a proper submodule of M_2 and $N_1 = M_1$. If both of N_1 and N_2 are proper submodules of M_1 and M_2 respectively, then we have $N_1\oplus N_2 \subseteq K_1\oplus K_2 \subseteq M_1\oplus M_2$ where $K_1\oplus K_2$ is semi essential submodule of $M_1\oplus M_2$, so by Lemma (1.15), K_1 is a semi essential submodule of M_1 and K_2 is a semi essential submodule of M_2. But both of M_1 and M_2 are SM-module, then $K_1 = M_1$ and $K_2 = M_2$, and this implies that $K = K_1\oplus K_2 = M_1\oplus M_2 = M$, hence M is an SM-module. If $N=N_1\oplus M_2$, and since M_1 is an SM-module, then $K = M_1$, hence $K = K_1\oplus K_2 = M_1\oplus M_2 = M$, hence M is an SM-module. Similarity for $N=M_1\oplus N_2$.

The following two examples are about the direct sum of two S-maximal submodules. The first one shows that for R-modules M_1 and M_2, if N_1 is an S-maximal submodule of M_1 and N_2 is an S-maximal submodule of M_2, then it is not necessarily that $N_1\oplus N_2$ is S-maximal submodule of $M_1\oplus M_2$.

Example (1.17): Consider the Z-module Z and the Z-module $M=Z\oplus Z$. It is clear that $N_1=2Z$ and $N_2=3Z$ are S-maximal submodules of Z. However, $N_1\oplus N_2 \subseteq Z\oplus 3Z \subseteq M$. Moreover, it is clear that $Z\oplus 3Z$ is a proper semi essential submodule of M, thus $N=N_1\oplus N_1$ is not S-maximal submodule of M.

The other example shows that if both of N_1 and N_2 are S-maximal submodules of an R-module M, then $N_1\oplus N_2$ is not necessarily S-maximal submodule of M.

Example (1.18): The submodules $N_1= (\bar{2})$ and $N_2= (\bar{3})$ are S-maximal submodules of the Z-module Z_6, but $N_1\oplus N_2 = Z_6$ is not S-maximal submodule since it is not proper submodule of Z_6.

2. S-maximal submodules and some other related concepts

In this section we study the relationships between S- maximal submodules and almost maximal submodules and some others classes of submodules such as semimaximal, weakly prime and nearly maximal submodules. Firstly, recall that a proper submodule N of an R-module M is called almost, maximal if whenever W is an essential submodule of M with N ⊊ W ⊆ M implies that W = M.

Remark (2.1): Every S-maximal submodule is almost maximal.

The converse of Remark (2.1) is not true in general as we see in the Z-module Z_{42}, the submodule $(\bar{6})$ is an almost maximal, since the only proper submodules which contains $(\bar{6})$ properly are $(\bar{2})$ and $(\bar{3})$ and each of them is not essential submodule of the Z-module Z_{42}. But $(\bar{6})$ is not S-maximal submodule of Z_{42}, since $(\bar{2})$ and $(\bar{3})$ are both semi essential submodules of Z_{42} and containing $(\bar{6})$ properly.

Recall that an R-module M is called chained, if for each submodules U, V of M, either U⊆V or V⊆U [11]. In order to prove the following theorems we need to give the following lemma.

Lemma (2.2):

1. Every chained module is a uniform module, so it is semi uniform module.

2. Every integral domain is a uniform module, so it is semi uniform module.

Theorem (2.3): Let N be a proper submodule of a chained module M, then the following statements are equivalent:

1. N is an S-maximal submodule.

2. N is a maximal submodule.

3. N is an almost maximal submodule.

Proof: **(1) ⇔ (2)** By Lemma (2.2) (1) and by Prop (1.6).
(2) ⇔ (3) [2, Cor (1.4)].
(3) ⇔ (1) By Remark (2.1) and [2, Cor (1.4)].

Theorem (2.4): Let I be a proper ideal of an integral domain R, then the following statements are equivalent.

