Journal of Mathematics Teacher Education
https://doi.org/10.1007/s10857-023-09578-3

Culturally crafted Lesson Study to improve teachers' professional development in mathematics: a case study in Italian secondary school

Roberto Capone[1] · **Maria Giuseppina Adesso**[2] · **Carola Manolino**[3] · **Riccardo Minisola**[3] · **Ornella Robutti**[3]

Accepted: 19 March 2023

Abstract
This paper describes a Lesson Study in which in-service mathematics secondary-school teachers, collaborating with researchers, involve grade 10 students in tessellation problems. The data are collected by an experiment carried out in the context of the "Liceo Matematico" project, with three volunteer teachers. The experiment goal was to craft a collaborative design of the research lesson between teachers and researchers. The research aim of the paper is to examine the use of Lesson Study in the institutional and cultural context of Italian secondary school with the use of Cultural Transposition as a theoretical framework. The research is qualitative with idiographic aims, based on video research. The educational aim of the research is to provide a solid basis for a revamped in-service teacher education first in the context of the project, then in curricular context. Semiotic mediation is used to provide, within Lesson Study, the conceptual framework for teachers and researchers collaborative design of the research lesson. The results show that Lesson Study, as a foreign practice, is an opportunity for teachers to confront their teaching practices, to enrich their professional development, resulting in more awareness on their didactical action in and outside the classroom.

Keywords Lesson Study · Teachers' professional development · Semiotics · Semiotic mediation · Cultural Transposition

Introduction

In recent years, practicing teachers' professional development has acquired a central position in the Italian and international debate on educational policies (Italian law 107/2015). In-service mathematics teacher education offers tools and opportunities to research, for

✉ Roberto Capone
Robertocapone69@gmail.com

[1] University of Bari, Bari, Italy

[2] Liceo Scientifico "Da Procida", Salerno, Italy

[3] University of Torino, Torino, Italy

investigating situations that encourage teachers to break out of routinised practices and move towards re-elaborating and re-planning their teaching (Brophy, 2006). Teachers agree that in-service professional development is the driving force of innovation and that the teaching–learning processes cannot remain static (Vermunt et al., 2019; Weber et al., 2018).

Among the various models of collaboration-based teachers' professional development, that combine effectiveness with sustainability, the Japanese 授業研究 (Jugyokenkyu), known in the Western world as *Lesson Study* (LS) (Yoshida, 1999), stands out.

Through and thanks to LS, several researchers around the world have identified active changes, consciously implemented by the teachers themselves, in teachers' beliefs or disposition towards work and learning, in their mathematics knowledge for teaching, and in their teaching practices (i.e., Huang & Shimizu, 2016; Xu & Pedder, 2015). A significant feature is that LS occurs within the school environment, where teachers play a central role, and is based on teachers and researchers collaboration. LS creates a context for teachers' collective reflection and develops the sense of confidence that is central to teachers' professional development (Nguyen & Tran, 2022; Pang, 2016). In February 2020, the ICMI Study 25 Conference focused on *Teachers of Mathematics working and learning in collaborative groups*, and LS was very present in the presented research works (both in one plenary lecture and in 14 out of 80 accepted papers cited "Lesson Study" in the title, e.g. Capone et al., 2020; Otaki et al., 2020; Shinno & Yanagimoto, 2020; Skott, 2020). The assumption underlying this ICMI Study is on teachers learning and working through collaboration; however, it can be challenging to investigate and explain how this learning occurs and gather evidence of what and how teachers learn.

In this paper, we present the (to our knowledge) first experiment of LS in Italian secondary school. The main aim of the paper is examining the use of LS in the institutional and cultural context of Italian secondary school to provide a solid basis for a revamped in-service teacher education. The LS experiment has been carried out in the context of the Liceo Matematico project (Capone et al., 2017; Capone, 2022)—a project of teacher professional development in which collaboration is a cornerstone, with 3 volunteer mathematics teachers.

The research is qualitative with idiographic aims, based on video research. Cultural Transposition (Mellone et al., 2019; Ramploud et al., 2022) is the main theoretical framework of the research and gives a tool to interpret data. Semiotic mediation is used to provide, within LS, the conceptual framework for teachers and researchers collaborative design of the research lesson.

Literature review and background

LS originated in Japan around 1870, to answer professional development needs for qualified teachers (Isoda et al., 2007), and gained attention within the years thanks to the TIMSS Video Study (Stigler et al., 1999) and the book *The Teaching Gap* (Stigler & Hiebert, 1999). In the last 30 years, it has been increasingly used around the world with teachers of different grades, including higher education: for example, LS was introduced in the United States thanks to Catherine Lewis' research (2000) and Makoto Yoshida's doctoral thesis (1999), creating an Anglo-American tradition; and in Europe it is present in a number of countries (e.g., Bartolini Bussi et al., 2017; Ní Shúilleabháin, 2018; Ponte et al., 2018;

Winsløw et al., 2018). LS has been effectively studied in teachers' professional development almost worldwide (e.g., Manolino, 2021a; Okubo, 2007; Ono & Ferreira, 2010; Presutti, 2022).

Lesson Study's main features

LS gained ground to gradually improve standard classroom practices (Stigler & Hiebert, 2016), and different specific features and conceptualizations of LS or associated forms of activity have developed in many countries, pursuing aims and assuming formats that appear very different from each other (Huang et al., 2019; Quaresma et al., 2018). Even within Japan itself LS can take multiple forms (Miyakawa & Winsløw, 2019). Identifying its essential generic characteristics can give insights when studying this model of professional development. A great deal of work in this sense has been coming from Western-cultured researchers, and the diffusion of LS in European countries, which mainly occurred in the last decade (Quaresma et al., 2018), contributed to its development. Mathematics Education teams all over the world are coming to an identification of the "essential generic" characteristics of LS (Buchard & Martin, 2017) through the detailed review of the existing literature and discussions in the WALS group, albeit there is still no complete academic agreement. Each country, each reality, has local cultural features.

One possible diagram is presented in Fig. 1 by Buchard and Martin (2017) from Switzerland: albeit slightly different from most used representation of LS (Lewis, 2000, in the USA), we believe it includes some elements of the process (such as the recruitment of participants or the sharing as an element of teacher education beyond the individual cycle) that are not trivial for the success of LS, especially when it is transposed in a context different from the Eastern one.

A further conceptualisation comes from Seleznyov (2018), who studies LS in the United Kingdom. She points out what she defines "the seven critical components" (ibid., p. 221) of LS:

1. Identify focus: the LS group compare long-term goals for learning;
2. Planning: collaborative work leading to the production of a collaboratively written plan (Lesson Plan) for research lesson;

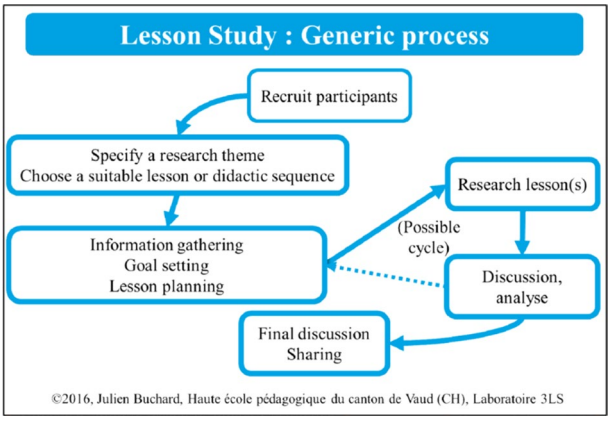


Fig. 1 The essential Lesson Study features (Buchard & Martin, 2017, p. 13)

3. Research lesson: taught by a nominated teacher—the implementing teacher, while the other group members act as observers;
4. Post-lesson discussion: the LS group meet to formally discuss the evidence gathered;
5. Repeated cycles of research: new lessons and not revisions nor re-teachings of previous research lessons are planned and taught;
6. Outside expertise: there is input from knowledgeable other;
7. Mobilising knowledge: opportunities of teacher learning are created, through observing and networking.

LS is usually modelled as a cycle. Figure 2 illustrates the diagram used to present LS to the teachers involved in our experiment, proposed by Joubert et al. (2020, p. 910), from South Africa.

A number of researchers define LS as a collaborative teachers' professional development model, focused on the participants' co-responsibility in the process of planning, implementation, and revision of lessons (in Mathematics, in our case), with exceptional attention to details such as time scheduling, planning of materials and instruments to be used (e.g., Bartolini Bussi et al., 2017; Fernandez & Yoshida, 2004; Huang et al., 2017), characteristics that are rooted in its Confucian origin. European researchers report that, as such, LS has offered to European teachers both a new approach to working on lesson planning, and an opportunity to confront with a different culture, sprouting reflections on the implicit assumptions that surround teaching practices (Clivaz & Miyakawa, 2020; Skott & Møller, 2020). In a review of Western studies on LS, Ponte (2017) from Portugal highlighted these potentialities of LS, while also stressing some difficulties related to the adaptation of LS in a context different from the original one. This study looks to contribute to the knowledge on LS adaptations in Western countries, hoping that the results obtained in the Italian context may prove useful for other contexts as well.

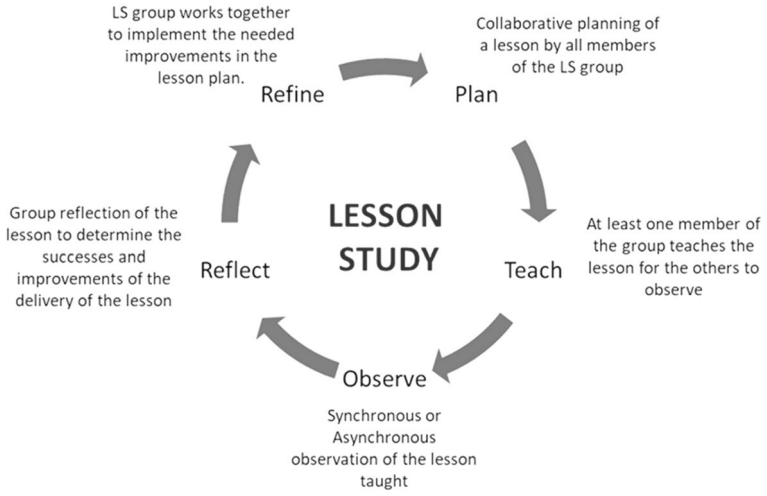


Fig. 2 The LS cyclical model (Joubert et al., 2020, p. 910)

Lesson Study in Italy

As outlined by Ponte and colleagues (2018), LS often starts from a detailed analysis of the teaching difficulties, and teachers research the tools to tackle such difficulties. Because of this, teachers participating in LS may be considered involved in research, without being "qualified" researchers. Hollingsworth (1995) calls them "teacher-researchers": "[they] are concerned simultaneously with (a) ways to improve their practices, (b) change the situations in which they work, and (c) understand their practices within the larger society" (ibid., p. 16). The idea of teachers as researchers (teacher-researchers) has been present in the Italian context since the 1960s (Arzarello & Bartolini Bussi, 1998), and is one of the many reasons that justifies the Italian interest about Lesson Study. In the following, we describe the two main paths through which the Italian research group in Mathematics Education became aware of LS:

1. The Department of Education and Human Sciences research group at the University of Modena and Reggio Emilia has approached Chinese teaching research experiences in international meetings. Lesson Studies have been observed in China (Bartolini Bussi et al., 2017). The researchers carried out several LS at the primary school level in Mathematics and Language. From teaching research experiences observed in China, they "identified some conflicts emerging between the cultures of teaching in China and Italy and explored the way to overcome the conflicts". For example, they "identified some additions to the Lesson Plan, suitable to make LS more compatible with Italian didactical context" (Ramploud et al., 2022, p. 154): one of them is a detailed analysis of classroom context, another – in line with the semiotic mediation framework—is the analysis of materials (Ramploud et al., 2022).
2. The Department of Mathematics research group at the University of Turin became interested in LS while working on the survey for ICME13 (Robutti et al. , 2016). The researchers implemented Japanese and Chinese LS with in-service and prospective primary and secondary school mathematics teachers (Manolino, 2021b).

At the present time, the introduction of LS in the Italian context is in its prime: i.e., we are currently dealing with issues around the management of time (as described in), establishing a shared non-confusing terminology to introduce LS to the teachers (Manolino, 2021b), and so on. Therefore, the characteristics that differentiate Japanese and Chinese LSs are not being considered to their full extent. We could say that Italian researchers are, at the moment, more interested in the general "philosophy" of LS, with special regards to the collaborative setting for planning a lesson in detail (Capone et al., 2020). From these studies, further experiments are being conducted in Piemonte, Lombardia, Valle d'Aosta, and more recently in southern Italy: Naples (Ribeiro et al., 2019) and Salerno (Capone et al., 2022a; this paper). An essential emphasis is provided to the cultural background in which the teaching processes occur (Bartolini Bussi & Martignone, 2013; Mellone et al., 2019).

The LS described in this paper is in continuity with Italian studies and with the contributions presented by the authors at WALS 2019 (https://www.walsnet.org/2019/; Capone et al., 2019), the Second International ACME Symposium on Mathematics Education (https://math.ecnu.edu.cn/academia/acme2019/en/index.html), and ICMI Study 25 (http://icmistudy25.ie.ulisboa.pt/; Capone et al., 2020).

Teachers' professional development in the Italian context and the Liceo Matematico project

As written before, Italy has a long tradition in collaboration between researchers and teachers, which has become a peculiarity of Mathematics Education Research (Arzarello & Bartolini Bussi, 1998), that has recently introduced LS as a topic of investigation (Minisola & Manolino, 2022).

One of the institutional contexts in which this collaboration is carried out is the Liceo Matematico project (https://www.liceomatematico.it/) (involving 25 Universities), characterised both by local teachers' professional development and implementations in schools, and national meetings in seminars/congresses to share issues and resources. At the local level, secondary school teachers collaborate in various ways with researchers, to design, solve, discuss, and implement (mathematical or interdisciplinary) activities for the students (Capone & Faggiano, 2022d). The novelties of the educational approach of Liceo Matematico project are manyfold: institutional, because schools and academies involved are officially related by institutional agreements; professional, because of the double role of teachers: as learners in a community of colleagues, in contact with researchers, and as teachers in their classes; didactical, because of the *mathematics laboratory* approach (Bartolini Bussi & Martignone, 2013) and other teaching practices coming from research studies (e.g., inquiry-based methods, mathematics argumentation). Researchers provide teachers with constant support, via online and face-to-face interactive meetings of professional development, giving them theories, methods, and activities to discuss and implement.

We decided to use the educational context of the Liceo Matematico project to implement LS at secondary school level, because of the already present collaboration among teachers. LS introduces in the community of teachers of this project something more: a structured sequence of collaborative planning and implementing of activities. The results will be useful in a double way: to understand if and how LS can be fruitful in this project and to study its implementation in a more general school and teacher education contexts (out of the project). And moreover, these results can give a more general approach to the introduction of LS, based on a cultural and institutional implementation.

Theoretical framework

In this section, we present the Cultural Transposition (Mellone et al., 2019; Ramploud et al., 2022) as a theoretical frame for investigating the use of LS in an institutional and cultural context different from the one in which it was created. The Theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) is also detailed, as it was an important element of the teacher education process: the researchers introduced it to teachers, as a theoretical tool for the planning and teaching phases and as a lens for reading experimental data from the research lesson with the teachers.

Cultural Transposition

Cultural Transposition is not only a theoretical but also an action perspective for teachers and teacher educators. It aims to frame the use of mathematical teaching practices from foreign cultural contexts as an opportunity to question one's own teaching practices and

educational intentionality (Mellone et al., 2021), in an "emancipatory" context for teacher learning (Mellone et al., 2020, p. 382). Within this framework, the study is situated in the context of teacher and teacher-educator learning literature (Jaworski & Huang, 2014), where teacher professional development increases teachers' knowledge and awareness (Andriano & Manolino, 2023) while reflecting in (and not only on) their teaching practices in a joint activity, with the aim of improving opportunities for learners of mathematics in the classroom. Here, teacher professional development framed in Cultural Transposition is innovative: culture becomes the trigger for development. Teachers, "interacting with foreignness" (Welsch, 1999, in Barton, 2008, p. 38; Manolino, 2021b) are led to become aware of their *unthoughts* (Jullien, 2006), namely, what is taken for granted and, in a sense, considered mandatory in their practice.

The idea of Cultural Transposition is not to "perform a comparative study or a slavish import–export of mathematics education methodologies and tools between different countries, but rather to open a dialogue between two different cultures in which every thought, meeting the other culture, questions its own unthought", as an emancipatory tool for teacher learning (Mellone et al., 2020, p. 382). An example is described in Di Paola & Buttitta (2020), where variation problems, as one of the most significant problem solving approaches in Chinese schools, is discussed as a useful methodology to let the students discover the relationship between pyramid and cone areas/volumes even in cultural contexts where relations between solids are not usually the focus of the teaching of this topic. In fact, "the transposed teaching practice should embed some of the cultural aspects of its original context, as this would allow teachers to come into contact with different values and beliefs on mathematics teaching/learning" (Mellone et al., 2021, p. 786).

Cultural Transposition (as conceptualised in Ramploud et al., 2022) attempts to hold together a dual purpose: on the one hand, to produce visible changes in teachers' teaching practices—the primary aim of teacher professional development; on the other hand, to produce teacher professional development practices that is culturally situated—namely, contextual and tailored to teachers' actual needs.

LS is just one of the possible teacher education models that can be framed in Cultural Transposition. Probably one of the most emblematic, and others have been shown in previous research (transposition of Chinese problems with variation: Mellone et al., 2019; of the El'konin-Davydov curriculum: Mellone et al, 2020; of the Thinking Classroom: Mellone et al., 2021; of the LS in Distance Education during the COVID pandemic: Ramploud et al., 2021; Capone et al., 2022a, Capone et al., 2022b), or for some other practices the need for Cultural Transposition has been postulated for the exportation of practice (transposition of the Reggio Emilia Approach: Landi & Pintus, 2022).

Embedded in the context of LS as an illustrative example, to attempt their dual purpose, Ramploud et al. (2022, p. 150–151) show how the three Key Phases of Cultural Transposition ((KP1): Contact with teaching practices of other cultural contexts; (KP2): deconstruction of teaching practices; and (KP3): Teacher education and development practices) lead to the development of different varieties of LS, always contextualised, as the LS group works. They are called transpositions, i.e. hybrid varieties (in the sense of Ribeiro et al., 2019) of the initial object.

Transposition significantly differs from "cultural adaptation", as meant by authors such as Stigler and Hiebert (2016): the purpose of Cultural Transposition is *not* to achieve the adaptation of LS suitable for the target cultural context. The word "adaptation" conveys the idea of something general, "universal". Rather, what the cultural transposition does is to stress, time after time, one or more hierarchical oppositions of antinomic concepts on which it is chosen to be overthrown, and depending on those decisions, different varieties

of LS could be developed suited to the concepts being worked on. Therefore, what Cultural Transposition describes are processes, not adaptations from one static model. Very local ones, that heavily depend on the choices made both, at the input, by the researcher and by the teachers implementing the LS, by the goals of the teacher professional development.

Those processes are carried out in the three Key Phases. KP1 is the encounter with the other/foreignness: teaching practice from another cultural context (here the LS). In KP2, teachers and knowledgeable others identify the deconstructive potential of the foreign practice, i.e. identify some hierarchical oppositions. KP3 is the experimentation of the foreign teaching practice. The three phases constitute an ongoing cyclic process. Throughout these phases it is carried out what Derrida (2002) calls *deconstruction*. The moves of deconstruction are two (Ramploud et al., 2022, p. 149):

1. "Overthrowing of hierarchical oppositions": addressing the conflict, considering its structure. It is not just a matter of "switching" the hierarchy between two opposite values.
2. "Hyperanalysis which, in the sign, points to the impossibility/unthinkability of the other": a continuous movement from one value to its opposite, which is not aimed at reaching a dialectical synthesis between them (the "third term"), but at continuously pointing the other.