1. I is an S-maximal ideal.

2. I is a maximal ideal.

3. I is an almost maximal ideal.

Proof: **(1) ⇔ (2)** By Lemma (2.2)(2).
(2) ⇔ (3) [2, Cor (1.5)].
(3) ⇔ (1) By Remark (2.1) and [2, Cor (1.5)].

Recall that a submodule N of M is called weakly prime, if whenever a, b ∈ R with 0 ≠ ab∈ I implies that a∈ I or b ∈ I [12]. It is well known that a non trivial proper ideal of a principle ideal domain (briefly PID), is prime if and only if it is maximal ideal, so we have the following.

Proposition (2.5): Let I be a non trivial proper ideal of a PID, R. Then the following statements are equivalent:

1. I is an S-maximal ideal.

2. I is an almost maximal ideal.

3. I is a maximal ideal.

4. I is a weakly prime ideal.

5. I is a prime ideal.

Proof: **(1) ⇒ (2)**, by Remark (2.1).
(2) ⇔ (3) ⇔ (4) ⇔ (5) [2].
(5) ⇒ (1), since I is a prime ideal and R is a PID, then I is a maximal and by Remark (1.2)(1), I is an S-maximal ideal.

Remark (2.6): Every S-maximal submodule is semimaximal submodule. In fact by Remark (2.1), every S-maximal submodule is almost maximal and every almost maximal submodule is semimaximal submodule [2].

The converse of Remark (2.6) is not true in general. In fact (6) is semimaximal submodule in the Z-module Z, but not S-maximal, since there exists a proper semi essential submodule (3) of Z such that (6) ⊊ (3).

However, under some condition the converse of Remark (2.6) is true as the following proposition shows.

Proposition (2.7): Let I be a prime ideal of a ring R. If I is a semimaximal ideal of R, then I is an S-maximal ideal.

Proof: It follows by using [2, Prop (1.15)] together with Rem and Ex (1.2)(1).

Recall that a proper submodule N of an R-module M is called nearly maximal, if whenever a submodule W of M containing N properly implies that W+J(M)=M, where J(M) is the Jacobson radical of M [5].

Remark (2.8): Nearly maximal submodules is not necessarily S-maximal submodule, for example: In the Z-module Q, the submodule Z is nearly maximal but not S-maximal submodule of Q, since there exists a semi essential submodule $\frac{1}{2}$ Z of Z such that Z $\subsetneq \frac{1}{2}$ Z. We think the two concepts are independent, but we can't find an example to complete this claim.

Proposition (2.9): Let M be a fully prime R-module and let $(0) \neq N \subseteq M$, then N is an S-maximal submodule of M if and only if N is an almost maximal submodule of M.

Proof: ⇒) Clear.

⇐) Assume that N is an almost maximal submodule of M, and let $N \subsetneq L \subseteq M$ where L is a nonzero semi essential submodule of M. Since M is a fully prime module, then by Lemma (1.14) L is an essential submodule of M. But N is an almost maximal submodule, thus L = M, that is N is an S-maximal submodule. .

We need to introduce the following definition.

Definition (2.10): A nonzero module M is called fully essential, if every nonzero semi essential submodule of M is an essential submodule of M. A ring R is called fully essential, if R is fully essential R-module.

Example (2.11):

1. Every integral domain is a fully essential module.
2. The Z-module Z_{p^∞} is fully essential module.
3. Both of Z_{36} and Z_{30} are not fully essential Z-module.
4. Z_4 is a fully essential module.
5. Every uniform module is a fully essential module.

Remark (2.12): In the Lemma (1.15) and Th (1.16) we can replace the condition "M is a fully prime module", by the condition "M is a fully essential module", and the proof is done in similar way.

Proposition (2.13): Let M be a fully essential R-module and let $N \subseteq M$, then N of M is S-maximal if and only if N is an almost maximal submodule of M.

In the following theorem, we prove analogous of Th (1.16), but without need to put the condition "fully prime" on an R-module M. Before that we need to introduce the following definition.