The whole process elicits reflection on one's own practices, prompting a choice of which hierarchies to overthrow and which to maintain, i.e. to become aware of and act on one's own unthoughts.

The context of the experimentation described in this paper is part of the ongoing cyclic process initiated by the two mentioned research groups (par. 2.2), which are working in the spirit of Cultural Transposition. Researchers activated several processes of deconstruction. We, as well as them, identified in this paper several hierarchical oppositions that can be overthrown keeping the internal articulation of the LS "critical components" almost unchanged. And through hyperanalysis process, we and them also identified some additions, suitable to make LS more compatible with Italian didactical context. An outstanding example is the adoption of a theory, namely the Theory of semiotic mediation, to provide the conceptual framework for teachers and researchers collaborative design of the research lesson. "Despite these changes, LS maintained its deconstructive potential with respect to several hierarchical oppositions" (Ramploud et al., 2022, p. 155), which allowed the reflection of teachers, highlighting new hierarchical opposition to be overthrown, some that have been maintained and others that LS was successful in overthrowing.

Theory of semiotic mediation

The expression "semiotic mediation" refers to the Vygotskian theoretical framework (Vygotsky, 1978). The process of "internalization" is described as a process of building knowledge and modes of thought derived from lived experiences. Two main aspects characterize this process of internalization: an external action, which is essentially social, and an internal action regulated by semiotic processes. These two aspects are interrelated because the systems of signs and the semiotic processes connected to their use are intrinsic to a social or individual activity. In particular, the Vygotskian hypothesis foresees a very close link between technical and psychological tools, thus creating a didactical path that, starting from the use of tools, aims to construct mathematically

significant meanings and concepts. In this sense, learning can be considered as a social activity mediated by the teacher (Vygotsky, 1978).

In Fig. 3, a scheme of the semiotic mediation, as shown in Bartolini Bussi and Mariotti (2008) is reported.

As Fig. 3 highlights, at the heart of semiotic mediation, an artifact embeds mathematical meanings but is not transparent to embedded meanings (Bartolini Bussi & Mariotti, 2008, p. 246). The teacher gives some tasks to the students. Students, interacting with the teacher-mediator, leave "traces" of their activities (through situated texts). These traces constitute a dynamic system of signs (gestures, words, drawings, sketches, …) and their relationships (e.g., the simultaneity of a gesture with an utterance) produced by one or more subjects who interact during the implementation of a task. Through a social interaction process, the situated texts become mathematical texts, and this cycle leads to the student's appropriation of mathematics knowledge.

Mathematics knowledge, tasks, and artifacts are the triangle of the semiotic potential of the artifact; it can favour, through concrete and significant actions, formal knowledge acquisition.

Therefore, the semiotic potential of an artifact is given by the set of signs that can arise when used as a tool when the student solves a task or a problem (Bartolini Bussi & Mariotti, 2008). The activities for the students and their organization are decided according to a preliminary analysis of the semiotic potential of artifacts so that the students' personal meanings can develop as they emerge in-class activities, reaching the status of mathematical meanings, thanks to the teacher's intervention.

The choice of artifacts is fundamental to the lesson's success, and the teacher's role is decisive in this choice (Bartolini Bussi & Mariotti, 2008).

In this LS, the Theory of semiotic mediation is an element of teacher education in the co-planning and teaching phases and a lens for reading experimental data with teachers within the reflecting phase. It is used to frame and then foster the deconstructive process to achieve greater awareness of teachers' *unthoughts*.

Given this theoretical framework, we can now re-formulate our naïve research questions.

We investigate how, according to the teachers, LS can contribute to the professional development of Italian in-service secondary-school teachers. To do so, we investigate teachers' deconstruction of teaching practices and the *unthoughts* emerging from their interaction with foreignness in professional development. Our re-formulated research questions are:

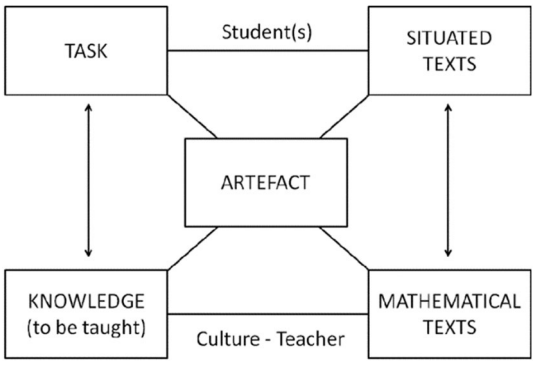


Fig. 3 The semiotic mediation scheme (Bartolini Bussi & Mariotti, 2008, p. 246)

1. Given the teachers' interaction with LS, what are the hierarchical oppositions that emerge?
2. What effects do these hierarchical oppositions have on teachers' professional development?

Methodology

In this section we present the context of the experiment, with the description of the setting and the participants involved, the data collected and the method.

Setting and participants

The LS experiment presented in this paper was carried out in a scientific-oriented secondary school in Avellino, a suburb of Naples, in the South of Italy. The school is involved in Liceo Matematico project with the University of Salerno.

The confidentiality of participants is maintained through comprehensive pseudonymization.

The headmaster invited two researchers (R_1 and R_2, two of the authors) from the Department of Mathematics at the University of Salerno to hold a teacher-oriented seminar on LS.

Three teachers (T_1, T_2 and T_3) volunteered to experiment with LS in their classes.

T1 has a degree in mathematics with about 10 years of teaching experience. She is a highly motivated teacher who has also proposed as the first to experiment with LS in her classroom. T2 has a degree in mathematics and has been teaching for 15 years. She has dealt extensively with children with special educational needs. T3 has a degree in physics and has a great deal of experience teaching both mathematics and physics. She has been teaching for 38 years and is close to retirement. The two researchers and the three teachers constituted the *project group* and R_1 acted as facilitator. R1 is currently a researcher in mathematics education but has a lot of teaching experience in high school because he taught in high school for 12 years. R2 is also currently a researcher in mathematics education but has about 20 years of teaching experience in high school.

Three different LS cycles were implemented in three grade-10 classes, one for each teacher. T_1 was the first to implement the research lesson (from now on, we will refer to her as the *implementing teacher*).

Grade-10 classes were chosen because in Italy these students are assessed through national standardised tests (INVALSI, 2017) and the project team wanted to work on the issues revealed by the tests (see par. 5.1). Out of all the data collected in the experiment, in this paper we refer as a convenient sample to what was gathered from the implementation of the only first cycle.

Data

Our data, concerning the first LS cycle only, are:

- video recording of the co-planning meetings (lasted a total of 4 h), the teaching phase (1h30'), and the reflecting phase (1h50');
- Lesson Plan: a 13-pages document;

- 8 observation grids: each observer (i.e., all members of the project group except T_1) filled out an observation grid twice, the first time in person during the research lesson, the second time by observing the video recording;
- a collection of photos in all the phases;
- 3 questionnaires: the three teachers answered a questionnaire at the end of the reflecting phase.

The tools used for data collection were:

1. three video cameras, placed initially in 3 different corners of the classroom, one fixed;
2. the Lesson Plan, designed by the Modena-Reggio Emilia research group (Manolino et al., 2020; Ramploud et al., 2022);
3. a semi-structured observation grid, jointly designed by the project group;
4. a final questionnaire, designed by the researchers and filled-in by the teachers.

Method

The research methodology is observational and interpretive (Baker, 2006), with the added value of video research, which allowed researchers to delve into verbal codes (conversational exchanges, oral reflections) and non-verbal, proxemic and interactional codes.

In fact, we approached the "investigator triangulation" (ibid., p. 283). Three authors of this paper independently watched the videos and transcribed the excerpts. All the authors then checked for interpretive validity, i.e. "accuracy in reporting the facts" (ibid., p. 285), by asking for participants' feedback and checking the use of "low-inference descriptors" (i.e. direct quotes) (ibid., p. 283; see also Adler & Adler, 1994). Finally, all the authors carried out a triangulated discussion on the meaning attributed to what is being studied.

The case study: an Italian Lesson Study with in-service secondary school teachers

In this section, we present the case study of the first cycle, taught by T_1. After this first cycle, two further LS cycles were planned and implemented with two other classes (not presented here). The first cycle involved twenty-five students.

In the following subsections, all phases of the LS cycle are described in detail.

To introduce LS to teachers, we used the cyclic model proposed by Joubert et al., (2020, Fig. 2). One of the reasons for choosing this model is that the observation phase does not coincide with the teaching phase. This served to give value to the two moments of observation: both synchronous, during the teaching phase, and asynchronous, analysing the videos of the research lesson within the reflecting phase.

Compared to the original model, we have made some minor changes (Fig. 4). The "plan" phase is called "co-planning" to emphasise the importance of collaborative aspects. Verbs in the gerund are used instead of nouns because they better represent the dynamism of the didactic action.

During the co-planning phase, the project group drafted the Lesson Plan, which includes a detailed description of the class, the activities to be carried out, and the materials to be used, and anticipates student responses, misconceptions, and lesson successes. Research objectives for active professional development are also chosen by the project group. For

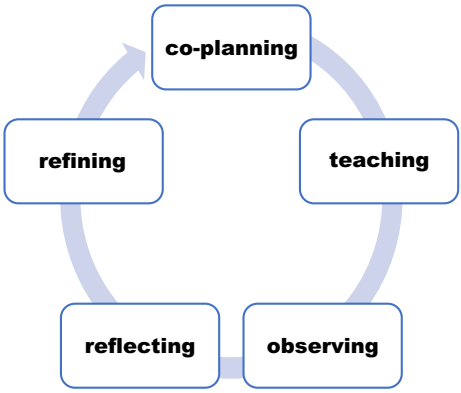


Fig. 4 The cyclical model used for Lesson Study in the Italian secondary school (adaptation from the Joubert et al. (2020) model of Fig. 2)

each activity of the research lesson, the classroom setting, and precise timing are planned and the teachers' educational intentionalities are described.

During the teaching phase, T_1 implemented the lesson. The lesson was video-recorded. By means of observation grids, observations were addressed on the elements chosen by the project group to be discussed during the reflecting phase. In the observing phase, the teacher's actions were observed. In the observing and reflecting phases, each project group component observed the teaching (synchronous observation), then analysed the recordings (asynchronous observation) and filled the grids. In the refining phase, the Lesson Plan was modified: changes were made collectively, based on the criticism that emerged during the reflecting phases to begin a new LS cycle, in which the lesson is taught by another teacher from the same group in a new class.

In the following, all the documents and empirical data are original in Italian language, and they have been translated by the authors.

Co-planning

The co-planning phase took place in three different meetings: the first two in-person at the university (1h30' each) and one via Skype (1 h). The first meeting started with the three teachers (T_1, T_2 and T_3) commenting the teacher-oriented seminar on LS conducted by R_1 and R_2 at school upon the invitation of the school headmaster:

> T_1: It seems a very distant teaching system from ours.
> T_2: Such a constructed and planned lesson seems too far removed from our freedom of teaching, guaranteed by law and by the National Curriculum.
> T_3: Thinking of being observed during the lesson… I don't think it is a good thing for the students, but also for us as teachers, who won't feel fully comfortable.
> [CO-P1]

During the rest of the meeting, in accordance with the literature on LS (Ponte et al., 2018), the project group focused on identifying students' main difficulties in mathematics. T_1 suggested focusing on Geometry content. The reasons given, with which all other participants agreed, were related to students' difficulties in Geometry at national level (Capone et al., 2022a): it is known from INVALSI (2017), shown in Table 1, that the percentage of

Table 1 INVALSI Report 2017. Percentage of correct answers in different content domains (adapted form INVALSI, 2017, p. 49)

Content domains	% Correct Answers
Numbers	49,78
Geometry (Space & Figures)	41,02
Data and Previsions	53,94
Relations and Functions	46,30

correct answers in Geometry in national standard tests in grade 10 is the lowest compared to the other content domains.

Teachers and researchers discussed some possible causes of these failures in Geometry, suggesting different strategies to overcome those difficulties. A discussion arose on the necessity of certain key concepts of the teaching of geometry and their importance for the development of logical thinking (the project group's *goal of the research lesson*). Transcripts of some fragments of the recordings are given:

T_1: In my opinion, students find more difficulties in Geometry because they have difficulty in reading and interpreting a text and sometimes even in graphically representing what is written.
R_1: I invite you to read some papers, which describe the difficulty that students encounter within the transition between different registers and representations.
T_2: I think that we could start from intuitive geometry rather than rational geometry.
T_1: But proofs are essential, as in the National Curriculum is highlighted!
R_2: So, the proposal could be to design units that start from intuitive geometry with immediate examples and an effective engagement, and then arrive at a formalization, i.e., to the rational geometry. What do you think?

T_2: We can try, but it is not effortless.
T_1: I need to choose a good engagement to motivate my students to study mathematics and to keep their attention during the lesson.
[CO-P2]

The teachers are strongly embedded in their institutional context, indeed they refer to specific topics of the National Curriculum (Indicazioni Nazionali, see Minisola & Manolino, 2022). They are also aware of the students' difficulties in approaching a problem text in general, and specifically in solving geometry problems, also for the difficulty they encounter in working in one representation (what Duval, 2006, calls treatments), or interpreting different representations and their corresponding transformations (what Duval, 2006, calls conversions). The role of the researcher here is to give teachers some hints from the research in Mathematics Education that include Duval and other authors, in order to link their experience coming from their practice to the theoretical studies in the field. Moreover, considering the National Curriculum and the three-year educational plan of the school (Minisola & Manolino, 2022), the teachers decided to plan their LS on Geometry in a long-term perspective, so that LS could be "sustainable", and not an isolated activity. In fact, according to Italian institutional context, teachers are aware of two equally fundamental elements of the planning: the plan of the research lesson, focused on one/more concepts of Geometry, and the connection of this lesson with a long-term vision, which takes into account the lesson inserted in the year/s (one or more) educational plan. Another important element to consider is the work of teachers, as made not only in collaboration, during the meetings, but also individually in asynchronous mode, to reflect and collect ideas, which

later are shared with the colleagues in the project group. Individual reflections are collected in the collaborative phase of planning, as the single teacher's contributions, which have a role in the construction of the shared plan.

During the second meeting, the teachers cleared the aim to design an activity named *The Art of Geometry*, which collects the short and long-term visions described above. Inside this activity, they selected Tessellation as the topic for the LS research lesson, under the suggestion of T_1. This topic is interdisciplinary, connecting Mathematics, Natural Sciences, and Arts, and for this reason it fits with the Liceo Matematico project's aims (see par. 2.3).

Coherently with their knowledge on LS, they wrote the Lesson Plan (Authors, 2019) for this topic, considering both the Italian secondary school context and the specific school features. In Table 2, the first row of the Lesson Plan is shown.

While choosing the topic and structure of the Lesson Plan, prepared as in Table 2, according to their institutional context, the teachers chose the artifacts to be used by students, and the prerequisites, the general educational goals (long-term vision), the specific learning objectives of the lesson (short-term), and the mathematical competencies associated with them.

The project group's choice for the observing phase (the *purpose of the observation*) was to concentrate on: teacher's communication (posture, tone, gestures, interaction with the class), ability to involve students with the use of scaffolding, and scheduling. The group of teachers and researchers produced a grid of observation (Table 3) as a tool to be used in class during the research lesson. During this meeting a criticism also emerged about the presence of the observers during the teaching: the private space of the class becomes a public space.

The project group, when writing the Lesson Plan, took into consideration not only the professional need to observe the teacher's teaching behaviour throughout the implementation of the research lesson, but also a number of didactical variables:

- the different students' learning styles;
- the context of the lesson, linked to realistic situations;
- the link between the lesson and the three-year school educational plan;
- the consistency of long-term and short-term learning objectives;
- the artifacts for introducing tessellation.

The artifacts chosen for the research lesson had been examined by the group in relation to their *semiotic potential* for the concepts to be introduced, according to the theory of semiotic mediation. As such, they are described in a specific section of the Lesson Plan.

Teaching

The teaching phase (lasted 1h30', despite a planned duration of 1 h) started with the introduction of the tessellation topic, as required by semiotic mediation (see contextualised scheme, Fig. 9). To do that, the implementing teacher, T_1, provided the students with a sheet where three tasks were shown, in reference to different artifacts: a sentence (in Fig. 5a); two images (an Escher picture in Fig. 5b, and the image of a beehive in Fig. 5c); and a technological tool (students' smartphone in Fig. 5d). These artifacts were chosen in the co-planning phase, for their *semiotic potential* and their meanings strictly connected to mathematics meaning, given respectively by the meaning of keywords in

Table 2 A row of the Lesson Plan, as co-planned by teachers and researchers

	Description of the activity	Grouping and class setting	Time	Educational intentionality (the reasons for the choices)
Introduction to the lesson and presentation of the topic	The topic is introduced using an artifact: the students are provided with a card containing a phrase from Hardy, an Escher's artwork, and a hive Students are invited to use their smartphone or tablet (in BYOD mode) to find further examples of tessellations and to grasp the concept of "tessellation"	The implementing teacher already arranged the students in small groups of 4/5.Within this setting all the students can look at the Interactive WhiteBoard (IWB) in case of face-to-face explanations	10'	We prefer to immediately use an artifact to "capture" students' attention and then get them into formalization

Table 3 Teacher behavioural indicators in the observation grid

Categories	Behavioural indicators
Communication	✓ Does he/she provide students with all the essential elements for the tasks? ✓ Does he/she provide explanations to the students during all the tasks?
Interaction with the class	✓ Does he/she use gestures to support tasks understanding? ✓ Does he/she use gestures as teaching support?
Class Management	✓ Does he/she participate in the student's activity? ✓ Does he/she intervene in the groups' presentations of their work?
Time management	✓ Does he/she manage the discussion times? ✓ oes he/she act in the scheduled times?

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas (Hardy, 1992).

(a)

(b)

(c)

(d)

Fig. 5 The artifacts of the semiotic mediation cycle

the sentence, the repetitive module represented in the two pictures, and the possible images of the same kind searchable on a smartphone. The different registers have been chosen in relation to the stimuli that can convey in the students, to understand the information in the semiotic representation, but also to connect different representations with the same meaning (according to Duval, 2006). For that reason, the students were asked to move from the words to the images.

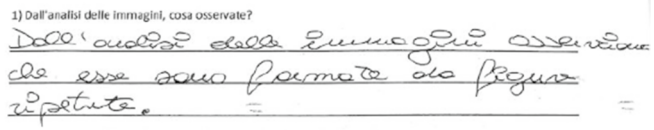
The tasks were:

1. *What do you observe concerning the shown images?*
2. *In Hardy's text we read "permanent", what adjective would you use instead of "permanent" to describe the observed images?*
3. *By using the smartphone, search for some "similar images" in art and in nature and try to represent them.*

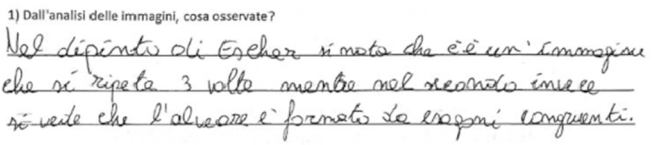
About Task 1, in Fig. 6, some students answered are shown. These answers are particularly interesting, since the students use the word "repeated", which conveys the idea to reuse the same shape to cover the space.

Again, in the second task, the students answered with similar adjectives: "repeated, regular, perpetual". And in the third task, the students referenced to some websites (eg. https://maretta89.wordpress.com/2010/11/24/le-tassellazioni/) and drew some "repeated and regular images", as in Fig. 7.

Group A



Group B



Group A (translated)
Analysing the images, it can be stated that they consist of repeated figures.

Group B (translated)
In Escher's painting there is an image repeated three times, while in the second image there are regular, congruent hexagons.