Definition (2.14): An R-module M is called AM-module, if every nonzero submodule of M is an almost maximal. And a ring R is called AM-ring, if every proper nonzero ideal of R is an almost maximal R-submodule.

Note that Z_6 as Z is an AM-module, and by using Remark (2.1) we can easily show that every SM-module is AM-module.

Theorem (2.15): Let M be an R-module, and $M=M_1 \oplus M_2$, where M_1 and M_2 be are modules, and let $annM_1 + annM_2 = R$. If M_1 and M_2 are AM-modules, then M is an AM-module.

Proof: Let N be a proper submodule of M, and let K be a submodule of M such that $N \subseteq K \subseteq M$ where K is an essential submodule of M. Since $annM_1 + annM_2 = R$, then $N=N_1 \oplus N_2$ for some submodules N_1 of M_1 and N_2 of M_2, also $K= K_1 \oplus K_2$ for some submodules K_1 of M_1 and K_2 of M_2 [10]. As the same argument of Th

(1.15), we have $N_1 \oplus N_2 \subseteq K_1 \oplus K_2 \subseteq M_1 \oplus M_2$ where $K_1 \oplus K_2$ is semi essential submodule of $M_1 \oplus M_2$. By [3], K_1 is an essential submodule of M_1 and K_2 is an essential submodule of M_2. But both of M_1 and M_2 are AM-module, then $K_1 = M_1$ and $K_2 = M_2$, and this implies that $K = K_1 \oplus K_2 = M$, hence M is an AM-module. .

3. S-maximal submodules and multiplication modules

In this section we will study the behavior of S-maximal submodules in the class of multiplication modules. Firstly, recall that an R-module is called multiplication, if for each submodule N of M, there exists an ideal I of R such that N=IM [13]. Equivalently, M is a multiplication module if and only if for each submodule N of M, $N = (N:_R M)\,M$ [14].

Remark (3.1): Every multiplication module contains an S-maximal submodule.

Proof: Since every multiplication has a maximal submodule then by Rem and Ex (1.2) (1) we are done.

Corollary (3.2): Every cyclic R-module has an S-maximal submodule.

Proof: The result follows from the fact that every cyclic module is multiplication module.

We need to give the following definition.

Definition (3.3): A nonzero R-module M is said to be S-local module if M has only S-maximal submodule which contains all proper submodules of M. A ring R is called S-local ring if R is an S-local R-module.

Example (3.4): The Z-module $(\bar{3})$ in the Z-module Z_{24} is an S-local module, since it has only S-maximal submodule which is $(\bar{6})$.

Proposition (3.5): Let M be a nonzero multiplication and S-local R-module, and let N be an S-maximal submodule of M. If $N \neq (0)$, then N is a semi essential submodule of M.

Proof: Let P be a prime submodule of M with $P \cap N = (0)$. Since M is a nonzero multiplication module, so by [14], P contained in some maximal (hence S-maximal) submodule of M. But M has only one S-maximal submodule which is N. Thus $P \subseteq N$. This implies that $P = (0)$, that is N is a semi essential submodule of M.

Corollary (3.6): Let R be an S-local, and let I be an S-maximal ideal of R. if $I \neq (0)$, then I is a semi essential ideal of R.

Theorem (3.7): Let N be a submodule of a faithful and multiplication R-module M. Consider the following statements:

1. N is an S-maximal submodule of M.

2. $(N:_R M)$ is an S-maximal ideal of R.

3. N =IM for some S-maximal ideal I of R.

Then: **(1) ⇒ (2) ⇒ (3)**, and if M is a finitely generated module then **(3) ⇒ (1).**

Proof **(1) ⇒ (2):** Suppose that $(N:_R M)$ is not S-maximal ideal of R. Then there exists a proper semi essential ideal J of R such that $(N:_R M) \subsetneq J \subseteq R$. Since M is a multiplication module, then $N = (N:_R M)\,M \subsetneq JM \subseteq M$ [14]. Since M is a faithful and multiplication module, then JM is a semi essential submodule of M [6]. Hence N is not S-maximal which a contradiction with our assumption is, thus $(N:_R M)$ is an S-maximal ideal. .