Fig. 6 Two students' answers to the first task

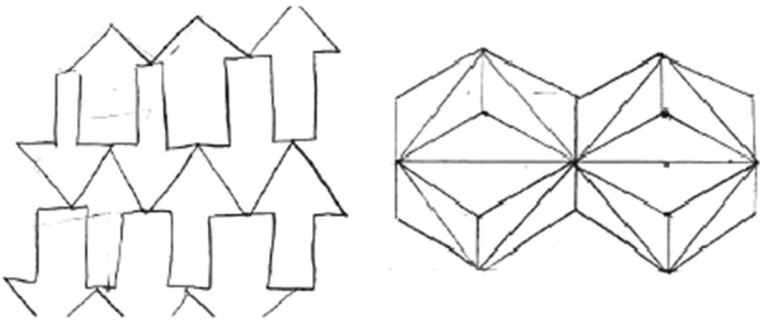
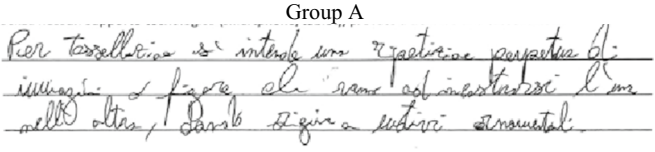
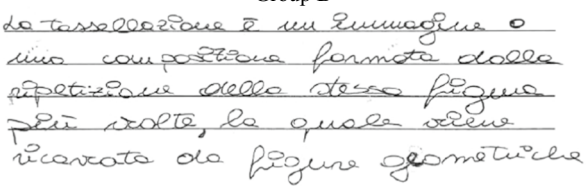


Fig. 7 Images drawn by students to answer the third task

Group A



Group B



Group A (translated)
Tessellation means a perpetual repetition of images or figures that interlock with one another, giving rise to ornamental motifs.

Group B (translated)
Tessellation is an image or composition formed by the repetition of the same figure several times, derived from geometric figures.

Fig. 8 Two groups' definitions of tessellation

In coherence with the co-planning and with the semiotic mediation framework, the texts and images produced by the students are *situated texts.* The word "tessellation" was not used during the tasks, and it was introduced by the implementing teacher to move from situated texts to mathematical texts: the students were asked to formalise

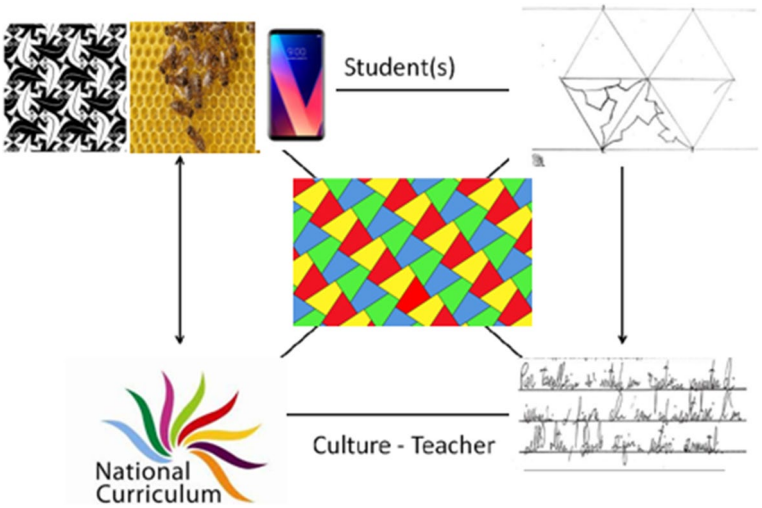


Fig. 9 The "tessellation" semiotic mediation scheme

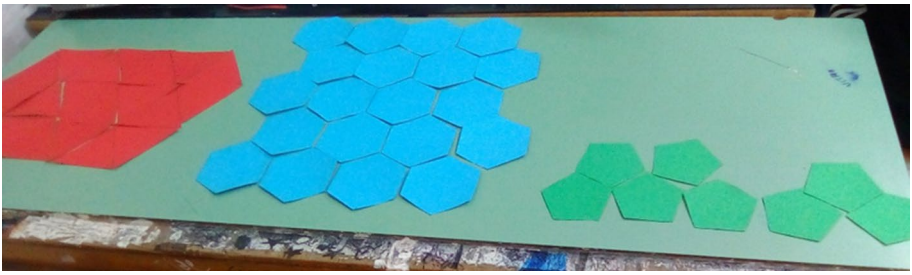


Fig. 10 A group's desk tessellation activity with different polygons

their observations in a *mathematical text*. Each group provided their own definition of "tessellation". In Fig. 8, two examples of students' *mathematical text* (group's definition of tessellation) are shown.

At the end of the activity, in line with the semiotic mediation scheme (in Fig. 9), the students identified the main features of tessellations: repeated, regular figures covering a plane. Indeed, the activity was aimed at the mathematical knowledge of tessellation in agreement with the Italian National Curriculum.

The subsequent task was chosen as a crafting reality problem:

> Tessellate your school desk with regular polygons of different types (equilateral triangles, pentagons, hexagons) that you have to build using coloured cards, scissors, pencils, ruler, and compass.

T_1 also provided written instructions on how to use straightedge and compass to realize regular polygons, specifying that smartphones could be used, if necessary, to search for additional information about instructions. An example of tessellation with regular triangles and hexagons is shown in Fig. 10, while pentagons tessellation is not possible with desk tessellation, as also shown in the same figure.

The students had to show and discuss the results of the problem with the class. T_1 guided the students to discover that not all tessellations with repeated and regular figures are possible. At the end of the class discussion, she formalised the concept of tessellation using some slides designed in the co-planning phase.

Observing

The observing phase took place at two different times. There was a synchronous observation during the implementation of the research lesson, and an asynchronous observation in which all the members of the project group analyzed the recordings of the student groups. Observers filled out the observation grid both times. As an example, the grid as fulfilled in-presence observation by T_3 is shown in Fig. 11.

Reflecting

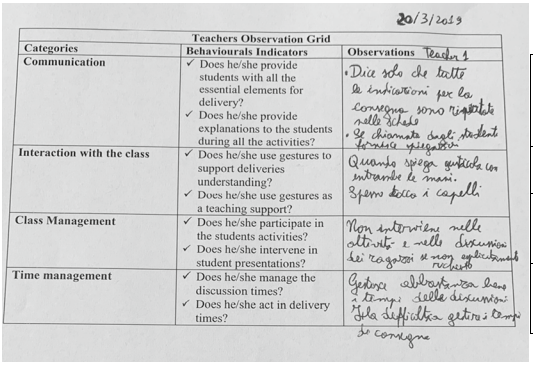
The reflecting phase took place after two weeks, due to researchers attending a conference and teachers' school commitments. The observing grids, as fulfilled both in a synchronous (in-presence observing phase) and asynchronous (video analysis observing phase) way, have been discussed.

It was noticed that the research lesson did not respect the planned time allocation and some possible causes were analysed. In T_2's observation grid there was written:

> Students waste too much time on the daily problem: each group had to build too many polygons.
> [RE1]

From the observational grids and the discussion, the project group deduces that mainly motivations for the time lacking were:

1. a bad planning:
2. some teacher's difficulties in managing delivery times;
3. lacking in organising the materials for the artefacts.



20/3/2019

Teachers Observation Grid

Categories	Behavioural Indicators	Observations Teacher 3
Communication	✓ Does he/she provide students with all the essential elements for delivery? ✓ Does he/she provide explanations to the students during all the activities?	• Dice solo che tutte le indicazioni per la consegna sono riportate sulla lavagna • Le chiamate degli studenti vengono spiegate
Interaction with the class	✓ Does he/she use gestures to support deliveries understanding? ✓ Does he/she use gestures as a teaching support?	Quando spiega gesticola con entrambe le mani. Spesso tocca i capelli
Class Management	✓ Does he/she participate in the students activities? ✓ Does he/she intervene in student presentations?	Non interviene nelle attività e nelle discussioni dei ragazzi se non espressamente richiesto
Time management	✓ Does he/she manage the discussion times? ✓ Does he/she act in delivery times?	Gestisce abbastanza bene i tempi delle discussioni Ha difficoltà gestire i tempi di consegna

Translation
T_1 merely says that all instructions for the task are given on the sheets. She only provides some explanations if requested.
She gestures with her hands when she explains. She often touches her hair.
She does not intervene in student activities and discussions unless expressly requested to do so.
She manages the timing of discussions quite well. Has difficulty managing delivery times.

Fig. 11 Observation grid, as fulfilled by T3

After the video reviewing, the project group observed:

T_1: I recognise that although I realised that the students were exceeding the time limit, I did not interrupt the discussion because I found it interesting how they were discussing with each other. I do not fully agree with these time constraints; I don't understand how Japanese teachers could interrupt an interesting activity.
T_2: I still think that we designed an activity that was too long. I suggest these changes: arrange the desks before the lesson; establish the groups before the research lesson; avoid students having to cut out the figures during the activity but prepare the materials beforehand.
T_3: We could modify the lesson planning: for example, each group could build just one type of polygon.
T_1: Perhaps we are not used to planning a single hour lesson with so many activities.
R_2: Let's try the changes suggested by T_2 and T_3, hoping to have a lesson on time.
[RE2]

During the reflecting meeting, concerning the value of jointly designing the activities to be provided to a specific group of students, T_1 stated:

T_1: The discussion between teachers in the choice of the activity was very important; in our teaching practices it is essential to choose a good engage to motivate our students to study mathematics and to keep their attention during the lesson; the laboratorial use of tessellations to introduce concepts of Geometry such as transformations, similitudes and comparison of areas was a nice idea that came out from the sharing of our experiences during many teaching years.
[RE3]

Finally, all the teachers answered a questionnaire, which included the following questions:

1. *Do you think that the Lesson Study experience has been helpful for your professional development? Write your motivations.*
2. *What do you think are the strengths and weaknesses of Lesson Study?*
3. *Do you think that Lesson Study is consistent with the requirements of the National Indications?*
4. *Do you think the confrontation with researchers in Mathematics Education about the planning of didactic activities was helpful?*
5. *Do you think your previous experience in collaborative design was significant in experimenting with Lesson Study?*
6. *Do you think that in Italian teaching, it is important to be able to involve (engage) more and more students in activities from which they generally flee, especially in challenging topics such as Geometry?*
7. *How important do you think the collaboration between teachers and researchers was choosing an appropriate artifact for effective experimentation of Lesson Study?*

As an empirical data, answering to the question 5, T_2 wrote:

T_2: During the co-planning phases, we realised that collaborative work is productive. It has been very important that mathematics teachers could discuss the planning of activities, but above all, it was an added value to be able to observe each other; our teaching practices will certainly be enriched and each one of us will bring a

piece of the other colleagues in our classroom practices.
[RE4]

About how useful it was to be able to discuss with researchers both in co-planning and teaching phases, T_3 answered to the question 4:

T_3: The researchers helped us to think about our teaching choices/intentionalities; often these choices are dictated by experience but the understanding that there are theories of Mathematics Education research that support teaching practices has been an added value of Lesson Study.
[RE5]

Refining

Considering the critical issues that emerged during the reflection phase, a revision of the Lesson Plan was carried out. Changes were made to both the tasks and the class discussion, e.g. in the subsequent classes, each group of students had to tessellate the desk with only one specific polygon instead of all four chosen polygons.

The LS cycle started again.

Theoretical discussion on data

We investigate how, according to the teachers, LS can contribute to the professional development of Italian in-service secondary-school teachers. To do so, we investigated teachers' deconstruction of teaching practices and the *unthoughts* emerging from their interaction with foreignness in professional development. Our re-formulated research questions were:

1. Given the teachers' interaction with LS, what are the hierarchical oppositions that emerge?
2. What effects do these hierarchical oppositions have on teachers' professional development?

In order to answer the first research question, we consider the three Key Phases of the Cultural Transposition, referred to as KP1 (Contact with teaching practices of other cultural contexts), KP2 (Deconstruction of teaching practices) and KP3 (Teacher education and development practices).

In this case study, the starting point of phase KP1 is the teacher-oriented seminar on LS conducted by R_1 and R_2 at school upon the invitation of the school headmaster (see par. 4.1). When the researchers emphasised the cyclical model and the LS seven critical components, by showing them videos of Japanese LS examples, many teachers at the school were sceptical. We have seen how even the three teachers (T_1, T_2 and T_3) who voluntarily chose to be involved in LS immediately expressed their doubts (from the transcripts of the first co-planning meeting): "*It seems a very distant teaching system from ours*"; "*seems too far removed from our freedom of teaching*".

The acceptance of the challenge and putting themselves at stake in LS, by the three teachers, triggers the KP2 phase. Some hierarchical oppositions have been overthrown keeping the internal articulation of the LS "critical components" almost unchanged. And

through hyperanalysis process, some additions have been identified, to make LS more compatible with Italian didactical context.

The empirical data, collected at the different LS phases and presented in Sect. 5, were analysed and 5 hierarchical oppositions are here highlighted and listed. For each hierarchical opposition, reference is made to at least one empirical representative datum.

HO1. Strict lesson planning/flexible lesson planning; or rather **actual planning/ customary "a-priori/theoretical" planning.**
T_2: Such a constructed and planned lesson seems **too far removed from our freedom** of teaching, guaranteed by law and by the National Curriculum.
[in CO-P1].

To be caught up in a "*constructed and planned*" planning, creates the perception among teachers that they are losing the freedom that is guaranteed by law in Italy. In Italy, in fact, the curriculum "contain[s] contents and aims for each subject, and its number of hours in a year. These contents are not prescriptive, but at the end of the 8th and 13th grades there are two national exams. Each teacher has the responsibility of the didactical plan for their classes, also according to the *Piano Triennale dell'Offerta Formativa* (Three-year Educational Plan – describing the cultural-pedagogical inspiration and the curricular, extracurricular, didactic and organisational design of the proposed activities). The contents of this document are specific to each school and decided by the collegiality of teachers and school staff (Minisola & Manolino, 2022, p. 5)".

Also for this reason, teachers are free to choose the purpose of the observation (see the layout of the observation grid, with details of what is the focus of the group's attention on, in Table 3 and Fig. 11) and the topic of the research lesson:

T_2: I think that we could start from intuitive geometry rather than rational geometry.
T_1: But proofs are essential, as in the National Curriculum is highlighted!
R_2: So, the proposal could be to design units that start from intuitive geometry with immediate examples and an effective engagement, and then arrive at a formalization, i.e., to the rational geometry. What do you think?
[in CO-P2].

Planning persists as a problem for some teachers, who are not used to detailed planning in a collective manner.

T_1: I recognise that although I realised that the students were exceeding the time limit, **I did not interrupt the discussion** because I found it interesting how they were discussing with each other. **I do not fully agree with these time constraints**; I don't understand how Japanese teachers could interrupt an interesting activity.
[…]
T_1: Perhaps **we are not used to planning a single hour lesson** with so many activities.
[in RE2].

Given that Japanese teachers would probably never interrupt an interesting discussion or activity when planned, this can be seen as a typical misunderstanding of what the Lesson Plan means. T1's statement points to the "disconnection" between the actual planning situation of working with a specific class and specific students in mind, anticipating student responses, misconceptions, lesson successes, and the customary "a-priori/theoretical" planning situation in teachers' practices, in which teachers broadly plan what they want to do, leaving the details of reaction to the students' answers completely to the "art of the

teacher" (Manolino, 2021b, p. 99) and to the day-ahead planning of the teaching materials of the single teacher who enters the classroom (alone).

Teachers, "forced" to plan in detail, are prompted to consciously prioritise (allocate time) certain aspects of the lesson. Indeed, teachers opt to make specific changes in the Lesson Plan:

> T_2: I still think that we designed an activity that was too long. I suggest these changes: arrange the desks before the lesson; establish the groups before the research lesson; avoid students having to cut out the figures during the activity but prepare the materials beforehand.
> T_3: We could modify the lesson planning: for example, each group could build **just one type of polygon**.
> [in RE2].

HO2. Teaching as individual work/teaching as collective work

> T_2: […] we realised that collaborative work is productive. It has been very important that mathematics teachers could discuss the planning of activities, […] our teaching practices will certainly be enriched and each one of us will bring a piece of the other colleagues in our classroom practices.
> [in RE4]

The T2 sentence "*It has been very important that mathematics teachers could discuss the planning of activities*" emphasises how it is a need for teachers to have the opportunity to discuss content/didactic issues. Teachers are often overwhelmed by logistical, institutional, or bureaucratic demands that detach them from their teaching role. Co-operation in schools often exists linked to educational, logistical, institutional, bureaucratic decisions. A professional development practice that allows teachers to sit at the same table and discuss mathematics contents is a valuable resource. Collaboration engenders this. It spares teachers the familiar feeling of loneliness when faced with teaching/mathematical content issues.

In the co-planning phase, T_1 stated:

> T_1: **I need** to choose a good engage to motivate **my students** to study mathematics and to keep their attention during the lesson.
> [in CO-P2].

In the reflecting phase, T_1 stated:

> T_1: The discussion between teachers in the choice of the activity was very important; in **our** teaching practices it is essential to choose a good engage to motivate **our students** to study mathematics and to keep their attention during the lesson; the laboratorial use of tessellations to introduce concepts of Geometry such as transformations, similitudes and comparison of areas was a nice idea that came out from the sharing of **our** experiences during many teaching years.
> [in RE3].

Moving from "*my students*" to "*our students*" was an impressive achievement, as teachers presented themselves as a community of teaching/learning professionals. Teaching as collective work allows for shared responsibility: all the teachers speak in the plural; it is never the fault of a single member of the group if things do not run as planned.

> T_2: I still think that **we designed** an activity that was too long. […]

[…]
T_1: Perhaps **we are not used** to planning a single hour lesson with so many activities.
[in RE2].
T_3: The researchers helped **us** to think about **our** teaching choices/intentionalities […].
[in RE5].

HO3. Class as a private space/class as a public space

In contrast to what is customary in China and Japan (Chen, 2017), in Italy it is not usual for more people than just students and the teacher to be present in the classroom, particularly at secondary school. The only exception may be in the presence of the support teacher of students with special needs. This conveys a dual feeling in the teachers: on the one hand of loneliness, and on the other hand of personal freedom and management of the classroom as a personal, private space. Space in which practices are habitual, contextual. Therefore, welcoming colleagues into one's space is seen as a violation. Teachers feel judged, assessed.

Given the common feeling of loneliness, most (at least Italian) teachers would not refuse to partially work collaboratively. Teachers might think that everything is fine in "a-priori" planning together, but then the actual practice will be in their classroom, they will have to make their own choices, as they know their students and what makes themselves feel competent.

T_3: Thinking of **being observed** during the lesson… I don't think it is a good thing **for the students**, but also **for us as teachers**, who **won't feel fully comfortable**.
[in CO-P1].

LS helps in "overthrows" such a fear. Due to the purpose of the observation, established by the teachers themselves—thus empowered—the preliminary scepticism turns into appreciation. Observation becomes the trigger element for deconstruction. Planning together is no longer the sole moment for everyone. Observation unlocks the space, from private to of the whole group:

T_2: […] above all, it was **an added value to be able to observe each other**; our teaching practices will certainly be enriched and each one of us will bring a piece of the other colleagues in our classroom practices.
[in RE4].

The added value of a class as a public space is to be enabled to enter the classroom as a peer. It is an uncommon practice, in teachers' context. Above all, it is uncommon with regard to a lesson "of everyone", which was designed by all the teachers, and everybody is responsible. Teachers, before LS, observed each other to learn "from" others, not "with" others. The value of collaboration emerges again; the classroom is a public space for collaboration. Not only collaboration among teachers, but also between the two communities of teachers and researchers, is recognised as an important pillar of teachers' professional development, especially when supported by the institutional synergy between university and school, as guaranteed in the Liceo Matematico project.

T_3: **The researchers helped** us to think about our **teaching** choices**/intentionalities** […].
[in RE5].