(2) ⇒ (3): Since N is a multiplication $N = (N:_R M)\,M$ [14], so by (2), $(N:_R M)$ is an S-maximal ideal of R and we are done.

(3) ⇒ (1): Suppose that N is not S-maximal submodule of M, then there exists a proper semi essential submodule U of M such that N ⊊ U ⊆ M. By assumption N = IM for some S-maximal ideal I of R and U = JM for some ideal J of R. Since M is a multiplication, thus IM ⊊ JM ⊆ RM = M and since M is a finitely generated faithful multiplication module, so by [14, Th (3.1)], I ⊊ J ⊆ R. But U = IM is a semi essential submodule of M and M is a faithful multiplication module, then J is a semi essential ideal of R [6] and thus I is not S-maximal ideal, which is a contradiction with (3). Therefore N is an S-maximal submodule of M. .

We end this section by the following theorem which gives the hereditary property between SM-module over ring and the ring R itself.

Theorem (3.8): Let M be a finitely generated faithful and multiplication module. Then M is an SM-module if and only if R is an SM-ring.

Proof: **⇒)** Assume that M is an SM-module, and let I be a proper ideal of R. Since M is a multiplication module then N=IM. But M is an SM-module, so N is an S-maximal submodule of M. By Th (3.7), I is an S-maximal ideal of R.

⇐) Suppose that R is an SM-ring and let N be a proper submodule of M. Since M is a multiplication module, so there exists an ideal I of R such that N=IM. By assumption I is an S-maximal ideal, and by Th (3.7) N is an S-maximal submodule of M, that is M is an SM-module.

4. S-Jacobson radical of rings and modules

In this section we introduce the concept of S-Jacobson radical of modules. We give some properties and other characterization for this type of radical. We start by the following definition.

Definition (4.1): Let M be an R-module. S-Jacobson radical of M is denoted by SJ(M), and we defined as follows:

SJ(M) = ∩{N, where N is an S-maximal submodules of M}.

If there is no S-maximal submodule in M, then we say that SJ(M)=M. An S-Jacobson radical of a ring R is the intersection of all S-maximal ideals of R.

Examples and Remarks (4.2):

1. SJ(M) ⊆ J(M).

2. If M is an SM-module, then SJ(M) = (0).

Proof (2): By assumption (0) is an S-maximal SJ(R) = ∩ {I, where I is an S-maximal ideal of R}.

submodule and hence SJ(M) ⊆ (0), and we are done.

3. If M is a S-semisimple module, then SJ(M) = (0).

Proof (3): Since M is an S-semisimple module, so by Cor (1.10) every proper submodule of M is an S-maximal, in particular (0) is an S-maximal submodule of M, thus SJ(M) ⊆ (0) and we are done.

Proposition (4.3): Let M be a faithful, finitely generated and multiplication R-module, then SJ(M) = ∩ {IM | I is an S-maximal ideal I of R}.

Proof: Put K= ∩{IM | I is an S-maximal ideal I of R}. If SJ(M)=M then clearly K⊆SJ(M), so assume that SJ(M)≠M and let N be an S-maximal submodule of M. Since M is a faithful and multiplication module then by Th (3.7), $(N:_R M)$ is an S-maximal ideal of R. By assumption K ⊆ $(N:_R M)$ M = N, and by definition of SJ(M) we have K ⊆ SJ(M)…(1). Now, let I be an S-maximal ideal of R. Since M is a faithful, finitely generated and multiplication module, then by Th (3.7), IM is an S-maximal submodule of M, hence SJ(M) ⊆ IM hence SJ(M) ⊆ K …(2). From (1) and (2) we get SJ(M) = K and we are done.

Corollary (4.4): Let M be a faithful, finitely generated and multiplication R-module. If R is an integral domain and M is a divisible, then SJ(M) = M.

Proof: Let I be an S-maximal ideal of R. Since M is a divisible module, then IM = M, but M is a faithful, finitely generated and multiplication module, so by Prop (4.3), SJ(M) = M.