HO4. **Teaching as guided by theoretical frameworks/teaching as a-theoretical**

According to Chen (2017), a number of cultural beliefs guide the practice of Chinese Lesson Study, two of which are relevant to us at this point: the unity of knowing and doing (知行合一) and the practical reasoning (实践推理) based on specific contexts. "Unity of knowing and doing as reflected in Chinese teachers' enactment of their understanding about teaching is mediated by their practical reasoning in adopting the most appropriate action in a specific context. […] in terms of their actions in LS, the Chinese teachers enact their understanding of teaching in public lessons through unity of knowing and doing more than conceptual explication. Second, with regard to their thinking about LS, the Chinese teachers use practical reasoning in deliberate practice of repeated teaching through group inquiry and reflection" (Chen, 2017, p. 285). In another way, "Westerners' way of thinking is inextricable from theoretical frameworks, so for us neither teaching can be conceived without them. […] What emerged was precisely this impossibility to implement LS without a theoretical framework. For the teaching research group, the designed lesson had no meaning without a semiotic mediation framework, […] [Researchers/teacher educators] maintained this possibility, and the semiotic mediation framework to frame activities with artifacts, for all LS experiments in the following years" (Ramploud et al., 2022, p. 155).

LS emerges as an instance where these two ways meet, i.e. this hierarchical opposition is maintained, but this does not prevent it from bringing out the unthoughts of the teachers.

> T_3: The researchers helped us to think about our teaching choices/intentionalities; often these choices are **dictated by experience** but the understanding that there are **theories of Mathematics Education research that support teaching practices has been an added value of Lesson Study**.
>
> [in RE5].

T_3 voices the added value of LS as a professional development process capable of bringing theory into practice. Western teachers do not know how to think without theory (Winsløw et al., 2018), but often rely on unaware "*experiences*". In LS, research in Mathematics Education, with its explicit and implemented theoretical frameworks, supports teachers' practice. It offers them awareness of their own unthoughts.

HO5. Teaching/Research

The project group's choice for the observing phase (the *purpose of the observation*) was to concentrate on: teacher's communication (posture, tone, gestures, interaction with the class), ability to involve students with the use of scaffolding, and scheduling (see Table 3 and Fig. 11). The teachers' focus for this LS is therefore all about their own professional development, and in particular the study of their own behaviour in interaction with students. The teaching phase, in the LS, becomes a locus of research. And the Lesson Plan (see the last column of Table 2) a locus of disclosure and reflection on one's customary teaching choices/intentionalities [in RE5]. The teaching phase and the Lesson Plan are the trigger elements for deconstruction. This prompts teachers to become more aware of teaching as research. Indeed, how to facilitate the development of teacher competences and which models of teacher education can best meet the needs of today's schools, is a recurring question nowadays and encouraged by the results of the 2018 OECD Teaching and Learning International Survey. If the answer, which can be ascribed to the strand of *participative* research (Nigris et al., 2020), can be found in the many positions that define teachers' professionalism in an attitude of self-reflection and in a "research mentality", this answer can be operationally declined in being protagonists of research actions and activities. Quoting again Chen (2017): "with regard to their thinking about LS, the Chinese teachers use practical reasoning in deliberate practice of repeated teaching through group

inquiry and reflection" (p. 285). This is why LS is proposed, but is not yet, in western contexts (and here it triggers deconstruction), a model to foster this protagonism.

Final discussion and future perspectives

Given the teachers' interaction with LS, these are the five hierarchical oppositions that we observed. In fact, we observed that the teachers were able to reflect on the unthought related to the usual practices that are connected with their professionalism. Of the five hierarchical oppositions that emerged, four were overthrown (thus enriching their teaching practices) and one was maintained (resulting in more awareness of their didactic action in and outside the classroom).

The teachers, by taking charge of their own professional development in collaboration with the researchers, have brought out what the LS has achieved on teacher education. Indeed, we see a critical reflection on the functioning of LS processes in the discussion raised by the final questionnaire. This enabled the phase KP3, and answers the second research question.

Going back to our starting goal, we can discuss how this study contribute to the general knowledge on LS. The data collected and studied in these experiments provide a new understanding on how to promote and design relevant professional development for mathematics teachers in secondary school, even outside of Italy. In fact, one may wonder if these results were only possible because of the peculiar cultural, scientific and professional background of the teachers involved in the experiment, but other research in Italy show that this may not be the case. Ramploud et al. (2022) identified mostly the same hierarchical opposition in a parallel analysis of a group of teachers with completely different backgrounds (primary school teachers). Moreover, some features of LS that appeared problematic for the Italian teachers were also identified as problematic in other experiments of LS in Europe (e.g., Ponte, 2018). This lets us wonder if similar results could be found in other cultural context, and what that would entail for the implementation of LS in another country. LS is an opportunity—in this case for Italian teachers—to come together with the unthoughts of their teaching practices, which is not an easy feat. This paper shows that it can be a valuable chance for the teachers to become more aware of the reasons that guide their professionalism, that can evolve over time thanks to LS, and that can become a guide for their practices. Teachers need professional development models that are immersed in the reality of the classroom and foster collaboration, and LS could be part of the answer to their request. In fact, the data show that LS is not "so far removed" from our teaching practices, which could bring teachers to "reject" it. Moreover, it can be effectively supported by the Italian studies in Mathematics Education research. One of the many aspects that we could not investigate in this paper is the effect that such professional development practices can have on the students. The teachers observed encouraging results, but more data is needed on this aspect of the research. We will tackle the issue in a future study.

Acknowledgements We are grateful to Prof. Ferdinando Arzarello for his suggestions in carrying out the research. We also give our thanks to teachers Flora Del Regno, Laura Lombardi and Oriana Fiore for their collaboration in implementing the experience.

Funding Open access funding provided by Università degli Studi di Bari Aldo Moro within the CRUI-CARE Agreement.

Declarations

Conflict of interest No potential conflict of interest was reported by the author(s).

References

Adler, P. A., & Adler, P. (1994). Observational techniques. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 377–392). Sage Publications Inc.

Andriano, V., & Manolino, C. (2023). Teachers' awareness of classroom interactions in hybrid distance education through lesson study. In Huang, R., et al. (Eds.), *Teacher professional learning through lesson study in virtual/hybrid environments: opportunities, challenges, and future directions*. WALS-Routledge Lesson Study Series.

Arzarello, F., & Bartolini Bussi, M. G. (1998). Italian trends in research in mathematical education: A national case study from an international perspective. In Sierpinska. A. & Kilpatrick. J. (Eds.) *Mathematics education as a research domain: A search for identity* (pp. 243–262). Kluwer Academic Publishers.

Arzarello, F., Funghi, S., Manolino, C., Ramploud, A., & Bartolini Bussi, M. G. (2022). Networking Hybridizations within the Semiosphere: a research trajectory for the Cultural Transposition of the Chinese Lesson Study within a Western context. *International Journal for Lesson & Learning Studies, 11*(4), 331–343. https://doi.org/10.1108/IJLLS-06-2022-0083

Baker, L. (2006). Observation: A complex research method. *Library Trends, 55*(1), 171–189. https://doi.org/10.1353/lib.2006.0045

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, et al. (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge.

Bartolini Bussi, M. G., & Martignone, F. (2013). Cultural issues in the communication of research on mathematics education. *For the Learning of Mathematics, 33*(1), 2–8.

Bartolini Bussi, M. G., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transposition of Chinese lesson study to Italy. *International Journal for Lesson and Learning Studies, 6*(4), 380–395. https://doi.org/10.1108/ijlls-12-2016-0057

Barton, B. (2008). *The Language of Mathematics*. Springer.

Brophy, J. (2006). History of Research on Classroom Management. In C. M. Evertson & C. S. Weinstein (Eds.), *Handbook of classroom management: Research, practice, and contemporary issues* (pp. 17–43). Lawrence Erlbaum Associates Publishers.

Buchard, J., & Martin, D. (2017). Lesson Study. . . and its effects. *Έρευνα Στην Εκπαίδευση*. https://doi.org/10.12681/hjre.14809

Capone, R., Rogora, E., & Tortoriello, F. S. (2017). La matematica come collante culturale nell'insegnamento. *Matematica, Cultura e Società*, *2*(1), 293-304.

Capone, R., Manolino, C., & Minisola, R. (2019). Polyphony on Lesson Study in the Italian Context. *Proceedings of WALS2019.*

Capone, R., Manolino, C., & Minisola, R. (2020). Networking of Theories for a Multifaceted Understanding on Lesson Study in the Italian Context. In H. Borko, & D. Potari (Eds.), *ICMI Study 25 Conference: Teachers of Mathematics Working and Learning in Collaborative Groups* (pp. 102-109). Lisbon, Portugal.

Capone, R. (2022). Interdisciplinarity in Mathematics Education: From Semiotic to Educational Processes. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(2). https://doi.org/10.29333/ejmste/11508

Capone, R., Adesso, M. G., & Fiore, O. (2022a). Distance Lesson Study in Mathematics: A Case Study of an Italian High School. *Frontiers in Education, 7:788418*. https://doi.org/10.3389/feduc.2022.788418

Capone, R., Adesso, M. G., & Fiore, O. (2022b). Lesson Study in Physics Education to Improve Teachers' Professional Development. In J. B. Mark, P. Galea, S. Gatt and D. Sands eds. *Physics Teacher Education: What Matters?* (pp. 125-136). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-031-06193-6_9

Capone, R., & Faggiano, E. (2022). Teachers as key players: from professional development to the design and use of digital resources and backward. *Proceedings of the Third Workshop on Technology Enhanced Learning Environments for Blended Education. CEUR Workshop Proceedings*

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

Derrida, J. (2002). The university without condition. In J. Derrida (Ed.), *Without Alibi* (pp. 202–237). Standford University Press.

Di Paola, B., & Buttitta, G. (2022). Problems with variation in teaching/learning Geometry: an example of Chinese Cultural Transposition. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the twelfth congress of european research in mathematics education (CERME12)*. (pp. 1704–1711). ERME / Free University of Bozen-Bolzano.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Routledge.

Hollingsworth, S. (1995). Teachers as researchers. In L. W. Anderson (Ed.), *International encyclopedia of teaching and teacher education* (2nd ed., pp.16–19). Cambridge University Press.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM, 48*(4), 393–409. https://doi.org/10.1007/s11858-016-0795-7

Huang, R., Fang, Y., & Chen, X. (2017). Chinese lesson study: A deliberate practice, a research methodology, and an improvement science. *International Journal for Lesson and Learning Studies, 6*(4), 270–282. https://doi.org/10.1108/ijlls-08-2017-0037

Huang, R., Takahashi, A., & Ponte, J. P. (Eds.). (2019). *Theory and practice of lesson study in mathematics: An international perspective.* Newyork: Springer.

INVALSI (2017). *Rilevazioni nazionali degli apprendimenti 2016–17. Rapporto risultati*. Retrieved 10 September 2022, from https://invalsi-areaprove.cineca.it/docs/file/rapporto_prove_invalsi_2017.pdf

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese Lesson Study in Mathematics: Its Impact.* World Scientific Publishing.

Jaworski, B., & Huang, R. (2014). Teachers and didacticians: Key stakeholders in the processes of developing mathematics teaching. *ZDM, 46*(2), 173–188. https://doi.org/10.1007/s11858-014-0574-2

Joubert, J., Callaghan, R., & Engelbrecht, J. (2020). Lesson study in a blended approach to support isolated teachers in teaching with technology. *ZDM, 52*(5), 907–925. https://doi.org/10.1007/s11858-020-01161-x

Jullien, F. (2006). *Si parler va sans dire. Du logos et d'autres ressources*. Seuil.

Landi, L., & Pintus, A. (2022). A critical approach to the Reggio Emilia Approach. *Educazione. Giornale di pedagogia critica, 11*(1).

Lewis, C. (2000). *Lesson study: The core of Japanese professional development*. Paper presented at the American Educational Research Association Annual Meeting.

Manolino, C., Minisola, R., Robutti, O., & Arzarello, F. (2020). Translating practices for reflecting on ourselves: Lesson Study. In B. Di Paola, & P. Palhares (Eds.), *Proceedings of CIEAEM71, connections and understanding in mathematics education: Making sense of a complex world. "Quaderni di Ricerca in Didattica (Mathematics)", 7*, 519–526. G.R.I.M. (Dipartimento di Matematica e Informatica, University of Palermo, Italy).

Manolino, C. (2021a). The Semiosphere Lens to look at Lesson Study practices in their Cultural Context: a case study. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 214-221). Khon Kaen, Thailand: PME.

Manolino, C. (2021b). *An elaboration of the Lotman's Semiosphere theoretical construct for Mathematics Education: analysis of the Chinese Mathematics Lesson Study Cultural Transposition within the Italian context*. [Doctoral dissertation]. University of Torino.

Minisola & Manolino (2022). Teachers' professional development: a cultural matter. How to describe cultural contexts?. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the*

Twelfth congress of European research in mathematics education (CERME12) (pp. 3650–3657). ERME/Free University of Bozen-Bolzano.

Mellone, M., Ramploud, A., Di Paola, B., & Martignone, F. (2019). Cultural transposition: Italian didactic experiences inspired by Chinese and Russian perspectives on whole number arithmetic. *ZDM, 51*(1), 199–212. https://doi.org/10.1007/s11858-018-0992-7

Mellone, M., Ramploud, A., & Carotenuto, G. (2020). An experience of cultural transposition of the El'konin-Davydov curriculum. *Educational Studies in Mathematics, 106*(3), 379–396. https://doi.org/10.1007/s10649-020-09942-7

Mellone, M., Pacelli, T., & Liljedahl, P. (2021). Cultural transposition of a thinking classroom: to conceive possible unthoughts in mathematical problem solving activity. *ZDM – Mathematics Education*, *53*(4), 785–798. https://doi.org/10.1007/s11858-021-01256-z

Miyakawa, T., & Winsløw, C. (2019). Paradidactic infrastructure for sharing and documenting mathematics teacher knowledge: A case study of "practice research" in Japan. *Journal of Mathematics Teacher Education, 22*(3), 281–303. https://doi.org/10.1007/s10857-017-9394-y

Nguyen, D. T., & Tran, D. (2022). High school mathematics teachers' changes in beliefs and knowledge during lesson study. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-022-09547-2

Ní Shúilleabháin, A. (2018). Enacting Curriculum Reform Through Lesson Study in the Irish Post-primary Mathematics Classroom. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 65–85). ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-75696-7_4

Nigris, E., Cardarello, R., Losito, B., & Vannini, I. (2020). Teacher professional development research and improvement of schools. CRESPI's perspective. *RicercAzione, 12*(2), 225–237. https://doi.org/10.32076/RA12210

Okubo, K. (2007). Lesson study in Egypt. *Japanese Lesson Study in Mathematics, 216–217*,. https://doi.org/10.1142/9789812707475_0046

Ono, Y., & Ferreira, J. (2010). A case study of continuing teacher professional development through lesson study in South Africa. *South African Journal of Education*, *30*(1), 59–74. https://doi.org/10.15700/saje.v30n1a320

Otaki, K., Asami-Johansson, Y., & Hakamata, R. (2020). Theoretical preparations for studying lesson study: within the framework of the anthropological theory of the didactic. In H. Borko & D. Potari (Eds.), *ICMI STUDY 25 conference proceedings: Teachers of mathematics working and learning in collaborative groups* (pp. 150–157). National and Kapodistrian University of Athens.

OECD (2021). Teachers as knowledge professionals. In Ulferts, H. (Ed.), Teaching as a Knowledge Profession: Studying Pedagogical Knowledge across Education Systems. OECD Publishing, Paris, https://doi.org/10.1787/e3428ee7-en

Pang, J. (2016). Improving mathematics instruction and supporting teacher learning in Korea through lesson study using five practices. *ZDM, 48*(4), 471–483. https://doi.org/10.1007/s11858-016-0768-x

Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181. https://doi.org/10.1108/IJLLS-08-2016-0021

Ponte, J. P., Quaresma, M., Mata-Pereira, J., & Baptista, M. (2018). Fitting Lesson Study to the Portuguese Context. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 87–103). ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-75696-7_5

Presutti, S. (2022). Relevance and adaptations of a lesson study process within the context of initial teacher education in the canton of Vaud, Switzerland. *Twelfth congress of the european society for research in mathematics education* (CERME12), Feb 2022, Bozen-Bolzano, Italy. ffhal-03744285

Quaresma, M., Winsløw, C., Clivaz, S., Ponte, J. P., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018). *Mathematics lesson study around the world*. ICME-13 Monographs. Springer.

Ramploud, A., Funghi, S., & Mellone, M. (2021). The time is out of joint. Teacher subjectivity during COVID-19. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-021-09506-3

Ramploud, A., Funghi, S., & Bartolini, M. G. (2022). Chinese lesson study: Critical aspects of transfer from China to Italy. *International Journal for Lesson and Learning Studies, 11*(2), 147–160. https://doi.org/10.1108/ijlls-04-2021-0031

Ramsay-Jordan, N. (2022). What collaborating teachers got to do with it? Understanding the importance of pre- and in-service teacher collaborations for enacting culturally responsive mathematics teaching through critical examinations of collaborating teachers' experiences. *International Journal of Educational Reform, 31*(4), 459–475. https://doi.org/10.1177/10567879221110513

Ribeiro, M., Mellone, M., & Esposito, S. (2019). Hybrid lesson study and cultural transposition: Intertwined requirements in and for improving teachers' knowledge and practices. In M. Graven, H. Venkat, A.

Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4). Pretoria, South Africa: PME.

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration: June 2016. *ZDM Mathematics Education, 48*(5), 651–690. https://doi.org/10.1007/s11858-016-0797-5

Seleznyov, S. (2018). Lesson study: An exploration of its translation beyond Japan. *International Journal for Lesson and Learning Studies, 7*(3), 217–229. https://doi.org/10.1108/ijlls-04-2018-0020

Shinno, Y., & Yanagimoto, T. (2020). An opportunity for Preservice Teachers to learn from Inservice Teachers' Lesson Study: Using Meta-Didactic Transposition. In H. Borko & D. Potari (Eds.), *ICMI STUDY 25 Conference Proceedings: Teachers of Mathematics Working and Learning in Collaborative Groups* (pp. 174–181). National and Kapodistrian University of Athens.

Skott, C. K. (2020). A social practice theoretical perspective on teacher collaboration in lesson study. In H. Borko & D. Potari (Eds.), *ICMI STUDY 25 Conference Proceedings: Teachers of Mathematics Working and Learning in Collaborative Groups* (pp. 182–189). National and Kapodistrian University of Athens.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eighth-Grade Mathematics Instruction in Germany, Japan, and the United States*. A Research and Development Report. U.S. Department of Education. National Center for Education Statistics.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. The Free Press.

Stigler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM, 48*(4), 581–587. https://doi.org/10.1007/s11858-016-0787-7

Vermunt, J. D., Vrikki, M., van Halem, N., Warwick, P., & Mercer, N. (2019). The impact of Lesson Study professional development on the quality of teacher learning. *Teaching and Teacher Education, 81*, 61–73. https://doi.org/10.1016/j.tate.2019.02.009

Vygotsky, L. S. (1978). Internalization of higher psychological process. In: M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society. The development of higher psychological process* (pp. 52–57). Harvard University Press.

Weber, K. E., Gold, B., Prilop, C. N., & Kleinknecht, M. (2018). Promoting pre-service teachers' professional vision of classroom management during practical school training: Effects of a structured online- and video-based self-reflection and feedback intervention. *Teaching and Teacher Education, 76*, 39–49. https://doi.org/10.1016/j.tate.2018.08.008

Winsløw, C., Bahn, J., Rasmussen, K. (2018). Theorizing lesson study: Two related frameworks and two Danish case studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 123–142). ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-75696-7_7

Xu, H., & Pedder, D. (2015). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 29–58). Routledge.

Yoshida, M. (1999). *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development*. [Doctoral dissertation]. University of Chicago.