Corollary (4.5): If R is an integral domain and divisible, then SJ(R) = R.

Corollary (4.6): If M is a faithful, finitely generated and multiplication R-module, then SJ(R) M ⊆ SJ(M).

Proof: Let I be an S-maximal ideal of R. By definition of SJ(R), SJ(R) ⊆ I, hence SJ(R) M ⊆ IM. Since M is a faithful, finitely generated and multiplication R-module so by Prop (4.3) SJ(R)M ⊆ SJ(M).

In the Cor (4.6), when SJ(R) is an S-maximal ideal of R, then the equality holds as the following shows.

Corollary (4.7): Let M be a faithful, finitely generated and multiplication R-module. If SJ(R) is an S-maximal ideal of R, then SJ(R) M = SJ(M).

Proof: Since SJ(R) is an S-maximal ideal of R and M is faithful and multiplication R-module, then by Th (3.7), SJ(R)M is an S-maximal submodule of M. This implies that SJ(M) ⊆ SJ(R)M. But M is a finitely generated module, then by Cor (4.6) we get the result.

Now, we study the S-Jacobson radical of submodules and the S-radical of ideals.

Definition (4.8): Let N be any submodule of an R-module M. The S-Jacobson radical of submodule N is denoted by SJ(N) and defined as follow: SJ(N) = ∩{K: K is an S-maximal submodule of M containing N) and the S-Jacobson radical ideal of A is defined by SJ(A) = ∩ { I | I is an S-maximal ideal of R containing A}.

Example (4.9): Consider the submodule ($\overline{12}$) of the Z-module Z_{24}. Not that SJ($\overline{12}$) = ∩ { K : K is an S-maximal submodules of Z_{24}}. The S-maximal submodules of Z_{24} containing ($\overline{12}$) are only ($\overline{2}$) and ($\overline{3}$), so ($\overline{2}$) ∩ ($\overline{3}$) = ($\overline{6}$). Thus SJ($\overline{12}$) = ($\overline{6}$).

The following proposition gives some properties of the S-Jacobson radical of submodules.

Proposition (4.10): Let N and L be two submodules of an R-module M, and let I be an ideal of R then:

1. N ⊆ SJ(N).

2. SJ(SJ(N)) = SJ(N).

3. SJ(N ∩ L) ⊆ SJ(N) ∩ SJ(L).

Proof: It is clear, so we omitted.

Proposition (4.11): If M is an SM-module, then SJ(N) = N, for each submodule N of M.

Proof: Let N be a submodule of M, SJ(N) = ∩{K | K is an S-maximal submodule of M such that N ⊆ K}. By assumption N is an S-maximal submodule of M, thus SJ(N) = N.

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المقاسات الجزئية العظمى من النمط S

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الخلاصة:

لتكن R حلقة إبدالية ذات عنصر محايد وليكن M مقاسا" أحاديا" أيسر على R. قدمنا في هذا البحث صنفاً من المقاسات الجزئية التي تمثل إعماماً للمقاسات الجزئية العظمى، وأطلقنا عليه أسم المقاس الجزئي الأعظم من النمط S، حيث يقال للمقاس الجزئي الفعلي N من M بأنه أعظم من النمط S، أذا تحقق الأتي: لكل مقاس جزئي واسع W في M بحيث أن $N \subset W \subseteq M$ يؤدي الى أن $W = M$. كما يقال للمثالي الفعلي I في R بأنه مثالي جزئي أعظم من النمط S، أذا كان I مقاساً جزئياً أعظم من النمط S على الحلقة R .

درسنا في هذا البحث خواص هذا الصنف من المقاسات الجزئية وعلاقته مع المقاسات الجزئية الأخرى ذات العلاقة، على سبيل المثال المقاسات الجزئية العظمى تقريباً والمقاسات الجزئية شبه العظمى. كما درسنا سلوك المقاسات الجزئية العظمى من النمط S في صنف المقاسات الجدائية. فضلاً عن ذلك، أعطينا مفهوم جذر جاكوبسن من النمط S للحلقات و للمقاسات.